Discrete Mathematics

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Contents

1 Basic formulas and operators

1.1 Factorials

Definition. Factorial. Factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n.

Definition. Falling Factorial. Falling factorial (sometimes called the descending factorial) is defined as the polynomial:

$$
\begin{aligned}
(x)_n = x^{\underline{n}} &= \overbrace{x(x-1)(x-2)\cdots(x-n+1)}^{n \text{ factors}} \\
&= \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k).
\end{aligned}
$$

Definition. Rising Factorial. Rising factorial (sometimes called the descending factorial) is defined as the polynomial:

$$\begin{aligned} x^{(n)} = x^{\overline{n}} &= \overbrace{x(x+1)(x+2)\cdots(x+n-1)}^{n \text{ factors}} \\ &= \prod_{k=1}^{n}(x+k-1) = \prod_{k=0}^{n-1}(x+k). \end{aligned}$$

1.2 Binomial Coefficient

Definition. Binomial Coefficient. Let $n, k \in \mathbb{N}$ and $n \geqslant k$. The binomial coefficient is the number of k-element subsets of an n-element set, and it is defined as:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$$

Furthermore let $x \in \mathbb{R}$, and again $k \in \mathbb{N}$. Then we define the binomial coefficient as:

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!}$$

1.3 Binomial Coefficient Identities

Identities. Binomial Coefficient. The binomial coefficient carries within itself a lot of identities, most of which can be easily observed in the Pascal's Triangle:

1. First identity.
$$\binom{n}{k} = \binom{n}{n-k}$$

2. Recursion for binomial coefficients.
$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$
alternatively re-indexed as $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k-1}$.

3. Another recursion.
$$\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$$

4. Another identity.
$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-k}{k-j}$$

5. Bookkeeper sum.
$$\sum_{k=2}^{n}\binom{k}{2} = \binom{n}{3}$$

6. Sum of coefficients.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k} k^p = n^p 2^{n-1}$$

7. Pascal diagonal sums.

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

8. Alternating Sums.

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k}$$

9. Strong sum.

$$\sum_{k=0}^{n} \binom{n}{k} k x^{k-1} = n(1+x)^{n-1}$$

1.4 Binomial Theorem

Theorem. Binomial Theorem. The expansion of any non-negative integer power $n \in \mathbb{Z}^+$ of the binomial $(x+y) : x, y \in \mathbb{R}$ is a sum of the form:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Notable example for when $y = 1$:

$$\begin{aligned}(1+x)^n &= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k.\end{aligned}$$

1.5 Vandermonde Convolution Identity

Theorem. Vandermonde's Convolution Identity. Let $m, n, k \in \mathbb{N}$. The identity states:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k} \binom{n}{r-k}$$

1.6 Binomial Coefficient Combinatorics

Information. Choosing k elements from n. Let $n, k \in \mathbb{N}, k \leqslant n$. Combinatorial formulas for choosing k elements from n:

Selection Method	Order	No Order
No Repetition	$n^{\underline{k}}$	$\binom{n}{k}$
Repetition	n^k	$\binom{n+k-1}{k}$

2 Combinatorical Principles

2.1 Inclusion–exclusion principle

Definition. Inclusion–exclusion principle. Inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets. Symbolically expressed as:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

For $n = 2, 3$. Or further in general $n \in \mathbb{N}$ by the formula:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} A_i - \sum_{1 \leqslant i \leqslant j \leqslant n}^{n} |A_i \cap A_j| + \sum_{1 \leqslant i \leqslant j \leqslant k \leqslant n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

2.2 Pigeonhole principle

Definition. Pigeonhole principle. Let S be a finite set. Let $s_1, s_2, \ldots, s_k$ be the subsets, which satisfy $(\forall i \neq j)\, i, j \in [k] s_i \cap s_j = \emptyset$ and $s_1 \dot\cup s_2 \dot\cup s_3 \dot\cup \ldots \dot\cup s_k = S$. Then:

$$(\exists i \in [k])\, |s_i| \geqslant \frac{|S|}{k}$$

3 Asymptotic Notation

1. $H_n \approx \ln(n)$
2. $\sum_{k=1}^{n} k^s \approx \frac{k^{s+1}}{s+1} \in O(k^{s+1})$

3.1 Big O

Definition. Big O Asymptotic Notation. Let $g : \mathbb{N} \to \mathbb{R}^+$ We define:

$$O\left(g(n)\right) = \{f : \mathbb{N} \to \mathbb{R}^+ : \left(\exists c \in \mathbb{R}^+\right) (\exists n_0 \in \mathbb{N})\, (\forall n > n_0)\, f(n) \leqslant c \cdot g(n)\}$$

For when $g : \mathbb{N} \to \mathbb{R}$ one can write $|f(n)| \leqslant |c \cdot g(n)|$.
Even though $O(g(n))$ is clearly a set we often write $f = O(g(n))$, instead of $f \in O(g(n))$.

Fact. Big O Limit. Let $f, g \in \mathbb{N} \to \mathbb{R}^+$. As a fact:

$$f(n) = O(g) \iff \limsup_{n \to \infty} \frac{f(n)}{g(n)} < \infty$$

3.2 Big Theta

Definition. Big Theta Asymptotic Notation. Let $g : \mathbb{N} \to \mathbb{R}^+$ We define:

$$\Theta\left(g(n)\right) = \{f : \mathbb{N} \to \mathbb{R}^+ : \left(\exists c_1, c_2 \in \mathbb{R}^+\right)\left(\exists n_0 \in \mathbb{N}\right)\left(\forall n > n_0\right) c_1 \cdot g(n) \leqslant f(n) \leqslant c_2 \cdot g(n)\}$$

Furthermore:

$$f(n) = \Theta(g(n)) \iff \begin{cases} f(n) = O(g(n)) \\ g(n) = O(f(n)) \end{cases}$$

Fact. Big Theta Limit. Let $f, g \in \mathbb{N} \to \mathbb{R}^+$. As a fact:

$$f(n) = \Theta(g) \iff \left(\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty\right) \wedge \left(\limsup_{n\to\infty} \frac{f(n)}{g(n)} > 0\right)$$

3.3 Approximate Notation

Definition. $\approx$ Notation. Let $f, g \in \mathbb{N} \to \mathbb{R}^+$. We define:

$$f(n) \approx g(n) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = c \in \mathbb{R}^+$$

4 Integral Sum Approximation

Theorem. Sum Approximation. Let $a, b \in \mathbb{N}$, $f : [a, b] \to \mathbb{R}$ **non-decreasing**, differentiable. Then:

$$f(a) + \int_a^b f(x)dx \leqslant \sum_{k=a}^{b} f(k) \leqslant \int_a^b f(x)dx + f(b)$$

Analogically. Let $a, b \in \mathbb{N}$, $f : [a, b] \to \mathbb{R}$ **non-increasing**, differentiable. Then:

$$f(a) + \int_a^b f(x)dx \geqslant \sum_{k=a}^{b} f(k) \geqslant \int_a^b f(x)dx + f(b)$$

4.1 Stirling formula

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

5 Stirling numbers of the second kind

We define $\left\{ {n \atop k} \right\}$ as the number of ways to partition a set of n objects into k non-empty subsets.

5.1 Basic values

1. $\left\{ {0 \atop 0} \right\} = 1, \left\{ {n \atop 0} \right\} = 0$
2. $\left\{ {n \atop n} \right\} = \left\{ {n \atop 1} \right\} = 1$
3. $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$
4. $\left\{ {n \atop 2} \right\} = \frac{2^n-2}{2} = 2^{n-1} - 1$

5.2 Properties

1. Explicit formula
$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j}(k-j)^n(-1)^j$$
2. Pascal identity:
$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$$
3. Expansion
$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$$
4. Boundary for triangle row inequality at k_n $\frac{n}{\ln(n)}$
$$\left\{ {n \atop 1} \right\} \leqslant \cdots \leqslant \left\{ {n \atop k_n} \right\} \geqslant ... \geqslant \left\{ {n \atop n} \right\}$$

5.3 Bell Numbers

Bell number B_n is the number of all partitions of an n-element set:

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

Bell numbers satisfiy the following recurrence relation:

$$\begin{cases} B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \\ B_0 = 1 \end{cases}$$

5.4 Stirling Number Combinatorics

Information. Choosing k elements from n. Let $n, k \in \mathbb{N}, k \leqslant n$. Combinatorial formulas for choosing k non empty subsets from a set of size n:

1. TOP - Elements
2. SIDE - Subsets

Selection Method	Distinguishable	Non-distinguishable
Distinguishable	$\left\{ {n \atop k} \right\} \cdot k!(\text{surj.})$	$\binom{n-1}{k-1}$
Non-distinguishable	$\left\{ {n \atop k} \right\}$	$\binom{n+k-1}{k}$

6 Permutations

6.1 Permutation

A **permutation** of a set A is a bijection from the set A to itself. A permutation σ can be written as:

$$\sigma : A \to A$$

where σ reorders the elements of A.
If $|A| = n$, without loss of generality we can assume: $A = \{1, 2, \ldots, n\}$.

6.2 Set of permutations

$S_n = \{f : [n] \xrightarrow[\text{bijection}]{} [n]\}$ and $|S_n| = n!$

6.3 Cycle

A **cycle** in a permutation σ is a subset of elements in S that are permuted among themselves, with each element mapping to the next element in the subset, and the last element mapping back to the first. A cycle of length k is written as:

$$\sigma = (a_1\, a_2\, \ldots\, a_k)$$

indicating that $\sigma(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $\sigma(a_k) = a_1$.

6.4 Two-Line Notation for Permutations

In **two-line notation**, a permutation σ is written as:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

where the top row lists the elements of the set S, and the bottom row lists their images under σ.
For example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

6.5 One-Line Notation for Permutations

In **one-line notation**, a permutation σ is written as a partition into disjoint cycles:

$$\sigma = (1\,2\,3)(4\,5)$$

6.6 Fixed point

Let σ be a permutation of a set S. A *fixed point* of σ is an element $x \in S$ such that $\sigma(x) = x$. For example *Id.* (identity) has n fixed points.

6.7 Derangement

A **derangement** is a permutation of a set where no element appears in its original position. More formally, for a set of n elements, a derangement is a permutation σ such that $\sigma(i) \neq i$ for all i in the set.

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

6.8 Transposition

A **transposition** is a cycle of length 2, i.e., it swaps two elements and leaves the others unchanged. It is written as:

$$\sigma = (a\,b)$$

indicating that $\sigma(a) = b$ and $\sigma(b) = a$, with $\sigma(x) = x$ for all $x \neq a, b$.

6.9 Inversion

Let $\sigma \in S_n$. An *inversion* is a pair $(\sigma(i), \sigma(j))$, which satisfies:

$$i < j \text{ and } \sigma(i) > \sigma(j)$$

One may think these two are "not in order".

6.10 Sign of a permutation (sgn)

The **sign** (or **parity**) of a permutation σ, denoted $\operatorname{sgn}(\sigma)$, is defined as number of inversions in a permutation. It satisfies the following property:

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$$

Where $N(\sigma)$ is number of transpositions in the decomposition of σ.
A permutation is called even if $\operatorname{sgn}(\sigma) = +1$ and odd if $\operatorname{sgn}(\sigma) = -1$.
For example:
Consider the permutation $\sigma = (1\,3\,2)$. This can be decomposed into transpositions as:

$$\sigma = (1\,3)(3\,2)$$

Since there are 2 transpositions, $\operatorname{sgn}(\sigma) = (-1)^2 = 1$. Therefore, σ is an even permutation.

6.11 Order of a permutation (ord)

The **order** of a permutation σ, denoted $\mathrm{ord}(\sigma)$, is the smallest positive integer k such that σ^k is the identity permutation. Formally,

$$\mathrm{ord}(\sigma) = \min\{k \in \mathbb{N} \mid \sigma^k = \mathrm{id}\}$$

For σ built of disjoint cycles of length $c_1, c_2, \ldots, c_k$, its order satisfies:

$$\mathrm{ord}(\sigma) = \mathrm{lcm}(c_1, c_2, \ldots, c_k)$$

For example:
Consider the permutation $\sigma = (1\,2\,3)$. Applying σ three times returns to the identity permutation:

$$\sigma = (1\,2\,3) \quad \sigma^2 = (1\,3\,2) \quad \sigma^3 = \mathrm{id}$$

Thus, $\mathrm{ord}(\sigma) = 3$.

7 Stirling numbers of the first kind

The Stirling numbers $\left[{n \atop k}\right]$ is the number of permutations in S_n, which have exactly k-disjoint cycles.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

with the initial conditions:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \quad \text{for} \quad n > 0.$$

and some interesting features:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \quad \text{and} \quad \begin{bmatrix} n \\ n \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$$

the following is also true:

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! \cdot H_{n-1}$$

7.1 Properties

1. Factorial correlation

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!.$$

2. Stirling relation

$$\begin{bmatrix} n \\ k \end{bmatrix} \geqslant \begin{Bmatrix} n \\ k \end{Bmatrix}$$

3. Relation for $x^{\underline{n}}$:

$$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{k+n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

4. Harmonic relation

$$n!H_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k$$

5. Weird Pascal recurrence

$$\begin{bmatrix} n+m+1 \\ n \end{bmatrix} = \sum_{k=0}^{m} (n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}$$

6. Another sum

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_{k=m}^{n} \binom{n}{k} \begin{bmatrix} k \\ m \end{bmatrix} (n-k)!$$

8 Fibonacci Numbers

8.1 Definition

The Fibonacci sequence (F_n) is defined as follows:

$$F_0 = 0, \quad F_1 = 1 \tag{1}$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geqslant 2 \tag{2}$$

8.2 Closed Form (Binet's Formula)

The n-th Fibonacci number can be expressed in closed form using Binet's formula:

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{3}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $\psi = \frac{1-\sqrt{5}}{2}$.

8.3 Matrix Representation

Fibonacci numbers can also be represented using matrices:

$$Q = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

9 Catalan Numbers

The n-th Catalan number C_n is the number of ways to triangulate a convex polygon with $n+2$ sides. C_n can be defined using the binomial coefficients:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \tag{4}$$

It can also be defined recursively as:

$$C_0 = 1 \tag{5}$$

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad \text{for} \quad n \geqslant 0 \tag{6}$$

9.1 Asymptotic growth

$$c_n = \frac{1}{n+1} \cdot \binom{2n}{n} \approx \frac{1}{n} \frac{4^n}{\pi n} \text{ (Stirling approx.)}$$

9.2 Alternate definitions

1. The number of ways to correctly parenthesize a product of $n+1$ factors is the n-th Catalan number.
2. The number of distinct binary trees with $n+1$ leaves (or n internal nodes) is the n-th Catalan number.
3. The number of mountain up-right, down-right paths of length $2n$ (paths from $(0,0)$ to $(2n,0)$ that do not dip below the x-axis) is given by the n-th Catalan number.

10 Generating Functions

A generating function for a sequence $\{a_n\}_{n=0}^{\infty}$ is a formal power series of the form:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

The coefficients a_n represent terms of the sequence.

10.1 Geometric series

The geometric series for $a_n = a_0 \cdot q^n$ is defined as:

$$A(x) = a_0 \cdot \sum_{n=1}^{\infty} (qx)^n = \frac{a_0}{1 - qx}$$

10.2 Exponential Generating Functions

The Taylor series for e^x is defined as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

10.3 Generating function for the Fibonacci sequence

Let $\{F_n\}$ denote the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. The generating function for the Fibonacci sequence is:

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

10.4 Generating function for binomial coefficient

The generating function for the binomial coefficient $\binom{n}{k}$ is:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

10.5 Generating function for n

Use derivation to find the generating function for the coefficient n is.

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{1 - x^2}$$

$$\sum_{n \geqslant 0} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$$

10.6 Generating function 1/(x+1)

The generating function $\frac{1}{1+x}$ is the sum:

$$\frac{1}{1+x} = \sum_{k=0}^{n} (-1)^k x^k$$

10.7 Identities

1. $A(x) + B(x)$ is the generating function for $c_n = a_n + b_n$
2. $cA(x)$ is the generating function for $c_n = c \cdot a_n$
3. $A(x)B(x)$ is the generating function for $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ (convolution)
4. $A'(x)$ is the generating function for $c_n = (n+1)a_{n+1}$
5. $\frac{A(x) - a_0}{x}$ is the generating function for $c_n = a_{n+1}$

11 Counting functions

11.1 Number of functions

1. Number of functions $|f : [k] \to [n]| = n^k$
2. Number of 1-1 functions $|f_{1-1} : [k] \to [n]| = n^{\underline{k}}$
3. Number of surjective functions $|f_{surj.} : [k] \to [n]| = \sum_{i=0}^{n} \binom{n}{i}(n-i)(-1)^i = k! \cdot \left\{ {n \atop k} \right\}$
4. Number of growing functions $|f_{grow.} : [k] \to [n]| = \binom{n}{k}$

11.2 Solutions to $\mathbf{x}_1 + x_2 + ... + x_k = n$

For when $x_i \geqslant 1$. We can write $x' = x_i - 1$, then $x' \in \{0, 1, 2, ..\}$. But now:

$$x'_1 + x'_2 + \cdots + x'_k = n - k$$

There are:

$$\binom{k + (n-k-1)}{k-1} = \binom{n-1}{k-1}$$

unique solutions to this equation.

11.3 Expansion coefficient

Coefficient for $a^{k_1} b^{k_2} \ldots$ in the expansion of $(a + b + c + \ldots)^n$

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \ldots k_m!}$$

12 Helpful integrals

1. $ln(x)$

$$\int \ln(x) dx = x \ln(x) - x + C$$