CS 261 Review Questions

Questions marked with an asterisk are more difficult or time consuming. These problems are: the fallout shelter, the up-down series, the fourteenth card, hockey playoff, Fibonacci thirds, deriving relations, and one of the finite automata.

1. Let p, q, and r be the following statements:
   p = "Grizzly bears have been seen in the area."
   q = "Hiking is safe on the trail."
   r = "Berries are ripe along the trail."
   Write the following statement using logic symbols:
   "Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail."

2. True or false: If a propositional statement is not a contradiction, then it is a tautology. Justify your answer.

3. Write the contrapositive of this statement: "If x is noble and fast, then x is a horse."

4. Consider the following statement: "If you are making a reservation, then you must pay a deposit."
   a. Rewrite the statement so that it contains the phrase "necessary condition"

   b. Write the converse, inverse and contrapositive of the statement.

   c. Rewrite the original statement without using "if…then". (i.e. convert to "and" or "or" statement, as appropriate.)

5. Consider this statement: "If Martha has a loaf of bread, then she also has peanut butter and jelly."
   a. Re-write the statement so that it uses the phrase "sufficient condition."

   b. Write the negation of the original statement.

6. Negate these statements:
   a. This is important tax information and is being furnished to the Internal Revenue Service.

   b. If you are required to file a return, a negligence penalty or other sanction may be imposed on you if this income is taxable and the IRS determines it has not been reported.

7. Consider this statement:
   P = "If Bob rolls the dice, then he can either move his token forward or accept a challenge."

a. Write the inverse of P.

b. Write the contrapositive of P.

c. Write the converse of P.

d. Write the negation of P.

8. Determine if each of the following statement forms is a tautology.
   a. $(((p \text{ or } q) \rightarrow r) \text{ and } (\sim p)) \rightarrow (q \rightarrow r)$

   b. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

   c. $(q \wedge r) \vee ((p \rightarrow q) \wedge \sim r) \vee \sim q$

   d. $(p \rightarrow \sim q) \vee (q \rightarrow \sim r) \vee (r \rightarrow p)$

9. Give an example of an argument that uses *modus ponens*. Is your argument valid?

10. Determine if the following arguments are valid or invalid. Explain your reasoning.
    a. $(p \rightarrow q) \text{ and } (r \rightarrow s)$
    $\sim q \text{ or } \sim s$

    $\sim p \text{ or } \sim r$

    b. If the program compiles, then I can ski.
    If I cannot find my goggles, then I cannot ski.
    Therefore, if I cannot find my goggles, then the program does not compile.

    c. There are three big stores in the mall: Eaton's, Holt Renfrew, and The Bay. Eaton's is such a rotten place. And I can't afford Holt Renfrew. So, we'll just go to The Bay.

    d. If you send me an e-mail message, then I will finish writing the program.
    If you do not send me an e-mail message, then I will go to sleep early.
    If I go to sleep early, then I will wake up feeling refreshed.
    Therefore, if I do not finish writing the program, then I will wake up feeling refreshed.

    e. If the robot can bake a cake, then I can leave the ship.
    If aliens are approaching, then I cannot leave the ship.
    Therefore, if aliens are not approaching, then the robot can bake a cake or I can leave the ship.

f. * If fallout shelters are built, other countries will feel endangered and our people will get a false sense of security. If other countries will feel endangered, they may start a preventive war. If our people will get a false sense of security, they will put less effort into preserving peace. If fallout shelters are not built, we run the risk of tremendous losses in the event of war. Hence, either other countries may start a preventive war and our people will put less effort into preserving peace, or we run the risk of tremendous losses in the event of war.

11. Consider the following statement.
"The sum of any two odd integers is even."

    Write this statement using logic symbols, using variables and quantifiers in the following format:
    "For all __________, if ___________ then ___________."

12. How would we write the following statements and into logic symbols? You should draw your variables from the set of all animals.
    a. Some dogs chase all rabbits.
    b. Only dogs chase rabbits.
    c. All dogs chase rabbits and geese.
    d. Only dogs chase rabbits and geese.
    e. Some dogs chase only squirrels and rabbits.
    f. All dogs chase some rabbits.
    g. Only cats and dogs play inside and sleep outside.
    h. Some dogs and alligators chase only geese.

13. Write the negation of each of the statements in the previous question.

14. Suppose S is a set containing some real numbers. Rewrite the following statement using logic symbols: "There is some number in S that is greater than all of the other numbers in S."

15. Give an example of a suitable predicate for each of the following situations:
    a. "$\exists x, P(x)$" is true and "$\exists x, \sim P(x)$" is true.
    b. "$\exists x, P(x)$" is true and "$\exists x, \sim P(x)$" is false.

c. "∃x, P(x)" is false and "∃x, ~P(x)" is true.

d. "∃x, P(x)" is false and "∃x, ~P(x)" is false.

16. Negate these statements:
   a. "For all integers n, n is prime or n is even."

   b. "For all integers a and b, if a does not divide b, then a > b or b does not divide a."

17. Consider the following statement: For all real numbers x, there exists a real number y such that xy = 4.
   a. Write the negation of the statement.

   b. Is the original statement true or false?

18. Let S = { −3, −1, 3, 5, 7, 9 }. Consider the following statement.
   For all x in S, if x is even, then x > 10.
   Is this statement true or false? Justify your answer.

19. Suppose F equals the XOR (exclusive or) of the Boolean variables x and y. Write a formula for F that uses inclusive OR, AND and/or NOT only, instead of XOR.

20. Let's practice DeMorgan's law.
   a. Re-write the Boolean function $F(x,y,z) = (x + y' + z)'$ in terms of AND and NOT operations only. Then, re-write it again with only NAND operations (the symbol for NAND is the vertical bar |).

   b. Re-write the Boolean function $F(x, y) = x + y'$ in terms of AND and NOT operators.

   c. Re-write the Boolean function $F(w, x, y, z) = w + x(y' + z)$ in terms of AND and NOT operators.

   d. Re-write the Boolean function $F(x, y, z) = x(yz)'$ using only OR and NOT operators.

   e. Re-write the Boolean function $F(x, y, z) = x(y + y'z)$ using only OR and NOT operators.

21. Suppose we are designing digital circuits that have three binary inputs. How many distinct Boolean functions exist? Justify your answer.

22. For the following problems, design a digital circuit and simplify it using a Karnaugh map.
    a. We have 3 inputs and 1 output. The circuit should output 1 if at most one of the inputs equals 1, and it should output 0 otherwise.

    b. The bottom light in a single-digit display. Note that input values 10-15 are don't cares.

    c. We have 4 inputs and 1 output. The output is true if and only if the binary value of the input, when interpreted as a 4-bit binary number, is less than our equal to 4.

23. Suppose a Boolean function F(w, x, y, z) consists of minterms 4 and 14.
    a. Write out the Boolean expression for F.

    b. Which additional minterms would be necessary so that minterms 4 and 14 can be combined into the same term?

24. Suppose F is a function of five boolean variables a, b, c, d, e. The minterms are numbered from 0 to 31, where minterm 0 represents a'b'c'd'e' and minterm 31 represents abcde. Then, the four minterms 9, 11, 25 and 27 can combine to form what single term?

25. Simplify these Boolean functions using a Karnaugh map:
    a. $F(x,y,z) = \Sigma\ (0, 4, 5)$

    b. $F(w,x,y,z) = \Sigma\ (0, 4, 5, 6, 8, 9)$

    c. $F(w,x,y,z) = \Sigma\ (2, 3, 6, 10, 11, 15)$

    d. Repeat part (c) but assume we also have don't cares = $\Sigma\ (5, 7)$.

    e. $F(w,x,y,z) = \Sigma\ (1, 3, 7, 11, 15)$ with don't cares = $\Sigma\ (0, 2, 5)$.

    f. $F(w,x,y,z) = \Sigma\ (3, 4, 14, 15)$ with don't cares = $\Sigma\ (1, 6, 7, 10, 12)$.

    g. $F(w,x,y,z) = \Sigma\ (0, 2, 5, 8, 13)$, with don't cares = $\Sigma\ (7, 10, 14, 15)$.

26. Use the Quine-McCluskey technique to simplify $\Sigma\ (2, 3, 18, 19)$.

27. Suppose x is a real number in the range $7 < x \leq 10$. Write a ceiling or floor expression that rounds all these values of x, and only these values of x, to the integer 5.

28. The floor of $(x+7)/4$ equals 6 for which real numbers x?

29. State the definition of "divides." In other words, the definition of the notation $a \mid b$.

30. Write direct proofs for the following:
    a. For all integers a, b, c and d, if $a \mid b$ and $c \mid d$ then $ac \mid (bc + ad)$.

    b. For all integers a, x and y, if $a \mid (x + y)$ and $a \mid (x - y)$, then $a \mid (5x + y)$.

    c. If a, b, c and d are nonzero integers, and $a \mid c$ and $b \mid d$, then $ab \mid cd$.

    d. The product of any two odd integers is odd.

    e. If a, b and c are odd integers, then $(2a - 4b + 5c)$ is also an odd integer.

    f. For all integers n, $(3n^2 + 5n) / 2$ is an integer.

    g. Let x and y be rational numbers where $x < y$. Also, let $m = (x + y) / 2$. Therefore, $x < m$.

31. The following statements are false. Show they are false by giving a counterexample.
    a. For boolean variables A, B and C, if $A \rightarrow B$ is true, $B \rightarrow C$ is true, and A is false, then C is false.

    b. The sum of two even numbers is an integer multiple of 4.

    c. The sum of two irrational numbers is irrational.

    d. If $n^2 \bmod 10 = 9$ then $n \bmod 10 = 3$.

32. Use proof by contradiction or proof by contraposition to show the following.
    a. The empty set is unique.

    b. If ab is even, then a is even or b is even.

    c. If n is an integer and $3n + 2$ is even, then n is even.

d. If n is an integer and n^3 is odd, then n is odd.

e. If x is irrational, and r is a rational number other than 0, then rx is irrational.

33. Prove or disprove:
   a. There exists a real number x, such that for all real numbers y, x = y + 1.

   b. $\forall x \in Z, \exists y \in Z: \; y > x^2$

   c. $\exists x \in Z, \forall y \in Z: \; x < y^2$

34. Write either a direct or indirect proof of this statement:
   For all integers x and y, if $x^4 + 4y^4$ is odd, then $x^2 + 2xy + 2y^2$ is also odd.

35. Use induction to prove the following statements.
   a. For all positive integers n,
      $(1)(2) + (2)(4) + (3)(6) + (4)(8) + \ldots + (n)(2n) = n(n+1)(2n+1)/3$.

   b. For all positive integers n, $4 + 8 + 12 + 16 + \ldots + 4n = 2n(n+1)$.

   c. For all positive integers n, $\sum_{i=1}^{n}(i^2 + 2i - 1)2^i = n^2 2^{n+1}$

   d. For all integers $n \geq 7$, $n^2 > 5n + 10$.

   e. For all positive integers n, $8 \mid (3^{2n} + 7)$.

   f. For all integers $n \geq 0$, $10^n + 3(4^{n+2}) + 5$ is divisible by 9.

   g. For all integers $n \geq 4$, $n^2 > n + 9$.

   h. For all integers $n \geq 6$, $n^2 > 4n + 9$.

   i. For all integers $n \geq 4$, $n^3 \geq 7n + 12$.

j. $n! < n^n$ for all integers $n \geq 2$.

k. Every positive integer is either even or odd.

l. $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all positive integers n.

m. For all positive integers n, 21 divides $4^{n+1} + 5^{2n-1}$.

n. Any postage amount of 35 cents or more can be accomplished using 5- and 9-cent stamps.

o. Repeat the previous part, but using the numbers 44, 5, and 12, respectively.

36. Consider the following code. How many times does the letter 'a' get printed?

```
for (i = 1; i <= 10; ++i)
   for (j = 1; j <= i*i; ++j)
      System.out.print("a");
```

37. Use Bernoulli formulas to determine the following.
    a. If the nth term of a sequence is $2n^2 + 5n$, then what is the sum of the first n terms?

    b. If the sum of the first n terms of a series is $2n^2 + 5n$, then write a formula for the nth term.

38. Use Bernoulli formulas to simplify $50^2 + 51^2 + 52^2 + \ldots + 100^2$.

39. Suppose that the following sum of consecutive integers from a through b, inclusive:

    $a + (a + 1) + (a + 2) + \ldots + b$

    equals $\frac{83(84)}{2} - \frac{56(57)}{2}$.

    What are the values of a and b?

40. * Consider the series $(1)(n) + (2)(n - 1) + (3)(n - 2) + \ldots + (n - 1)(2) + (n)(1)$. Notice that there are n terms, and each term consists of two factors. Each factor can be expressed as a linear function of i, where i is the term number, and i ranges from 1 to n inclusive. Thus, each term of the series has the form f(i)g(i) and we could write the series as follows:
    $f(1)g(1) + f(2)g(2) + f(3)g(3) + \ldots + f(n - 1)g(n - 1) + f(n)g(n)$.
    a. First, write formulas for the linear functions f and g.

    b. Use Bernoulli formulas to derive the sum of the above series as a polynomial in n.

c. Use induction to show that the formula you get in part (b) is correct.

d. How can you generalize the formula you came up with in part (b) to handle more general types of series where the functions f(i) and g(i) are any linear function?

41. Consider the following code. Assume that n is a positive number.

```
sum = 0;
for (i = 1; i <= n; ++i)
   sum += (i + 4) * (i - 4);
```

a. Use Bernoulli formulas to determine the value of sum after the loop is finished.

b. A loop invariant property of this loop is that after k iterations, the value of sum will be whatever you get if you substitute k in for n in the summation formula you just found in part a. Show that this loop invariant property is satisfied.

c. How many operations are performed when the code executes?

42. True or false… For any sets A, B and C: $A - (B - C) = (A - B) - C$. And explain how you arrived at your answer. (Hint: the most elegant way is to use a Karnaugh map. Or, you could re-write this statement about sets as a statement from propositional logic, and then see if the two expressions are logically equivalent.)

43. Given that $(A - C) \cup (B - D) \subseteq (A \cup B) - (C \cup D)$ for sets A, B, C and D, then what set intersection(s) must be empty?

44. In a survey of 100 people, 35 said that they liked Bach, 15 said they liked both Bach and Mozart, while 20 people said they liked neither Bach nor Mozart. How many people liked Mozart?

45. A friend has given us a small cookbook. After perusing the ingredients of each recipe, we determine the following quantities:
   - n = total number of recipes
   - a = number of recipes that contain meat
   - b = number of recipes that contain onion
   - c = number of recipes that contain barbeque sauce
   - d = number of recipes that contain meat and onion
   - e = number of recipes that contain onion and barbeque sauce
   - f = number of recipes that contain meat and barbeque sauce
   - g = number of recipes that contain meat, onion, and barbeque sauce

   In terms of the variables, how many recipes do not contain any meat, onion, or barbeque sauce? (In other words, devoid of all three ingredients)

46. Suppose X = { 1, 2, 3, 4 } and Y = { 3, 4, 5 }. How many elements are in the set $(X \times Y) \cap (Y \times X)$?

47. Suppose x and y have the following 8-bit representations.

    x = 1010 1101
    y = 1111 0000

    What are the results of these bitwise operations? Assume they run independently of one another.

    a. x & y

    b. x | y

    c. x ^ y

48. Suppose x is a bit vector (i.e. an integer with a binary representation). Explain what bitwise operation we should perform in order to invert the rightmost 3 bits of x, and leave the remaining bits of x unchanged.

49. Let n be an integer variable. Write an assignment statement using bitwise operators that will perform the following modification on the bits of n: the rightmost 2 bits should become zero, and the 3rd and 5th bits from the right end should become 1. The remaining bits should be unchanged.

50. What is the overall effect of each of the following Java statements on the bits of x?

    a. `x &= ~(1 << 8);`

    b. `x |= 0x1e;`

51. Consider the binary number 0000 1111 1100 0000, which may be used as a mask for some bitwise operation.

    a. The value of this binary number is the sum of consecutive powers of two, namely $2^x + 2^{x+1} + 2^{x+2} + \ldots + 2^y$. Therefore, x = _____ and y = _____.

    b. This summation can be simplified as a difference of powers of 2, namely $2^a - 2^b$. Therefore, a = _____ and b = _____.

52. Let's warm up with some counting:

    a. Suppose in your foreign language class, you are studying for a vocabulary test. You need to study the forms of 10 verbs. For each verb you need to know how it is conjugated in the past, present, and future, and for first, second, and third person forms both in the singular and the plural. How many total words do you need to know?

    b. Suppose you have designed an instruction cache simulator. It is set up to run any one of 30 different programs. When you run the simulator, the cache configuration can be set as follows.

The set associativity can be any power of two from 1 to 16, inclusive. The number of lines in cache can be any power of two from 32 to 2^{20}. And the line size can be 4, 8, 16 or 32 bytes. How many different ways can your simulator be run?

53. Suppose a social group consisting of five families of three, six families of four, and four families of five all go out to a ball game. In how many ways can they sit in a row of seats so that members of the same family all sit together?

54. In how many ways can 104 children be divided into four 16-member football teams, four 6-member hockey teams and four 4-member curling teams, such that teams playing the same game are indistinguishable? What if the teams are distinguishable?

55. A CD contains 10 songs and it has a total running time of 44 minutes. How long would it take to play all possible permutations of the songs?

56. In how many ways can 2n men and 2n women pair up to play mixed doubles tennis? Assume that the players are only interested in who their partner is and who their two opponents are. They don't care which court or which end of the court they play on.

57. From a group of 20 married couples, how many ways are there to pick:
    a. a man and a woman who are not married to each other?

    b. two people who are not married to each other?

58. Consider the set of all 4-digit telephone numbers (0000-9999). How many such telephone numbers are there satisfying these criteria?
    a. All the digits are different.

    b. No two consecutive digits are the same.

59. How many permutations of the letters ABCDEFG contain the substring "CAFE" ?

60. How many 5-letter strings (from aaaaa to zzzzz) can be created, subject to the restriction that the letter 'i' may not appear immediately after a 'g'?

61. Consider the set of all four-letter strings of capital letters from AAAA to ZZZZ.
    a. How many contain exactly one M?

    b. How many contain at least one M?

62. On a certain computer, an integer is internally stored as 32 bits. How many possible representations are there in which:
    a. Exactly 5 of the bits are 1's?

    b. At most 5 of the bits are 1's?

    c. The first 5 bits are 1's?

    d. There is an equal number of 0's and 1's?

    e. There are more 1's than 0's?

    f. Exactly 5 of the bits are 1's, and the two leftmost bits are the same?

    g. Let's look at some of our results. In part (a) the five 1's could be anywhere in the bit string. But in part (c) the five 1's had to be at the beginning of the bit string. Logically, can we say that we should expect the answer to part (a) to be larger than part (c)? Explain.

63. Let's count some binary strings.
    a. How many bit strings of length 10 either begin with three 0's, or end with two 0's, or both?

    b. How many bit strings of length 16 contain 14 or more consecutive zeros?

    c. How many bit strings of length 16 begin with 4 zeros and end with 4 ones?

64. The Leaning Tower Pizza Parlor makes just three kinds of pizza. Medium, Large, and Sicilian. In how many ways can someone order six pizzas?

65. * A deck of cards is dealt out. What is the probability that:
    a. The fourteenth card is an ace?

    b. The first ace occurs on the fourteenth card?

    c. The second ace occurs on the fourteenth card?

    d. The third ace occurs on the fourteenth card?

e. The last ace occurs on the fourteenth card?

f. What mathematical relationship exists among the values you calculated in parts (a) through (e)?

g. What is the probability that the last ace is dealt before reaching the fourteenth card?

h. What is the probability that the first ace is dealt after the fourteenth card?

i. Use a calculator to help you verify that b+c+d+e+g+h < 1. This result means that these six events do not completely describe the sample space of all possible events. What event is not accounted for?

j. Create a computer program or spreadsheet that computes the probability that ace #a occurs on card #c. Use the output of your program to determine the most likely times when each ace should occur, and when we reach the point where we have at least a 50% chance of having seen each ace.

66. How many 5-card poker hands have:

a. a full house containing at least 1 ace?

b. a full house containing 3 aces?

c. exactly four face cards (jack, queen, king)?

d. as many diamonds as clubs?

e. more diamonds than clubs?

f. cards from all four suits?

g. at least 3 hearts?

h. exactly 2 clubs and exactly 2 kings?

i. the same number of diamonds, hearts and clubs?

j. no kings?

k. exactly one king?

l. at least one king?

m. at least two kings?

n. exactly two kings, exactly two queens, or both?

o. Which of the above combinations of answers should sum to C(52, 5)?

67. Find the probability of each of the following events when rolling five dice.
    a. Five-of-a-kind: All dice show the same number.

    b. Four-of-a-kind: Exactly 4 of the 5 dice show the same number.

    c. A full house: three dice show the same number, and the other two match some other number.

    d. Straight: The dice show the numbers 1-5 or 2-6.

    e. What is the most likely sum of the dice? What is the probability of achieving this sum?

68. Registration PINs consist of 4 characters: a letter, a digit, and then 2 more letters. If each Furman student receives 3 different PINs during the year, and the registrar attempts to give unique PINs every term, how long will it take for all possible PINs to be used up?

69. In how many ways can we divide twenty people into teams of sizes 4, 5, 5 and 6?

70. How many distinct strings can be formed in each of the following situations?
    a. Using all the letters of WORKING, but the letter K is not in the middle position

    b. Using all the letters of FEELING, but the letter E is not in the middle position

    c. Using five of the letters taken from ELEMENT

71. Getting in line…

a. A bookshelf contains 4 physics books, 5 chemistry books, 3 history books and 6 economics books. In how many ways can these books be arranged on the shelf so that books on the same subject are grouped together, and the science books are grouped together as well?

b. In how many ways can 5 boys and 5 girls get on a lunch line at school in such a way that the boys and girls alternate positions? If there were 6 boys and 5 girls, how many ways can they line up alternating?

c. How many ways can 8 men and 5 women seat themselves in a line of chairs so that no two women sit together? (In other words, between every two women, there must be at least one man.)

72. A bookshelf contains 7 books written in German, 6 books written in French, and 5 books written in Swedish. In how many ways can the books be arranged on the shelf if …

a. The books are all distinct, and books of the same language are grouped together?

b. The books do not need to be grouped by language, but books of the same language are identical?

73. Fill in the blanks.

a. There are ______ ways to arrange n books on a bookshelf.

b. There are ______ ways to arrange n people around a dining room table.

c. There are ______ ways to arrange n keys around a key chain.

d. The number of ways to select 5 objects from a set of 20 is equal to the number of ways to select ______ objects from the same set.

74. Suppose a set has ten elements. How many ways are there to select a subset of elements, under each of the following scenarios?

a. There are no restrictions on the size or membership in the subset.

b. The subset must contain exactly 4 elements.

c. The subset must contain exactly 7 elements, and element “A” must be included in the subset.

75. Suppose S is a set and S = { 2, 4, 6, 8, 25, 50, 75, 100, 200, 300, 400 }. How many subsets of S contain exactly two 1-digit numbers, exactly two 2-digit numbers, and any quantity of 3-digit numbers?

76. Let’s flip some coins.

a. Suppose you flip a coin seven times. Each flip of the coin is considered distinct. In how many ways is it possible to get at least three heads and at least two tails?

b. A coin is flipped 7 times. What is the probability that exactly 4 outcomes are heads and 3 are tails?

c. If you flip a coin 10 times, what is the probability that you will see heads come up at least once?

77. It's time for a committee meeting!
    a. A social club consists of 10 men and 20 women. A finance committee of 5 individuals needs to be formed. How many ways can this be done if at least 2 members of the committee must be women?

    b. A company employs 28 men and 30 women. A committee of at least 6 but no more than 10 employees needs to be formed. In how many ways can this be done if the number of men and women on the committee must be the same?

78. Suppose you run a theater company, and you need to plan on which days to schedule performances. For the month of April there are 22 weekdays (Mon-Fri) and 8 weekend days (Sat/Sun). How many ways can you choose exactly 10 of these days to perform given that at least 6 of these performance days must be weekend days?

79. How many different images exist? How much storage space would be needed to store them all? Assume that each image can be accommodated on your computer screen. What other assumptions do you need to make?

80. For this problem write both exact answers and decimal approximations. A lottery drawing consists of six different numbers in the range 1-50. What is the probability of winning the lottery if you play with:
    a. One ticket?

    b. One thousand tickets?

81. A bag contains 20 balls: 8 blue, 6 red and 6 green. Assume that balls of the same color are identical.
    a. How many ways can we select 5 balls from the bag?

    b. How many ways can we select 5 or fewer balls from the bag?

    c. How many ways can we select 10 balls from the bag?

    d. If we select 5 balls from the bag, what is the probability that we will get no green balls?

82. Suppose you have a bucket of 50 fish and your job is to feed 10 hungry dolphins. If you randomly toss fish into the water, what is the probability that all the dolphins get at least one fish?

83. A bagel shop sells 5 kinds of bagels. You want to buy at least 1 of each type of bagel, and you have enough money to buy 20 bagels. Therefore, in how many ways can you place an order of bagels that you can afford?

84. A cookie jar contains 40 cookies: 10 chocolate chip, 10 oatmeal, 10 linzer, and 10 peanut butter. Cookies of the same type are indistinguishable. You want to select 8 cookies to eat. In how many ways can you make the selection if at least 2 cookies must be oatmeal?

85. A bowl contains an unlimited supply of each of 21 varieties of Halloween candy. You want to take ten pieces of candy. How many ways could you make the selection if …
    a. The order in which you take the candy does <u>not</u> matter?

    b. The order in which you take the candy <u>does</u> matter?

    c. The order in which you take the candy does not matter, and two of the 21 varieties each have only six pieces available (the other 19 varieties being unlimited)?

86. Suppose you have two dice. To roll "snake eyes" means to roll double ones. If you roll the pair of dice 100 times, what is the probability that you will roll snake eyes at least once?

87. Suppose you are an Olympic athlete, and at the Olympics there are 2800 athletes, and 500 medals will be given out.
    a. What is the probability that you will receive a medal if the medals are awarded at random, and no one is allowed to receive more than one?

    b. Re-do part (a), but assume that there is no restriction on the number of medals one may receive.

    c. Use a calculator to approximate your answers to parts (a) and (b) to the nearest thousandth. Explain why your answers make sense.

    d. If no athlete may receive more than one medal, then clearly the number of athletes who receive a medal is 500. But if we remove this restriction, what is the expected number of athletes who receive at least one medal?

88. A university has 30 computer labs, each with 25 computers. Let p be the probability that a given machine is not functioning.
    a. What is the probability that the university has at least one computer lab where all its machines are not functioning?

    b. What is the probability that all of the computer labs have at least one computer that is not functioning?

    c. Let p = 0.2. Use a calculator to evaluate the answers you found in parts (a) and (b). Explain why we should have expected one answer to be much larger than the other.

    d. Re-work parts (a) and (b), this time assuming that we have n computer labs and n computers in each lab. To what values do your two probabilities approach as n approaches infinity?

89. Suppose that the probability of an event occurring once is 1/n, and there are n trials. We are interested in computing the probability that the event will occur at least once over the n trials.
    a. What is the formula for this probability in terms of n?

    b. Use a calculator to determine this probability (to the nearest millionth) if n = 1 million.

    c. Use Wolfram Alpha to help you determine the probability when n is arbitrarily large. Your answer will make use of a famous mathematical constant.

90. * In the NHL, the format of a playoff series is 2-2-1-1-1. This means that if your team has home ice advantage, then you play your first two games at home, then the next two games away, and then, as necessary, the fifth game is home, the sixth game is away, and the seventh game is home. The series immediately ends whenever a team has won its fourth game of the series. As a result, the length of a series is 4, 5, 6 or 7 games. An individual game cannot end in a tie.

    We are interested in determining the probability that a team having home ice advantage will win a series. Assume that a team playing in its home arena has a 60% probability of winning a single game. In other words, a team playing at home is the favorite to win that individual game. A spreadsheet may be helpful in organizing your solution to this problem.

    a. Use a combinatorical argument to explain why there are exactly 20 ways that the series could be played to 7 games.

    b. What is the probability that the team with home ice advantage (i.e. the team that plays the first game of the series at home) will win the series? Express your answer to the nearest millionth.

    c. What is the probability that the series will last a total of n games, where n = 4, 5, 6, 7? Express your answers to the nearest millionth. What are the most likely and second most likely lengths for the series?

91. Use combinations to determine the value of count after the following code executes.

```
count = 0;
for (a = 0; a <= 100; ++a)
  for (b = 0; b <= 200; ++b)
    for (c = 0; c <= 300; ++c)
       for (d = 0; d <= 400; ++d)
          if (a + b + c + d == 100)
            ++count;
```

92. In Pascal's triangle,
    a. What is the fourth number in the tenth row?

    b. Does the number 13 appear anywhere? Explain.

    c. The number 1 appears an infinite number of times. Does any other positive integer appear an infinite number of times in Pascal's triangle? Prove your answer.

    d. Let PT(r, c) be the number in row r and column c of Pascal's triangle. What is the recursive rule that defines PT(r, c)? In other words, express PT(r, c) in terms of smaller numerical parameters.

93. Consider this list of eight numbers: L = (1, 1, 2, 2, 3, 3, 4, 4). We wish to know how many ways there are to make a selection of numbers from L that has a sum of 10. Show how to solve this problem using the generating function technique.

94. Consider this list of numbers: (0, 0, 1, 1, 2, 2, 3, 3, 4, 4). Show how we can use generating functions to determine how many ways there are to make a selection of 4 numbers having a sum of 10.

95. Let L be this list of numbers: (1, 1, 2, 2, 3, 3). Explain how generating functions can be used to determine how many ways there are to:
    a. Make a selection of 4 numbers.

    b. Make a selection of numbers that adds up to 7.

    c. Make a selection of 2 odd and 2 even numbers.

96. Explain how generating functions can be used to determine how many 5-digit numbers have a sum of digits of 20. (Note: the first digit of the number cannot be 0.)

97. A bowl contains 5 apples, 5 bananas, 4 oranges and 4 pears. Assume that fruits of the same type are identical. Show how we can use generating functions to answer this question: how many ways can we make a selection of 10 fruits so that we have at least 2 apples, no more than 3 bananas, an even number of oranges, and an odd number of pears?

98. Suppose you are planning a wedding reception and need to distribute the 12 bottles in a case of champagne among 4 dining room tables. How many ways can it be done under each of the following scenarios?
    a. There are no restrictions.

    b. Each table must receive at least one bottle.

    c. Each table may receive no more than 3 bottles.

    d. Each table may receive no more than 4 bottles.

    e. The first table must have no more than 1 bottle, the second table must receive at least 1 bottle, the third table must receive 0, 2 or 4 bottles, and the last table must receive either 2 or 3 bottles?

99. Use generating functions to solve this problem: In how many ways can a committee of 7 people be chosen out of a larger group of 16?

100. A bag contains 60 nickels, 20 dimes and 20 quarters. Assuming that the coins of any one denomination are indistinguishable, in how many ways can 10 coins be selected from the bag?
    a. Solve this problem using the ball-in-urn approach.

    b. Solve this problem using generating functions.

    c. Redo parts (a) and (b) assuming that we select 25 coins.

    d. Redo parts (a) and (b) assuming that we select 50 coins.

101. How many ways are there to make 15 cents change from pennies minted in 1952 and 1959 and nickels minted in 1964?
    a. Solve this problem using the ball-in-urn approach.

    b. Solve this problem using generating functions.

102. Using the fact that $(x + x^2 + x^3 + x^4 + x^5 + x^6)^4$ equals $x^4 + 4x^5 + 10x^6 + 20x^7 + 38x^8 + 56x^9 + 80x^{10} + 104x^{11} + 125x^{12} + 140x^{13} + 146x^{14} + 140x^{15} + 125x^{16} + 104x^{17} + 80x^{18} + 56x^{19} + 35x^{20} + 20x^{21} + 10x^{22} + 4x^{23} + x^{24}$, what question can we ask about rolling dice to which the answer is 125?

103. How many terms are in the expansion of $(1 + ax + a^2x^2)(1 + bx)(cx + c^2x^2 + c^3x^3)(1 + d^3x^3 + d^5x^5)$? What combinatorical questions can this generating function answer?

104. In the online game Apterous there is a numbers round where we have 24 numbered tiles face down, and six of these tiles are to be selected at random. The 24 tiles consist of the following: two each of the "small" values 1-10, plus one each of the "large" values 25, 50, 75 and 100. How many ways are there to select…
    a. Any six numbers?
    b. Six small numbers?
    c. Five small and one large?
    d. Four small and two large?
    e. Three small and three large?
    f. Two small and all four of the large?
    g. Look at your answers so far, and verify that a = b+c+d+e+f. Why should we expect this to be true?
    h. Six numbers whose sum is 40?
    i. Four odd and two even numbers?

105. Determine whether the following functions are one-to-one and/or onto.
    a. The sum function defined as sum : $Z^2 \rightarrow Z$, sum (a, b) = a+b.
    b. The function f defined as f: $X \rightarrow X$ where X = { 0, 1, 2, 3, 4 } and the rule is
       f(x) = 4x mod 5
    c. The function F for which the domain is (0+1)* and the codomain is { true, false }. The rule for F is: F(x) = true if x has an equal number of 0's and 1's; and F(x) = false otherwise.

d. The function s defined as s: $Z^+ \rightarrow Z^+$ where s(x) is the sum of digits of x.

e. The function f with domain $Z \times Z$ and co-domain Z, where $f(x, y) = |x| - |y|$. The vertical bars mean absolute value.

f. The function F whose domain is the set of binary strings. The co-domain is the set of non-negative integers. The rule of F is: F(x) = the number of zeros that appear in x.

106. How many onto functions are there mapping X to Y (i.e. $X \rightarrow Y$) if X has 4 elements and Y has 3 elements?

107. Suppose set A has seven elements and set B has three elements. Consider all functions that use A as the domain and B as the co-domain.
   a. How many such functions exist?

   b. How many of these functions are onto?

108. Is the concatenation of binary strings a one-to-one operation? Is it onto?

109. Design finite automata that accept the following sets of bit strings (draw one FA for each part):
   a. The input begins with 00.

   b. The input ends with 00.

   c. The input has two consecutive zeros.

   d. The input does not have two consecutive zeros.

   e. The input begins with 0 and contains exactly two 1's.

   f. * Every block of 4 consecutive digits contains at least 2 zeros. In other words, bits 1-4 of the input contain at least 2 zeros, and so do bits 2-5, 3-6, 4-7, etc.

110. Let L be the set of all binary strings that begin with 1 and end with 0.
   a. How many words in L have a length of 16?

   b. Write a regular expression for L.

c. Write a grammar for L.

d. Draw a finite automaton that recognizes L.

111. Let L be the set of binary strings that begin with 0 and contains exactly one 1.
   a. Draw a finite automaton for L.

   b. Write a regular expression for L.

112. Let S be the set of binary strings that begin with 0 and end with 11.
   a. Write a regular expression for S

   b. Write a grammar for S.

113. Explain why it is not possible to design a finite automaton that accepts the set of bit strings that have the same number of 0's and 1's.

114. Consider this definition of a Boolean expression:

A Boolean expression is defined by the following rules.

I. A statement variable (e.g. p, q, r, …) standing alone is a Boolean expression.
II. If E is Boolean expression, then ~(E) is a Boolean expression.
III. If E and F are Boolean expression, then so are (E) + (F) and (E) • (F).
IV. A string of symbols is a Boolean expression if and only if it derives from finitely many applications of rules I, II, III. (i.e. Nothing else is a Boolean expression.)

Determine if the following strings of symbols satisfy the above definition.

a. ((p) + (q)) + ((p) • (r))

b. ((p) + q)

c. ((p) + (q) + ~(r))

115. Let's write explicit formulas for the sequence: 1, 5, 1, 5, 1, 5, ….
   a. Write a formula that uses (–1) raised to an integer power.

   b. Write a formula that uses the cosine of an integer multiple of pi.

116. For this sequence of numbers: 10, 5, 0, −5, −10, −15, …, assume the first number of the sequence is called a_1.
   a. Write a recursive definition of the sequence.

   b. Write an explicit definition of the sequence.

117. Write recursive definitions for:
   a. The sequence of positive odd integers.

   b. This sequence of perfect squares: 0, 1, 4, 9, 16, …

   c. The sequence 17, −22, 27, −32, 37, −42, 47, −52, 57, …

   d. The sequence −3, 13, −23, 33, −43, 53, −63, 73, …

118. For the following sets of binary strings, write both a regular expression and a grammar (i.e. recursive definition) for the set.
   a. L = the set of binary strings that begin with 0.

   b. L = the set of binary strings that have two consecutive 1's.

   c. L = the set of binary strings that begin with 0, and contain two consecutive 1's.

   d. L = the set of binary strings that begin and end with 0, and contain two consecutive 1's.

   e. L = the set of binary strings that begin with 0 and end with 1.

   f. Write a grammar only for this language: L = the set of binary strings consisting of n 0's followed by 2n 1's.

119. Consider the following definition of integer exponentiation:

   exp (a, n) equals 1 if n = 0; or a * exp (a, n − 1) if n >= 1.

   Give another recursive definition that would require fewer recursive calls for large values of n.

120. Ackermann's function is a recursive function defined by the following rules.

$A(0, y) = 1$
$A(1, 0) = 2$
$A(x, 0) = x + 2 \quad \text{for } x \geq 2$

$A(x, y) = A(A(x - 1, y), y - 1)$

Compute the value of A(2, 2). In your solution, you distinguish when you are using a base rule or a recursive rule.

121. Let f(x) be defined for all real numbers as follows.
$f(x) = 0$, if $x <= 0$
$f(x) = 1 + f(\log(x))$, if $x > 0$
Assume that the log is taken to the base 10. For which values of x does f(x) return 3?

122. Suppose Fibonacci numbers are calculated by the following function. How many times is F called when we evaluate F(5)?

```
int F(int n)
{
  if (n <= 2)
    return 1;
  else
    return F(n-1) + F(n-2);
}
```

123. Let f(n) represent the Fibonacci sequence. Prove the following for all positive integers n.
a. $f(n + 4) = 3 f(n + 2) - f(n)$ [Hint: don't use induction for this part]

b. * Based on your answer to part (a), derive a formula for $f(n + 6)$ in terms of $f(n + 3)$ and $f(n)$.

c. $f(n) < (7/4)^n$

d. The sum of the first n Fibonacci numbers is equal to $f(n+2) - 1$.

124. In each of the following cases, use induction to show that the indicated explicit formula solves the given recurrence relation.
a. $a_n = 2^n + 2(3^n)$ solves the recurrence relation given by: $a_0 = 3$, $a_1 = 8$ and $a_n = 5 a_{n-1} - 6 a_{n-2}$ for $n \geq 2$.

b. $a_n = 4(2^n) - 3$ solves the recurrence relation given by: $a_0 = 1$, $a_n = 2a_{n-1} + 3$ for $n \geq 1$.

125. Consider the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 10$ and $a_1 = 41$.
a. Write an explicit formula for a_n.

b. Use induction to verify that your explicit formula is correct.

126. For the following recurrence relations, derive an explicit formula for a_n.
   a. $a_1 = 32$ and $a_n = 16a_{n-1}$ for $n \geq 2$.

   b. $a_0 = 2$, $a_1 = 2$, $a_n = 4\,a_{n-1} - 4\,a_{n-2}$ for $n \geq 2$.

   c. $a_0 = 2$, $a_1 = 6$, and $a_n = 3a_{n-1} + 10a_{n-2}$ for $n \geq 2$.

   d. $a_0 = 2$, $a_n = 5\,a_{n-1} + 3$ for $n \geq 1$.

   e. $a_n = 3a_{n-1} + n$, $a_0 = 1$

   f. $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$, $a_0 = 0$ and $a_1 = 2$.

   g. $a_n = 5a_{n-1} - 6a_{n-2} + n$, $a_0 = 19/4$ and $a_1 = 41/4$.

   h. $a_0 = 1$, $a_n = 2a_{n-1} + 3n$ for $n \geq 1$.

127. Give an example of an equivalence relation (having more than one equivalence class) that can be defined on a deck of cards. Show that your relation is in fact an equivalence relation. How does your relation partition the deck of cards?

128. Consider this relation R defined on binary strings. xRy if the last digit of x is the same as the first digit of y. Determine if the following properties hold for R: reflexive, symmetric, transitive, antisymmetric, definite, equivalence relation, partial order, total order. (Note that a "definite" relation is one in which xRy or yRx for all x and y.)

129. * For each part below, give an example of a relation on the set { 1, 2, 3, 4 } that has the given collection of properties. In each case, explain why your relation satisfies all of the given properties. Hint: a computer program might help you solve this problem.
   a. Not reflexive, not symmetric, not antisymmetric, transitive.

   b. Reflexive, not symmetric, not antisymmetric, transitive.

   c. Reflexive, symmetric, not antisymmetric, not transitive.

   d. Reflexive, symmetric, not antisymmetric, transitive.

130. Draw a simple graph that has the degree sequence indicated or explain why such a graph does not exist.
    a. 1, 1, 1, 1, 1, 1, 1, 1
    b. 1, 3, 3, 3
    c. 1, 2, 2, 2, 3
    d. 1, 1, 1, 1, 2, 2, 3, 3
    e. 2, 3, 3, 3, 4, 5
    f. 1, 1, 1, 1, 2, 2, 5

131. Suppose a graph G has 6 vertices. How many subgraphs of G contain no edges?

132. If a graph is isomorphic to its complement, how many vertices should it have?

133. Suppose a (simple) graph has n vertices.
    a. What are the maximum and minimum number of edges the graph may have?
    b. If the graph has 6n edges, what does this tell us about the size of the graph (i.e. the value of n)?
    c. If the graph has 6n edges, how many edges are in the complement?

134. If a regular polygon has n vertices, how many diagonals does it have?

135. Is the following statement true or false (and explain why) ? "If G is a cyclic graph, and G′ is obtained from G by deleting some edge, then G′ is connected."

136. Draw all simple graphs having 4 vertices and 3 edges. For each graph that you draw, find its complement. What do you notice about the graphs and their complements?

137. Four couples have come to a party given by Ken and Mary. At the beginning of the evening, there is quite a lot of handshaking as people get acquainted. Of course, Ken and Mary don't shake hands with each other, and no one else shakes hands with his/her spouse. Later, Ken asks each person (including Mary) how many hands they had shaken. As it happens, no two of the answers

are the same. What was Mary's answer? Hint: Draw a graph that depicts this situation. The degree of each vertex will correspond to the number of handshakes each person made.

138. A Hamiltonian cycle is a cycle that visits every vertex of a graph exactly once. Draw a graph that has a Hamiltonian cycle and whose degree sequence is 3, 3, 4, 4, 4, 4, 4, 4.

139. A "free" tree is a tree in which there is no distinct root vertex, and there is no limit on the degree of any vertex. Give an example of two free trees that have the same degree sequence but are not isomorphic.

140. Draw all non-isomorphic free trees having 5 vertices. How many non-isomorphic *binary* trees with 5 vertices are there? (Note – in a binary tree, the left and right children are considered distinct.)

141. Place the following words into a binary search tree according to alphabetical order:
Once a jolly swagman camped beside the billabong

142. Draw a binary tree whose preorder traversal is C R B M Y E T H N A G Q and inorder traversal is B M R E Y C H T A G N Q.

143. Give an example of two binary trees that have matching preorder and postorder traversals, but whose inorder traversals are not the same.

144. Practice Kruskal's and Prim's algorithms for finding a minimal spanning tree for the weighted graph given in problem 10.6.5 in the textbook. If you don't have the book, just begin by drawing a graph having 7 vertices and 10 edges, and label the edges with the weight values 1-10.

145. For the arithmetical expression $a + b * c + (d - a) / (b + c) * a - b$,
draw a tree that represents this expression, and then rewrite the expression in two ways: using prefix notation, and using postfix notation.

146. For the following questions, write a formula in terms of n.
   a. What is the sum of the first n positive integers?

   b. At a party with n people, each person shakes hands with everybody else. How many handshakes are there?

   c. How many binary functions exist with n input variables?

   d. How many relations exist on a set of n elements?

   e. How many functions exist, assuming that the domain and codomain have n elements? Is this number more or less than the number of relations you found in the previous part?