# INDUCTION EXAMPLES (sums)

$P_n$ says that for all $n \geq 1$:

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

First we show $P_1$ is true, $P_1$ says

$$1^3 = \left[\frac{1(1+1)}{2}\right]^2$$

$$1 = \left(\frac{2}{2}\right)^2 \text{ which is true.}$$

Now assume $P_k$ is true for some $k \geq 1$. In other words, assume:

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

add next term to both sides:

$$+ (k+1)^3 \qquad + (k+1)^3$$

now we have:

$$1^3 + 2^3 + 3^3 + \dots + k^3 + \boxed{(k+1)^3} = \left[\frac{k(k+1)}{2}\right]^2 + \boxed{(k+1)^3}$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

This note is not part of the proof, but remember our goal is for the right side to look like $P_{k+1}$: In other words, plug $k+1$ in for $n$:

$$\left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

and this last equation is $P_{k+1}$.

Since $P_1$ is true and $P_k$ implies $P_{k+1}$, $P_n$ is true for all $n \geq 1$.

$P_n$ says that for all $n \geq 1$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

$P_1$ says that:

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$ which is true.

Now assume $P_k$ is true. In other words, assume that for some $k \geq 1$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

we add next term to both sides to obtain

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \boxed{\frac{1}{(k+1)(k+2)}} = \frac{k}{k+1} + \boxed{\frac{1}{(k+1)(k+2)}}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

Scratch work, not part of proof...
Our goal is for right side to look like $\frac{n}{n+1}$ when $n = k+1$
which is $\frac{k+1}{k+2}$.

This last equation is $P_{k+1}$.

Since $P_1$ is true and $P_k$ implies $P_{k+1}$,
$P_n$ is true for all $n \geq 1$.

$P_n$ says that for all $n \geq 0$:

$$1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \dots + (n+1)2^{n+1} = n\cdot 2^{n+2} + 2.$$

The base case is $n=0$.

$P_0$ says: $\quad (0+1)2^{0+1} = 0\cdot 2^{0+2} + 2$

$2^1 = 2$, which is true.

Now assume $P_K$ is true. That is, assume that for some integer $k \geq 0$:

$$1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \dots + (k+1)2^{k+1} = k\cdot 2^{k+2} + 2$$

Add the next term to both sides:

$$1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \dots + (k+1)2^{k+1} + \boxed{(k+2)2^{k+2}} = k\cdot 2^{k+2} + 2 + \boxed{(k+2)2^{k+2}}$$

$$= \underbrace{k\cdot 2^{k+2} + k\cdot 2^{k+2}} + 2\cdot 2^{k+2} + 2$$

$$= k\left(2^{k+2} + 2^{k+2}\right) + 2^1\cdot 2^{k+2} + 2$$

$$= \underbrace{k\cdot 2^{k+3} + 2^{k+3}} + 2$$

$$= (k+1)\,2^{k+3} + 2$$

Scratch work, not part of proof.
Our goal is for right side to look like $n\cdot 2^{n+2} + 2$ when $n = k+1$ which is:
$(k+1)2^{k+3} + 2$

This last equation is $P_{K+1}$.

Since $P_0$ is true and $P_K$ implies $P_{K+1}$

$P_n$ is true for all $n \geq 0$.

Showing an inequality is true —

For all (positive integers) $n \geq 3$, $2n+1 < 2^n$.

Step 1: $P(3)$ says $2(3)+1 < 2^3$

$7 < 8$ which is true.

Step 2: Assume $P(k)$ is true for some $k \geq 3$.

In other words, assume $2k+1 < 2^k$. (1)

Our goal is to show that $2k+3 < 2^{k+1}$.

Consider the difference between these 2 inequalities:

consider $(2k+3)-(2k+1) < 2^{k+1} - 2^k$

$$2 < 2^{k+1} - 2^k$$

$$2 < 2 \cdot 2^k - 2^k$$

$$2^1 < 2^k$$

Since $k \geq 3$, we can definitely say $2^1 < 2^k$, and this is equivalent to

$(2k+3)-(2k+1) < 2^{k+1} - 2^k$ (2)

Now that we know inequalities (1) and (2) are true, we can add them to obtain

$$2k+3 < 2^{k+1}$$

which is $P(k+1)$.

Show that for all $n \geq 0$, $2^n < (n+2)!$

"Base case" — $P(0)$ says that $2^0 < (0+2)!$

$1 < 2$ which is true.

"Inductive step" — Assume $P(k)$ is true for some $k \geq 0$.

In other words, assume $2^k < (k+2)!$ ①

We need to show that $2^{k+1} < (k+3)!$

Consider the quotient of these 2 inequalities:

consider $\dfrac{2^{k+1}}{2^k} < \dfrac{(k+3)!}{(k+2)!}$

$$2 < k+3$$

$$-1 < k$$

Since $k \geq 0$, we can definitely say $-1 < k$, and this is equivalent to $\dfrac{2^{k+1}}{2^k} < \dfrac{(k+3)!}{(k+2)!}$ ②

Since inequalities ① and ② are true (and all represent positive numbers), we can multiply them to obtain —

$$2^{k+1} < (k+3)!$$

which is $P(k+1)$.

For all $n \geq 2$, $\quad \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{n}{n+1} < \frac{n^2}{n+1}$

"Base case" is $P(2)$ which says

$$\frac{1}{2} + \frac{2}{3} < \frac{2^2}{2+1}$$

$$\frac{7}{6} < \frac{4}{3} \qquad \text{which is true.}$$

"Inductive step" — Assume $P(k)$ is true for some $k \geq 2$.

in other words assume

$$\frac{1}{2} + \frac{2}{3} + \dots + \frac{k}{k+1} < \frac{k^2}{k+1} \qquad (1)$$

We need to show $p(k+1)$ is therefore true:

$$\frac{1}{2} + \frac{2}{3} + \dots + \frac{k}{k+1} + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$$

Consider the difference between these 2 inequalities, which is

$$\frac{k+1}{k+2} < \frac{(k+1)^2}{k+2} - \frac{k^2}{k+1} \qquad (2)$$

(Since $k \geq 2$, all denoms positive)

$$(k+1)(k+1) < (k+1)^3 - k^2(k+2)$$

$$k^2 + 2k + 1 < k^3 + 3k^2 + 3k + 1 - (k^3 + 2k^2)$$

$$k^2 + 2k + 1 < k^2 + 3k + 1$$

$$0 < k$$

Since $k \geq 2$, we can definitely say $0 < k$, and this is equivalent to (2).

Since (1) and (2) are true, we can add them to obtain

$$\frac{1}{2} + \frac{2}{3} + \dots + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$$

which is $P(k+1)$.