Problem 2.11

Differentating the Dirichlet Distribution:

$$
\begin{aligned}
\frac{\partial Dir(\mu|\alpha)}{\partial \alpha_j} &= \partial(\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}\prod_{k=1}^{K}\mu_k^{\alpha_k-1})/\partial\alpha_j \\
&= \frac{\partial\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}}{\partial\alpha_j}\prod_{k=1}^{K}\mu_k^{\alpha_k-1} + \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}\frac{\partial\prod_{k=1}^{K}\mu_k^{\alpha_k-1}}{\partial\alpha_j}
\end{aligned}
$$

We will handle with second term:

$$
\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}\frac{\partial\prod_{k=1}^{K}\mu_k^{\alpha_k-1}}{\partial\alpha_j} = ln(\alpha_j)Dir(\mu|\alpha)
$$

With above result, we integrate both sides of equation and handle the left side firstly:

$$
\int\frac{\partial Dir(\mu|\alpha)}{\partial\alpha_j}d\mu = \int\frac{\partial 1}{\partial\alpha_j}d\mu = 0
$$

The Right Side:

$$
\begin{aligned}
&\int\frac{\partial\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}}{\partial\alpha_j}\prod_{k=1}^{K}\mu_k^{\alpha_k-1}d\mu + \int ln(\alpha_j)Dir(\mu|\alpha)d\mu \\
&= \frac{\partial\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}}{\partial\alpha_j}\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}{\Gamma(\alpha_0)} + E[ln(\alpha_j)] \\
&= \frac{\partial ln(\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)})}{\partial\alpha_j} + E[ln(\alpha_j)]
\end{aligned}
$$

Combining both sides, we obtain:

$$
\begin{aligned}
E[ln(\mu)] &= -\frac{\partial ln(\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)})}{\partial\alpha_j} \\
&= -\frac{\partial ln(\Gamma(\alpha_0)) - ln(\Gamma(\alpha_1)\cdots\Gamma(\alpha_K))}{\partial\alpha_j} \\
&= \frac{\partial ln(\Gamma(\alpha_j))}{\partial\alpha_j} - \frac{\partial ln(\Gamma(\alpha_0))}{\partial\alpha_j} \\
&= \frac{\partial ln(\Gamma(\alpha_j))}{\partial\alpha_j} - \frac{\partial ln(\Gamma(\alpha_0))}{\partial\alpha_0}\frac{\partial\alpha_0}{\partial\alpha_j} = \psi(\alpha_j) - \psi(\alpha_0)
\end{aligned}
$$