Problem 2.10

Based on Dirichlet Distribution given by 2.38, we have:

$$
\begin{aligned}
E[\mu_j] &= \int Dir(\mu|\alpha)\mu_j d\mu \\
&= \int \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)} \prod_{k=1}^{K} \mu_k^{\alpha_{k-1}} \mu_j du \\
&= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)} \int \prod_{k=1}^{K} \mu_k^{\alpha_{k-1}} \mu_j du \\
&= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)} \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_j+1)\cdots\Gamma(\alpha_K)}{\Gamma(\alpha_0+1)} \\
&= \frac{\Gamma(\alpha_0)\Gamma(\alpha_j+1)}{\Gamma(\alpha_0+1)\Gamma(\alpha_j)}
\end{aligned}
$$

We also have $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Hence:

$$
E[\mu_j] = \frac{\alpha_j}{\alpha_0}
$$

We will solve as above with $E[\mu_j^2]$:

$$
\begin{aligned}
E[\mu_j^2] &= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)} \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_j+2)\cdots\Gamma(\alpha_K)}{\Gamma(\alpha_0+2)} \\
&= \frac{\Gamma(\alpha_0)\Gamma(\alpha_j+2)}{\Gamma(\alpha_0+2)\Gamma(\alpha_j)} \\
&= \frac{\alpha_j(\alpha_j+1)}{\alpha_0(\alpha_0+1)}
\end{aligned}
$$

Bring the above result into equation $var[\mu_j]$:

$$
\begin{aligned}
var[\mu_j] &= E[\mu_j^2] - E[\mu_j]^2 \\
&= \frac{\alpha_j(\alpha_j+1)}{\alpha_0(\alpha_0+1)} - \frac{\alpha_j}{\alpha_0}^2 \\
&= \frac{\alpha_j(\alpha_0+1)}{\alpha_0^2(\alpha_0+1)}
\end{aligned}
$$

In case of Covariance:

$$\begin{aligned}
cov[\mu_j\mu_l] &= E[(\mu_j - E[\mu_j])(\mu_l - E[\mu_l])] \\
&= \int (\mu_j - E[\mu_j])(\mu_l - E[\mu_l]) Dir(x|\mu) d\mu \\
&= \int (\mu_j\mu_l - E[\mu_l]\mu_j - E[\mu_j]\mu_l + E[\mu_j]E[\mu_l]) Dir(x|mu) d\mu \\
&= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)} \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_j+1)\Gamma(\alpha_l+1)\cdots\Gamma(\alpha_K)}{\Gamma(\alpha_0+2)} - E[\mu_l]E[\mu_j] - E[\mu_j]E[\mu_l] + E[\mu_j]E[\mu_l] \\
&= \frac{\alpha_j\alpha_l}{\alpha_0(\alpha_0+1)} - E[\mu_j]E[\mu_l] \\
&= \frac{\alpha_j\alpha_l}{\alpha_0(\alpha_0+1)} - \frac{\alpha_j\alpha_l}{\alpha_0^2} \\
&= -\frac{\alpha_j\alpha_l}{\alpha_0^2(\alpha_0+1)}
\end{aligned}$$