Problem 2.8

Proof of equation (2.270):

$$\begin{aligned}
E_y[E_x[x|y]] &= E_y[\int xp(x|y)dx] \\
&= \int\int xp(y)p(x|y)dxdy \\
&= \int\int xp(x,y)dxdy \\
&= \int xp(x)dx = E[x] && \text{(Solved)}
\end{aligned}$$

Proof of equation (2.271):

$$var_x[x] = E_x[x^2] - E_x[x]^2$$

Separating the above equation into two terms. The First Term:

$$\begin{aligned}
E_x[x^2] &= E_y[E_x[x^2|y]] \\
&= E_y[var_x[x|y] + E_x[x|y]^2] \\
&= E_y[var_x[x|y]] + E_y[E_x[x|y]^2] \\
&= E_y[var_x[x|y]] + var_y[E_x[x|y]] + E_y[E_x[x|y]]^2
\end{aligned}$$

The Second Term:

$$E_x[x]^2 = E_y[E_x[x|y]]^2 \qquad \text{(Based on equation 2.270)}$$

Combining two terms, we will obtain:

$$\begin{aligned}
var[x] &= E_y[var_x[x|y]] + var_y[E_x[x|y]] + E_y[E_x[x|y]]^2 - E_y[E_x[x|y]]^2 \\
&= E_y[var_x[x|y]] + var_y[E_x[x|y]] && \text{(Solved)}
\end{aligned}$$