Problem 2.3

Transforming the equation (2.262) based on the definition (2.10) of the number of combinations, we will obtain:

$$\frac{N!}{(N-m)!m!} + \frac{N!}{(N-m+1)!(m-1)!} = \frac{(N+1)!}{(N-m+1)!m!}$$

So we will solve the problem with above equation:

$$\begin{aligned}\frac{N!}{(N-m)!m!} + \frac{N!}{(N-m+1)!(m-1)!} &= \frac{N!(N-m+1)}{(N-m+1)!m!} + \frac{N!m}{(N-m+1)!m!}\\ &= \frac{N!(N+1)}{(N-m+1)!m!} = \frac{(N+1)!}{(N-m+1)!m!} \qquad (\mathit{Solved})\end{aligned}$$

We will prove equation (2.263) by induction and easy to see that the equation is true if N = 1. Assuming the equation (2.263) holds true on N then we will prove that equation also holds:

$$\begin{aligned}(1+x)^{N+1} = (1+x)\sum_{m=0}^{N} C_N^m x^m &= x\sum_{m=0}^{N} C_N^m x^m + \sum_{m=0}^{N} C_N^m x^m\\ = \sum_{m=0}^{N} C_N^m x^{m+1} + \sum_{m=0}^{N} C_N^m x^m &= \sum_{m=1}^{N+1} C_N^{m-1} x^{m+1} + \sum_{m=0}^{N} C_N^m x^m\\ &= \sum_{m=1}^{N}(C_N^m + C_N^{m-1})x^m + x^{N+1} + x^0\\ &= \sum_{m=1}^{N} C_{N+1}^m x^m + x^{N+1} + x^0\\ &= \sum_{m=0}^{N+1} C_{N+1}^m x^m \qquad (\mathit{Solved})\end{aligned}$$

Then we will show that the equation (2.264) holds true by induction like above. Subtituting $y = (1-x)$, we obtain:

$$
\begin{aligned}
(x+y)^{N+1} &= (x+y)\sum_{m=0}^{N} C_N^m x^m y^{N-m} \\
&= x\sum_{m=0}^{N} C_N^m x^m y^{N-m} + y\sum_{m=0}^{N} C_N^m x^m y^{N-m} \\
&= \sum_{m=0}^{N} C_N^m x^{m+1} y^{N-m} + y\sum_{m=0}^{N} C_N^m x^m y^{N-m+1} \\
&= \sum_{m=1}^{N+1} C_{N+1}^{m-1} x^m y^{N-m} + \sum_{m=0}^{N} C_N^m x^m y^{N-m+1} \\
&= \sum_{m=1}^{N} (C_{N+1}^{m-1} + C_N^m) x^m y^{N-m} + x^{N+1} + y^{N+1} \\
&= \sum_{m=1}^{N} C_{N+1}^m x^m y^{N-m} + x^{N+1} + y^{N+1} \\
&= \sum_{m=0}^{N+1} C_{N+1}^m x^{m+1} y^{N-m+1}
\end{aligned}
$$

We have solved the equation by inducntion then subtitute again $y = (1-x)$, we will prove the binomial distribution is normalized.

$$
\sum_{m=0}^{N} C_N^m x^m y^{N-m} = (x+1-x)^N = 1^N = 1
$$