Applying the Klein-Gordon Theory to Gravitation

Modelling Newtonian gravitation as a classical scalar field theory obeying Klein-Gordon structure

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Towards Classical Field Theory

The Inverse Square Law

- Gravitational force:

$$F_m = -G\frac{Mm}{r^2}$$

- Electrostatic force:

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{Q_e q_e}{r^2}$$

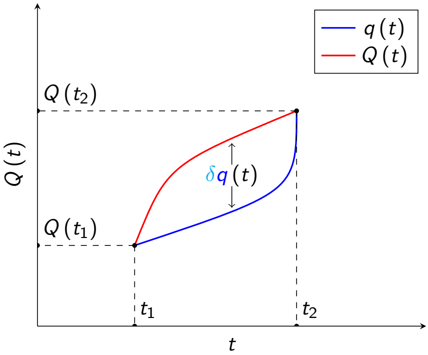
- Magnetic force:

$$F_b = \frac{\mu_0}{4\pi}\frac{Q_b q_b}{r^2}$$

Formal Analogies Between the Gravitational and Electrostatic Forces

	Gravitation	Static electricity
Newton's second law	$a^i = \underbrace{-\partial^i V}_{-\vec{\nabla} V}$	$E^i = \underbrace{-\partial^i \phi}_{-\vec{\nabla}\phi}$
Gauss' law	$\underbrace{\sum_{i=1}^{3} \nabla_i a^i}_{\vec{\nabla}\cdot\vec{a}} = -4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i E^i}_{\vec{\nabla}\cdot\vec{a}} = \frac{1}{\epsilon_0}\rho_e$
Poisson's equation	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i V}_{\nabla^2 V} = 4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i \phi}_{\nabla^2 \phi} = -\frac{1}{\epsilon_0}\rho_e$

Lagrangian Mechanics



- Nature 'selects' the unique on-shell trajectory $q(t)$ given the boundary conditions $(t_1, Q(t_1))$ and $(t_2, Q(t_2))$ for a system.

$$\underbrace{Q(t)}_{\text{Off-shell}} = \underbrace{q(t)}_{\text{On-shell}} + \underbrace{\delta q(t)}_{\text{Variation}}$$

$$\delta q(t_1) = \delta q(t_2) = 0$$

- Each trajectory $Q(t)$ between the endpoints is associated with a corresponding number called the action.

$$\boxed{S[Q(t)](t_1, t_2) = \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right)}$$

The integrand $L\left(Q(t), \dot{Q}(t), t\right)$ is known as the Lagrangian of the system being modelled and encodes the dynamics of the system.

- In general, the action S maps $Q(t)$ to a real number determined by the above integral. Therefore, it is a functional, i.e. a higher-order function which takes in infinite values of the form $\{(t, Q(t)) : t \in \mathbb{R}\}$ and spits out a real.

$$S : \begin{cases} \mathbb{R}^{\mathbb{R}} & \to \mathbb{R} \\ Q(t) & \mapsto \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right) \end{cases}$$

Principle of Stationary Action

Lagrange's principle of stationary action

Suppose we vary $q(t)$ about its on-shell evolution as, $q(t) \to q(t) + \delta q(t)$. Then, the variation in the action satisfies,

$$\delta S \in \mathcal{O}\left(\delta q^2\right)$$

Corollary (First-order approximation)

For very small $\delta q(t)$ i.e.,

$$\forall\ \delta q(t) = \lim_{\epsilon \to 0} \epsilon \eta(t) : \eta(t_1) = \eta(t_2) = 0 :$$

$$\delta S \in \mathcal{O}\left(\epsilon^2 \eta(t)\right) = \{0\}$$

$$\implies \boxed{\delta S = 0}$$

Euler-Lagrange Equation

Lemma (Fundamental lemma of calculus of variations)

The former is possible if and only if the latter is,

$$\forall\, \delta q : \int_{t_1}^{t_2} dt\, \delta q\, f(q, \dot{q}, t) = 0$$
$$\iff \forall\, t \in (t_1, t_2) : f(q, \dot{q}, t) = 0$$

Theorem

An on-shell $q(t)$ obeying the principle of stationary action for a given $L(q, \dot{q}, t)$ must also obey the Euler-Lagrange equation of motion:

$$\boxed{\underbrace{\frac{\partial L}{\partial q}}_{\textit{Generalized force}} = \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}}}_{\textit{Conjugate momentum}} = \frac{dp}{dt}}$$

Proof.

$$\delta S = 0 \quad \text{[Principle of stationary action]}$$

$$\delta \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) = 0$$

$$\int_{t_1}^{t_2} dt\, \delta L(q, \dot{q}, t) = 0 \quad \text{[Additivity of variations]}$$

$$\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \cancel{\delta t} \frac{\partial L}{\partial t} \right] = 0 \quad \text{[Chain rule for variations]}$$

$$\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + (\dot{\delta q}) \frac{\partial L}{\partial \dot{q}} \right] = 0 \quad \text{[Commutativity of derivatives]}$$

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \int_{t_1}^{t_2} dt\, (\dot{\delta q}) \frac{\partial L}{\partial \dot{q}} = 0$$

Proof.

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \int_{t_1}^{t_2} dt\, (\dot{\delta q}) - \int_{t_1}^{t_2} dt \left[\int dt\, (\dot{\delta q}) \right] \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

[Integration by parts]

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \cancel{\frac{\partial L}{\partial \dot{q}} [\delta q]_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt\, \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$[\delta q\,(t_1) = \delta q\,(t_2)]$

$$\forall\, \delta q : \int_{t_1}^{t_2} dt \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0$$

$$\Longleftrightarrow \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$\Longleftrightarrow \frac{\partial L}{\partial q} - \frac{d p}{dt} = 0 \quad \square$$

[Fundamental lemma of the calculus of variations]

Noether's Theorem

Theorem (Noether's first theorem)

If the action $S\left[q\left(t\right)\right]$ *remains invariant under perturbations of the following form,*

$$q \rightarrow q + \delta q$$

then the following quantity is conserved,

$$j = p\, \delta q$$
$$\frac{dj}{dt} = 0$$

Proof.

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \\
&= \dot{p}\delta q + p\delta \dot{q} && \text{[Euler-Lagrange equation]} \\
&= \frac{d}{dt}\left(p\delta q\right) \\
\text{But } \delta L &= 0 \\
\implies \frac{d}{dt}\left(p\delta q\right) &= 0
\end{aligned}$$

$\square$

Example

If $S\left[q\left(t\right)\right]$ is symmetric (i.e. conserved) under a small time-independent translation $q \to q + \epsilon$, we obtain the invariant $j = p\epsilon$. Since $\frac{dj}{dt} = 0$, $\frac{d\epsilon}{dt} = 0$, we get $\frac{dp}{dt} = 0$.

Classical Mechanics

- The Lagrangian for classical mechanics is of the form,

$$\begin{aligned} L(q,\dot{q},t) &= T(\dot{q}) - V(q) \\ &= \frac{1}{2} mg\dot{q}^2 - V(q) \\ &= \frac{1}{2} mv^2 - V(q) \end{aligned}$$

- The equation of motion obtained by applying the Euler-Lagrange equation to the above Lagrangian is,

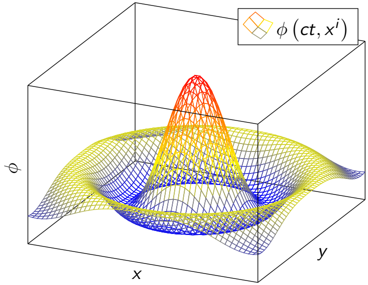
$$\frac{d}{dt}(mv) + \frac{\partial V}{\partial q} = 0$$

This is Newton's second law. If the entire system concerned is symmetric under small translations on q, we have $\frac{\partial V}{\partial q} = 0$ implying $\frac{d}{dt}(mv) = 0$. This is Newton's third law.

Classical Field Theory

- A classical field is a tensor field on spacetime (which is a pseudo-Riemannian manifold obeying dynamical field equations such as the Einstein field equations). Therefore, a classical field is some rank (p, q) tensor $\phi^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x^\alpha)$ at each point x^α in space and time with $\alpha \in (0, 1, 2, 3)$.
- A classical field obeys the following principles:
  1. Principle of stationary action
  2. Local Lorentz invariance
  3. Locality
  4. Gauge invariance
- The simplest classical field theory is that of rank $(0, 0)$ tensor fields i.e. scalar fields $\phi(x^\alpha)$, in a flat spacetime $\mathcal{M}$. We will study such fields in the following slides.

Principle of Stationary Action for Classical Fields



- To construct the action for a particle, we integrated its Lagrangian between endpoints in time. A field such as $\phi\left(x^\alpha\right)$, however, lives in space and time. Therefore, its action is a *volume* integral of a Lagrangian *density* $\mathcal{L}$, in a 4-dimensional region of spacetime $\Omega \subset \mathcal{M}$,

$$\boxed{S\left[\phi\left(x^\alpha\right)\right] = \int_\Omega d^4x \, \mathcal{L}\left(\phi\left(x^\alpha\right), \partial_\mu \phi\left(x^\alpha\right), x^\nu\right)}$$

- The Lagrangian density is so-called as it looks like a Lagrangian (integrable over some time interval $\Omega^{(1)}$) when integrated over a region of space $\Omega^{(3)}$:

$$\begin{aligned} L\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) &= \int_{\Omega^{(3)}} d^{3}x\, \mathcal{L}\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \\ S\left[\phi\left(x^{\alpha}\right)\right] &= \int_{\Omega} d^{4}x\, \mathcal{L}\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \\ &= \int_{\Omega^{(1)}} cdt\, L\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \end{aligned}$$

- The principle of stationary action for fields states that for small variations $\delta\phi$ of a field ϕ in its on-shell configuration, the action remains stationary,

$$\boxed{\delta S = 0}$$

Euler-Lagrange Equation for Classical Fields

Lemma (Fundamental lemma of multivariable calculus of variations)

$$\forall\, \delta\phi : \int_\Omega d^4x \delta\phi\, f\left(\phi, \partial_\mu\phi, x^\nu\right) = 0$$
$$\Longleftrightarrow \forall\, x^\alpha \in \Omega \backslash \partial\Omega : f\left(\phi, \partial_\mu\phi, x^\nu\right) = 0$$

Einstein summation convention

Dummy indices, i.e. pairs of upper and lower tensor indices, are implicitly summed over.

Example

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu$$

Theorem

A field ϕ obeys the principle of stationary action if and only if it also satisfies,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \nabla_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)}}_{\textit{Conjugate momentum tensor}} = \nabla_\mu \pi^\mu}$$

Proof.

$$\begin{aligned} \delta S &= 0 && \text{[Principle of stationary action]} \\ \delta \int_\Omega d^4x\, \mathcal{L} &= 0 \\ \int_\Omega d^4x\, \delta\mathcal{L} &= 0 && \text{[Additivity of variations]} \end{aligned}$$

$$\int_\Omega d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta\left(\partial_\mu \phi\right) \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}}_{\pi^\mu} + \cancel{\delta x^\mu} \partial_\mu \mathcal{L} \right] = 0$$

[Multivariable chain rule for variations]

$$\int_\Omega d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \left(\partial_\mu \delta\phi\right) \pi^\mu \right] = 0$$

[Commutativity of variations and covariant derivatives]

$$\int_\Omega d^4x \, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \pi^\mu \underbrace{\int_\Omega d^4x \, \partial_\mu \delta\phi}_{\text{Constant surface term}} - \int_\Omega d^4x \left[\int d^4x \, \partial_\mu \delta\phi \right] \nabla_\mu \pi^\mu = 0$$

[Volume integration by parts]

Using Stokes' theorem, the constant surface term can be set to 0. We then find,

$$
\begin{aligned}
\int_\Omega d^4x\, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} - \int_\Omega d^4x\, \delta\phi\, \nabla_\mu \pi^\mu &= 0 \\
\int_\Omega d^4x\, \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \pi^\mu \right) &= 0 \\
\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \pi^\mu &= 0 \\
\iff \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} &= 0 \quad \square
\end{aligned}
$$

[Fundamental lemma of multivariable calculus of variations]

Noether's Theorem for Classical Fields