Applying the Klein-Gordon Theory to Gravitation

Modelling Newtonian gravitation as a classical scalar field theory obeying Klein-Gordon structure

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Towards Classical Field Theory

The Inverse Square Law

- Gravitational force:

$$F_m = -G\frac{Mm}{r^2}$$

- Electrostatic force:

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{Q_e q_e}{r^2}$$

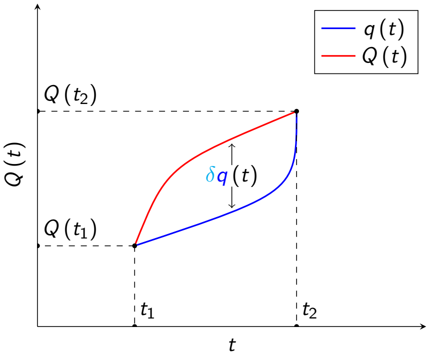
- Magnetic force:

$$F_b = \frac{\mu_0}{4\pi}\frac{Q_b q_b}{r^2}$$

Formal Analogies Between the Gravitational and Electrostatic Forces

	Gravitation	Static electricity
Newton's second law	$a^i = \underbrace{-\partial^i V}_{-\vec{\nabla} V}$	$E^i = \underbrace{-\partial^i \phi}_{-\vec{\nabla}\phi}$
Gauss' law	$\underbrace{\sum_{i=1}^{3} \nabla_i a^i}_{\vec{\nabla}\cdot\vec{a}} = -4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i E^i}_{\vec{\nabla}\cdot\vec{a}} = \frac{1}{\epsilon_0}\rho_e$
Poisson's equation	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i V}_{\nabla^2 V} = 4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i \phi}_{\nabla^2 \phi} = -\frac{1}{\epsilon_0}\rho_e$

Lagrangian Mechanics



- Nature 'selects' the unique on-shell trajectory $q(t)$ given the boundary conditions $(t_1, Q(t_1))$ and $(t_2, Q(t_2))$ for a system.

$$\underbrace{Q(t)}_{\text{Off-shell}} = \underbrace{q(t)}_{\text{On-shell}} + \underbrace{\delta q(t)}_{\text{Variation}}$$

$$\delta q(t_1) = \delta q(t_2) = 0$$

- Each trajectory $Q(t)$ between the endpoints is associated with a corresponding number called the action.

$$\boxed{S[Q(t)](t_1, t_2) = \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right)}$$

  The integrand $L\left(Q(t), \dot{Q}(t), t\right)$ is known as the Lagrangian of the system being modelled and encodes the dynamics of the system.

- In general, the action S maps $Q(t)$ to a real number determined by the above integral. Therefore, it is a functional, i.e. a higher-order function which takes in infinite values of the form $\{(t, Q(t)) : t \in \mathbb{R}\}$ and spits out a real.

$$S : \begin{cases} \mathbb{R}^{\mathbb{R}} & \to \mathbb{R} \\ Q(t) & \mapsto \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right) \end{cases}$$

Principle of Stationary Action

Lagrange's principle of stationary action

Suppose we vary $q(t)$ about its on-shell evolution as, $q(t) \to q(t) + \delta q(t)$. Then, the variation in the action satisfies,

$$\delta S = \mathcal{O}\left(\delta q^2\right)$$

Corollary (First-order approximation)

For very small $\delta q(t)$ i.e.,

$$\forall\ \delta q(t) = \lim_{\epsilon \to 0} \epsilon \eta(t) : \eta(t_1) = \eta(t_2) = 0 :$$
$$\delta S = \mathcal{O}\left(\cancel{\epsilon^2}(\eta(t))^2\right) = 0$$

Euler-Lagrange Equation

Lemma (Fundamental lemma of calculus of variations)

The former is possible if and only if the latter is,

$$\forall\, \delta q : \int_{t_1}^{t_2} dt\, \delta q\, f\left(q, \dot{q}, t\right) = 0$$
$$\iff \forall\, t \in \left(t_1, t_2\right) : f\left(q, \dot{q}, t\right) = 0$$

Theorem

An on-shell $q\left(t\right)$ obeying the principle of stationary action for a given $L\left(q, \dot{q}, t\right)$ must also obey the Euler-Lagrange equation of motion:

$$\boxed{\underbrace{\frac{\partial L}{\partial q}}_{\textit{Generalized force}} = \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}}}_{\textit{Conjugate momentum}} = \frac{dp}{dt}}$$

Proof.

$$\delta S = 0 \quad \text{[Principle of stationary action]}$$

$$\delta \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) = 0$$

$$\int_{t_1}^{t_2} dt\, \delta L(q, \dot{q}, t) = 0 \quad \text{[Additivity of variations]}$$

$$\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \cancel{\delta t} \frac{\partial L}{\partial t}\right] = 0 \quad \text{[Chain rule for variations]}$$

$$\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + (\dot{\delta q}) \frac{\partial L}{\partial \dot{q}}\right] = 0 \quad \text{[Commutativity of derivatives]}$$

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \int_{t_1}^{t_2} dt\, (\dot{\delta q}) \frac{\partial L}{\partial \dot{q}} = 0$$

Proof.

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \int_{t_1}^{t_2} dt\, (\dot{\delta q}) - \int_{t_1}^{t_2} dt \left[\int dt\, (\dot{\delta q}) \right] \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

[Integration by parts]

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \cancel{[\delta q]_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt\, \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$[\delta q\,(t_1) = \delta q\,(t_2)]$

$$\forall\, \delta q : \int_{t_1}^{t_2} dt \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{dp}{dt} = 0 \quad \square$$

[Fundamental lemma of the calculus of variations]

Noether's Theorem

Theorem (Noether's first theorem)

If the action $S[q(t)]$ *remains invariant under perturbations of the following form,*

$$q \rightarrow q + \delta q$$

then the following quantity is conserved,

$$j = p\,\delta q$$

$$\frac{dj}{dt} = 0$$

Proof.

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \\
&= \dot{p}\delta q + p\delta \dot{q} \qquad\qquad \text{[Euler-Lagrange equation]} \\
&= \frac{d}{dt}\left(p\delta q\right) \\
\text{But } \delta L &= 0 \\
\implies \frac{d}{dt}\left(p\delta q\right) &= 0
\end{aligned}$$

$\square$

Example

If $S\left[q\left(t\right)\right]$ is symmetric (i.e. conserved) under a small time-independent translation $q \to q + \epsilon$, we obtain the invariant $j = p\epsilon$. Since $\frac{dj}{dt} = 0$, $\frac{d\epsilon}{dt} = 0$, we get $\frac{dp}{dt} = 0$.

Classical Mechanics

- The Lagrangian for classical mechanics is of the form,

$$\begin{aligned} L(q, \dot{q}, t) &= T(\dot{q}) - V(q) \\ &= \frac{1}{2} mg\dot{q}^2 - V(q) \\ &= \frac{1}{2} mv^2 - V(q) \end{aligned}$$

- The equation of motion obtained by applying the Euler-Lagrange equation to the above Lagrangian is,

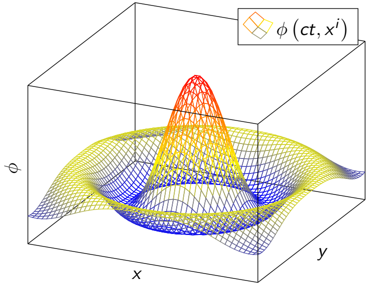
$$\frac{d}{dt}(mv) + \frac{\partial V}{\partial q} = 0$$

  This is Newton's second law. If the entire system concerned is symmetric under small translations on q, we have $\frac{\partial V}{\partial q} = 0$ implying $\frac{d}{dt}(mv) = 0$. This is Newton's third law.

Classical Field Theory

- A classical field is a tensor field on spacetime (which is a pseudo-Riemannian manifold obeying dynamical field equations such as the Einstein field equations). Therefore, a classical field is some rank (p, q) tensor $\phi^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}(x^\alpha)$ at each point x^α in space and time with $\alpha \in (0, 1, 2, 3)$.
- A classical field obeys the following principles:
  1. Principle of stationary action
  2. Local Lorentz invariance
  3. Locality
  4. Gauge invariance
- The simplest classical field theory is that of rank $(0, 0)$ tensor fields i.e. scalar fields $\phi(x^\alpha)$, in a flat spacetime $\mathcal{M}$. We will study such fields in the following slides.

Principle of Stationary Action for Classical Fields



- To construct the action for a particle, we integrated its Lagrangian between endpoints in time. A field such as $\phi\left(x^\alpha\right)$, however, lives in space and time. Therefore, its action is a *volume* integral of a Lagrangian *density* $\mathcal{L}$, in a 4-dimensional region of spacetime $\Omega \subset \mathcal{M}$,

$$\boxed{S\left[\phi\left(x^\alpha\right)\right] = \int_\Omega d^4x\, \mathcal{L}\left(\phi\left(x^\alpha\right), \partial_\mu \phi\left(x^\alpha\right), x^\nu\right)}$$

- The Lagrangian density is so-called as it looks like a Lagrangian (integrable over some time interval $\Omega^{(1)}$) when integrated over a region of space $\Omega^{(3)}$:

$$
\begin{aligned}
L\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) &= \int_{\Omega^{(3)}} d^{3}x\, \mathcal{L}\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \\
S\left[\phi\left(x^{\alpha}\right)\right] &= \int_{\Omega} d^{4}x\, \mathcal{L}\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \\
&= \int_{\Omega^{(1)}} cdt\, L\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right)
\end{aligned}
$$

- The principle of stationary action for fields states that for small variations $\delta\phi$ of a field ϕ in its on-shell configuration, the action remains stationary,

$$\boxed{\delta S = 0}$$

Euler-Lagrange Equation for Classical Fields

Lemma (Fundamental lemma of multivariable calculus of variations)

$$\forall\, \delta\phi : \int_{\Omega} d^4x \delta\phi\, f\left(\phi, \partial_\mu \phi, x^\nu\right) = 0$$
$$\Longleftrightarrow \forall\, x^\alpha \in \Omega \backslash \partial\Omega : f\left(\phi, \partial_\mu \phi, x^\nu\right) = 0$$

Einstein summation convention

Dummy indices, i.e. pairs of upper and lower tensor indices, are implicitly summed over.

Example

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu$$

Theorem

A field ϕ obeys the principle of stationary action if and only if it also satisfies,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \nabla_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}_{\text{Conjugate momentum tensor}} = \nabla_\mu \pi^\mu}$$

Proof.

$$\begin{aligned} \delta S &= 0 && \text{[Principle of stationary action]} \\ \delta \int_\Omega d^4x\, \mathcal{L} &= 0 \\ \int_\Omega d^4x\, \delta\mathcal{L} &= 0 && \text{[Additivity of variations]} \end{aligned}$$

$$\int_\Omega d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta\left(\partial_\mu \phi\right) \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}}_{\pi^\mu} + \cancel{\delta x^\mu} \partial_\mu \mathcal{L} \right] = 0$$

[Multivariable chain rule for variations]

$$\int_\Omega d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \left(\partial_\mu \delta\phi\right) \pi^\mu \right] = 0$$

[Commutativity of variations and covariant derivatives]

$$\int_\Omega d^4x \, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \pi^\mu \quad \underbrace{\int_\Omega d^4x \, \partial_\mu \delta\phi}_{\text{Constant surface term}} \quad - \int_\Omega d^4x \left[\int d^4x \, \partial_\mu \delta\phi \right] \nabla_\mu \pi^\mu = 0$$

[Volume integration by parts]

Using Stokes' theorem, the constant surface term can be set to 0. We then find,

$$\begin{aligned} \int_\Omega d^4x\, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} - \int_\Omega d^4x\, \delta\phi\, \nabla_\mu \pi^\mu &= 0 \\ \int_\Omega d^4x\, \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \pi^\mu \right) &= 0 \\ \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \pi^\mu &= 0 \\ \iff \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} &= 0 \quad \square \end{aligned}$$

[Fundamental lemma of multivariable calculus of variations]

Noether's Theorem for Classical Fields

Theorem (Field-theoretic Noether's theorem)

If under a small perturbation $x^\alpha \to x^\alpha + \delta x^\alpha$, the action of a field ϕ remains invariant, then the following quantity is conserved i.e. has a vanishing divergence,

$$
\begin{aligned}
j^\mu &= \pi^\mu \delta\phi - \mathcal{L}\delta x^\mu \\
\nabla_\mu j^\mu &= 0
\end{aligned}
$$

Proof.

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\left(\partial_\mu\phi\right) \\
&= \left(\nabla_\mu\pi^\mu\right)\delta\phi + \pi^\mu\partial_\mu\delta\phi \qquad \text{[E-L equation]} \\
&= \nabla_\mu\left(\pi^\mu\delta\phi\right) \\
\therefore \left(\nabla_\mu\mathcal{L}\right)\delta x^\mu &= \nabla_\mu\left(\pi^\mu\delta\phi\right) \\
\implies \nabla_\mu\left(\pi^\mu\delta\phi - \mathcal{L}\delta x^\mu\right) &= 0 \\
\iff \nabla_\mu j^\mu &= 0
\end{aligned}$$

□

Energy-momentum Tensor

Corollary (Conservation of energy-momentum tensor)

Dividing both sides of the above equation by δx^ν, we find an explicit conserved tensor called the energy-momentum tensor,

$$
\begin{aligned}
T^\mu{}_\nu &= \pi^\mu \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L} \\
T^{\mu\nu} &= \underbrace{\eta^{\nu\alpha}}_{\textit{Inverse Minkowski metric}} T^\nu{}_\alpha \\
&= \eta^{\nu\alpha}\left(\pi^\mu \partial_\alpha \phi - \delta^\mu{}_\alpha \mathcal{L}\right) \\
&= \pi^\mu \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}
\end{aligned}
$$

$$
\boxed{\nabla_\mu T^\mu{}_\nu = \nabla_\mu T^{\mu\nu} = 0}
$$

- The energy-momentum tensor $T^{\mu\nu}$ physically represents the flux of π^{μ} through the surface form $\bigwedge_{\alpha\neq\nu} dx^{\alpha}$. This corresponds to the flow of the field's energy($\mu = 0$)/momentum($\mu = 1, 2, 3$) along x^{ν}.
- But if we have flow of π^{μ} in the x^{ν} direction, then it implies an energy/momentum in the x^{ν} direction. The world line correpsonding to this flow must intersect with the former through a hypersurface of simulteinity, giving rise to equal $T^{\nu\mu}$.
- Thus, the energy-momentum tensor, being a geometric object with the mentioned physical meaning (motivated by particle and continuum dynamics), turns out to be symmetric,

$$\boxed{T^{\mu\nu} = T^{\nu\mu}}$$

Klein-Gordon Theory

Klein-Gordon Lagrangian

- The classical-field theoretic construction of Klein-Gordon theory begins by asking which Lagrangian yields a symmetric energy-momentum tensor. Such a theory, by virtue of respecting the physical meaning of the energy-momentum tensor, successfully describes many 'physically valid' systems.
- It turns out that the Klein-Gordon theory is deeply rooted in nature. In quantum mechanics, it is the theory for spin-0 particles. In quantum electrodynamics, the Klein-Gordon theory can be used to construct that of Dirac spinor fields, which describe all massive spin-1/2 particles such as electrons.

- Recall the energy-momentum tensor for a scalar field ϕ and its symmetry,

$$T^{\mu\nu} = \pi^{\mu}\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L}$$
$$T^{\mu\nu} = T^{\nu\mu}$$

where $\pi^{\mu} = \dfrac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}$. Thus,

$$\pi^{\mu}\partial^{\nu}\phi - \cancel{\eta^{\mu\nu}\mathcal{L}} = \pi^{\nu}\partial^{\mu}\phi - \cancel{\eta^{\nu\mu}\mathcal{L}} \qquad [\eta^{\mu\nu} = \eta^{\nu\mu}]$$
$$\pi^{\mu}\partial^{\nu}\phi = \pi^{\nu}\partial^{\mu}\phi$$

- The above is true in the most general case when π^{μ} is equal to $\partial^{\mu}\phi$, allowing us to exchange the product of conjugate momentum π^{μ} and generalized velocity $\partial^{\nu}\phi$ as above (by commutativity of component multiplication).

Therefore, we have,

$$
\begin{aligned}
\pi^{\mu} &= \partial^{\mu}\phi \\
\implies \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)} &= \partial^{\mu}\phi
\end{aligned}
$$

'Integrating' over $\partial_{\mu}\phi$, we find that the Lagrangian is constrained to be of the form,

$$
\boxed{\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V\left(\phi\right)}
$$

- This is the Klein-Gordon Lagrangian $\mathcal{L}_{KG}$. Notice that it is analogous to the Lagrangian for classical mechanics, if we interpret $\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi$ as a kinetic term $T\left(\partial_{\mu}\phi\right)$ and $V\left(\phi\right)$ as a potential term.

Indeed,

$$
\begin{aligned}
\pi^{\mu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)} \\
&= \frac{\partial}{\partial\left(\partial_{\mu}\phi\right)}\left[\frac{1}{2}\partial_{\alpha}\phi\partial^{\alpha}\phi - V\left(\phi\right)\right] \\
&= \frac{1}{2}\frac{\partial}{\partial\left(\partial_{\mu}\phi\right)}\left[\partial_{\alpha}\phi\partial^{\alpha}\phi\right] \\
&= \partial^{\alpha}\phi\frac{\partial}{\partial\left(\partial_{\mu}\phi\right)}\partial_{\alpha}\phi \\
&= \partial^{\alpha}\phi\delta^{\mu}{}_{\alpha} \\
&= \partial^{\mu}\phi
\end{aligned}
$$

Klein-Gordon Equation

- Let us find the equation of motion for a Klein-Gordon field by plugging its conjugate momentum into the Euler-Lagrange equation:

$$\nabla_\mu \pi^\mu - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Longrightarrow \boxed{\nabla_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0}$$

$$\Longleftrightarrow \Box \phi + \frac{\partial V}{\partial \phi} = 0$$

- This is the celebrated Klein-Gordon equation [for a scalar field in a potential]. In the absence of a potential, we obtain the wave equation $\Box \phi = 0$.

- For small oscillations of ϕ about local minima of the potential $V(\phi)$, only differences in ϕ physically matter. In the series expansion for $\frac{\partial V}{\partial \phi}$, we can set a vanishing first power term. Hence, in the series for $V(\phi)$, the first power term for ϕ vanishes, and so do cubic and higher terms,

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

- Plugging the above potential into the Klein-Gordon equation, we get the Klein-Gordon equation for a scalar field in a potential whose effects locally vanish:

$$\boxed{\nabla_\mu \partial^\mu \phi + m^2 \phi = 0}$$

- This is analogous to Hooke's law for harmonic oscillators, with mass assuming the role of the 'spring constant'. That's no coincidence – solutions to the above equation *are* systems of infinite harmonic oscillators at each point in spacetime!

Gauge Invariance

- Classical fields admit the structure of gauge invariance – wherein different definitions of the field represent the same physical situation. This happens when the field can encode more information than the physical system being represented.

Example

In electromagnetism, it is possible to add an arbitrary 4-gradient $\partial^\mu s$ to the electromagnetic 4-potential field A^μ, but get the same electromagnetic tensor, which encodes the physics,

$$
\begin{aligned}
A^\mu &\to A^\mu + \partial^\mu s && \text{[Gauge transformation]} \\
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
\therefore F^{\mu\nu} &\to \partial^\mu \left(A^\nu + \partial^\nu s\right) - \partial^\nu \left(A^\mu + \partial^\mu s\right) \\
&= \partial^\mu A^\nu + \cancel{\partial^\mu \partial^\nu s} - \partial^\nu A^\mu - \cancel{\partial^\nu \partial^\mu s} \\
&= F^{\mu\nu}
\end{aligned}
$$

- In general, we want physics to remain the same under a gauge transformation $\phi \to \widetilde{\phi}$ where this transformation belongs to the symmetry group of the gauge theory in question.
  This makes us expect that the Euler-Lagrange equation should stay the same under such a transformation – except it does not!
  This is fixed by allowing gauge structure and modifying the Euler-Lagrange equation by replacing usual covariant derivatives with gauge covariant derivatives ($=$ covariant derivatives $+$ gauge connection term),

$$\partial_\mu \phi \to D_\mu \phi = \quad \partial_\mu \phi - \phi \underbrace{\frac{\partial \phi}{\partial \widetilde{\phi}} \partial_\mu \frac{\partial \widetilde{\phi}}{\phi}}_{\text{Gauge connection coefficient}}$$

$$\pi^\mu \to \widetilde{\pi}^\mu = \qquad \underbrace{\frac{\partial \phi}{\partial \widetilde{\phi}}}_{\text{Inverse gauge Jacobian}} \pi^\mu$$

- Furthermore, if we have multiple similar scalar fields, they start behaving like abstract indexed quantities i.e. abstract tensors. Since by definition the theory of a real-valued classical field is equivalent to the theory of a single-index classical field theory, we can, for instance, make the correspondances,

$$\phi^{2n} \leftrightarrow (\phi_a \phi^a)^n$$
$$\phi^{2n+1} \leftrightarrow (\phi_a \phi^a)^n \phi_b$$

- In the theory of indexed classical fields, the potential must act on a gauge scalar, and the simplest gauge scalar which only depends on the indexed fields ϕ_a is $\phi_a \phi^a$. Thus,

$$V = V(\phi_a \phi^a)$$

In the real-valued classical field theory, this corresponds to asserting,

$$V = V(\phi^2)$$

- Therefore, in a series expansion of the potential, odd powers vanish,

$$V\left(\phi^2\right) = \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{g_n}_{\text{Coupling constants}} \phi^{2n}$$

- Since we know that in Klein-Gordon theory, $V\left(\phi\right) = \frac{1}{2}m^2\phi^2 + \mathcal{O}\left(\phi^3\right)$, in the above, we set $g_1 = \frac{1}{2}m^2\phi^2$ and write the potential as,

$$V\left(\phi\right) = \frac{1}{2}m^2\phi^2 + \sum_{n=2}^{\infty} \frac{g_n}{n!}\phi^{2n}$$

- Thus, the full Klein-Gordon Lagrangian resembles

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \sum_{n=2}^{\infty}\frac{g_n}{n!}\phi^{2n}}$$

- From the above, the full Klein-Gordon equation resembles,

$$\boxed{\nabla_\mu\partial^\mu\phi + m^2\phi - 2\sum_{n=1}^{\infty}\frac{g_{n+1}}{n!}\phi^{2n+1} = 0}$$

This completes our Klein-Gordon analysis.

Newtonian Gravitation

As a Classical Field Theory

- Newtonian gravitation has two aspects:
  1. How matter, encoded in the mass density field ρ, affects the gravitational potential field ϕ.
  2. How the gravitational potential field ϕ causes matter to move and hence change the matter density field ρ.

  Field theory concerns itself with the latter aspect.
- Unfortunately, Newtonian gravitation is not a classical field theory, as it does not obey local Lorentz invariance and locality. Fixing this requires Einstein's general theory of relativity, where the gravitational field is not a separate tensor field living in spacetime but the metric tensor itself, capturing the geometry of spacetime.

- Fortunately, Newtonian gravitation is *almost* a classical field theory, as it still obeys the principle of stationary action, global Galilean invariance and gauge invariance.
- The gauge structure of the gravitational potential field ϕ is affine, i.e. $\phi \to \phi + s$ where s is a constant, does not change physics, encoded in the gravitational field $g^i = -\partial^i \phi$.
- As we shall see, the gravitational potential field ϕ obeys the principle of stationary action, as its equation of motion, Poisson's equation, can be written as an Euler-Lagrange equation. In fact, it is an example of the Klein-Gordon equation in $D = 3$ dimensions!

Non-relativistic Klein-Gordon Theory in $D = 3$

- The Klein-Gordon theory not only applies in $D = 4$ dimensions with special/general relativity, but also $D = 3$ dimensions with Galilean relativity!
  This is because the way classical fields are constructed in $D = 3$ is analogous to that in $D = 4$, except we replace local Lorentz invariance + locality with global Galilean invariance. Furthermore, we replace the Minkowski metric for $D = 1 + 3$ spacetime with the Euclidean metric for $D = 3$ space.
- Except the above, much of the mathematical machinery for fields remains the same in the non-relativistic $D = 3$ case, for example – tensor fields, principle of stationary action, Euler-Lagrange equation, Noether's theorem, energy-momentum tensor and its conservation, etc.

Constructing the Lagrangian

- Recall Poisson's equation for the gravitational potential field ϕ,

$$\nabla_i \partial^i \phi + 4\pi G \rho = 0$$

In comparison, here's the Klein-Gordon equation in a potential:

$$\nabla_i \partial^i \phi + \frac{\partial V}{\partial \phi} = 0$$

- Clearly, Poisson's equation can be written as a Klein-Gordon equation by setting $\frac{\partial V}{\partial \phi} = 4\pi G \rho$. The simplest solution for such a $V(\phi)$ is,

$$V(\phi) = 4\pi G \rho \phi$$

Thus, the Lagrangian for Newtonian gravitation is,

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_i\phi\partial^i\phi - 4\pi G\rho\phi}$$

- A gauge structure inherent to any classical field theory is linearity. I.e., linear transformations of the Lagrangian $\mathcal{L} \to \alpha\mathcal{L} + \beta$ (α, β are constants) do not change the corresponding Euler-Lagrange equation.
- By dividing the obtained Lagrangian by $4\pi G$, we get an equally valid Lagrangian, which is more commonly used in the physics and mathematics literature,

$$\boxed{\mathcal{L} = \frac{1}{8\pi G}\underbrace{\partial_i\phi\partial^i\phi}_{\vec{\nabla}\phi\cdot\vec{\nabla}\phi} - \rho\phi}$$

Mass Density Field as a Classical Field

- Notice that if we vary the Lagrangian we constructed with respect to the mass density field, we get the [nonsensical] equation:

$$\phi = 0$$

- Classical field theory deals with this problem by adding a new term $\mathcal{L}_{\mathcal{M}}$ to the Lagrangian, representing the unspecified theory of matter.

$$\boxed{\mathcal{L} = \frac{1}{8\pi G}\partial_i\phi\partial^i\phi - \rho\phi + \mathcal{L}_{\mathcal{M}}}$$

Additionally, we want $\dfrac{\delta\mathcal{L}_{\mathcal{M}}}{\delta\phi} = \rho$. Such a matter field Lagrangian would respect both Newtonian gravitation and the matter field theory.

References

1. **Classical Mechanics:The Theoretical Minimum** by Leonard Susskind
2. **From Special Relativity to Feynman Diagrams** by Riccardo D'Auria, Mario Trigiante
3. **The Lagrangian of Newtonian gravity** (https://www.vttoth.com/CMS/physics-notes/269-the-lagrangian-of-newtonian-gravity) by Viktor T. Toth

Final Notes

- Big thank you to Pure Math Club for hosting this exciting event and to everyone attending this talk!
- More on this presentation can be found on my blog in the classical field theory section (https://booodaness.github.io/tempus-spatium/categories/classical-field-theory/).
- Source code, pdf for this presentation available at: https://github.com/Booodaness/scientific-documents
- Feel free to contact me on Discord at Booodaness#1464 if you have any questions!

THANK YOU