Applying the Klein-Gordon Theory to Gravitation

Modelling Newtonian gravitation as a classical scalar field theory obeying Klein-Gordon structure

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Towards Classical Field Theory

The Inverse Square Law

- Gravitational force:

$$F_m = -G\frac{Mm}{r^2}$$

- Electrostatic force:

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{Q_e q_e}{r^2}$$

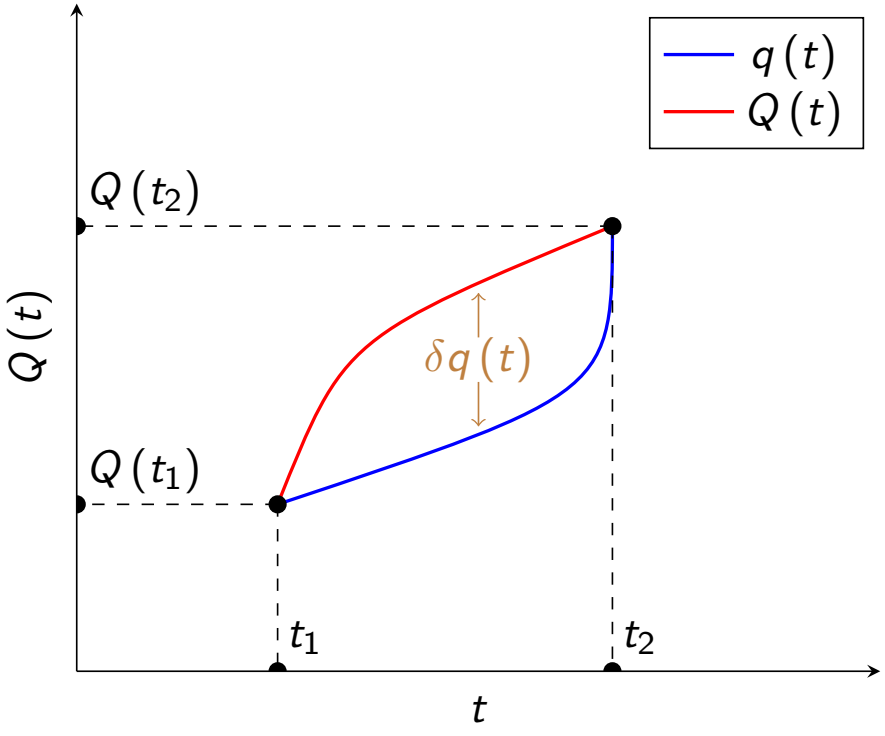
- Magnetic force:

$$F_b = \frac{\mu_0}{4\pi}\frac{Q_b q_b}{r^2}$$

Formal Analogies Between the Gravitational and Electrostatic Forces

	Gravitation	Static electricity
Newton's second law	$a^i = \underbrace{-\partial^i V}_{-\vec{\nabla} V}$	$E^i = \underbrace{-\partial^i \phi}_{-\vec{\nabla}\phi}$
Gauss' law	$\underbrace{\sum_{i=1}^{3} \nabla_i a^i}_{\vec{\nabla}\cdot\vec{a}} = -4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i E^i}_{\vec{\nabla}\cdot\vec{a}} = \frac{1}{\epsilon_0}\rho_e$
Poisson's equation	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i V}_{\nabla^2 V} = 4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i \phi}_{\nabla^2 \phi} = -\frac{1}{\epsilon_0}\rho_e$

Lagrangian Mechanics



- Nature 'selects' the unique on-shell trajectory $q(t)$ given the boundary conditions $(t_1, Q(t_1))$ and $(t_2, Q(t_2))$ for a system.

$$\underbrace{Q(t)}_{Off-shell} = \underbrace{q(t)}_{On-shell} + \underbrace{\delta q(t)}_{Variation}$$

- Each trajectory $Q(t)$ between the endpoints is associated with a corresponding number called the action.

$$S[Q(t)](t_1, t_2) = \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right)$$

The integrand $L\left(Q(t), \dot{Q}(t), t\right)$ is known as the Lagrangian of the system being modelled and encodes the dynamics of the system.

- In general, the action S maps $Q(t)$ to a real number determined by the above integral. Therefore, it is a functional, i.e. a higher-order function which takes in infinite values of the form $\{(t, Q(t)) : t \in \mathbb{R}\}$ and spits out a real.

$$S : \begin{cases} \mathbb{R}^{\mathbb{R}} & \to \mathbb{R} \\ Q(t) & \mapsto \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right) \end{cases}$$

Principle of Stationary Action