Applying the Klein-Gordon Theory to Gravitation

Modelling Newtonian gravitation as a classical scalar field theory obeying Klein-Gordon structure

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Towards Classical Field Theory

The Inverse Square Law

- Gravitational force:

$$F_m = -G\frac{Mm}{r^2}$$

- Electrostatic force:

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{Q_e q_e}{r^2}$$

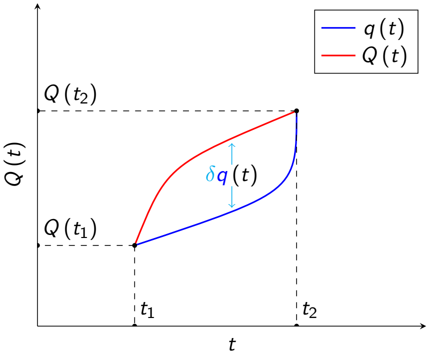
- Magnetic force:

$$F_b = \frac{\mu_0}{4\pi}\frac{Q_b q_b}{r^2}$$

Formal Analogies Between the Gravitational and Electrostatic Forces

	Gravitation	Static electricity
Newton's second law	$a^i = \underbrace{-\partial^i V}_{-\vec{\nabla} V}$	$E^i = \underbrace{-\partial^i \phi}_{-\vec{\nabla}\phi}$
Gauss' law	$\underbrace{\sum_{i=1}^{3} \nabla_i a^i}_{\vec{\nabla}\cdot\vec{a}} = -4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i E^i}_{\vec{\nabla}\cdot\vec{a}} = \frac{1}{\epsilon_0}\rho_e$
Poisson's equation	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i V}_{\nabla^2 V} = 4\pi G \rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i \phi}_{\nabla^2 \phi} = -\frac{1}{\epsilon_0}\rho_e$

Lagrangian Mechanics



- Nature 'selects' the unique on-shell trajectory $q(t)$ given the boundary conditions $(t_1, Q(t_1))$ and $(t_2, Q(t_2))$ for a system.

$$\underbrace{Q(t)}_{\text{Off-shell}} = \underbrace{q(t)}_{\text{On-shell}} + \underbrace{\delta q(t)}_{\text{Variation}}$$

$$\delta q(t_1) = \delta q(t_2) = 0$$

- Each trajectory $Q(t)$ between the endpoints is associated with a corresponding number called the action.

$$\boxed{S[Q(t)](t_1, t_2) = \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right)}$$

The integrand $L\left(Q(t), \dot{Q}(t), t\right)$ is known as the Lagrangian of the system being modelled and encodes the dynamics of the system.

- In general, the action S maps $Q(t)$ to a real number determined by the above integral. Therefore, it is a functional, i.e. a higher-order function which takes in infinite values of the form $\{(t, Q(t)) : t \in \mathbb{R}\}$ and spits out a real.

$$S : \begin{cases} \mathbb{R}^{\mathbb{R}} & \to \mathbb{R} \\ Q(t) & \mapsto \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right) \end{cases}$$

Principle of Stationary Action

Lagrange's principle of stationary action

Suppose we vary $q(t)$ about its on-shell evolution as, $q(t) \to q(t) + \delta q(t)$. Then, the variation in the action satisfies,

$$\delta S \in \mathcal{O}\left(\delta q^2\right)$$

Corollary (First-order approximation)

For very small $\delta q(t)$ i.e.,

$$\forall\ \delta q(t) = \lim_{\epsilon \to 0} \epsilon \eta(t) : \eta(t_1) = \eta(t_2) = 0 :$$

$$\delta S \in \mathcal{O}\left(\epsilon^2 \eta(t)\right) = \{0\}$$

$$\implies \boxed{\delta S = 0}$$

Euler-Lagrange Equation

Lemma (Fundamental lemma of the calculus of variations)

The former is possible if and only if the latter is,

$$\forall\, \delta q : \int_{t_1}^{t_2} dt\, \delta q\, f(q, \dot{q}, t) = 0 \tag{1}$$

$$\iff \forall\, q, \dot{q}, t : f(q, \dot{q}, t) = 0 \tag{2}$$

Theorem

An on-shell $q(t)$ *obeying the principle of stationary action for a given* $L(q, \dot{q}, t)$ *must also obey the Euler-Lagrange equation of motion:*

$$\boxed{\underbrace{\frac{\partial L}{\partial q}}_{\text{Generalized force}} = \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}}}_{\text{Generalized momentum}} = \frac{dp}{dt}}$$

Proof.

$$
\begin{aligned}
\delta S &= 0 \quad \text{[Principle of stationary action]} \\
\delta \int_{t_1}^{t_2} dt\, L\left(q, \dot{q}, t\right) &= 0 \\
\int_{t_1}^{t_2} dt\, \delta L\left(q, \dot{q}, t\right) &= 0 \quad \text{[Additivity of variations]} \\
\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \cancel{\delta t} \frac{\partial L}{\partial t}\right] &= 0 \quad \text{[Chain rule for variations]} \\
\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \dot{(\delta q)} \frac{\partial L}{\partial \dot{q}}\right] &= 0 \quad \text{[Commutativity of derivatives]} \\
\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \int_{t_1}^{t_2} dt\, \dot{(\delta q)} \frac{\partial L}{\partial \dot{q}} &= 0
\end{aligned}
$$

Proof.

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \int_{t_1}^{t_2} dt\, (\dot{\delta q}) - \int_{t_1}^{t_2} dt \left[\int dt\, (\dot{\delta q}) \right] \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

[Integration by parts]

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \cancel{[\delta q]_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt\, \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$[\delta q\,(t_1) = \delta q\,(t_2)]$

$$\forall\, \delta q : \int_{t_1}^{t_2} dt \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{dp}{dt} = 0 \quad \square$$

[Fundamental lemma of the calculus of variations]

Noether's theorem

Theorem (Noether's first theorem)

If the action $S\left[q\left(t\right)\right]$ remains invariant under perturbations of the following form,

$$t \rightarrow t + \delta t$$
$$q \rightarrow q + \delta q\left(t\right)$$

then the following quantity is conserved,

$$j = \left(p\dot{q} - L\right)\delta t - p\,\delta q$$
$$\frac{dj}{dt} = 0$$

Proof.

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial t}\delta t \\
&= \dot{p}\delta q + p\delta \dot{q} + \frac{\partial L}{\partial t}\delta t \\
&= \frac{d}{dt}\left(p\delta q\right) + \frac{\partial L}{\partial t}\delta t
\end{aligned}$$