Applying the Klein-Gordon Theory to Gravitation

Modelling Newtonian gravitation as a classical scalar field theory obeying Klein-Gordon structure

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Towards Classical Field Theory

The Inverse Square Law

- Gravitational force:

$$F_m = -G\frac{Mm}{r^2}$$

- Electrostatic force:

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{Q_e q_e}{r^2}$$

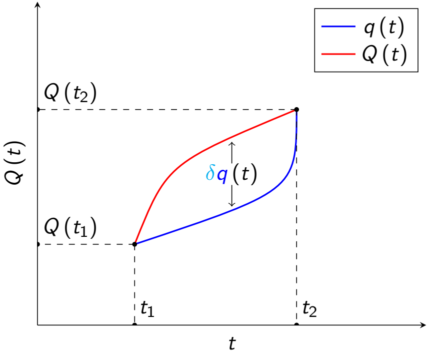
- Magnetic force:

$$F_b = \frac{\mu_0}{4\pi}\frac{Q_b q_b}{r^2}$$

Formal Analogies Between the Gravitational and Electrostatic Forces

	Gravitation	Static electricity
Newton's second law	$a^i = \underbrace{-\partial^i V}_{-\vec{\nabla} V}$	$E^i = \underbrace{-\partial^i \phi}_{-\vec{\nabla}\phi}$
Gauss' law	$\underbrace{\sum_{i=1}^{3} \nabla_i a^i}_{\vec{\nabla}\cdot\vec{a}} = -4\pi G\rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i E^i}_{\vec{\nabla}\cdot\vec{a}} = \frac{1}{\epsilon_0}\rho_e$
Poisson's equation	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i V}_{\nabla^2 V} = 4\pi G\rho_m$	$\underbrace{\sum_{i=1}^{3} \nabla_i \partial^i \phi}_{\nabla^2\phi} = -\frac{1}{\epsilon_0}\rho_e$

Lagrangian Mechanics



- Nature 'selects' the unique on-shell trajectory $q(t)$ given the boundary conditions $(t_1, Q(t_1))$ and $(t_2, Q(t_2))$ for a system.

$$\underbrace{Q(t)}_{\text{Off-shell}} = \underbrace{q(t)}_{\text{On-shell}} + \underbrace{\delta q(t)}_{\text{Variation}}$$

$$\delta q(t_1) = \delta q(t_2) = 0$$

- Each trajectory $Q(t)$ between the endpoints is associated with a corresponding number called the action.

$$\boxed{S[Q(t)](t_1, t_2) = \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right)}$$

The integrand $L\left(Q(t), \dot{Q}(t), t\right)$ is known as the Lagrangian of the system being modelled and encodes the dynamics of the system.

- In general, the action S maps $Q(t)$ to a real number determined by the above integral. Therefore, it is a functional, i.e. a higher-order function which takes in infinite values of the form $\{(t, Q(t)) : t \in \mathbb{R}\}$ and spits out a real.

$$S : \begin{cases} \mathbb{R}^{\mathbb{R}} & \to \mathbb{R} \\ Q(t) & \mapsto \int_{t_1}^{t_2} dt\, L\left(Q(t), \dot{Q}(t), t\right) \end{cases}$$

Principle of Stationary Action

Lagrange's principle of stationary action

Suppose we vary $q(t)$ about its on-shell evolution as, $q(t) \to q(t) + \delta q(t)$. Then, the variation in the action satisfies,

$$\delta S \in \mathcal{O}\left(\delta q^2\right)$$

Corollary (First-order approximation)

For very small $\delta q(t)$ i.e.,

$$\forall\ \delta q(t) = \lim_{\epsilon \to 0} \epsilon \eta(t) : \eta(t_1) = \eta(t_2) = 0 :$$

$$\delta S \in \mathcal{O}\left(\epsilon^2 \eta(t)\right) = \{0\}$$

$$\implies \boxed{\delta S = 0}$$

Euler-Lagrange Equation

Lemma (Fundamental lemma of calculus of variations)

The former is possible if and only if the latter is,

$$\forall\, \delta q : \int_{t_1}^{t_2} dt\, \delta q\, f(q, \dot{q}, t) = 0$$
$$\iff \forall\, t \in (t_1, t_2) : f(q, \dot{q}, t) = 0$$

Theorem

An on-shell $q(t)$ obeying the principle of stationary action for a given $L(q, \dot{q}, t)$ must also obey the Euler-Lagrange equation of motion:

$$\boxed{\underbrace{\frac{\partial L}{\partial q}}_{\text{Generalized force}} = \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}}}_{\text{Conjugate momentum}} = \frac{dp}{dt}}$$

Proof.

$$
\begin{aligned}
\delta S &= 0 \quad \text{[Principle of stationary action]}\\
\delta \int_{t_1}^{t_2} dt\, L\left(q, \dot{q}, t\right) &= 0\\
\int_{t_1}^{t_2} dt\, \delta L\left(q, \dot{q}, t\right) &= 0 \quad \text{[Additivity of variations]}\\
\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \cancel{\delta t} \frac{\partial L}{\partial t}\right] &= 0 \quad \text{[Chain rule for variations]}\\
\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + (\dot{\delta q}) \frac{\partial L}{\partial \dot{q}}\right] &= 0 \quad \text{[Commutativity of derivatives]}\\
\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \int_{t_1}^{t_2} dt\, (\dot{\delta q}) \frac{\partial L}{\partial \dot{q}} &= 0
\end{aligned}
$$

Proof.

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \int_{t_1}^{t_2} dt\, (\dot{\delta q}) - \int_{t_1}^{t_2} dt \left[\int dt\, (\dot{\delta q}) \right] \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

[Integration by parts]

$$\int_{t_1}^{t_2} dt\, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \cancel{[\delta q]_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt\, \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$[\delta q\,(t_1) = \delta q\,(t_2)]$

$$\forall\, \delta q : \int_{t_1}^{t_2} dt \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{dp}{dt} = 0 \quad \square$$

[Fundamental lemma of the calculus of variations]

Noether's Theorem

Theorem (Noether's first theorem)

If the action $S\left[q\left(t\right)\right]$ *remains invariant under perturbations of the following form,*

$$q \to q + \delta q$$

then the following quantity is conserved,

$$\begin{aligned} j &= p\,\delta q \\ \frac{dj}{dt} &= 0 \end{aligned}$$

Proof.

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \\
&= \dot{p}\delta q + p\delta \dot{q} && \text{[Euler-Lagrange equation]} \\
&= \frac{d}{dt}\left(p\delta q\right) \\
\text{But } \delta L &= 0 \\
\implies \frac{d}{dt}\left(p\delta q\right) &= 0 && \square
\end{aligned}$$

Example

If $S\left[q\left(t\right)\right]$ is symmetric (i.e. conserved) under a small time-independent translation $q \to q + \epsilon$, we obtain the invariant $j = p\epsilon$. Since $\frac{dj}{dt} = 0$, $\frac{d\epsilon}{dt} = 0$, we get $\frac{dp}{dt} = 0$.

Classical Mechanics

- The Lagrangian for classical mechanics is of the form,

$$
\begin{aligned}
L(q, \dot{q}, t) &= T(\dot{q}) - V(q) \\
&= \frac{1}{2} mg\dot{q}^2 - V(q) \\
&= \frac{1}{2} mv^2 - V(q)
\end{aligned}
$$

- The equation of motion obtained by applying the Euler-Lagrange equation to the above Lagrangian is,

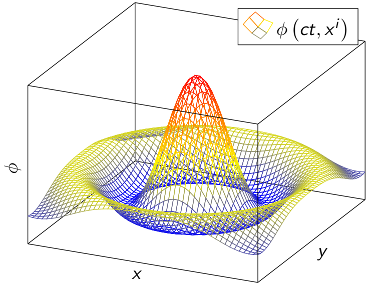
$$
\frac{d}{dt}(mv) + \frac{\partial V}{\partial q} = 0
$$

  This is Newton's second law. If the entire system concerned is symmetric under small translations on q, we have $\frac{\partial V}{\partial q} = 0$ implying $\frac{d}{dt}(mv) = 0$. This is Newton's third law.

Classical Field Theory

- A classical field is a tensor field on spacetime (which is a pseudo-Riemannian manifold obeying dynamical field equations such as the Einstein field equations). Therefore, a classical field is some rank (p, q) tensor $\phi^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x^\alpha)$ at each point x^α in space and time with $\alpha \in (0, 1, 2, 3)$.
- A classical field obeys the following principles:
  1. Principle of stationary action
  2. Local Lorentz invariance
  3. Locality
  4. Gauge invariance
- The simplest classical field theory is that of rank $(0, 0)$ tensor fields i.e. scalar fields $\phi(x^\alpha)$, in a flat spacetime $\mathcal{M}$. We will study such fields in the following slides.

Principle of Stationary Action for Classical Fields



- To construct the action for a particle, we integrated its Lagrangian between endpoints in time. A field such as $\phi\left(x^\alpha\right)$, however, lives in space and time. Therefore, its action is a *volume* integral of a Lagrangian *density* $\mathcal{L}$, in a 4-dimensional region of spacetime $\Omega \subset \mathcal{M}$,

$$\boxed{S\left[\phi\left(x^\alpha\right)\right] = \int_\Omega d^4x\, \mathcal{L}\left(\phi\left(x^\alpha\right), \partial_\mu \phi\left(x^\alpha\right), x^\nu\right)}$$

- The Lagrangian density is so-called as it looks like a Lagrangian (integrable over some time interval $\Omega^{(1)}$) when integrated over a region of space $\Omega^{(3)}$:

$$\begin{aligned} L\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) &= \int_{\Omega^{(3)}} d^{3}x\, \mathcal{L}\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \\ S\left[\phi\left(x^{\alpha}\right)\right] &= \int_{\Omega} d^{4}x\, \mathcal{L}\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \\ &= \int_{\Omega^{(1)}} cdt\, L\left(\phi\left(x^{\alpha}\right), \partial_{\mu}\phi\left(x^{\alpha}\right), x^{\nu}\right) \end{aligned}$$

- The principle of stationary action for fields states that for small variations $\delta\phi$ of a field ϕ in its on-shell configuration, the action remains stationary,

$$\boxed{\delta S = 0}$$

Euler-Lagrange Equation for Classical Fields

Lemma (Fundamental lemma of multivariable calculus of variations)

$$\forall\, \delta\phi : \int_\Omega d^4x \delta\phi\, f\left(\phi, \partial_\mu \phi, x^\nu\right) = 0$$
$$\Longleftrightarrow \forall\, x^\alpha \in \Omega \backslash \partial\Omega : f\left(\phi, \partial_\mu \phi, x^\nu\right) = 0$$

Einstein summation convention

Dummy indices, i.e. pairs of upper and lower tensor indices, are implicitly summed over.

Example

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu$$

Theorem

A field ϕ obeys the principle of stationary action if and only if it also satisfies,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \nabla_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}}_{\textit{Conjugate momentum tensor}} = \nabla_\mu \pi^\mu}$$

Proof.

$$\begin{aligned}
\delta S &= 0 && \text{[Principle of stationary action]} \\
\delta \int_\Omega d^4x\, \mathcal{L} &= 0 \\
\int_\Omega d^4x\, \delta\mathcal{L} &= 0 && \text{[Additivity of variations]}
\end{aligned}$$

$$\int_\Omega d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta\left(\partial_\mu \phi\right) \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}}_{\pi^\mu} + \cancel{\delta x^\mu} \partial_\mu \mathcal{L}\right] = 0$$

[Multivariable chain rule for variations]

$$\int_\Omega d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \left(\partial_\mu \delta\phi\right) \pi^\mu\right] = 0$$

[Commutativity of variations and covariant derivatives]

$$\int_\Omega d^4x\, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \pi^\mu \underbrace{\int_\Omega d^4x\, \partial_\mu \delta\phi}_{\text{Constant surface term}} - \int_\Omega d^4x \left[\int d^4x\, \partial_\mu \delta\phi\right] \nabla_\mu \pi^\mu = 0$$

[Volume integration by parts]

Using Stokes' theorem, the constant surface term can be set to 0. We then find,

$$\begin{aligned}
\int_\Omega d^4x\, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} - \int_\Omega d^4x\, \delta\phi\, \nabla_\mu \pi^\mu &= 0 \\
\int_\Omega d^4x\, \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \pi^\mu \right) &= 0 \\
\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \pi^\mu &= 0 \\
\iff \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} &= 0 \quad \square
\end{aligned}$$

[Fundamental lemma of multivariable calculus of variations]

Noether's Theorem for Classical Fields

Theorem (Field-theoretic Noether's theorem)

If under a small perturbation $x^\alpha \to x^\alpha + \delta x^\alpha$, the action of a field ϕ remains invariant, then the following quantity is conserved i.e. has a vanishing divergence,

$$\begin{aligned} j^\mu &= \pi^\mu \delta\phi - \mathcal{L}\delta x^\mu \\ \nabla_\mu j^\mu &= 0 \end{aligned}$$

Proof.

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\left(\partial_\mu\phi\right) \\
&= \left(\nabla_\mu\pi^\mu\right)\delta\phi + \pi^\mu\partial_\mu\delta\phi \qquad \text{[E-L equation]} \\
&= \nabla_\mu\left(\pi^\mu\delta\phi\right) \\
\therefore \left(\nabla_\mu\mathcal{L}\right)\delta x^\mu &= \nabla_\mu\left(\pi^\mu\delta\phi\right) \\
\implies \nabla_\mu\left(\pi^\mu\delta\phi - \mathcal{L}\delta x^\mu\right) &= 0 \\
\iff \nabla_\mu j^\mu &= 0
\end{aligned}
$$

$\square$

Energy-momentum Tensor

Corollary (Conservation of energy-momentum tensor)

Dividing both sides of the above equation by δx^ν, we find an explicit conserved tensor called the energy-momentum tensor,

$$
\begin{aligned}
T^{\mu}{}_{\nu} &= \pi^{\mu}\partial_{\nu}\phi - \delta^{\mu}{}_{\nu}\mathcal{L} \\
T^{\mu\nu} &= \underbrace{\eta^{\nu\alpha}}_{\textit{Inverse Minkowski metric}} T^{\nu}{}_{\alpha} \\
&= \eta^{\nu\alpha}\left(\pi^{\mu}\partial_{\alpha}\phi - \delta^{\mu}{}_{\alpha}\mathcal{L}\right) \\
&= \pi^{\mu}\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
$$

$$\boxed{\nabla_{\mu} T^{\mu}{}_{\nu} = \nabla_{\mu} T^{\mu\nu} = 0}$$

- The energy-momentum tensor $T^{\mu\nu}$ physically represents the flux of π^μ through the surface form $\bigwedge_{\alpha \neq \nu} dx^\alpha$. This corresponds to the flow of the field's energy($\mu = 0$)/momentum($\mu = 1, 2, 3$) along x^ν.
- But if we have flow of π^μ in the x^ν direction, then it implies an energy/momentum in the x^ν direction. The world line correpsonding to this flow must intersect with the former through a hypersurface of simulteinity, giving rise to equal $T^{\nu\mu}$.
- Thus, the energy-momentum tensor, being a geometric object with the mentioned physical meaning (motivated by particle and continuum dynamics), turns out to be symmetric,

$$\boxed{T^{\mu\nu} = T^{\nu\mu}}$$

Klein-Gordon Theory

Klein-Gordon Lagrangian

- The classical-field theoretic construction of Klein-Gordon theory begins by asking which Lagrangian yields a symmetric energy-momentum tensor. Such a theory, by virtue of respecting the physical meaning of the energy-momentum tensor, successfully describes many 'physically valid' systems.
- It turns out that the Klein-Gordon theory is deeply rooted in nature. In quantum mechanics, it is the theory for spin-0 particles. In quantum electrodynamics, the Klein-Gordon theory can be used to construct that of Dirac spinor fields, which describe all massive spin-1/2 particles such as electrons.

- Recall the energy-momentum tensor for a scalar field ϕ and its symmetry,

$$
\begin{aligned}
T^{\mu\nu} &= \pi^{\mu}\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L} \\
T^{\mu\nu} &= T^{\nu\mu}
\end{aligned}
$$

where $\pi^{\mu} = \dfrac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}$. Thus,

$$
\begin{aligned}
\pi^{\mu}\partial^{\nu}\phi - \cancel{\eta^{\mu\nu}\mathcal{L}} &= \pi^{\nu}\partial^{\mu}\phi - \cancel{\eta^{\nu\mu}\mathcal{L}} \qquad [\eta^{\mu\nu} = \eta^{\nu\mu}] \\
\pi^{\mu}\partial^{\nu}\phi &= \pi^{\nu}\partial^{\mu}\phi
\end{aligned}
$$

- The above is true in the most general case when π^{μ} is equal to $\partial^{\mu}\phi$, allowing us to exchange the product of conjugate momentum π^{μ} and generalized velocity $\partial^{\nu}\phi$ as above (by commutativity of component multiplication).

Therefore, we have,

$$\pi^{\mu} = \partial^{\mu}\phi$$
$$\implies \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)} = \partial^{\mu}\phi$$

'Integrating' over $\partial_{\mu}\phi$, we find that the Lagrangian is constrained to be of the form,

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V\left(\phi\right)}$$

- This is the Klein-Gordon Lagrangian $\mathcal{L}_{KG}$. Notice that it is analogous to the Lagrangian for classical mechanics, if we interpret $\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi$ as a kinetic term $T\left(\partial_{\mu}\phi\right)$ and $V\left(\phi\right)$ as a potential term.

Indeed,

$$\begin{aligned}
\pi^{\mu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)} \\
&= \frac{\partial}{\partial\left(\partial_{\mu}\phi\right)}\left[\frac{1}{2}\partial_{\alpha}\phi\partial^{\alpha}\phi - V\left(\phi\right)\right] \\
&= \frac{1}{2}\frac{\partial}{\partial\left(\partial_{\mu}\phi\right)}\left[\partial_{\alpha}\phi\partial^{\alpha}\phi\right] \\
&= \partial^{\alpha}\phi\frac{\partial}{\partial\left(\partial_{\mu}\phi\right)}\partial_{\alpha}\phi \\
&= \partial^{\alpha}\phi\delta^{\mu}{}_{\alpha} \\
&= \partial^{\mu}\phi
\end{aligned}$$

Klein-Gordon Equation

- Let us find the equation of motion for a Klein-Gordon field by plugging its conjugate momentum into the Euler-Lagrange equation:

$$\nabla_\mu \pi^\mu - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\implies \boxed{\nabla_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0}$$

$$\iff \Box \phi + \frac{\partial V}{\partial \phi} = 0$$

- This is the celebrated Klein-Gordon equation [for a scalar field in a potential]. In the absence of a potential, we obtain the wave equation $\Box \phi = 0$.

- For small oscillations of ϕ about local minima of the potential $V(\phi)$, only differences in ϕ physically matter. In the series expansion for $\frac{\partial V}{\partial \phi}$, we can set a vanishing first power term. Hence, in the series for $V(\phi)$, the first power term for ϕ vanishes, and so do cubic and higher terms,

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

- Plugging the above potential into the Klein-Gordon equation, we get the Klein-Gordon equation for a scalar field in a potential whose effects locally vanish:

$$\boxed{\nabla_\mu \partial^\mu \phi + m^2 \phi = 0}$$

- This is analogous to Hooke's law for harmonic oscillators, with mass assuming the role of the 'spring constant'. That's no coincidence – solutions to the above equation *are* systems of infinite harmonic oscillators at each point in spacetime!