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Bundles in Classical Gauge Field Theory

Canadian Undergraduate Mathematics Conference, University of Toronto, 2023

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Definition

- Let \mathcal{M} be a pseudo-Riemannian manifold with a metric g . A classical field ϕ of rank (p, q) is a differentiable tensor field living on \mathcal{M} i.e.,

As a differentiable tensor field

- $$\phi : \mathcal{M} \rightarrow \left(\prod_{i=1}^p V^* \times \prod_{j=1}^q V \rightarrow \mathbb{R} \right)$$

- $$\phi \in C(\mathcal{M})$$

where V is a vector space with \mathbb{R} as the base field.

- This is the starting point for defining classical fields. Additionally, they obey some physical properties discussed below.

Physical properties

1. Stationary-action principle
2. Local Lorentz invariance
3. Gauge invariance

Stationary-action Principle

- Let the function space of ϕ , i.e.

$$\left[\mathcal{M} \rightarrow \left(\prod_{i=1}^p V^* \times \prod_{j=1}^q V \rightarrow \mathbb{R} \right) \right] \cap C(\mathcal{M}), \text{ be denoted as } \mathcal{F}.$$

Definition (Lagrangian)

The Lagrangian [density] \mathcal{L} of a classical field ϕ is a differentiable map $\mathcal{L} : \mathcal{F} \times T^*\mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$.

Here, $T^*\mathcal{M}$ denotes the cotangent bundle of \mathcal{M} .

- However, we have not yet motivated bundles. Therefore, for now, we will think of $T^*\mathcal{M}$ as being set-theoretically isomorphic to the set of covariant derivatives of ϕ along every continuous curve γ in \mathcal{M} ,

$$T^*\mathcal{M} \cong_{\text{set}} \{ \nabla_{\gamma} \phi \mid \gamma : [0, 1] \rightarrow \mathcal{M} \text{ is continuous} \}$$

- By continuous curves, we refer to the topological notion of the continuity of maps from the topological space $\left([0, 1], \mathcal{O}_{\mathbb{R}}|_{[0,1]} \right)$ to $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$.

Here, $\mathcal{O}_{\mathbb{R}}|_{[0,1]}$ is the subspace topology induced on the unit interval by the Euclidean topology on \mathbb{R} and $\mathcal{O}_{\mathcal{M}}$ is the manifold topology on \mathcal{M} .

Definition (Action)

The action for a tensor field ϕ in a compact neighbourhood $U \subset \mathcal{M}$ is the linear functional,

$$S[\phi] := \int_{x \in U} \varepsilon \mathcal{L}(\phi(x), T_x^* \mathcal{M}, x)$$

where ε is the Riemannian volume form which in local coordinates can be written as,

$$\varepsilon := \sqrt{|\det(g)|} \bigwedge_{\mu} dx^{\mu}$$

In local coordinates, using index notation, the action can be covariantly written in terms of components as,

$$S[\phi(x^{\alpha})] = \int_U \varepsilon \mathcal{L}(\phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, \nabla_{\mu} \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, x^{\alpha})$$

where $\phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} = \bigcirc_{i=1}^p dx^{\rho_i} \circ \bigcirc_{j=1}^q \partial_{\lambda_j}(\phi)$.

Postulate (Stationary-principle action)

For on-shell trajectories $\phi \in \mathcal{F}$, we have the following for all compact neighbourhoods $U \subset \mathcal{M}$,

$$\delta S[\phi] = 0$$

i.e.,

$$\delta \int_U \varepsilon \mathcal{L} \left(\phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, x^\alpha \right) = 0$$

Theorem (Euler-Lagrange equations)

A classical field ϕ is on-shell i.e. obeys the principle of stationary action if and only if it satisfies the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} = 0$$

with summation over dummy indices implied (Einstein summation convention).

Proof.

$$\delta S = 0$$

$$\delta \int_U \varepsilon \mathcal{L} = 0$$

$$\int_U \varepsilon \left[\delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} + \delta \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \right] = 0$$

$$\int_U \varepsilon \left[\delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} + \nabla_\mu \left(\delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \right] = 0$$

Proof (continued).

$$\begin{aligned}
 & \int_U \varepsilon \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} \\
 & + \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \int_U \varepsilon \nabla_\mu \left(\delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \\
 & - \int_U \varepsilon \left[\nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \int \varepsilon \nabla_\mu \left(\delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \right] = 0
 \end{aligned}$$

$$\begin{aligned}
 & \int_U \varepsilon \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} \\
 & - \int_U \varepsilon \left[\delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \right] = 0
 \end{aligned}$$

Proof (continued).

$$\int_U \varepsilon \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left[\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q})} \right] = 0$$

Since the above is true for all compact neighbourhoods $U \subset \mathcal{M}$, by the fundamental lemma of the calculus of variations,

$$\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q})} = 0 \quad \square$$

The power of the above functional-analytic manipulations and notions is that the above statements are all logically equivalent, therefore proving 'S-A principle iff E-L equations'.

Local Lorentz Invariance

- ❖ Local Lorentz invariance is the idea that at each $p \in \mathcal{M}$, the action of the restricted Lorentz group $SO^+(1, 3)$ on tensorial objects living on $T_p\mathcal{M}$, leaves them invariant.
- ❖ This means that the components of a rank (p, q) tensor field T with components $T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}$ must transform covariantly with respect to the restricted Lorentz group.
In other words, we require that for any pair of primed and unprimed coordinate systems related by some transformation $\Lambda \in SO^+(1, 3)$, the following principle applies:

Postulate (Local Lorentz invariance)

$$T = T'$$

This simple principle has far-reaching consequences in theoretical physics, such as severe restriction induced on the form of physical laws and equations.

Theorem (Tensor component transformation law)

Invariance holds if and only if for a tensor field T , its components transform under any $\Lambda \in \text{SO}^+(1, 3)$ represented by (in terms of its action on the concerned tangent space) a Jacobian with components $\Lambda^{\mu'}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}$ as,

$$T^{\rho_1' \dots \rho_{p'}}_{\lambda_1' \dots \lambda_{q'}} = \left(\prod_{i=1}^p \Lambda^{\rho_i'}_{\rho_i} \right) T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left(\prod_{j=1}^q \Lambda^{\lambda_j}_{\lambda_j'} \right)$$

Proof.

By local Lorentz invariance,

$$\begin{aligned} T^{\rho_1' \dots \rho_{p'}}_{\lambda_1' \dots \lambda_{q'}} &:= T \left(dx^{\rho_1'}, \dots, dx^{\rho_{p'}}, \partial_{\lambda_1'}, \dots, \partial_{\lambda_{q'}} \right) \\ &= T \left(\frac{\partial x^{\rho_1'}}{\partial x^{\rho_1}} dx^{\rho_1}, \dots, \frac{\partial x^{\rho_{p'}}}{\partial x^{\rho_p}} dx^{\rho_p}, \frac{\partial x^{\lambda_1}}{\partial \lambda_1'} \partial_{\lambda_1}, \dots, \frac{\partial x^{\lambda_q}}{\partial \lambda_{q'}} \partial_{\lambda_q} \right) \end{aligned}$$

Proof (continued).

Since a tensor is a multilinear map,

$$\begin{aligned}
 T^{\rho_1' \dots \rho_{p'}'}_{\lambda_1' \dots \lambda_{q'}} &= \left(\prod_{i=1}^p \frac{\partial x^{\rho_{i'}}}{\partial x^{\rho_i}} \right) T(d x^{\rho_1}, \dots, d x^{\rho_p}, \partial_{\lambda_1}, \dots, \partial_{\lambda_q}) \left(\prod_{j=1}^q \frac{\partial x^{\lambda_j}}{\partial \lambda_{j'}} \right) \\
 &= \left(\prod_{i=1}^p \frac{\partial x^{\rho_{i'}}}{\partial x^{\rho_i}} \right) T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left(\prod_{j=1}^q \frac{\partial x^{\lambda_j}}{\partial \lambda_{j'}} \right) \\
 &= \left(\prod_{i=1}^p \Lambda^{\rho_{i'}}_{\rho_i} \right) T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left(\prod_{j=1}^q \dots \Lambda^{\lambda_j}_{\lambda_{j'}} \right)
 \end{aligned}$$

□

Gauge Invariance

Observational equivalence

In classical field theory, observational equivalence is the idea that two classical fields ψ and ϕ yielding identical physical quantities give rise to identical physical predictions.

- ✦ Typically, these physical quantities are geometric objects such as the curvature form $\Omega = d\phi + \phi \wedge \phi$ associated with ϕ .
- ✦ This gives rise to gauge freedom, wherein a classical field can contain physically redundant information in its representation as a differentiable tensor field.
- ✦ Therefore, given actual physical quantities in some context, such as the curvature form, there arise multiple ways to write the underlying classical field, each representation said to be a 'gauge' of the field.

Definition (Gauge of a classical field)

Formally, a gauge of a classical field ϕ can be thought of as some representative of the equivalence class $[\phi]$ defined by some equivalence relation (gauge invariance) of the form,

$$\forall \psi, \phi \in \mathcal{F} : \psi \sim \phi : \Longleftrightarrow \exists f \in G : f \cdot \psi = \phi$$

where (G, \cdot) is some group (called the gauge group of the concerned field) which preserves relevant physical quantities such as curvature.

- ✚ e.g. Consider the Newtonian gravitational field ϕ , which is a real-valued scalar field on a 3-dimensional *pseudo*-Riemannian manifold \mathcal{M} . Its curvature form is,

$$\begin{aligned}\Omega &= d\phi + \phi \wedge \phi \\ &= d\phi\end{aligned}$$

In local coordinates, the components of $\Omega = d\phi$ are $\Omega_i = \partial_i \phi$. This is identical (up to scaling) to the dual of the gravitational force field F^* . I.e.,

$$\begin{aligned}F^* &= -m d\phi \\ F_i &= -m \partial_i \phi\end{aligned}$$

- Since the force field is a physical entity, any gauge transformation of ϕ leaving its curvature form invariant, must be observationally equivalent to ϕ . An example of such a transformation is a translation dictated by the additive group of closed 1-forms ω ,

$$\begin{aligned}
 \phi &\mapsto \tilde{\phi} = \phi + \omega \\
 \Omega &\mapsto \tilde{\Omega} = d\tilde{\phi} \\
 &= d(\phi + \omega) \\
 &= d\phi + \cancel{d\omega} \\
 &= \Omega
 \end{aligned}$$

- Similarly, in electromagnetism, a gauge transformation of the potential 1-form A resembles translation under the additive group of 1-forms. This leaves the curvature form $F = dA$ invariant,

$$\begin{aligned}
 A &\mapsto \tilde{A} = A + d\alpha \\
 F &\mapsto \tilde{F} = d\tilde{A} \\
 &= d(A + d\alpha) \\
 &= dA + \cancel{d^2\alpha} \\
 &= F
 \end{aligned}$$

Fibres

Definition (Fibre)

The fibre $F(p)$ associated with a classical field ϕ , at a point $p \in \mathcal{M}$ is defined as,

$$F(p) := \bigcup_{\psi \in [\phi]} \{(p, \psi(p))\}$$

Intuitively, the fibre at a point is simply the set of values of the classical field in all its gauges, at that point.

Total Space

Definition (Total space)

The total space E associated with a classical field ϕ living on a spacetime \mathcal{M} is defined as,

$$E := \bigcup_{p \in \mathcal{M}} F(p)$$

Remark

$$\begin{aligned} E &= \bigcup_{p \in \mathcal{M}} F(p) \\ &= \bigcup_{p \in \mathcal{M}} \bigcup_{\psi \in [\phi]} \{(p, \psi(p))\} \\ &= \bigcup_{\psi \in [\phi]} \bigcup_{p \in \mathcal{M}} \{(p, \psi(p))\} \\ &\subseteq \mathcal{M} \times \mathbb{R} \end{aligned}$$

Projections

- ❖ Consider the following projection:

Projections $E \rightarrow \mathcal{M}$

$$\pi : \begin{cases} E & \rightarrow \mathcal{M} \\ (p, \psi(p)) & \mapsto p \\ \quad \in [\phi] & \end{cases}$$

- ❖ So far, we have been trying to build bundle-related notions algebraically rather than topologically. In this light, a projection $\pi : E \rightarrow \mathcal{M}$ can be viewed as an idempotent map from E to its subset \mathcal{M} ,

$$\pi \circ \pi = \pi$$

'Baby' Bundles

- ❖ A bundle formalizes the notion of a space living on another space (or a space parameterized by another space).
- ❖ Informally, we may imagine a bundle captures the idea of the graphs $\bigcup_{p \in \mathcal{M}} \{(p, \psi(p))\}$ of multiple fields ψ in the same gauge $[\phi]$, living on a spacetime \mathcal{M} .
- ❖ Such a structure (which we will call a 'baby' bundle as it does not yet incorporate topology :) is the tuple (E, π, \mathcal{M}) , often simply denoted as $E \xrightarrow{\pi} \mathcal{M}$.

Visualizing Bundles

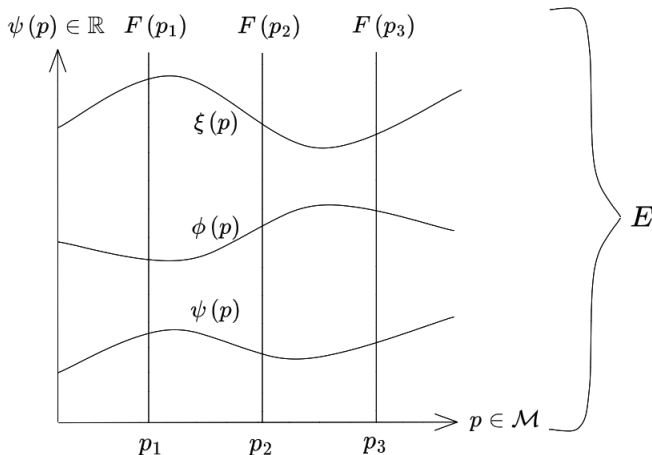


Figure: A bundle $E \xrightarrow{\pi} \mathcal{M}$. Note that $\psi \sim \phi \sim \xi$.

Topological Bundles

- ❖ In topology, a bundle is constructed by considering a total [topological] space (E, \mathcal{O}_E) , a base space (B, \mathcal{O}_B) and a continuous surjection $\pi : E \rightarrow B$.
 (E, π, B) or $E \xrightarrow{\pi} B$ is then said to be a [topological] bundle.
- ❖ The fibre at a point $p \in B$ is defined as,

$$\begin{aligned} F(p) &:= \text{preim}_{\pi}(\{p\}) \\ &:= \{x \in E : \pi(x) = p\} \end{aligned}$$

- ❖ A fibre bundle (E, B, π, F) or $E \rightarrow B \xleftarrow{\pi} F$ is a structure where $E \xrightarrow{\pi} B$ is a bundle and every fibre is homeomorphic to a manifold F , called the typical fibre of the fibre bundle,

$$\forall x \in E : \text{preim}_{\pi}(\{x\}) \cong_{\text{top}} F$$

Total Space

- ❖ In the field-theoretic situation we considered earlier, the total space associated with a rank (p, q) field on a spacetime \mathcal{M} is typically homeomorphic to a manifold of dimension $\dim(\mathcal{M}) + p + q$.
- ❖ We will consider Newtonian gravitation and classical electrodynamics on 3-dimensional Euclidean, and 4-dimensional Minkowski space, respectively.
- ❖ In the case of the Newtonian gravitational field ϕ , the total space is $\mathbb{R}^3 \times \mathbb{R}$ and this can be equipped with the Euclidean topology $\mathcal{O}_{\mathbb{R}^4}$.
- ❖ For the electromagnetic 4-potential A , the total space is $\mathbb{R}^4 \times \mathbb{R}^4$. This is Lorentzian, but we can make it Euclidean after a Wick rotation. In other words, the total space is isomorphic to \mathbb{R}^8 , which can then be equipped with the Euclidean topology $\mathcal{O}_{\mathbb{R}^8}$.

Product Bundle Structure

- ❖ With the above constructions, we find that the canonical projection $E \rightarrow \mathcal{M}$ we defined earlier is indeed continuous and surjective, for both the gravitational potential and electromagnetic 4-potential fields.
- ❖ Therefore, $(\mathbb{R}^3 \times \mathbb{R}, \pi_{\mathbb{R}^3}, \mathbb{R})$ is a bundle, known as a product bundle. The same goes for $(\mathbb{R}^4 \times \mathbb{R}^4, \pi_{\mathbb{R}^4}, \mathbb{R}^4)$ in the case of the electromagnetic field in flat spacetime.
- ❖ Furthermore, in each case, the fibres are isomorphic to \mathbb{R} and \mathbb{R}^4 , respectively. This means that the product bundles above are also fibre bundles.

Sections

Definition (Section)

A [cross-]section s of a bundle $E \xrightarrow{\pi} B$ is as a continuous inverse of π ,

$$\pi \circ s = \text{id}_B$$

Sections can be visualized in the following manner:

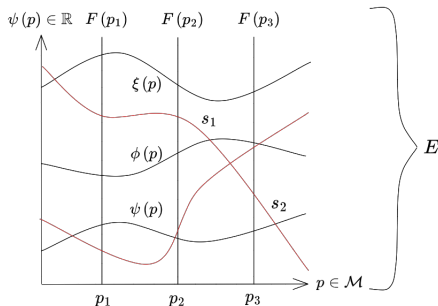


Figure: s_1 and s_2 are sections of the bundle $E \xrightarrow{\pi} B$.

- ❖ In the modern, geometric construction of classical field theory, classical fields are defined as sections of fibre bundles.
- ❖ The typical fibres of these fibre bundles are usually Lie groups (which are manifolds, as required).

References

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