

Applying the Klein-Gordon Theory to Gravitation

Modelling Newtonian gravitation as a classical scalar field
theory obeying Klein-Gordon structure

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SASMS, Feb 10, 2023

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Towards Classical Field Theory

The Inverse Square Law

- ▶ Gravitational force:

$$F_m = -G \frac{Mm}{r^2}$$

- ▶ Electrostatic force:

$$F_e = \frac{1}{4\pi\epsilon_0} \frac{Q_e q_e}{r^2}$$

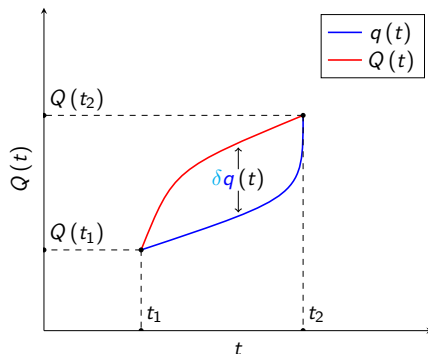
- ▶ Magnetic force:

$$F_b = \frac{\mu_0}{4\pi} \frac{Q_b q_b}{r^2}$$

Formal Analogies Between the Gravitational and Electrostatic Forces

	Gravitation	Static electricity
Newton's second law	$a^i = \underbrace{-\partial^i V}_{-\vec{\nabla} V}$	$E^i = \underbrace{-\partial^i \phi}_{-\vec{\nabla} \phi}$
Gauss' law	$\underbrace{\sum_{i=1}^3 \nabla_i a^i}_{\vec{\nabla} \cdot \vec{a}} = -4\pi G \rho_m$	$\underbrace{\sum_{i=1}^3 \nabla_i E^i}_{\vec{\nabla} \cdot \vec{a}} = \frac{1}{\epsilon_0} \rho_e$
Poisson's equation	$\underbrace{\sum_{i=1}^3 \nabla_i \partial^i V}_{\nabla^2 V} = 4\pi G \rho_m$	$\underbrace{\sum_{i=1}^3 \nabla_i \partial^i \phi}_{\nabla^2 \phi} = -\frac{1}{\epsilon_0} \rho_e$

Lagrangian Mechanics



- Nature 'selects' the unique on-shell trajectory $q(t)$ given the boundary conditions $(t_1, Q(t_1))$ and $(t_2, Q(t_2))$ for a system.

$$\underbrace{Q(t)}_{\text{Off-shell}} = \underbrace{q(t)}_{\text{On-shell}} + \underbrace{\delta q(t)}_{\text{Variation}}$$
$$\delta q(t_1) = \delta q(t_2) = 0$$

- ▶ Each trajectory $Q(t)$ between the endpoints is associated with a corresponding number called the action.

$$S[Q(t)](t_1, t_2) = \int_{t_1}^{t_2} dt L(Q(t), \dot{Q}(t), t)$$

The integrand $L(Q(t), \dot{Q}(t), t)$ is known as the Lagrangian of the system being modelled and encodes the dynamics of the system.

- ▶ In general, the action S maps $Q(t)$ to a real number determined by the above integral. Therefore, it is a functional, i.e. a higher-order function which takes in infinite values of the form $\{(t, Q(t)) : t \in \mathbb{R}\}$ and spits out a real.

$$S : \begin{cases} \mathbb{R}^{\mathbb{R}} & \rightarrow \mathbb{R} \\ Q(t) & \mapsto \int_{t_1}^{t_2} dt L(Q(t), \dot{Q}(t), t) \end{cases}$$

Principle of Stationary Action

Lagrange's principle of stationary action

Suppose we vary $q(t)$ about its on-shell evolution as,
 $q(t) \rightarrow q(t) + \delta q(t)$. Then, the variation in the action satisfies,

$$\delta S \in \mathcal{O}(\delta q^2)$$

Corollary (First-order approximation)

For very small $\delta q(t)$ i.e.,

$$\forall \delta q(t) = \lim_{\epsilon \rightarrow 0} \epsilon \eta(t) : \eta(t_1) = \eta(t_2) = 0 :$$

$$\delta S \in \mathcal{O}(\epsilon^2 \eta(t)) = \{0\}$$

$$\implies \boxed{\delta S = 0}$$

Euler-Lagrange Equation

Lemma (Fundamental lemma of calculus of variations)

The former is possible if and only if the latter is,

$$\begin{aligned}\forall \delta q : \int_{t_1}^{t_2} dt \delta q f(q, \dot{q}, t) &= 0 \\ \iff \forall t \in (t_1, t_2) : f(q, \dot{q}, t) &= 0\end{aligned}$$

Theorem

An on-shell $q(t)$ obeying the principle of stationary action for a given $L(q, \dot{q}, t)$ must also obey the Euler-Lagrange equation of motion:

$\frac{\partial L}{\partial q}$	$=$	$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$	$=$	$\frac{dp}{dt}$
$\underbrace{\hspace{1cm}}$		$\underbrace{\hspace{1cm}}$		
Generalized force		Conjugate momentum		

Proof.

$$\delta S = 0 \quad [\text{Principle of stationary action}]$$

$$\delta \int_{t_1}^{t_2} dt L(q, \dot{q}, t) = 0$$

$$\int_{t_1}^{t_2} dt \delta L(q, \dot{q}, t) = 0 \quad [\text{Additivity of variations}]$$

$$\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \cancel{\delta t} \frac{\partial L}{\partial t} \right] = 0 \quad [\text{Chain rule for variations}]$$

$$\int_{t_1}^{t_2} dt \left[\delta q \frac{\partial L}{\partial q} + (\delta \dot{q}) \frac{\partial L}{\partial \dot{q}} \right] = 0 \quad [\text{Commutativity of derivatives}]$$

$$\int_{t_1}^{t_2} dt \delta q \frac{\partial L}{\partial q} + \int_{t_1}^{t_2} dt (\delta \dot{q}) \frac{\partial L}{\partial \dot{q}} = 0$$

Proof.

$$\int_{t_1}^{t_2} dt \, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \int_{t_1}^{t_2} dt \, (\delta \dot{q}) - \int_{t_1}^{t_2} dt \left[\int dt \, (\delta \dot{q}) \right] \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

[Integration by parts]

$$\int_{t_1}^{t_2} dt \, \delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} [\cancel{\delta q}]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \, \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$[\delta q(t_1) = \delta q(t_2)]$$

$$\forall \delta q : \int_{t_1}^{t_2} dt \, \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$\iff \frac{\partial L}{\partial q} - \frac{dp}{dt} = 0 \quad \square$$

[Fundamental lemma of the calculus of variations]

Noether's Theorem

Theorem (Noether's first theorem)

If the action $S[q(t)]$ remains invariant under perturbations of the following form,

$$q \rightarrow q + \delta q$$

then the following quantity is conserved,

$$j = p \delta q$$
$$\frac{dj}{dt} = 0$$

Proof.

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \\ &= \dot{p} \delta q + p \delta \dot{q} \quad [\text{Euler-Lagrange equation}] \\ &= \frac{d}{dt} (p \delta q)\end{aligned}$$

But $\delta L = 0$

$$\implies \frac{d}{dt} (p \delta q) = 0$$

□

Example

If $S[q(t)]$ is symmetric (i.e. conserved) under a small time-independent translation $q \rightarrow q + \epsilon$, we obtain the invariant $j = p\epsilon$. Since $\frac{dj}{dt} = 0$, $\frac{d\epsilon}{dt} = 0$, we get $\frac{dp}{dt} = 0$.

Classical Mechanics

- ▶ The Lagrangian for classical mechanics is of the form,

$$\begin{aligned} L(q, \dot{q}, t) &= T(\dot{q}) - V(q) \\ &= \frac{1}{2} m g \dot{q}^2 - V(q) \\ &= \frac{1}{2} m v^2 - V(q) \end{aligned}$$

- ▶ The equation of motion obtained by applying the Euler-Lagrange equation to the above Lagrangian is,

$$\frac{d}{dt}(mv) + \frac{\partial V}{\partial q} = 0$$

This is Newton's second law. If the entire system concerned is symmetric under small translations on q , we have $\frac{\partial V}{\partial q} = 0$ implying $\frac{d}{dt}(mv) = 0$. This is Newton's third law.

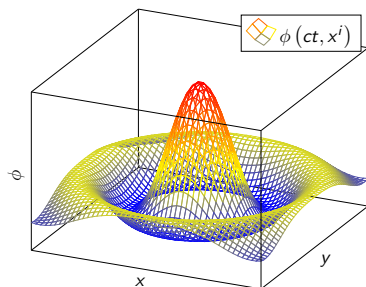
Classical Field Theory

- ▶ A classical field is a tensor field on spacetime (which is a pseudo-Riemannian manifold obeying dynamical field equations such as the Einstein field equations).

Therefore, a classical field is some rank (p, q) tensor $\phi^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}(x^\alpha)$ at each point x^α in space and time with $\alpha \in (0, 1, 2, 3)$.

- ▶ A classical field obeys the following principles:
 1. Principle of stationary action
 2. Local Lorentz invariance
 3. Locality
 4. Gauge invariance
- ▶ The simplest classical field theory is that of rank $(0, 0)$ tensor fields i.e. scalar fields $\phi(x^\alpha)$, in a flat spacetime \mathcal{M} . We will study such fields in the following slides.

Principle of Stationary Action for Classical Fields



- To construct the action for a particle, we integrated its Lagrangian between endpoints in time. A field such as $\phi(x^\alpha)$, however, lives in space and time. Therefore, its action is a *volume* integral of a Lagrangian *density* \mathcal{L} , in a 4-dimensional region of spacetime $\Omega \subset \mathcal{M}$,

$$S[\phi(x^\alpha)] = \int_{\Omega} d^4x \mathcal{L}(\phi(x^\alpha), \partial_\mu \phi(x^\alpha), x^\nu)$$

- ▶ The Lagrangian density is so-called as it looks like a Lagrangian (integrable over some time interval $\Omega^{(1)}$) when integrated over a region of space $\Omega^{(3)}$:

$$\begin{aligned}
 L(\phi(x^\alpha), \partial_\mu \phi(x^\alpha), x^\nu) &= \int_{\Omega^{(3)}} d^3x \mathcal{L}(\phi(x^\alpha), \partial_\mu \phi(x^\alpha), x^\nu) \\
 S[\phi(x^\alpha)] &= \int_{\Omega} d^4x \mathcal{L}(\phi(x^\alpha), \partial_\mu \phi(x^\alpha), x^\nu) \\
 &= \int_{\Omega^{(1)}} c dt L(\phi(x^\alpha), \partial_\mu \phi(x^\alpha), x^\nu)
 \end{aligned}$$

- ▶ The principle of stationary action for fields states that for small variations $\delta\phi$ of a field ϕ in its on-shell configuration, the action remains stationary,

$$\boxed{\delta S = 0}$$

Euler-Lagrange Equation for Classical Fields

Lemma (Fundamental lemma of multivariable calculus of variations)

$$\begin{aligned}\forall \delta\phi : \int_{\Omega} d^4x \delta\phi f(\phi, \partial_{\mu}\phi, x^{\nu}) &= 0 \\ \iff \forall x^{\alpha} \in \Omega \setminus \partial\Omega : f(\phi, \partial_{\mu}\phi, x^{\nu}) &= 0\end{aligned}$$

Einstein summation convention

Dummy indices, i.e. pairs of upper and lower tensor indices, are implicitly summed over.

Example

$$A_{\mu}B^{\mu} = \sum_{\mu=0}^3 A_{\mu}B^{\mu}$$

Theorem

A field ϕ obeys the principle of stationary action if and only if it also satisfies,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \nabla_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}}_{\text{Conjugate momentum tensor}} = \nabla_\mu \pi^\mu$$

Proof.

$$\delta S = 0 \quad [\text{Principle of stationary action}]$$

$$\delta \int_{\Omega} d^4x \mathcal{L} = 0$$

$$\int_{\Omega} d^4x \delta \mathcal{L} = 0 \quad [\text{Additivity of variations}]$$

$$\int_{\Omega} d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta(\partial_{\mu}\phi) \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}}_{\pi^{\mu}} + \cancel{\delta x^{\mu}} \partial_{\mu} \mathcal{L} \right] = 0$$

[Multivariable chain rule for variations]

$$\int_{\Omega} d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + (\partial_{\mu}\delta\phi) \pi^{\mu} \right] = 0$$

[Commutativity of variations and covariant derivatives]

$$\int_{\Omega} d^4x \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \pi^{\mu} \underbrace{\int_{\Omega} d^4x \partial_{\mu}\delta\phi}_{\text{Constant surface term}} - \int_{\Omega} d^4x \left[\int d^4x \partial_{\mu}\delta\phi \right] \nabla_{\mu} \pi^{\mu} = 0$$

[Volume integration by parts]

Using Stokes' theorem, the constant surface term can be set to 0.
We then find,

$$\begin{aligned}\int_{\Omega} d^4x \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} - \int_{\Omega} d^4x \delta\phi \nabla_{\mu} \pi^{\mu} &= 0 \\ \int_{\Omega} d^4x \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_{\mu} \pi^{\mu} \right) &= 0 \\ \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_{\mu} \pi^{\mu} &= 0 \\ \iff \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} &= 0 \quad \square\end{aligned}$$

[Fundamental lemma of multivariable calculus of variations]

Noether's Theorem for Classical Fields