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## **Bundles in Classical Gauge Field Theory**

Canadian Mathematics Undergraduate Conference, University of Toronto, 2023

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## Definition

- Let  $\mathcal{M}$  be a pseudo-Riemannian manifold with a metric  $g$ . A classical field  $\phi$  of rank  $(p, q)$  is a differentiable tensor field living on  $\mathcal{M}$  i.e.,

As a differentiable tensor field

- $$\phi : \mathcal{M} \rightarrow \left( \prod_{i=1}^p V^* \times \prod_{j=1}^q V \rightarrow \mathbb{R} \right)$$

- $$\phi \in C(\mathcal{M})$$

where  $V$  is a vector space with  $\mathbb{R}$  as the base field.

- This is the starting point for defining classical fields. Additionally, they obey some physical properties discussed below.

### Physical properties

1. Stationary-action principle
2. Local Lorentz invariance
3. Gauge invariance

# Stationary-action Principle

- Let the function space of  $\phi$ , i.e.

$$\left[ \mathcal{M} \rightarrow \left( \prod_{i=1}^p V^* \times \prod_{j=1}^q V \rightarrow \mathbb{R} \right) \right] \cap C(\mathcal{M}), \text{ be denoted as } \mathcal{F}.$$

## Definition (Lagrangian)

The Lagrangian [density]  $\mathcal{L}$  of a classical field  $\phi$  is a differentiable map  $\mathcal{L} : \mathcal{F} \times T^*\mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ .

Here,  $T^*\mathcal{M}$  denotes the cotangent bundle of  $\mathcal{M}$ .

- However, we have not yet motivated bundles. Therefore, for now, we will think of  $T^*\mathcal{M}$  as being set-theoretically isomorphic to the set of covariant derivatives of  $\phi$  along every continuous curve  $\gamma$  in  $\mathcal{M}$ ,

$$T^*\mathcal{M} \cong_{\text{set}} \{ \nabla_{\gamma} \phi \mid \gamma : [0, 1] \rightarrow \mathcal{M} \text{ is continuous} \}$$

- By continuous curves, we refer to the topological notion of the continuity of maps from the topological space  $\left( [0, 1], \mathcal{O}_{\mathbb{R}}|_{[0,1]} \right)$  to  $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ .

Here,  $\mathcal{O}_{\mathbb{R}}|_{[0,1]}$  is the subspace topology induced on the unit interval by the Euclidean topology on  $\mathbb{R}$  and  $\mathcal{O}_{\mathcal{M}}$  is the manifold topology on  $\mathcal{M}$ .

## Definition (Action)

The action for a tensor field  $\phi$  in a compact neighbourhood  $U \subset \mathcal{M}$  is the linear functional,

$$S[\phi] := \int_{x \in U} \varepsilon \mathcal{L}(\phi(x), T_x^* \mathcal{M}, x)$$

where  $\varepsilon$  is the Riemannian volume form which in local coordinates can be written as,

$$\varepsilon := \sqrt{|\det(g)|} \bigwedge_{\mu} dx^{\mu}$$

In local coordinates, using index notation, the action can be covariantly written in terms of components as,

$$S[\phi(x^{\alpha})] = \int_U \varepsilon \mathcal{L}(\phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, \nabla_{\mu} \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, x^{\alpha})$$

where  $\phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} = \bigcirc_{i=1}^p dx^{\rho_i} \circ \bigcirc_{j=1}^q \partial_{\lambda_j}(\phi)$ .

## Postulate (Stationary-principle action)

*For on-shell trajectories  $\phi \in \mathcal{F}$ , we have the following for all compact neighbourhoods  $U \subset \mathcal{M}$ ,*

$$\delta S[\phi] = 0$$

i.e.,

$$\delta \int_U \varepsilon \mathcal{L} \left( \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}, x^\alpha \right) = 0$$

## Theorem (Euler-Lagrange equations)

A classical field  $\phi$  is on-shell i.e. obeys the principle of stationary action if and only if it satisfies the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} = 0$$

with summation over dummy indices implied (Einstein summation convention).

Proof.

$$\delta S = 0$$

$$\delta \int_U \varepsilon \mathcal{L} = 0$$


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$$\int_U \varepsilon \left[ \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} + \delta \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \frac{\partial \mathcal{L}}{\partial \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \right] = 0$$


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$$\int_U \varepsilon \left[ \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} + \nabla_\mu \left( \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \frac{\partial \mathcal{L}}{\partial \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \right] = 0$$

Proof (continued).

$$\begin{aligned}
 & \int_U \varepsilon \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} \\
 & + \frac{\partial \mathcal{L}}{\partial \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \int_U \varepsilon \nabla_\mu \left( \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \\
 & - \int_U \varepsilon \left[ \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \int \varepsilon \nabla_\mu \left( \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right) \right] = 0
 \end{aligned}$$


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$$\begin{aligned}
 & \int_U \varepsilon \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} \\
 & - \int_U \varepsilon \left[ \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left( \nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \right)} \right] = 0
 \end{aligned}$$



Proof (continued).

$$\int_U \varepsilon \delta \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left[ \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q})} \right] = 0$$

Since the above is true for all compact neighbourhoods  $U \subset \mathcal{M}$ , by the fundamental lemma of the calculus of variations,

$$\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q})} = 0 \quad \square$$

The power of the above functional-analytic manipulations and notions is that the above statements are all logically equivalent, therefore proving 'S-A principle iff E-L equations'.

## Local Lorentz Invariance

- ❖ Local Lorentz invariance is the idea that at each  $p \in \mathcal{M}$ , the action of the restricted Lorentz group  $SO^+(1, 3)$  on tensorial objects living on  $T_p\mathcal{M}$ , leaves them invariant.
- ❖ This means that the components of a rank  $(p, q)$  tensor field  $T$  with components  $T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q}$  must transform covariantly with respect to the restricted Lorentz group.  
In other words, we require that for any pair of primed and unprimed coordinate systems related by some transformation  $\Lambda \in SO^+(1, 3)$ , the following principle applies:

Postulate (Local Lorentz invariance)

$$T = T'$$

This simple principle has far-reaching consequences in theoretical physics, such as severe restriction induced on the form of physical laws and equations.

## Theorem (Tensor component transformation law)

Invariance holds if and only if for a tensor field  $T$ , its components transform under any  $\Lambda \in \text{SO}^+(1, 3)$  represented by (in terms of its action on the concerned tangent space) a Jacobian with components  $\Lambda^{\mu'}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}$  as,

$$T^{\rho_1' \dots \rho_{p'}}_{\lambda_1' \dots \lambda_{q'}} = \left( \prod_{i=1}^p \Lambda^{\rho_i'}_{\rho_i} \right) T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left( \prod_{j=1}^q \Lambda^{\lambda_j}_{\lambda_j'} \right)$$

## Proof.

By local Lorentz invariance,

$$\begin{aligned} T^{\rho_1' \dots \rho_{p'}}_{\lambda_1' \dots \lambda_{q'}} &:= T \left( dx^{\rho_1'}, \dots, dx^{\rho_{p'}}, \partial_{\lambda_1'}, \dots, \partial_{\lambda_{q'}} \right) \\ &= T \left( \frac{\partial x^{\rho_1'}}{\partial x^{\rho_1}} dx^{\rho_1}, \dots, \frac{\partial x^{\rho_{p'}}}{\partial x^{\rho_p}} dx^{\rho_p}, \frac{\partial x^{\lambda_1}}{\partial \lambda_1'} \partial_{\lambda_1}, \dots, \frac{\partial x^{\lambda_q}}{\partial \lambda_{q'}} \partial_{\lambda_q} \right) \end{aligned}$$

Proof (continued).

Since a tensor is a multilinear map,

$$\begin{aligned}
 T^{\rho_1' \dots \rho_{p'}'}_{\lambda_1' \dots \lambda_{q'}} &= \left( \prod_{i=1}^p \frac{\partial x^{\rho_{i'}}}{\partial x^{\rho_i}} \right) T(d x^{\rho_1}, \dots, d x^{\rho_p}, \partial_{\lambda_1}, \dots, \partial_{\lambda_q}) \left( \prod_{j=1}^q \frac{\partial x^{\lambda_j}}{\partial \lambda_{j'}} \right) \\
 &= \left( \prod_{i=1}^p \frac{\partial x^{\rho_{i'}}}{\partial x^{\rho_i}} \right) T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left( \prod_{j=1}^q \frac{\partial x^{\lambda_j}}{\partial \lambda_{j'}} \right) \\
 &= \left( \prod_{i=1}^p \Lambda^{\rho_{i'}}_{\rho_i} \right) T^{\rho_1 \dots \rho_p}_{\lambda_1 \dots \lambda_q} \left( \prod_{j=1}^q \dots \Lambda^{\lambda_j}_{\lambda_{j'}} \right) \quad \square
 \end{aligned}$$

# Gauge Invariance

## Observational equivalence

In classical field theory, observational equivalence is the idea that two classical fields  $\psi$  and  $\phi$  yielding identical physical quantities give rise to identical physical predictions.

- ✦ Typically, these physical quantities are geometric objects such as the curvature form  $\Omega = d\phi + \phi \wedge \phi$  associated with  $\phi$ .
- ✦ This gives rise to gauge freedom, wherein a classical field can contain physically redundant information in its representation as a differentiable tensor field.
- ✦ Therefore, given actual physical quantities in some context, such as the curvature form, there arise multiple ways to write the underlying classical field, each representation said to be a 'gauge' of the field.

## Definition (Gauge of a classical field)

Formally, a gauge of a classical field  $\phi$  can be thought of as some representative of the equivalence class  $[\phi]$  defined by some equivalence relation (gauge invariance) of the form,

$$\forall \psi, \phi \in \mathcal{F} : \psi \sim \phi : \Longleftrightarrow \exists f \in G : f \cdot \psi = \cdot \phi$$

where  $(G, \cdot)$  is some group (called the gauge group of the concerned field) which preserves relevant physical quantities such as curvature.

- ✚ e.g. Consider the Newtonian gravitational field  $\phi$ , which is a real-valued scalar field on a 3-dimensional *pseudo*-Riemannian manifold  $\mathcal{M}$ . Its curvature form is,

$$\begin{aligned}\Omega &= d\phi + \phi \wedge \phi \\ &= d\phi\end{aligned}$$

In local coordinates, the components of  $\Omega = d\phi$  are  $\Omega_i = \partial_i \phi$ . This is identical (up to scaling) to the dual of the gravitational force field  $F^*$ . I.e.,

$$\begin{aligned}F^* &= -m d\phi \\ F_i &= -m \partial_i \phi\end{aligned}$$

- Since the force field is a physical entity, any gauge transformation of  $\phi$  leaving its curvature form invariant, must be observationally equivalent to  $\phi$ . An example of such a transformation is a translation dictated by the additive group of closed 1-forms  $\omega$ ,

$$\begin{aligned}
 \phi &\mapsto \tilde{\phi} = \phi + \omega \\
 \Omega &\mapsto \tilde{\Omega} = d\tilde{\phi} \\
 &= d(\phi + \omega) \\
 &= d\phi + \cancel{d\omega} \\
 &= \Omega
 \end{aligned}$$

- Similarly, in electromagnetism, a gauge transformation of the potential 1-form  $A$  resembles translation under the additive group of 1-forms. This leaves the curvature form  $F = dA$  invariant,

$$\begin{aligned}
 A &\mapsto \tilde{A} = A + d\alpha \\
 F &\mapsto \tilde{F} = d\tilde{A} \\
 &= d(A + d\alpha) \\
 &= dA + \cancel{d^2\alpha} \\
 &= F
 \end{aligned}$$

# Fibres

## Definition (Fibre)

The fibre  $F(p)$  associated with a classical field  $\phi$ , at a point  $p \in \mathcal{M}$  is defined as,

$$F(p) := \bigcup_{\psi \in [\phi]} \{(p, \psi(p))\}$$

Intuitively, the fibre at a point is simply the set of values of the classical field in all its gauges, at that point.



# Total Space

## Definition (Total space)

The total space  $E$  associated with a classical field  $\phi$  living on a spacetime  $\mathcal{M}$  is defined as,

$$E := \bigcup_{p \in \mathcal{M}} F(p)$$

## Remark

$$\begin{aligned} E &= \bigcup_{p \in \mathcal{M}} F(p) \\ &= \bigcup_{p \in \mathcal{M}} \bigcup_{\psi \in [\phi]} \{(p, \psi(p))\} \\ &= \bigcup_{\psi \in [\phi]} \bigcup_{p \in \mathcal{M}} \{(p, \psi(p))\} \\ &\subseteq \mathcal{M} \times \mathbb{R} \end{aligned}$$

# Projections

- ❖ Consider the following projection:

Projections  $E \rightarrow \mathcal{M}$

$$\pi : \begin{cases} E & \rightarrow \mathcal{M} \\ (p, \psi(p)) & \mapsto p \\ \quad \in [\phi] & \end{cases}$$

- ❖ So far, we have been trying to build bundle-related notions algebraically rather than topologically. In this light, a projection  $\pi : E \rightarrow \mathcal{M}$  can be viewed as an idempotent map from  $E$  to its subset  $\mathcal{M}$ ,

$$\pi \circ \pi = \pi$$

## 'Baby' Bundles

- ❖ A bundle formalizes the notion of a space living on another space (or a space parameterized by another space).
- ❖ Informally, we may imagine a bundle captures the idea of the graphs  $\bigcup_{p \in \mathcal{M}} \{(p, \psi(p))\}$  of multiple fields  $\psi$  in the same gauge  $[\phi]$ , living on a spacetime  $\mathcal{M}$ .
- ❖ Such a structure (which we will call a 'baby' bundle as it does not yet incorporate topology :) is the tuple  $(E, \pi, \mathcal{M})$ , often simply denoted as  $E \xrightarrow{\pi} \mathcal{M}$ .

# Visualizing Bundles

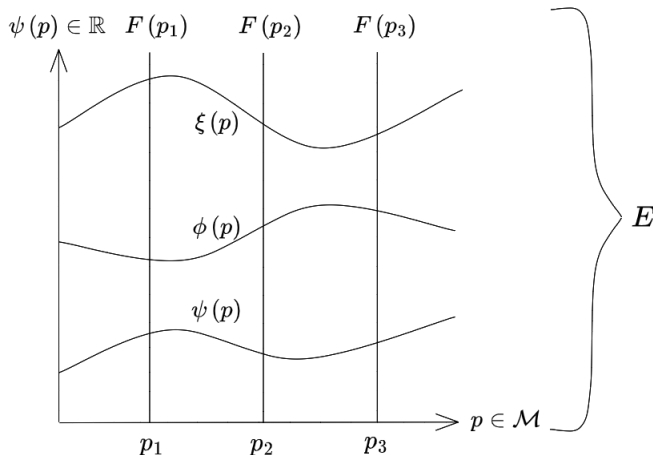


Figure: A bundle  $E \xrightarrow{\pi} \mathcal{M}$ . Note that  $\psi \sim \phi \sim \xi$ .

# Topological Bundles

- ❖ In topology, a bundle is constructed by considering a total [topological] space  $(E, \mathcal{O}_E)$ , a base space  $(B, \mathcal{O}_B)$  and a continuous surjection  $\pi : E \rightarrow B$ .  
 $(E, \pi, B)$  or  $E \xrightarrow{\pi} B$  is then said to be a [topological] bundle.
- ❖ The fibre at a point  $p \in B$  is defined as,

$$\begin{aligned} F(p) &:= \text{preim}_{\pi}(\{p\}) \\ &:= \{x \in E : \pi(x) = p\} \end{aligned}$$

- ❖ A fibre bundle  $(E, B, \pi, F)$  or  $E \rightarrow B \xleftarrow{\pi} F$  is a structure where  $E \xrightarrow{\pi} B$  is a bundle and every fibre is homeomorphic to a manifold  $F$ , called the typical fibre of the fibre bundle,

$$\forall x \in E : \text{preim}_{\pi}(\{x\}) \cong_{\text{top}} F$$

# Total Space

- ❖ In the field-theoretic situation we considered earlier, the total space associated with a rank  $(p, q)$  field on a spacetime  $\mathcal{M}$  is typically homeomorphic to a manifold of dimension  $\dim(\mathcal{M}) + p + q$ .
- ❖ We will consider Newtonian gravitation and classical electrodynamics on 3-dimensional Euclidean, and 4-dimensional Minkowski space, respectively.
- ❖ In the case of the Newtonian gravitational field  $\phi$ , the total space is  $\mathbb{R}^3 \times \mathbb{R}$  and this can be equipped with the Euclidean topology  $\mathcal{O}_{\mathbb{R}^4}$ .
- ❖ For the electromagnetic 4-potential  $A$ , the total space is  $\mathbb{R}^4 \times \mathbb{R}^4$ . This is Lorentzian, but we can make it Euclidean after a Wick rotation. In other words, the total space is isomorphic to  $\mathbb{R}^8$ , which can then be equipped with the Euclidean topology  $\mathcal{O}_{\mathbb{R}^8}$ .

## Product Bundle Structure

- ❖ With the above constructions, we find that the canonical projection  $E \rightarrow \mathcal{M}$  we defined earlier is indeed continuous and surjective, for both the gravitational potential and electromagnetic 4-potential fields.
- ❖ Therefore,  $(\mathbb{R}^3 \times \mathbb{R}, \pi_{\mathbb{R}^3}, \mathbb{R})$  is a bundle, known as a product bundle. The same goes for  $(\mathbb{R}^4 \times \mathbb{R}^4, \pi_{\mathbb{R}^4}, \mathbb{R}^4)$  in the case of the electromagnetic field in flat spacetime.
- ❖ Furthermore, in each case, the fibres are isomorphic to  $\mathbb{R}$  and  $\mathbb{R}^4$ , respectively. This means that the product bundles above are also fibre bundles.

# Sections

## Definition (Section)

A [cross-]section  $s$  of a bundle  $E \xrightarrow{\pi} B$  is as a continuous inverse of  $\pi$ ,

$$\pi \circ s = \text{id}_B$$

Sections can be visualized in the following manner:

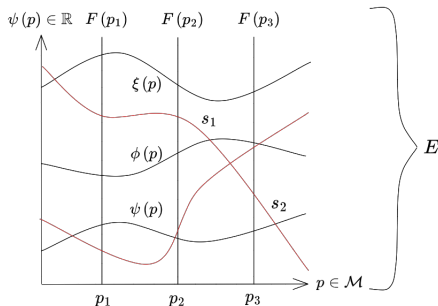


Figure:  $s_1$  and  $s_2$  are sections of the bundle  $E \xrightarrow{\pi} B$ .



- ❖ In the modern, geometric construction of classical field theory, classical fields are defined as sections of fibre bundles.
- ❖ The typical fibres of these fibre bundles are usually Lie groups (which are manifolds, as required).

# References

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