

# Loop Quantum Gravity Classical Mechanics: Algebraic Topology for Non-holonomic Mechanics

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There is no clear-cut distinction between example and theory.

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## $n$ -Simplices (Triangles!)

The **standard  $n$ -simplex**  $\Delta^n \subsetneq \mathbb{R}^{n+1}$  is the *convex hull* of the standard basis vectors  $\{e_0, \dots, e_n\}$ ,

$$\Delta^n = \left\{ \sum_i t_i e_i : \sum_i t_i = 1; t_0, \dots, t_n \geq 0 \right\}$$

$(t_0, \dots, t_n)$  are the **barycentric coordinates** of  $\Delta^n$ . Associated with a standard  $n$  simplex are its **face (inclusion) maps** (opposite to the  $e_i$ ) which are *affine linear maps*  $\phi_i^n : \Delta^{n-1} \hookrightarrow \Delta^n$  with,

$$\phi_i^n = (e_0, \dots, \widehat{e_i}, \dots, e_n)^\flat : (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

where the widehat denotes the omission of the  $i$ -th index. The  $i$ -th **face** of  $\Delta^n$  is the *subsimplex*  $\phi_i^n(\Delta^{n-1})$ . The boundary of a non-oriented  $n$ -simplex then consists (up to permutation) the union of all its faces.

Standard  $n$ -simplices are a well-understood and surprisingly rich class of topological spaces. It is therefore convenient to understand an arbitrary topological space  $X$  and its topological invariants via continuous maps from  $n$ -simplices.

## Singular $n$ -simplices (Curved triangles!)

A **singular  $n$ -simplex in  $X$**  is a  $C^1$  map  $\sigma_n : \Delta^n \rightarrow X$ . The set of all (singular)  $n$ -simplices in  $X$  is denoted as  $\text{Sing}_n(X)$ . Analogous to  $n$ -simplices, singular  $n$ -simplices are associated with **face maps**  $d_i^n : \text{Sing}_n(X) \rightarrow \text{Sing}_{n-1}(X)$ ,  $\sigma_n \mapsto \sigma_n^{(i)} = \sigma_n \circ \phi_i^n$  with the  $i$ -th **face** of  $\sigma_n$  being  $d_i^n(\sigma_n) = \sigma_n^{(i)}$ .

The above information can be captured as an example commutative diagram,

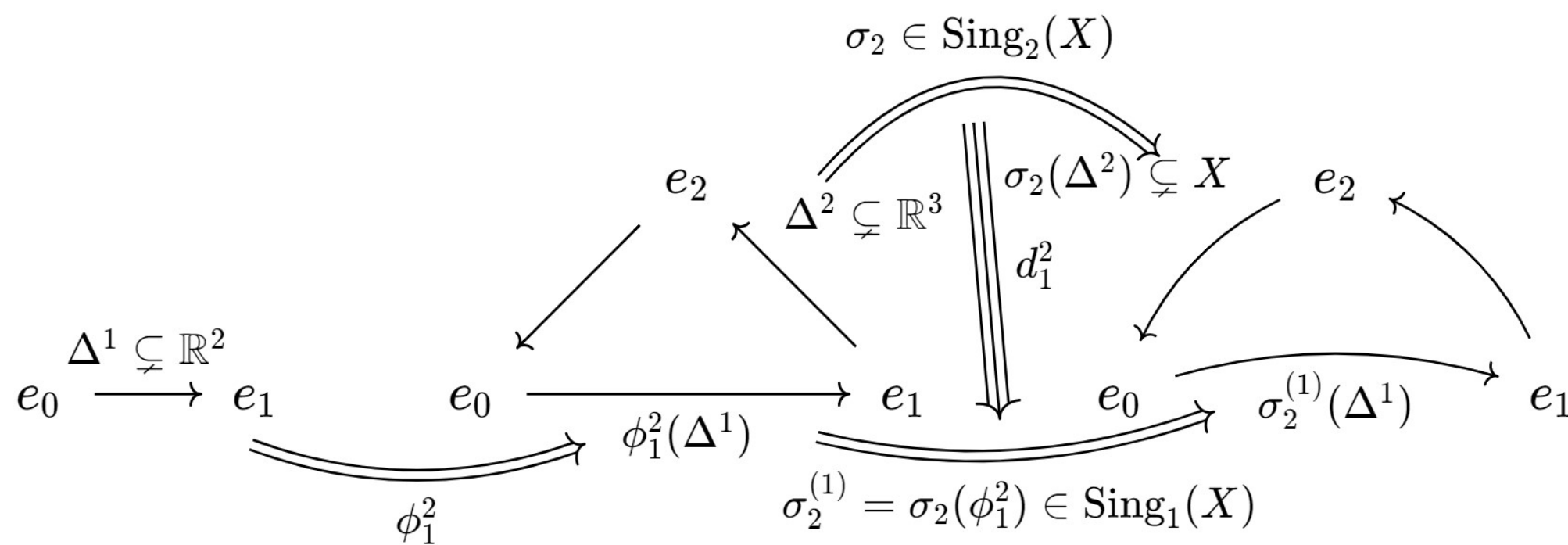


Figure 1. The standard 1-simplex  $\Delta^1$ ; face maps  $\phi_i^2$  to the standard 2-simplex  $\Delta^2$ ; and a singular 2-simplex  $\sigma_2$  in  $X$ .

To talk about the boundary of  $n$ -simplices in  $X$ , we will need to construct a *free Abelian group* structure on simplices.

## Singular $n$ -chains (Triangles form modules, yay!)

The **singular  $n$ -chains** of  $X$  are members of the *free Abelian group* generated by singular  $n$ -simplices,

$$S_n(X) = \mathbb{Z} \text{Sing}_n(X)$$

Therefore, an  $n$ -chain is a *finite  $\mathbb{Z}$ -linear combination of simplices*,  $\sum_{i \in I \subsetneq \mathbb{N}} a_i \sigma_i$  s.t. for all  $i \in I$ , we have,  $a_i \in \mathbb{Z}$ ,  $\sigma_i \in \text{Sing}_n(X)$ . This construction is instrumental as  $S_n(X)$  behaves as a  $\mathbb{Z}$ -module. It is *free* in the sense that  $\text{Sing}_n(X)$  is a  $\mathbb{Z}$ -basis for it. The **rank** of  $S_n(X)$  is defined to be  $\dim(\text{Sing}_n(X))$ .

Now, the **boundary operator**  $\partial_n : \text{Sing}_n(X) \rightarrow S_{n-1}(X)$  is defined as,

$$\partial_n(\sigma_n) = \sum_{i=0}^n (-1)^i d_i^n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_n^{(i)}$$

This canonically generates a *homomorphism* between chains,  $\partial_n : S_n(X) \rightarrow S_{n-1}(X)$  with

$$\partial_n \left( \sum_{k=0}^r a_k \sigma_k \right) = \sum_{k=0}^r a_k \partial_n(\sigma_k)$$

## $n$ -cycles and $n$ -boundaries (Loops and surfaces go brrr!)

An  $n$ -**cycle** in  $X$  is an  $n$ -chain  $c \in S_n(X)$  with a *vanishing boundary* i.e.,  $\partial_n c = 0$ . The **group of  $n$ -cycles** in  $X$  is the subset (of  $S_n(X)$ ),

$$Z_n(X) = \ker(\partial_n : S_n(X) \rightarrow S_{n-1}(X))$$

Similarly, an  $n$ -**dimensional boundary** in  $X$  is an  $n$ -chain  $c \in S_n(X)$  s.t. there exists an  $(n+1)$ -chain  $b \in S_{n+1}(X)$  satisfying  $\partial_{n+1} b = c$ . The **group of  $n$ -boundaries** of  $X$  is consequently,

$$B_n(X) = \text{im}(\partial_{n+1} : S_{n+1}(X) \rightarrow S_n(X))$$

A celebrated theorem is that boundaries have no boundaries,  $\partial_n \circ \partial_{n+1} = 0$ , which follows from the antisymmetric nature of  $\partial$ . In other words, boundaries are always cycles,  $B_n(X) \subseteq Z_n(X)$ . This motivates us to define a **chain complex**, which is a sequence of graded ( $\mathbb{Z}$ -indexed) Abelian groups  $\{A_n\}$  together with homomorphisms  $\partial_n : A_n \rightarrow A_{n-1}$  satisfying  $\partial_n \circ \partial_{n+1} = 0$ .

The chain complex associated with a topological space  $X$  is its **singular chain complex**,

$$\dots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

## Singular homology (Cycles with no interior measure holes!!!)

The  $n$ -th **singular homology group** of  $X$  is the *quotient group*,

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \{[c] : c \in Z_n(X)\} = \{c + B_n(X) : c \in Z_n(X)\}$$

Formally,  $H_n(X)$  identifies cycles differing by boundaries i.e. **homologous** cycles. Therefore, each homology class is represented by a distinct cycle that contains no boundaries — which are  $n$ -dimensional holes! Intuitively, homology ‘forgets’ boundaries to detect topological invariants like holes of a given dimension. For example, let  $\sigma_n(\Delta^n) = \partial_{n-1} \sigma_{n-1}(\Delta^{n-1})$ . Informally, since  $\sigma_n(\Delta^n)$  is a convex region, it can be continuously *contracted* to a point, which is ‘uninteresting’ from the point of view of detecting holes. This is why modding out by  $B_n(X)$  allows us to ‘see’ cycles that cannot be contracted to inner regions, thereby counting holes.

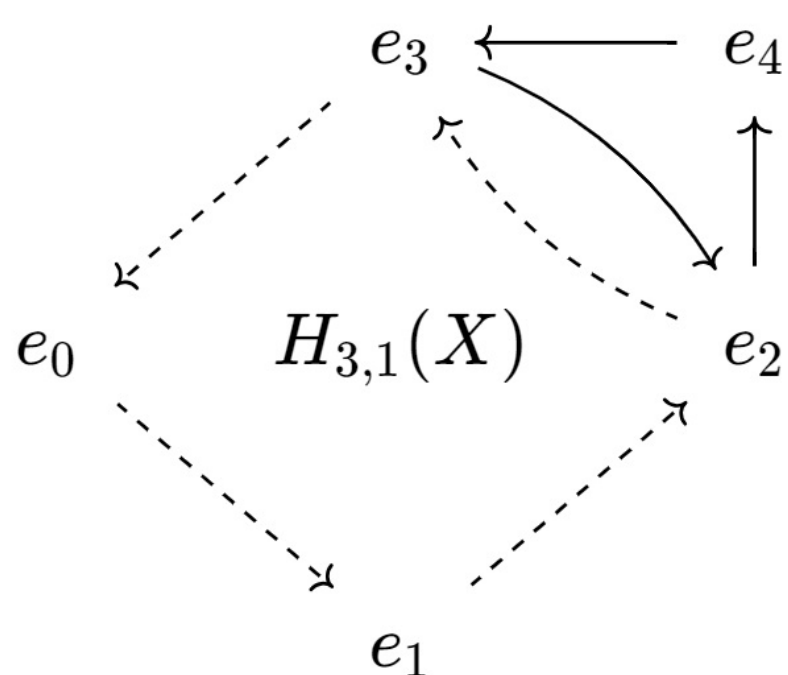


Figure 2. A 3-hole  $H_{3,1}(X)$ , represented by the (dotted) non-boundary cycle  $(e_0, e_1, e_2, e_3)$ . The (undotted) boundary  $b = (e_4, e_3, e_2)$  does not affect the hole, so that  $H_{3,1}(X)$  is homologous to the cycle  $b + H_{3,1}(X)$ .

## Homotopy (Non-Abelianized Loops!)

Homotopy is a vast discipline so we will only cover it very briefly. A  $n$ -**loop**  $\gamma$  in a *pointed topological space*  $X$  with base  $p \in X$  is an  $n$ -cycle i.e.  $\partial_n \gamma = 0$ . The  $n$ -**loop space**  $\mathcal{L}_n(X)$  is the set of all  $n$ -loops containing the base i.e.,  $\mathcal{L}_n(X) = \{\gamma \in \text{Sing}_n(\mathbb{R}^{n+1}) : p \in \gamma(\Delta^n), \partial_n \gamma = 0\}$ . We say that two loops  $\gamma, \delta \in \mathcal{L}_n(X)$  are **homotopic** i.e.  $\gamma \sim \delta$  iff there exists a  $C^1$  map called a **homotopy**  $h : \Delta^1 \times \Delta^n \rightarrow \mathcal{L}_n(X)$  s.t.  $h(e_0, \cdot) = \gamma$  and  $h(e_1, \cdot) = \delta$ , allowing us to define the  $n$ -**th homotopy group** of  $X$ ,

$$\pi_n(X) = \mathcal{L}_n(X) / \sim$$

To complete the notion of a homotopy group, we define the **concatenation** of  $n$ -loops as the  $\Psi$ -composition of the **wedge sum** of their domains,  $x \in \Delta^n \vee \Delta^n \implies (\gamma * \delta)(x) = \sigma_n(\Psi(x))$  where  $\Delta^n \vee \Delta^n = (\Delta^n \sqcup \Delta^n) / \equiv$  with  $\Delta^n \sqcup \Delta^n$  being their **disjoint union**  $\Delta^n \times \{0\} \cup \Delta^n \times \{1\}$  and  $\equiv$  identifying the faces  $\phi_0^n(\Delta^{n-1})$  and  $\phi_1^n(\Delta^{n-1})$ . Furthermore,  $\Psi$  is the map  $\Delta^n \vee \Delta^n \rightarrow \Delta^n$ . Informally, this corresponds to gluing the said faces together.  $(\pi_n(X), *)$  are *non-Abelian groups* with the identity element being the **constant curve**  $\gamma_e = \pi_{\{p\}}$ . Finally, a loop  $\gamma \in \mathcal{L}(X)$  is **contractible** if it is **null-homotopic** i.e. homotopic to  $\gamma_e$ .

Therefore, *each non-contractible homotopy class in  $\pi_n(X)$  represents a hole with an  $n$ -dimensional boundary!*

## Holonomies (States are loop-like, not point-like!!!)

Let  $\mu$  be a **Borel measure** on  $X$  i.e. any measure on its  *$\sigma$ -algebra of Borel sets*,  $\mathfrak{B}$  (which is the smallest  $\sigma$ -algebra on  $X$  containing its *open sets*). Let  $A$  be, for brevity, a scalar field  $X \rightarrow \mathbb{R}$  representing a **physical observable** with  $X$  being the *configuration space* and  $\mathbb{R}$  being the *state space*. Then, the **holonomy** corresponding to  $A$  along an  $n$ -loop  $\gamma \in \mathcal{L}_n(X)$  is given by the  $\mathbb{Z}$ -linear functional,

$$W_A[\gamma] = \oint_{x \in \gamma(\Delta^n)} d\mu A(x)$$

Now, by the Fundamental Theorem of Calculus, the state space  $\mathcal{S}(X)$  corresponding to  $p \in X$  is the set of holonomies for all loops based at  $p$ ,

$$\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \bigcup_{\gamma \in \mathcal{L}_n(X)} \{W_A[\gamma]\}$$

As a result, **if we forget paths which do not update  $A$ , the state space is identical to the modded loop space!** This is an unexpected physical consequence of the first isomorphism theorem:

$$\mathcal{S}(X) \cong_{\text{Set}} \mathcal{L}(X) / \ker(W_A)$$

$A$  is said to be **holonomic** if its holonomy is *trivial* i.e.  $\ker(W_A) = \mathcal{L}(X)$ . Then,  $\mathcal{S}(X)$  is the singleton  $\{A(p)\}$ , allowing us to use the usual functional treatment to manipulate states. However, in general,  $A$  is otherwise i.e. **non-holonomic**, and the best we can treat states is as sections of the bundle  $(\mathcal{S}(X), \varphi \circ \pi_{\{p\}}, \{p\})$  which is bundle-isomorphic to  $(\mathcal{L}(X), \pi_{\{p\}}, \{p\})$  where  $\varphi : \mathcal{S}(X) \rightarrow \mathcal{L}(X)$ .

## References

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