Loop Quantum Gravity Classical Mechanics: Algebraic Topology for Non-holonomic Mechanics

Siddhartha Bhattacharjee

University of Waterloo

There is no clear-cut distinction between example and theory.

Michael Atiyah

n-Simpices (*Triangles!*)

The **standard n-simplex** $\Delta^n \subsetneq \mathbb{R}^{n+1}$ is the *convex hull* of the standard basis vectors $\{e_0, \ldots, e_n\}$,

$$\Delta^n = \left\{ \sum_i t_i e_i : \sum_i t_i = 1; t_0, \ldots, t_n \geq 0 \right\}$$

$(t_0, \ldots, t_n)$ are the **barycentric coordinates** of Δ^n. Associated with a standard n simplex are its **face (inclusion) maps** (opposite to the e_i) which are *affine linear maps* $\phi_i^n : \Delta^{n-1} \hookrightarrow \Delta^n$ with,

$$\phi_i^n = (e_0, \ldots, \bar{e}_i, \ldots, e_n)^\flat : (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

where the widehat denotes the omission of the i-th index. The i-th **face** of Δ^n is the *subsimplex* $\phi_i^n(\Delta^{n-1})$. The boundary of a non-oriented n-simplex then consists (up to permutation) the union of all its faces.

Standard n-simplices are a well-understood and surprisingly rich class of topological spaces. It is therefore convenient to understand an arbitrary topological space X and its topological invariants via continuous maps from n-simplices.

Singular n-simplices (*Curved triangles!*)

A **singular n-simplex in** X is a C^1 map $\sigma_n : \Delta^n \to X$. The set of all (singular) n-simplices in X is denoted as $\mathsf{Sing}_n(X)$. Analogous to n-simplices, singular n-simplices are associated with **face maps** $d_i^n : \mathsf{Sing}_n(X) \to \mathsf{Sing}_{n-1}(X), \sigma_n \mapsto \sigma_n^{(i)} = \sigma_n \circ \phi_i^n$ with the i-th **face** of σ_n being $d_i^n(\sigma_n) = \sigma_n^{(i)}$.

The above information can be captured as an example commutative diagram,

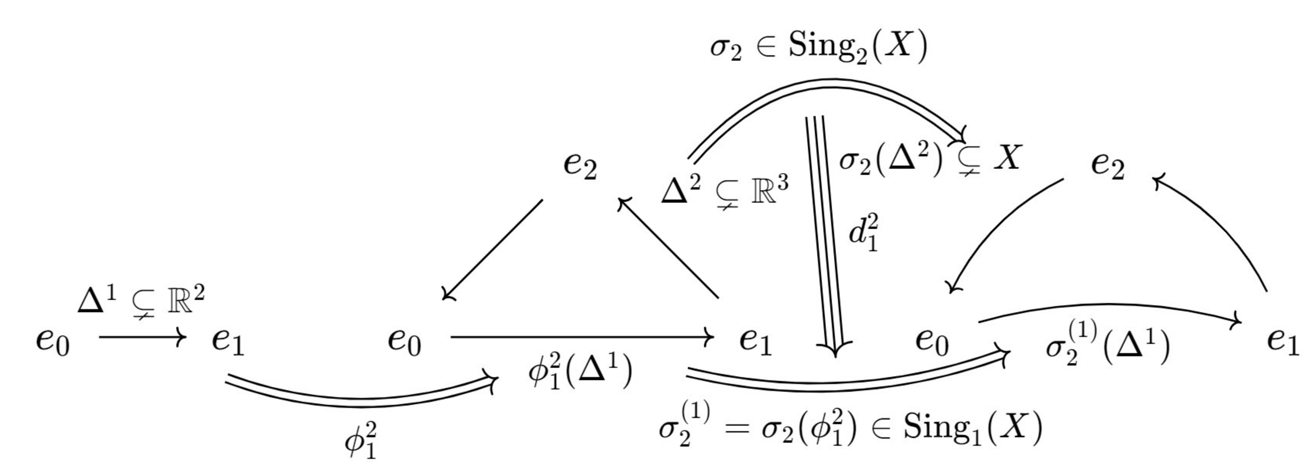

Figure 1. The standard 1-simplex Δ^1; face maps ϕ_i^2 to the standard 2-simplex Δ^2; and a singular 2-simplex σ_2 in X.

To talk about the boundary of n-simplices in X, we will need to construct a *free Abelian group* structure on simplices.

Singular n-chains (*Triangles form modules, yay!*)

The **singular n-chains** of X are members of the *free Abelian group* generated by singular n-simplices,

$$S_n(X) = \mathbb{Z}\, \mathsf{Sing}_n(X)$$

Therefore, an n-chain is a *finite $\mathbb{Z}$-linear combination of simplices*, $\sum_{i \in I \subsetneq \mathbb{N}} a_i \sigma_i$ s.t. for all $i \in I$, we have, $a_i \in \mathbb{Z}, \sigma_i \in \mathsf{Sing}_n(X)$. This construction is instrumental as $S_n(X)$ behaves as a $\mathbb{Z}$-module. It is *free* in the sense that $\mathsf{Sing}_n(X)$ is a $\mathbb{Z}$-basis for it. The **rank** of $S_n(X)$ is defined to be $\dim(\mathsf{Sing}_n(X))$.

Now, the **boundary operator** $\partial_n : \mathsf{Sing}_n(X) \to S_{n-1}(X)$ is defined as,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^n d_i^n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n^{(i)}$$

This canonically generates a *homomorphism* between chains, $\partial_n : S_n(X) \to S_{n-1}(X)$ with

$$\partial_n \left(\sum_{k=0}^{r} a_k \sigma_k \right) = \sum_{k=0}^{r} a_k \partial_n(\sigma_k)$$

n-cycles and n-boundaries (*Loops and surfaces go brrr!*)

An **n-cycle** in X is an n-chain $c \in S_n(X)$ with a *vanishing boundary* i.e., $\partial_n c = 0$. The **group of n-cycles** in X is the subset (of $S_n(X)$),

$$Z_n(X) = \ker(\partial_n : S_n(X) \to S_{n-1}(X))$$

Similarly, an **n-dimensional boundary** in X is an n-chain $c \in S_n(X)$ s.t. there exists an $(n+1)$-chain $b \in S_{n+1}(X)$ satisfying $\partial_{n+1} b = c$. The **group of n-boundaries** of X is consequently,

$$B_n(X) = \mathsf{im}(\partial_{n+1} : S_{n+1}(X) \to S_n(X))$$

A celebrated theorem is that boundaries have no boundaries, $\partial_n \circ \partial_{n+1} = 0$, which follows from the antisymmetric nature of ∂. In other words, boundaries are always cycles, $B_n(X) \subseteq Z_n(X)$. This motivates us to define a **chain complex**, which is a sequence of graded ($\mathbb{Z}$-indexed) Abelian groups $\{A_n\}$ together with homomorphisms $\partial_n : A_n \to A_{n-1}$ satisfying $\partial_n \circ \partial_{n+1} = 0$.
The chain complex associated with a topological space X is its **singular chain complex**,

$$\ldots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

Singular homology (*Cycles with no interior measure holes!!!*)

The n-th **singular homology** group of X is the *quotient group*,

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \{[c] : c \in Z_n(X)\} = \{c + B_n(X) : c \in Z_n(X)\}$$

Formally, $H_n(X)$ identifies cycles differing by boundaries i.e. **homologous** cycles. Therefore, each homology class is represented by a distinct cycle that contains no boundaries — which are n-dimensional holes! Intuitively, homology 'forgets' boundaries to detect topological invariants like holes of a given dimension. For example, let $\sigma_n(\Delta^n) = \partial_{n-1}\sigma_{n-1}(\Delta^{n-1})$. Informally, since $\sigma_n(\Delta^n)$ is a convex region, it can be continuously *contracted* to a point, which is 'uninteresting' from the point of view of detecting holes. This is why modding out by $B_n(X)$ allows us to 'see' cycles that cannot be contracted to inner regions, thereby counting holes.

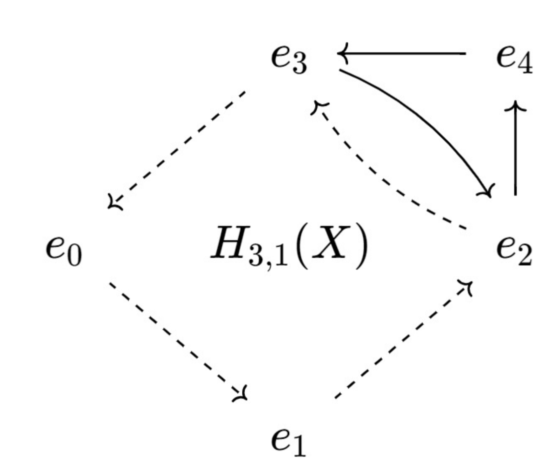

Figure 2. A 3-hole $H_{3,1}(X)$, represented by the (dotted) non-boundary cycle (e_0, e_1, e_2, e_3). The (undotted) boundary $b = (e_4, e_3, e_2)$ does not affect the hole, so that $H_{3,1}(X)$ is homologous to the cycle $b + H_{3,1}(X)$.

Homotopy (*Non-Abelianized Loops!*)

Homotopy is a vast discipline so we will only cover it very briefly. A **n-loop** γ in a *pointed topological space* X with base $p \in X$ is an n-cycle i.e. $\partial_n \gamma = 0$. The **n-loop space** $\mathcal{L}_n(X)$ is the set of all n-loops containing the base i.e., $\mathcal{L}_n(X) = \{\gamma \in \mathsf{Sing}_n(\mathbb{R}^{n+1}) : p \in \gamma(\Delta^n), \partial_n \gamma = 0\}$. We say that two loops $\gamma, \delta \in \mathcal{L}_n(X)$ are **homotopic** i.e. $\gamma \sim \delta$ iff there exists a C^1 map called a **homotopy** $h : \Delta^1 \times \Delta^n \to \mathcal{L}_n(X)$ s.t. $h(e_0, \cdot) = \gamma$ and $h(e_1, \cdot) = \delta$, allowing us to define the **n-th homotopy group** of X,

$$\pi_n(X) = \mathcal{L}_n(X)/\sim$$

To complete the notion of a homotopy group, we define the **concatenation** of n-loops as the Ψ-composition of the **wedge sum** of their domains, $x \in \Delta^n \vee \Delta^n \implies (\gamma * \delta)(x) = \sigma_n(\Psi(x))$ where $\Delta^n \vee \Delta^n = (\Delta^n \sqcup \Delta^n)/\equiv$ with $\Delta^n \sqcup \Delta^n$ being their **disjoint union** $\Delta^n \times \{0\} \cup \Delta^n \times \{1\}$ and $\equiv$ identifying the faces $\phi_n^n(\Delta^{n-1})$ and $\phi_1^n(\Delta^{n-1})$. Furthermore, Ψ is the map $\Delta^n \vee \Delta^n \to \Delta^n$. Informally, this corresponds to gluing the said faces together. $(\pi_n(X), *)$ are *non-Abelian groups* with the identity element being the **constant curve** $\gamma_e = \pi_{\{p\}}$. Finally, a loop $\gamma \in \mathcal{L}(X)$ is **contractible** if it is **null-homotopic** i.e. homotopic to γ_e.

Therefore, *each non-contractible homotopy class in $\pi_n(X)$ represents a hole with an n-dimensional boundary!*

Holonomies (*States are loop-like, not point-like!!!*)

Let μ be a **Borel measure** on X i.e. any measure on its *σ-algebra of Borel sets*, $\mathfrak{B}$ (which is the smallest σ-algebra on X containing its *open sets*. Let A be, for brevity, a scalar field $X \to \mathbb{R}$ representing a **physical observable** with X being the *configuration space* and $\mathbb{R}$ being the *state space*. Then, the **holonomy** corresponding to A along an n-loop $\gamma \in \mathcal{L}_n(X)$ is given by the $\mathbb{Z}$-linear functional,

$$W_A[\gamma] = \oint_{x \in \gamma(\Delta^n)} d\mu\, A(x)$$

Now, by the Fundamental Theorem of Calculus, the state space $\mathcal{S}(X)$ corresponding to $p \in X$ is the set of holonomies for all loops based at p,

$$\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \bigcup_{\gamma \in \mathcal{L}_n(X)} W_A[\gamma]$$

As a result, **if we forget paths which do not update A, the state space is identical to the modded loop space!** This is an unexpected physical consequence of the first isomorphism theorem:

$$\mathcal{S}(X) \cong_{\mathsf{Set}} \mathcal{L}(X)/\ker(W_A)$$

A is said to be **holonomic** if its holonomy is *trivial* i.e. $\ker(W_A) = \mathcal{L}(X)$. Then, $\mathcal{S}(X)$ is the singleton $\{A(p)\}$, allowing us to use the usual functional treatment to manipulate states. However, in general, A is otherwise i.e. **non-holonomic**, and the best we can treat states is as sections of the bundle $(\mathcal{S}(X), \varphi \circ \pi_{\{p\}}, \{p\})$ which is bundle-isomorphic to $\mathcal{L}(X), \pi_{\{p\}}, \{p\})$ where $\varphi : \mathcal{S}(X) \to \mathcal{L}(X)$.

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