Homogeneity from additivity for operators on $\mathbb{R}^m$

Statement

Consider the 2 well-known properties of a linear operator $T : \mathbb{R}^m \to \mathbb{R}^n : m, n \in \mathbb{N}$,

$$T\left(\sum_a \boldsymbol{u}_a\right) = \sum_a \boldsymbol{u}_a \qquad \forall\, \boldsymbol{u}_a \in \mathbb{R}^m \qquad (1)$$
$$T(c\boldsymbol{u}) = cT(\boldsymbol{u}) \qquad \forall\, c \in \mathbb{R}, \boldsymbol{u} \in \mathbb{R}^m \qquad (2)$$

Proposition

Property (2) (homogeneity) can be derived from property (1) (additivity) using the underlying structure of $\mathbb{R}$ if we introduce a weaker version of (2) with $c = 0$,

$$T(\mathbf{0}_m) = \mathbf{0}_n \qquad (3)$$

where $\mathbf{0}_k$ is the null vector in $\mathbb{R}^k$.

Thus, we propose that linearity can be unambiguously defined using the axiomatic system $\{(1), (3)\}$ which is simpler than the original system $\{(1), (2)\}$. In other words, (2) is equivalent to (3) under (1) while also being more general than (3).

Proof

Represent an arbitrary $c \in \mathbb{R}$ as an infinite series of rationals, using Dedekind cuts. Plug the result into (1) to obtain (2). Elaborated below.

Reals as infinite series of rationals

Suppose we are given any $c \in \mathbb{R}$. The construction of $\mathbb{R}$ from $\mathbb{Q}$ using Dedekind cuts guarantees the following:

$$\forall\, c \in \mathbb{R} : \exists\ \{a_n : n \in \mathbb{N}\} : c = \lim_{n \to \infty} a_n, a_n \in \mathbb{Q}\, \forall\, n \in \mathbb{N}$$

Now, we can define another sequence of rationals $\{b_k : k \in \mathbb{N}\} : \sum_{k=1}^{n} b_k = a_n$. It follows,

$$\begin{aligned} b_n &= \sum_{k=1}^{n} b_k - \sum_{k=1}^{n-1} b_k \\ &= a_n - a_{n-1} \end{aligned}$$

Thus, every real number can be represented as an infinite series of rationals,

$$\implies \forall\, c \in \mathbb{R} : \exists\ \{b_k : k \in \mathbb{N}\} : c = \sum_{k=1}^{\infty} b_k, b_k \in \mathbb{Q}\, \forall\, k \in \mathbb{N}$$

Furthermore, by definition, every rational is some integer divided by some non-zero integer,

$$\forall\, a \in \mathbb{Q} : \exists\, p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} : a = \frac{p}{q}$$

Let us proceed to derive homogeneity from additivity for the special case $c \in \mathbb{Q}$ (however, we will still have $\boldsymbol{u} \in \mathbb{R}^m$). Then, we will apply the result above to extend our observations to $c \in \mathbb{R}$.

Suppose $c \in \mathbb{Q}, \boldsymbol{u} \in \mathbb{R}^m$. Consider an operator $T : \mathbb{R}^m \to \mathbb{R}^n$ which obeys additivity i.e. property (1). Let us express c as $c = \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}$. The signature of c can be encoded in either the numerator p, which, pedantically speaking, is preferred over q as the domain of the former is the entirety of $\mathbb{Z}$.

For $c \in \mathbb{Z}$

We will first derive homogeneity from additivity for the simple case $q = 1, p \in \mathbb{Z} \implies c \in \mathbb{Z}$. To do so, we will investigate the individual scenarios $p \in \mathbb{Z}^+, p = 0, p \in \mathbb{Z}^-$.

For $c \in \mathbb{Z}^+ = \mathbb{N}$

We have,

$$
\begin{aligned}
T(c\boldsymbol{u}) &= T(p\boldsymbol{u}) : p \in \mathbb{N} \\
&= T\left(\sum_{a=1}^{p} u\right)
\end{aligned}
$$

By additivity,

$$
\begin{aligned}
T(c\boldsymbol{u}) &= \sum_{a=1}^{p} T(\boldsymbol{u}) \\
&= pT(\boldsymbol{u}) \\
&= cT(\boldsymbol{u}) \qquad \square
\end{aligned}
$$

For $c = 0$

For $c = p = 0$, we can use property (3) to get,

$$
\begin{aligned}
T(0 \cdot \boldsymbol{u}) &= T(\mathbf{0}_m) \\
&= \mathbf{0}_n \\
&= 0 \cdot T(\boldsymbol{u}) \qquad \square
\end{aligned}
$$

For $c \in \mathbb{Z}^-$

Let $p = -n : n \in \mathbb{Z}^+$,

$$
\begin{aligned}
\mathbf{0}_n &= T(\mathbf{0}_m) \\
&= T(n\boldsymbol{u} - n\boldsymbol{u})
\end{aligned}
$$

By additivity,

$$
\begin{aligned}
\mathbf{0}_n &= T(n\boldsymbol{u}) + T(-n\boldsymbol{u}) \\
&= nT(\boldsymbol{u}) + T(-n\boldsymbol{u}) \qquad [\because n \in \mathbb{Z}^+] \\
\implies T(p\boldsymbol{u}) &= T(-n\boldsymbol{u}) \\
&= \mathbf{0}_n - nT(\boldsymbol{u}) \\
&= -nT(\boldsymbol{u}) \\
&= pT(\boldsymbol{u}) \qquad \square
\end{aligned}
$$

Combining the different scenarios above, we have indeed found,

$$
\forall\, c \in \mathbb{Z} : (1), (3) \implies (2) \qquad (*)
$$

For $c = \frac{1}{q} : q \in \mathbb{Z} \setminus \{0\}$

We will now see how the above result is also true for $c = \frac{p}{q} : p = 1, q \in \mathbb{Z} \setminus \{0\}$. Ultimately, we will combine this result with the corresponding one for $c \in \mathbb{Z}$ to generalize it for all $c \in \mathbb{Q}$ which in turn will let us generalize it to $c \in \mathbb{R}$ via Dedekind cuts as previously stated.

We begin with the statement,

$$\begin{aligned}
T\left(\sum_{a=1}^{q}\frac{1}{q}\boldsymbol{u}\right) &= T\left(\boldsymbol{u}\right)\\
\sum_{a=1}^{q}T\left(\frac{1}{q}\boldsymbol{u}\right) &= T\left(\boldsymbol{u}\right)\\
qT\left(\frac{1}{q}\boldsymbol{u}\right) &= T\left(u\right)\\
T\left(\frac{1}{q}\boldsymbol{u}\right) &= \frac{1}{q}T\left(u\right) \qquad \square
\end{aligned}$$

Thus,

$$\forall\, c = \frac{1}{q} : q \in \mathbb{Z}\backslash\left\{0\right\} : (1), (3) \implies (2) \qquad (\dagger)$$

For $c \in \mathbb{Q}$

Let $c = \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}\backslash\left\{0\right\}$. Using the previous results $(*)$ and $(\dagger)$,

$$\begin{aligned}
T\left(c\boldsymbol{u}\right) &= T\left(\frac{p}{q}\boldsymbol{u}\right)\\
&= T\left(p\cdot\frac{1}{q}\boldsymbol{u}\right)\\
&= pT\left(\frac{1}{q}\boldsymbol{u}\right) && [(*)]\\
&= \frac{p}{q}T\left(\boldsymbol{u}\right) && [(\dagger)]\\
&= cT\left(\boldsymbol{u}\right) && \square
\end{aligned}$$

$$\forall\, c \in \mathbb{Q} : (1), (3) \implies (2) \qquad (\sim)$$

For $c \in \mathbb{R}$

Consider an arbitrary $c \in \mathbb{R}$. From the section on reals as infinite series of rationals, there exists a sequence of rationals $\left\{b_k : k \in \mathbb{N}\right\} : b_k \in \mathbb{Q}\,\forall\, k \in \mathbb{N}$ such that it adds up to c,

$$c = \sum_{k=1}^{\infty} b_k$$

Hence,

$$T\left(c\boldsymbol{u}\right) = T\left(\sum_{k=1}^{\infty} b_k\boldsymbol{u}\right)$$

By additivity,

$$\begin{aligned} T(c\boldsymbol{u}) &= \sum_{k=1}^{\infty} T(b_k\boldsymbol{u}) \\ &= \sum_{k=1}^{\infty} b_k T(\boldsymbol{u}) && [(\sim)] \\ &= cT(\boldsymbol{u}) && \blacksquare \end{aligned}$$

$$\forall\, c \in \mathbb{R} : (1), (3) \implies (2)$$

Summary

To summarize the above approach, we derived the homogeneity of operators on vector spaces built on the base field $\mathbb{R}$, from simpler given statements, namely their additivity and mapping of null vectors to null vectors. We realized this by using the construction of $\mathbb{R}$ from $\mathbb{Q}$ which is in turn constructed from $\mathbb{Z}$.

The advantage of expressing reals in terms of integers is that integers can fundamentally be used for counting, which is implicitly applied in property (1), additivity. Thus, this bridge from (1) to (2) allows us to logically proceed in the same direction, using only a second axiom in addition to (1), i.e. (3).

Moreover, (3) is a special case of (2), which is neat as it means that the axiomatic system $\{(1), (3)\}$ is informationally more compact than $\{(1), (2)\}$ despite both leading to equivalent formalisms.