Linearized Quantum Gravity Using the Projection Operator Formalism

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Abstract

The theory of canonical linearized gravity is quantized using the Projection Operator formalism, in which no gauge or coordinate choices are made. The ADM Hamiltonian is used and the canonical variables and constraints are expanded around a flat background. As a result of the coordinate independence and linear truncation of the perturbation series, the constraint algebra surprisingly becomes partially second-class in both the classical and quantum pictures after all secondary constraints are considered. While new features emerge in the constraint structure, the end result is the same as previously reported: the (separable) physical Hilbert space still only depends on the transverse-traceless degrees of freedom.

1 Introduction

In the early days of canonical quantum gravity (CQG), it was widely thought that the advent of a consistent ADM-Hamiltonian description of general relativity would herald the successful merger between quantum mechanics and general relativity, subsequently providing a useful description of Planck-scale physics. Instead of the Schrödinger equation, CQG had the Wheeler-DeWitt equation (WdW) [4], widely regarded as the functional analogue of the Schrödinger equation. However, the promise of solutions to the Wheeler-DeWitt equation has been indefinitely postponed due to the many illnesses plaguing the procedure. The diseases of non-renormalizability, constraint

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consistency, and the problem of time have caused many physicists to abandon the thought of canonical quantum gravity altogether; or, at the very least, formulate CQG in terms of a completely different set of variables.

One of these persistent problems has been the problem of constraint quantization. How can one consistently promote the constraints of general relativity to self-adjoint quantum operators, and at the same time be certain these constraints are still satisfied by the quantum dynamics? An entire constraint classification system was invented by Dirac to address this very issue. In his exploration [5], it was noticed that, while the constraints of gravity are closed under the classical Poisson algebra, when the constraints are promoted to quantum operators, a subset of the constraints mutated into a different class of constraints altogether. Or, in the parlance of Dirac, the first class constraint functions transmuted into second class constraint operators (see, e.g., [7] or [24] for a modern treatment). For a time, it was believed that with a consistent factor-ordering of the Hamiltonian constraint of general relativity, that constraint consistency would be maintained [26].

To the present day, the CQG program, even in its modern versions [3], has encountered numerous difficulties associated with problem of constraint quantization, the anomalous constraint algebra being only one such problem. Another problematic aspect of quantizing gravitational constraints is that they are infinite in number, as each constraint is needed per point in space. Any modern canonical quantum gravity program using metric variables must therefore be equipped to quantize constraints which are both infinite in number and partially second class in their commutator algebra.

The Projection Operator method of quantization provides the machinery in which the Affine Quantum Gravity (AQG) program, a variation of the CQG Program, proposes to quantize the constraints of general relativity. Prior work has demonstrated that, for a multitude of finite dimensional toy models [13, 10] including one which mimics the anomalous constraint algebra of gravity [20], the Projection Operator provides an unambiguous reduction from the unconstrained Hilbert space to the physical Hilbert space. It suffices to say that much of the success involved in using the Projection Operator to quantize these systems comes from the ability to quantize first and second class systems of constraints within one general formulation. The key to this unified treatment of constraints is that no gauge choices are to be made at the classical level. One must quantize the theory before any systematic reduction takes place.

The initial goal of this work was to further examine the feasibility of quantizing an infinite number of constraints with the Projection Operator quantization formalism. Linearized general relativity, a theory whose quantized form is well known, was selected as the ideal candidate for testing the techniques that the Projection Operator formalism proposes for the full, nonlinear theory of gravity. However, the stipulation of gauge independence at the classical level necessitated the inclusion of secondary constraints into the theory. These new constraints then led to a second class constraint sector in both the classical and quantum theories. The final result of this work shows that the Projection Operator can quantize infinite numbers of (non-anomalous) first and second class constraints in a quantum theory.

The next section briefly reviews the classical theory of linearized gravity, and highlights how coordinate-independence changes the constraint structure. The third section quantizes the system following the Projection Operator formalism, discussing salient features of reproducing kernels, Projection Operators and coherent states along the way. The last section of the text concludes and compares our work to some past work done in the CQG program. Finally, there are three appendices: one appendix which provides a simple toy model analogue of the analysis required in the paper, another which proves a theorem used to show the reduction of reproducing kernels of quantum, first-class constraints, and a third appendix which lists the fundamental algebras used in the main text.

2 Classical theory

This section consists of a review of linearized gravity, the perturbation of the metric field around a flat background; see [2] or [25]. We follow the prescription of quantizing the theory before reducing the space. Hence, no gauge is chosen, and the flat, canonical phase space Γ is not mapped to a reduced phase space Γ_R . The physical degrees of freedom will only become apparent in the quantum world after the Hilbert space \mathfrak{H} is reduced to the physical Hilbert space \mathfrak{H}_P with the help of a suitable projection operator. There is a good reason for quantization before reduction since there are counterexamples which show that reduction and quantization do not necessarily commute [10]. In all work to follow, surface terms will be discarded.

2.1 Geometrodynamics

In order to craft general relativity into a Hamiltonian formalism, it is well known that the manifest covariance of the Einstein-Hilbert action can be (3+1)-decomposed and encoded into the ADM [2] action. The resulting theory describes the evolution of three-dimensional hypersurfaces embedded in four-dimensional spacetime. The dynamical variables are the symmetric

3-metric of the hypersurface and its canonical momentum density, meaning that the phase space Γ for this theory is \mathbb{R}^{12} at each spatial point. The initial data of a metric, the symmetric tensor $g_{ab}(x)^1$, with its canonical momentum density tensor $\pi^{ab}(x)$, are specified and a set of constraints are satisfied by this data. The evolution of the hypersurface is thereafter restricted to being causal and invariant to diffeomorphisms on the hypersurface, a spacelike 3-surface.

The ADM action is defined as

$$I[\pi, g] = \int dt \int d^3x \left[\pi^{ab} \dot{g}_{ab} - N^i H_i - NH \right],$$
 (2.1)

where the lapse N=N(x) and shift vector $N^i=N^i(x)$ are the Lagrange multipliers. The constraints are then given by H(x)=0 and $H_i(x)=0$, and are defined respectively as the Hamiltonian constraint and the set of diffeomorphism constraints. With $\pi\equiv\pi^a_a$, the Hamiltonian constraint term is given by

$$C[N] \equiv \int d^3x \ N(x) \ H(x)$$

$$= \int d^3x \ N\left\{-\sqrt{g}\left[R + \frac{1}{g}\left(\frac{\pi^2}{2} - \pi^{ab}\pi_{ab}\right)\right]\right\}$$
(2.2)

where R is the three-dimensional Ricci scalar and follows the same convention of index contraction as [21]. Equation (2.2) provides a fundamental link between the intrinsic and extrinsic curvature of the evolving hypersurfaces, one that must remain constant throughout time [19]. Infinitesimal diffeomorphisms in the initial data are generated by the diffeomorphism constraint term

$$C[N^{a}] \equiv \int d^{3}x \ N^{a}(x) \ H_{a}(x) = \int d^{3}x \ \left[-2N^{a}\pi_{a}^{k}|_{k} \right]$$

$$= \int d^{3}x \ N^{a} \left[-2\pi_{a}^{k}|_{k} - (2g_{al,m} - g_{lm,a})\pi^{lm} \right].$$
(2.3)

These diffeomorphisms are the generators of small tangential displacements on the spacelike hypersurfaces.

¹In each of these dynamical variables, and those that follow, lower case Latin indices indicate Euclidean, spatial indices (e. g. $i \in \{1, 2, 3\}$), and repeated indices in a term are summed over. Greek indices, such as $\mu \in \{0, i\}$, are of the Minkowski, spacetime variety.

2.2 Perturbation of metric variables

The metric tensor, and its conjugate momenta can be expanded around a flat background according to

$$g_{ab}(x) \rightarrow \delta_{ab} + \epsilon h_{ab}^{(1)}(x) + \epsilon^2 h_{ab}^{(2)}(x) + O(\epsilon^3),$$

 $\pi_{ab}(x) \rightarrow 0 + \epsilon p_{ab}^{(1)}(x) + \epsilon^2 p_{ab}^{(2)}(x) + O(\epsilon^3),$ (2.4)

where ϵ is merely an order parameter for the perturbation analysis, and δ_{ab} is the Euclidean 3-metric. The lapse and shift are similarly expanded according to

$$N(x) \rightarrow 1 + \epsilon N^{(1)}(x) + \epsilon^2 N^{(2)}(x) + O(\epsilon^3),$$

$$N_a(x) \rightarrow 0 + \epsilon N_a^{(1)}(x) + \epsilon^2 N_a^{(2)}(x) + O(\epsilon^3),$$
(2.5)

c.f., [18].

Implementing an orthogonal decomposition for the symmetric tensors into their transverse-traceless, transverse, and longitudinal components provides an elegant, reduced expression for these constraints. Each p_{ab} and h_{ab} tensor is decomposed into a set of orthogonal components [2] represented by

$$f_{ab}(x) = \mathbf{f}_{ab}^{TT}(x) + \mathbf{f}_{ab}^{T}(x) + 2\mathbf{f}_{(a,b)}^{L}(x), \tag{2.6}$$

where $f_{ab}(x)$ stands for either the metric or momentum density tensor. Also note the use of different fonts to denote the various tensor components; as we proceed, we shall continue to use the different fonts and drop the capital superscripts.

In (2.6), each set of components has the same definition as in [2]. The components of which are uniquely determined by (2.6) and the relations

$$\mathbf{f}_{ab,b} \equiv 0, \quad \mathbf{f}_{aa} \equiv 0 \tag{2.7}$$

$$f_{ab} = \frac{1}{2} \left(f \delta_{ab} - \frac{\partial_a \partial_b}{\nabla^2} f \right)$$
 (2.8)

$$\mathfrak{f}_a = \frac{1}{\nabla^2} \left(f_{ab,b} - \frac{f_{bc,bca}}{2\nabla^2} \right), \tag{2.9}$$

yielding two degrees of freedom in \mathbf{f}_{ab} , one degree of freedom in $\mathbf{f} = \mathbf{f}_{aa}$, and three in the vector \mathbf{f}_a . This type of symmetric tensor decomposition allows the linear constraints to be written in a more streamlined fashion.

Using such a decomposition along with the expansion strategy given in (2.4) and (2.5), the integrand in (2.2) may be written to second order as

$$N(x) \ H(x) \ \rightarrow \ -\nabla^2 \mathsf{h}^{(2)} + R^{(1)} + h_{aa} \ R^{(1)}/2 + R^{(2)}$$
 (2.10)

$$+p_{ab}p^{ab} - p^2/2 - N^{(1)}\nabla^2 \mathbf{h}^{(1)}.$$
 (2.11)

The factor $N^{(1)}(x)$ is to be interpreted as one of the new Lagrange multipliers of the linearized theory. The linear Hamiltonian constraint density can then be immediately read off as the term multiplying $N^{(1)}(x)$, namely,

$$H^{(1)}(x) = -\nabla^2 \mathsf{h}^{(1)}(x). \tag{2.12}$$

The rest of the terms in (2.11) can then be interpreted as the negative of an unconstrained Hamiltonian density, defined by

$$\mathcal{H}[p,h] = -R^{(1)} - h_{aa} R^{(1)}/2 - R^{(2)} - p_{ab} p^{ab} + p^2/2. \tag{2.13}$$

Using the decomposition of (2.6)-(2.9), it has been shown that the only transverse-traceless terms in (2.13) combine to form a harmonic oscillator piece [18]:

$$\mathcal{H}_{TT}[\mathbf{p}, \mathbf{h}] = \mathbf{p}_{ab}^{(1)} \mathbf{p}_{ab}^{(1)} + \frac{1}{4} \mathbf{h}_{ab,c}^{(1)} \mathbf{h}_{ab,c}^{(1)}.$$
 (2.14)

However, if no gauge is fixed, unconstrained Hamiltonians for the transverse and longitudinal variables arise from \mathcal{H} and must also be considered in the dynamics.

Temporarily postponing the discussion about the remaining terms in \mathcal{H} , the symmetric tensor decomposition and expansions are repeated on the term in the action containing the diffeomorphism constraint. This constraint depends only on the longitudinal portion of p_{ab} and h_{ab} , denoted by the 3-vectors \mathfrak{p}_a and \mathfrak{h}_a . These three degrees of freedom may be isolated further by splitting each longitudinal vector into transverse $(\mathfrak{p}_a^t, \mathfrak{h}_a^t)$ and longitudinal pieces $(\mathfrak{p}^l, \mathfrak{h}^l)$. With this secondary, vector decomposition, the expansion of the integrand of (2.3) becomes

$$N^{a}(x) H_{a}(x) \to -2N^{a} {}^{(1)}(x) \nabla^{2} \left(\mathfrak{p}_{a}^{(1)t} + \mathfrak{p}_{,a}^{(1)l} \right)$$
 (2.15)

through quadratic order in the expansion. The linearized diffeomorphism constraint is therefore

$$H_a^{(1)}(x) = -2\nabla^2 \left(\mathfrak{p}_a^{(1)t}(x) + \mathfrak{p}_{,a}^{(1)l}(x) \right) = 2\nabla^2 \mathfrak{p}_a(x). \tag{2.16}$$

Since all the terms in the ADM action have now been expanded through quadratic order, and only the first order terms contributed, the superscript ⁽¹⁾ will be discarded to allow for shorter expressions. A step that can be taken to further simplify the analysis is to work in momentum space with the Fourier transform pair

$$f_{ab}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \widetilde{f}_{ab}(t, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$
 (2.17)

$$\widetilde{f}_{ab}(t,\mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}} f_{ab}(x) e^{-i\mathbf{k}\cdot\mathbf{x}}.$$
 (2.18)

In momentum space, the constraints of (2.12) and (2.16) transparently give

$$\widetilde{\mathbf{h}}(t, \mathbf{k}) = 0 = \widetilde{\mathfrak{p}}_a(t, \mathbf{k}). \tag{2.19}$$

The next step is to find out to which Dirac constraint class the constraints (2.16) and (2.16) belong. This algebra depends on the equal-time canonical Poisson bracket expression

$$\{h_{ab}(x), p^{cd}(x')\} = \delta^c_{(a}\delta^d_{b)}\delta(x - x').$$
 (2.20)

To check that the time derivatives of the constraints are zero, the Dirac check on constraint consistency, we first must know how to compute the Poisson brackets of the tensor components. While using the tensor decomposition of (2.6)-(2.9) will give the same results, we choose to work with a more calculationally friendly form in **k**-space. The complex, null vector m_a , its conjugate \bar{m}_a , and the longitudinally pointing k_a form a basis in **k**-space, such that

$$m_a m_a = 0 = \bar{m}_a \bar{m}_a = k_a m_a = k_a \bar{m}_a, \quad m_a \bar{m}_a = 1.$$
 (2.21)

This allows for the expansion of each Fourier-transformed canonical variable into its basic tensor components according to

$$\widetilde{f}_{ab}(t,\mathbf{k}) = \widetilde{\mathbf{f}}_{ab} + m_{(a}\overline{m}_{b)}\widetilde{\mathbf{f}} - 2i\left(k_{(a}m_{b)}\widetilde{\mathbf{f}}_{1}^{t} + k_{(a}\overline{m}_{b)}\widetilde{\mathbf{f}}_{2}^{t} - ik_{(a}k_{b)}\widetilde{\mathbf{f}}^{t}\right). \quad (2.22)$$

Contractions of combinations of k_a , m_a , \bar{m}_a into (2.20) lead to the algebra in Appendix C. Using the decomposition of (2.22) allows the linearized constraints of (2.12) and (2.16) to be respectively written in **k**-space as

$$\widetilde{H}(t, \mathbf{k}) = k^2 \widetilde{\mathsf{h}}(t, \mathbf{k})$$
 (2.23)

$$\widetilde{H}_a(t, \mathbf{k}) = 2k^2 \left(\widetilde{\mathfrak{p}}_a^{\ t}(t, \mathbf{k}) - ik_a \widetilde{\mathfrak{p}}^{\ l}(t, \mathbf{k}) \right).$$
 (2.24)

Lastly, the expansion (2.22) allows the Fourier transform of the Hamiltonian density of (2.13) to be written as

$$\widetilde{\mathcal{H}}(t, \mathbf{k}) = \widetilde{\mathcal{H}}_{TT} + \widetilde{\mathcal{H}}_{T} + \widetilde{\mathcal{H}}_{L} + \widetilde{\mathcal{H}}_{int}.$$
 (2.25)

The \mathcal{H}_T and \mathcal{H}_L terms are given by

$$\widetilde{\mathcal{H}}_T(t, \mathbf{k}) = |\widetilde{\mathbf{p}}|^2 / 2$$
 (2.26)

$$\widetilde{\mathcal{H}}_L(t, \mathbf{k}) = 4k^2 |\widetilde{\mathfrak{p}}_a^t|^2 + 2k^4 |\widetilde{\mathfrak{p}}^t|^2 - k^4 |\widetilde{\mathfrak{h}}_a|^2, \tag{2.27}$$

while $\widetilde{\mathcal{H}}_{TT}$ is the transform of (2.14), or

$$\widetilde{\mathcal{H}}_{TT}(t, \mathbf{k}) = \bar{\widetilde{\mathbf{p}}}_{ab}\widetilde{\mathbf{p}}_{ab} + \frac{k^2}{4}\bar{\widetilde{\mathbf{h}}}_{ab}\widetilde{\mathbf{h}}_{ab}.$$
(2.28)

Finally, the interaction Hamiltonian density is named as such because it contains interaction amongst the constrained variables in the form of

$$\widetilde{\mathcal{H}}_{int}(t, \mathbf{k}) = k^2 \left(\overline{\widetilde{\mathfrak{p}}}^{\,l} \widetilde{\mathbf{p}} + \widetilde{\mathfrak{p}}^{\,l} \, \overline{\widetilde{\mathbf{p}}} \right). \tag{2.29}$$

Next the time derivatives of the constraints are calculated in a straightforward manner using (2.22), (2.25), and the algebra in Appendix C. For the system to continuously remain on the constraint hypersurface as time evolves, all time derivatives of the constraints must be equal to zero [5]. The results of the consistency requirement on the linearized constraints are given by

$$\dot{\widetilde{H}}(t, \mathbf{k}) = \int \frac{d^3k'}{(2\pi)^{3/2}} \left\{ \widetilde{H}(t, \mathbf{k}), \widetilde{\mathcal{H}}(t, \mathbf{k}') \right\}$$

$$= k^2 \left(\widetilde{p}(t, \mathbf{k}) + 2k^2 \widetilde{p}^l(t, \mathbf{k}) \right) = 0, \tag{2.30}$$

for the evolution of (2.23), and

$$\dot{\widetilde{H}}_{a}(t, \mathbf{k}) = \int \frac{d^{3}k'}{(2\pi)^{3/2}} \left\{ \widetilde{H}_{a}(t, \mathbf{k}), \widetilde{\mathcal{H}}(t, \mathbf{k}') \right\}$$

$$= \frac{k^{4}\widetilde{\mathfrak{h}}_{a}^{t}(t, \mathbf{k})}{4} = 0 \tag{2.31}$$

for the time derivative of the linearized diffeomorphism constraint (2.24). Equations (2.30) and (2.31) define secondary constraints. These new constraints must be retroactively placed into the action and, again the consistency requirement must be checked. The total Hamiltonian (constraint terms plus $\widetilde{\mathcal{H}}$) is then transformed to

$$\widetilde{\mathcal{H}}_{tot}(t, \mathbf{k}) = \widetilde{\mathcal{H}} - N\widetilde{H} - N^a\widetilde{H}_a - M\dot{\widetilde{H}} - M^a\dot{\widetilde{H}}_a, \tag{2.32}$$

where the (1) superscripts have now been completely suppressed and $M(t, \mathbf{k})$, $M^a(t, \mathbf{k})$ are the new Lagrange multipliers to go with the secondary constraints $\dot{\tilde{H}}(t, \mathbf{k})$, $\dot{\tilde{H}}_a(t, \mathbf{k})$. Normally in the literature on linearized gravity [2], the secondary constraints are automatically satisfied by the gauge conditions (or time-slicing) chosen in the classical analysis. Only when one does not make a coordinate or gauge choice, does one have these extra constraints.

The story, however, does not end with (2.30) and (2.31). The time derivatives of these equations must again be calculated and set equal to zero. The constraint-constraint terms in the Poisson brackets must now be considered as they now longer commute. Demanding that the double time derivative of (2.23) be zero,

$$\ddot{\widetilde{H}}(t,\mathbf{k}) = \int \frac{d^3k'}{(2\pi)^{3/2}} \left\{ \dot{\widetilde{H}}(t,\mathbf{k}), \mathcal{H}(t,\mathbf{k}') \right\} + \int \frac{d^3k'}{(2\pi)^{3/2}} N \left\{ \dot{\widetilde{H}}(t,\mathbf{k}), \widetilde{H}(t,\mathbf{k}') \right\} = 0$$
(2.33)

implies that N is determined by this equation. The N^a are determined by evaluating the consistency condition

$$\ddot{\widetilde{H}}_{a}(t,\mathbf{k}) = \int \frac{d^{3}k'}{(2\pi)^{3/2}} \left\{ \dot{\widetilde{H}}_{a}(t,\mathbf{k}), \widetilde{\mathcal{H}}(t,\mathbf{k}') \right\} + \int \frac{d^{3}k'}{(2\pi)^{3/2}} N^{b} \left\{ \dot{\widetilde{H}}_{a}(t,\mathbf{k}), \widetilde{H}_{b}(t,\mathbf{k}') \right\} = 0.$$
(2.34)

Solutions for M, M^a are in turn be found by recursively plugging in the equations for N, N^a into (2.23) and (2.24). The solutions

$$M = 0, (2.35)$$

$$M_a^t = -32k^2 \widetilde{\mathfrak{h}}_a^t(t, \mathbf{k}), \qquad N_a = 0 \qquad , \qquad (2.36)$$

are uncovered by this final step. It should be noticed that the Lagrange multipliers, except for M^l , have been determined by the equations of motion.

The fact that the Lagrange multipliers for $(\widetilde{H}(t,\mathbf{k}),\ \dot{\widetilde{H}}(t,\mathbf{k});\ \widetilde{H}_a^{\ t}(t,\mathbf{k}),\ \dot{\widetilde{H}}_a^{\ t}(t,\mathbf{k}),$ $\dot{\widetilde{H}}_a^{\ t}(t,\mathbf{k})$ are determined by the equations of motion leads us to conclude that all constraints except $\widetilde{H}_a^{\ l}(t,\mathbf{k})$ are second-class constraints [10]. We reemphasize the point that this additional, second-class constraint structure for linearized gravity only comes from the fact that we are not choosing a gauge. Conversely, $\widetilde{H}^l(t,\mathbf{k})$ is a first-class constraint since its Poisson algebra is Abelian and its Lagrange multiplier M^l is not determined by the dynamics.

Before promoting the canonical, phase-space variables $\{\mathbf{p}^{ab}, \mathbf{h}_{ab}; \mathbf{p}, \mathbf{h}; \mathbf{p}^{a}, \mathbf{h}_{a}\}$ to operators, one further step is taken to rewrite the classical action. An additional measure of convenience may be afforded by rewriting the constraints so that they are each functions of only one phase space variable. For a constrained action, one is free to add and subtract linear combinations of the constraints. Instead of using $\hat{H}(t, \mathbf{k}), \hat{H}_a(t, \mathbf{k})$, the new constraints, the

ones we will actually quantize, are gained by adding and subtracting linear combinations of $\{\widetilde{H},\widetilde{H},\widetilde{H}_a,\widetilde{H}_a\}$:

$$\widetilde{\psi}(t, \mathbf{k}) \equiv k\widetilde{\mathsf{h}}(t, \mathbf{k})$$
 (2.37)

$$\widetilde{\psi}(t, \mathbf{k}) \equiv k\widetilde{\mathbf{h}}(t, \mathbf{k}) \qquad (2.37)$$

$$\widetilde{\psi}_{a}(t, \mathbf{k}) \equiv k\widetilde{\mathbf{p}}_{a}^{t}(t, \mathbf{k}) \qquad (2.38)$$

$$\widetilde{\phi}(t, \mathbf{k}) \equiv \widetilde{\mathbf{p}}(t, \mathbf{k}) \qquad (2.39)$$

$$\widetilde{\phi}_{a}(t, \mathbf{k}) \equiv k^{2}\widetilde{\mathbf{h}}_{a}^{t}(t, \mathbf{k}) \qquad (2.40)$$

$$\phi(t, \mathbf{k}) \equiv \widetilde{\mathbf{p}}(t, \mathbf{k})$$
(2.39)

$$\widetilde{\phi}_a(t, \mathbf{k}) \equiv k^2 \widetilde{\mathfrak{h}}_a^t(t, \mathbf{k})$$
 (2.40)

$$\widetilde{\varsigma}(t, \mathbf{k}) \equiv k^2 \widetilde{\mathfrak{p}}^l(t, \mathbf{k}),$$
 (2.41)

where the lone first class constraint is $\widetilde{\varsigma}(t, \mathbf{k})$. The fact that the phase-space variables $\{p, h; p^a, h_a\}$ are constrained is manifest in the final version of the total Hamiltonian

$$\widetilde{\mathcal{H}}_{tot}(t, \mathbf{k}) = \widetilde{\mathcal{H}} - N\widetilde{\phi} - N^a \widetilde{\phi}_a - M\widetilde{\psi} - M^a \widetilde{\psi}_a - L\widetilde{\varsigma}, \tag{2.42}$$

where each factor has had its functional dependence on (t, \mathbf{k}) suppressed.

To summarize, the above procedure found a classical action describing linearized gravity most amenable to the Projection Operator Method of quantization. Insistence on a gauge-independent classical theory and the perturbation scheme of (2.4)-(2.5) led to the necessity of incorporating new constraints into the theory. These new constraints came about as a result of the non-commutativity of the primary constraints with the unconstrained Hamiltonian, leading to a second-class designation for all but one of the original, primary constraints—the longitudinal part of H_a that we call $\widetilde{\varsigma}(t, \mathbf{k})$. While further investigation into this algebraic structure is left to future work, after uncovering all secondary constraints and adding and subtracting linear combinations of these constraints with the primary ones, a convenient, quantum-ready form of the classical theory was found (2.37)-(2.42).

3 Quantization

As a consequence of not choosing a gauge, any reduction of the unconstrained theory to a physical set of quantities must be done after quantization. The reduction is accomplished by use of a projection operator (\mathbb{E}) , which maps states in the unconstrained Hilbert space $(|\psi\rangle \in \mathfrak{H})$ to states in the physical Hilbert space $(|\psi_P\rangle \in \mathfrak{H}_P)$, according to

$$\mathbb{E}|\psi\rangle = |\psi_P\rangle. \tag{3.1}$$

As a preliminary step, the Projection Operator is a function of the sum of squares of the constraint operators $(\Sigma\Phi^2)$ and keeps their spectrum small,

as indicated by

$$\mathbb{E} = \mathbb{E}[\Sigma \Phi^2 \le \delta^2],\tag{3.2}$$

for some small parameter $\delta > 0$, whose exact value and behavior depends on the constraints under consideration. At this level, $\Sigma \Phi^2$ could be a mixture of both first and second class constraints.

Quantum dynamics will then take place in this physical Hilbert space using only the $|\psi_P\rangle$'s. In effect this reduces \mathfrak{H} to \mathfrak{H}_P , symbolically written as

$$\mathbb{E}\mathfrak{H} = \mathfrak{H}_P. \tag{3.3}$$

Of course, this projection operator obeys all of the usual properties of a projection operator, namely

$$\mathbb{E}^{\dagger} = \mathbb{E} \quad , \quad \mathbb{E}^2 = \mathbb{E}. \tag{3.4}$$

The coherent state matrix elements of \mathbb{E} define a reproducing kernel and are the key to constructing the physical Hilbert space [10, 13]. This is done in a two part step—the reproducing kernel, like the Projection Operator, is first regularized by the small parameter δ , and then δ is reduced to its smallest size consistent with the spectrum of the constraint operators.

Generally, the reproducing kernel [10, 13] involving the Projection Operator is given by

$$\langle\!\langle p', q' | p, q \rangle\!\rangle \equiv \frac{\langle p', q' | \mathbb{E} | p, q \rangle}{\langle 0, 0 | \mathbb{E} | 0, 0 \rangle},$$
 (3.5)

where $|p,q\rangle$ are suitable canonical coherent states. The re-scaling introduced by the reduction procedure, as indicated by the denominator of (3.5), simply involves the coherent state matrix element of \mathbb{E} for which all labels are zero. Vectors in the physical Hilbert space are given by linear superpositions of the reproducing kernel² (3.5), such as

$$\psi_P(p,q) = \sum_{m=1}^{M} \alpha_m \langle \langle p, q | p_m, q_m \rangle \rangle.$$
 (3.6)

For finite M, vectors of this sort form a dense set $D_{\mathfrak{H}_P} \subset \mathfrak{H}_P$. In addition, inner products of vectors in the dense set are determined by

$$(\psi, \phi)_P = \sum_{m,n=1}^{M,N} \alpha_m^* \beta_n \langle \langle p_m, q_m | p_n, q_n \rangle \rangle.$$
 (3.7)

²Vectors in the original Hilbert space \mathfrak{H} are given by similar expressions; except, in that case, the reproducing kernel is the coherent state overlap function.

The physical Hilbert space is completed by including the limit points of all Cauchy sequences in the norm $\|\psi\|_P = (\psi, \psi)_P^{1/2}$, completing the construction of the reproducing kernel Hilbert space.

While the fundamental structure of the Projection Operator Method treats first and second class constraints equally, the subspaces over which the constraint operators act reduce in an entirely different manner as $\delta \to 0$. In short, the reduction depends on if the constraint operators in the argument of \mathbb{E} are first or second class. For the first class constraint of linearized gravity, $\hat{\varsigma}$ has zero in its continuum, and a Projection Operator involving only these constraints permits the limit $\delta \to 0$. However, a δ -dependent rescaling of the reproducing kernel is incorporated into the projection operator in order to prevent $\mathbb{E} \to 0$ as $\delta \to 0$. The rest of the constraint operators, being second class in nature, do not have zero in their continuum and a projection operator containing only these constraints would forbid the limit of $\delta \to 0$. One might expect the Projection Operator of linearized gravity to exhibit both types of reduction since it contains both types of constraints.

Concerning the above statements: if we can find an appropriate functional expression for the reproducing kernel, we can characterize the Hilbert space in question. Therefore, the subsections to follow focus on finding suitable definitions of the Projection Operator and coherent states so that we can determine the exact form the reduced reproducing kernel assumes in linearized quantum gravity.

3.1 Operators and coherent states

Quantization begins by the promotion of the Γ coordinates $(p^{cd}(x), h_{ab}(y))$ to sets of local, self-adjoint operators $(\hat{p}^{cd}(x), \hat{h}_{ab}(y))$, whose canonical commutation relations³, in units where $\hbar = 1$, read

$$\left[\hat{h}_{ab}(x), \hat{p}^{cd}(x')\right] = i\delta^c_{\ (a}\delta_b)^d\delta^3(x - x'),\tag{3.8}$$

where parentheses denote index symmetrization. As they stand in the above expression, the quantities $\hat{h}_{ab}(x)$ and $\hat{p}^{cd}(y)$ are ill-defined as operators.

To introduce a regularization for the canonical operators, let the vector $\mathbf{n} = \{n_1, n_2, n_3\}$ label nodes on a finite, three-dimensional lattice of volume

³Henceforth, operators will be represented by hatted symbols such as \hat{h}_{ab} . In addition, capital superscripts on operators and labels referring to subspaces will be suppressed in favor of their respective font denominations with the exception being the lowercase t and l referring to the transverse and longitudinal parts of the longitudinal vector subcomponents.

 L^3 so that band-limited Fourier coefficients for $\hat{f}_{ab}(x)$ may be found by

$$\hat{f}_{ab}^{\mathbf{n}} = \mathcal{V}^{-1/2} \int_{L^3} d^3x \, e^{-i\mathbf{k}^{\mathbf{n}} \cdot \mathbf{x}} \hat{f}_{ab}(x), \qquad (3.9)$$

where $\mathcal{V} = (L/2\pi)^3$ and $\mathbf{k}^{\mathbf{n}} \equiv 2\pi \mathbf{n}/L$. The operators themselves may then be regularized and approximated by expansions based on these coefficients:

$$\hat{f}_{ab}(x) \equiv \mathcal{V}^{-1/2} \sum_{\mathbf{n} \in \mathcal{I}} \hat{f}_{ab}^{\mathbf{n}} e^{i\mathbf{k}^{\mathbf{n}} \cdot \mathbf{x}},$$
 (3.10)

where the truncation of the sum is implicit on the left hand side. The Fourier series has been truncated in the above equation to include only the lattice points in the set

$$\mathcal{I} = \left\{ \mathbf{n} | \mathbf{n} \in \mathbb{Z}^3, -N \le n_a \le N, \mathbf{n} \ne 0, N < \infty \right\}, \tag{3.11}$$

which means that the set of wave vectors $\mathbf{k}^{\mathbf{n}}$ is band-limited to

$$\mathcal{K} = \left\{ \mathbf{k}^{\mathbf{n}} | \mathbf{k}^{\mathbf{n}} \in \mathbb{R}^{3}, \ 0 < |\mathbf{k}^{\mathbf{n}}| \le k_{max}, \ k_{max} = 2\pi\sqrt{3}N/L \right\}.$$
 (3.12)

The self-adjoint requirement for $\hat{f}_{ab}(x)$ is met as long as the reality condition $\hat{f}_{ab}^{\dagger \mathbf{n}} \equiv \hat{f}_{ab}^{\mathbf{n}} = \hat{f}_{ab}^{-\mathbf{n}}$ is implemented. Lattice indices will reside in the middle of the alphabet, will be kept raised and in bold font, and will not be automatically summed if repeated. (**Remark**: In nonperturbative quantum gravity, such a simple Fourier series truncation is not available. Instead of summing over $e^{i\mathbf{k}^{\mathbf{n}} \cdot \mathbf{x}}$, the metric variables may be smeared over real, orthonormal test functions of rapid decrease [12]).

The next step is to express each operator equation in terms of its Fourier components. Consequently, the momentum space CCR's⁴ may be written as

$$\left[\hat{\bar{h}}_{ab}^{\mathbf{m}}, \hat{p}_{cd}^{\mathbf{n}}\right] = i\delta_{c(a}\delta_{b)d}\delta^{\mathbf{m},\mathbf{n}} \mathcal{V}.$$
(3.13)

Using the above expression and the expansion of (2.22) with the $\tilde{\ }$'s replaced by $\tilde{\ }$'s, the commutation relations of Appendix B may be derived. These commutation relations define the algebras of each component in the factorized Hilbert space $\mathfrak{H} = \mathfrak{H}_{TT} \otimes \mathfrak{H}_{T} \otimes \mathfrak{H}_{L}$.

With the canonical commutation relations in hand, the ${\bf k}$ -space forms being found in Appendix B, the algebraic structure of the quantum constraints

⁴Henceforth, operators will be represented by hatted symbols such as \hat{h}_{ab} . In addition, superscripts on operators and labels will be suppressed except for those involving longitudinal and transverse vector parts of the longitudinal tensor subcomponents.

may now be examined. Promoting the constraint functions in (2.37)-(2.41) to operators, the non-vanishing sector in the commutator algebra may still be found:

$$\left[\hat{\phi}(x), \hat{\psi}(y)\right] = i\delta^3(x-y) \tag{3.14}$$

$$\left[\hat{\phi}_a(x), \hat{\psi}_b(y)\right] = i\delta^3(x-y)\delta_{ab}. \tag{3.15}$$

As in the classical case, the constraint quantum constraint algebra can be classified as second class.

The CCR's given in (3.15) also allow canonical coherent states for each subspace to be defined. Coherent states for this problem admit a factorization in the form of

$$|p,h\rangle = \exp\left\{-i(2\pi/L)^3 \sum_{\mathbf{n}\in\mathcal{I}} \left[\bar{h}_{ab}^{\mathbf{n}} \hat{p}_{ab}^{\mathbf{n}} - \bar{p}_{ab}^{\mathbf{n}} \hat{h}_{ab}^{\mathbf{n}}\right]\right\} |\eta\rangle$$

$$\equiv |\mathbf{p},\mathbf{h}\rangle^{TT} |\mathbf{p},\mathbf{h}\rangle^{T} |\mathbf{p},\mathbf{h}\rangle^{L}, \qquad (3.16)$$

where a general fiducial vector $|\eta\rangle$ has been used. As the above expression for the coherent states in \mathfrak{H} is vital to our analysis, the rest of this section seeks to explain each of the factors in (3.16).

The first term in (2.22) represents the transverse-traceless components $(\hat{\mathbf{p}}_{ab}, \hat{\mathbf{h}}_{ab})$, which can in turn be expanded in terms of the two independent polarization states as

$$\hat{\mathbf{f}}_{ab}^{\mathbf{n}} = m_a m_b \hat{\mathbf{f}}_+^{\mathbf{n}} + \bar{m}_a \bar{m}_b \hat{\mathbf{f}}_-^{\mathbf{n}}.$$
(3.17)

This subspace has no associated constraint and represents the two independent degrees of freedom in linearized gravity. In terms of only the transverse-traceless variables, the Hamiltonian density in (2.14) may be interpreted as the operator

$$\hat{\mathcal{H}}_{TT}[\mathbf{p}, \mathbf{h}] = \hat{\bar{\mathbf{p}}}_{ab}^{\mathbf{n}} \hat{\mathbf{p}}_{ab}^{\mathbf{n}} + \frac{k^2}{4} \hat{\bar{\mathbf{h}}}_{ab,c}^{\mathbf{n}} \hat{\mathbf{h}}_{ab,c}^{\mathbf{n}}.$$
 (3.18)

The transverse-traceless coherent states [17] then assume the form

$$|\mathbf{p}, \mathbf{h}\rangle^{TT} = \exp\left\{-i(2\pi/L)^3 \sum_{\mathbf{n} \in \mathcal{I}} \left[\bar{\mathbf{h}}_{ab}^{\mathbf{n}} \hat{\mathbf{p}}_{ab}^{\mathbf{n}} - \bar{\mathbf{p}}_{ab}^{\mathbf{n}} \hat{\mathbf{h}}_{ab}^{\mathbf{n}}\right]\right\} |O^{TT}\rangle,$$
 (3.19)

where $|O^{TT}\rangle$ is the ground state for (3.18). The coherent states in (3.19) comprise an overcomplete basis for the transverse-traceless Hilbert space

 \mathfrak{H}_{TT} . The Weyl operator in (3.19) contains un-hatted quantities, which serve dual purpose as being both smooth test functions on the lattice and the coherent state labels.

The transverse operators (\hat{p}, \hat{h}) are represented by the second term in (3.16) and act on states in the transverse Hilbert space \mathfrak{H}_T . To keep things sufficiently general, coherent states, alá [17] or [14], in \mathfrak{H}_T can be formed by the Weyl operator acting on the transverse fiducial vector $|\eta^T\rangle$, i.e.,

$$|\mathsf{p},\mathsf{h}\rangle^T = \exp\left\{-i(2\pi/L)^3 \sum_{\mathbf{n}\in\mathcal{I}} \left[\bar{\mathsf{h}}^{\mathbf{n}} \hat{\mathsf{p}}^{\mathbf{n}} - \bar{\mathsf{p}}^{\mathbf{n}} \hat{\mathsf{h}}^{\mathbf{n}} \right] \right\} |\eta^T\rangle.$$
 (3.20)

However, the quantum structure of \mathfrak{H}_T is restricted by the constraint

$$\hat{H}(x) = \sum_{\mathbf{n} \in \mathcal{I}} k_{\mathbf{n}}^2 e^{i\mathbf{k}^{\mathbf{n}} \cdot \mathbf{x}} \hat{\mathbf{h}}^{\mathbf{n}}, \tag{3.21}$$

where $k_{\mathbf{n}}^2 \equiv |\mathbf{k}^{\mathbf{n}}|^2 = k_a^{\mathbf{n}} k_a^{\mathbf{n}}$.

The other constrained Hilbert space is the longitudinal (vector) Hilbert space \mathfrak{H}_L . As with the case for $|\eta^T\rangle$, the fiducial vector of the longitudinal coherent states, $|\eta^L\rangle$, is the longitudinal fiducial vector. These fiducial vectors, left as general states for now, will be determined in the next subsection. The Weyl operator version of the coherent states in \mathfrak{H}_L is conveniently derived when

$$\mathfrak{f}_a^t \equiv \frac{\bar{m}_a \mathfrak{f}_1^t + m_a \mathfrak{f}_2^t}{2} \tag{3.22}$$

is inserted into the definition of (3.16). The operator version of the expansion of (2.22), applied to (3.16), then allows us to extract

$$|\mathfrak{p},\mathfrak{h}\rangle^{L} = \exp\left\{-i(2\pi/L)^{3} \sum_{\mathbf{n}\in\mathcal{I}} 4k_{\mathbf{n}}^{2} \left[\bar{\mathfrak{h}}_{a}^{t} \mathbf{n} \hat{\mathfrak{p}}_{a}^{t} \mathbf{n} - \bar{\mathfrak{p}}_{a}^{t} \mathbf{n} \hat{\mathfrak{h}}_{a}^{t} \mathbf{n} + k_{\mathbf{n}}^{2} \left(\bar{\mathfrak{h}}^{l} \mathbf{n} \hat{\mathfrak{p}}^{l} \mathbf{n} - \bar{\mathfrak{p}}^{l} \mathbf{n} \hat{\mathfrak{h}}^{l} \mathbf{n}\right)\right]\right\} |\eta^{L}\rangle$$

$$= \exp\left\{-i(2\pi/L)^{3} \sum_{\mathbf{n}\in\mathcal{I}} 4k_{\mathbf{n}}^{2} \left[\bar{\mathfrak{h}}_{a}^{t} \mathbf{n} \hat{\mathfrak{p}}_{a}^{t} \mathbf{n} - \bar{\mathfrak{p}}_{a}^{t} \mathbf{n} \hat{\mathfrak{h}}_{a}^{t} \mathbf{n}\right]\right\} |\eta^{t}\rangle$$

$$\otimes \exp\left\{-i(2\pi/L)^{3} \sum_{\mathbf{n}\in\mathcal{I}} 4k_{\mathbf{n}}^{4} \left[\bar{\mathfrak{h}}_{a}^{t} \mathbf{n} \hat{\mathfrak{p}}_{a}^{l} \mathbf{n} - \bar{\mathfrak{p}}_{a}^{l} \mathbf{n} \hat{\mathfrak{h}}_{a}^{l} \mathbf{n}\right]\right\} |\eta^{l}\rangle$$

$$= |\mathfrak{p},\mathfrak{h}\rangle^{t} |\mathfrak{p},\mathfrak{h}\rangle^{l}$$

$$(3.23)$$

such that $|\eta^t\rangle \otimes |\eta^l\rangle = |\eta^L\rangle$. This means our starting fiducial vector $|\eta\rangle$ may be expressed in factorized, direct product form as

$$|\eta\rangle = |O^{TT}\rangle \otimes |\eta^T\rangle \otimes (|\eta^t\rangle \otimes |\eta^l\rangle).$$
 (3.24)

To give a brief overview of the above discussion, convenient forms for the coherent states and operators were found. In promoting the position space versions of (2.37)-(2.41) to self-adjoint constraint operators, a second-class constraint structure is unavoidable. Kinematical coherent states in \mathfrak{H} are given by (3.16), defined as a direct product of states in \mathfrak{H}_{TT} , \mathfrak{H}_{T} , and \mathfrak{H}_{L} . Each of the (canonical) coherent states provides an overcomplete basis for their corresponding subspace in \mathfrak{H} . This unconstrained Hilbert space may be expressed in direct product form as

$$\mathfrak{H} = \mathfrak{H}_{TT} \otimes \mathfrak{H}_T \otimes \mathfrak{H}_L. \tag{3.25}$$

Of the three subspaces, the transverse and longitudinal Hilbert spaces are constrained when acted upon by the Projection Operator. We seek to examine how this works in the next section.

3.2 The reduced reproducing kernels

With the quantum operators, constraints, and coherent states defined, the only item left remaining to do is to determine the reduced reproducing kernel, as this will uncover the nature of the physical Hilbert space \mathfrak{H}_P . The projection operator is constructed so that $\mathbb{E} = \mathbb{E}\left[\Sigma\Phi^2 \leq \delta^2\right]$, where δ is a small, but nonzero, parameter. As introduced in earlier works, $\Sigma\Phi^2$ can be defined as

$$\Sigma \Phi^{2} = \sum_{\mathbf{n} \in \mathcal{I}} \left(|\hat{\psi}^{\mathbf{n}}|^{2} + |\hat{\phi}^{\mathbf{n}}|^{2} \right) + \sum_{\mathbf{n} \in \mathcal{I}} \left(|\hat{\psi}^{\mathbf{n}}_{a}|^{2} + |\hat{\phi}^{\mathbf{n}}_{a}|^{2} \right) + \sum_{\mathbf{n} \in \mathcal{I}} |\hat{\varsigma}^{\mathbf{n}}|^{2}$$

$$= \sum_{\mathbf{n} \in \mathcal{I}} \left(|\hat{\mathbf{p}}^{\mathbf{n}}|^{2} + k_{\mathbf{n}}^{2} |\hat{\mathbf{h}}^{\mathbf{n}}|^{2} \right)$$

$$+ \sum_{\mathbf{n} \in \mathcal{I}} k_{\mathbf{n}}^{2} \left(|\hat{\mathbf{p}}^{t}_{a}|^{2} + k_{\mathbf{n}}^{2} |\hat{\mathbf{h}}^{t}_{a}|^{2} \right) + \sum_{\mathbf{n} \in \mathcal{I}} k_{\mathbf{n}}^{4} |\hat{\mathbf{p}}^{l}|^{\mathbf{n}} |^{2}, \qquad (3.26)$$

where position and momentum operators conjugate to one another have been suggestively grouped together with parentheses. The constraint operators in (3.26) are quantized from the constraint functions in (2.37)-(2.41). The first two parenthetical terms in (3.26) can be interpreted heuristically as, being quadratic in $\{\hat{p}^{\mathbf{n}}, \hat{h}^{\mathbf{n}}\}$ and $\{\hat{p}^{t\,\mathbf{n}}, \hat{h}^{t\,\mathbf{n}}\}$, a sum of harmonic-oscillator Hamiltonians on the momentum lattice in both the transverse and longitudinal subspaces. With this insight, it is anticipated that δ cannot be taken to zero in any limit without \mathbb{E} vanishing everywhere. Thus, δ must have a positive-valued minimum so as to capture only the ground state eigenvalue of each oscillator, a situation typical in second-class systems with the

Projection Operator Method [10]. However preventing $\delta \to 0$ for the spectrum of the $|\hat{\varsigma}^{\mathbf{n}}|^2$ operator amounts to a violation of the quantum constraint. Therefore, the appropriate δ to use for this mixed case of first and second class constraints is

$$\delta^2 \to \delta^2 + 3\left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{n}\in\mathcal{I}} \hbar k_{\mathbf{n}}.$$
 (3.27)

The Projection Operator is formed by taking into account the fact that \mathfrak{H} is formed by a direct product of Hilbert spaces

$$\mathfrak{H} = \mathfrak{H}_{TT} \otimes \mathfrak{H}_T \otimes \mathfrak{H}_L \tag{3.28}$$

$$= \mathfrak{H}_{TT} \otimes \mathfrak{H}_{T} \otimes (\mathfrak{H}_{t} \otimes \mathfrak{H}_{l}), \qquad (3.29)$$

where the second line above exhibits the independence of the vector degrees of freedom of \mathfrak{H}_L . This suggests that we can split the projection operator according to

$$\mathbb{E} = \mathbb{E}\left[\Sigma\Phi^2 \le \delta^2\right] \tag{3.30}$$

$$= \mathbb{E}\left[\Sigma(|\hat{p}|^2 + k^2|\hat{h}|^2) + \Sigma(|\hat{p}|^2 + k^2|\hat{h}|^2) \le \delta^2 + 3\Sigma\hbar k\right]$$
(3.31)

$$= \mathbb{E}^{T} \left[\Sigma(|\hat{\mathbf{p}}|^{2} + k^{2}|\hat{\mathbf{h}}|^{2}) \leq \Sigma \hbar k \right] \otimes \mathbb{E}^{t} \left[\Sigma k^{2} (|\hat{\mathbf{p}}^{t}|^{2} + k^{2}|\hat{\mathbf{h}}^{t}|^{2}) \leq 2\Sigma \hbar k \right]$$

$$\otimes \mathbb{E}^{l} \left[\Sigma k^{4} |\hat{\mathbf{p}}^{l}|^{2} \leq \delta^{2} \right],$$
(3.32)

using the shorthand Σ for $\mathcal{V}^{-1}\sum_{\mathbf{n}\in\mathcal{I}}$. Since the operator arguments of \mathbb{E}^T and \mathbb{E}^t may be interpreted as the Hamiltonians of harmonic oscillators, the transition from (3.31) to (3.32) is made by assigning the appropriate multiplicative factors of $\Sigma\hbar k$ to capture the associated ground state of each oscillator.

More specifically, the transverse Projection Operator \mathbb{E}^T in (3.32) confines the eigenvalues of $\Sigma(\hat{\mathsf{p}}^2 + k^2\hat{\mathsf{h}}^2)$ to being in their ground state values for each point on the lattice in \mathfrak{H}_T . Its counterpart, \mathbb{E}^t , similarly restricts the two degrees of freedom per lattice point in \mathfrak{H}_t . The only nonzero states which will meet the inequality conditions of \mathbb{E}^T and \mathbb{E}^t in (3.32) are the ground states of each operator. We use the familiar projection operator notation of quantum mechanics to abbreviate (3.32) by

$$\mathbb{E} = |O^T\rangle\langle O^T| \otimes |O^t\rangle\langle O^t| \otimes \mathbb{E}^l \left[\Sigma |\hat{\varsigma}|^2 \le \delta^2 \right]. \tag{3.33}$$

The projection operator \mathbb{E}^l is constructed in a fundamentally different manner from those of the \mathfrak{H}_T and \mathfrak{H}_t subspaces, as the $\hat{\varsigma}^{\mathbf{n}}$ constraints have

zeros in their continuous spectrum. The projection operator \mathbb{E}^l can be formally written as

$$\mathbb{E}^{l} = \mathbb{E}^{l} \left[\Sigma |\hat{\varsigma}|^{2} \leq \delta^{2} \right]$$

$$= \int_{-\infty}^{\infty} d\lambda \, \exp \left[-i\lambda \Sigma |\hat{\varsigma}|^{2} \right] \, \frac{\sin(\delta^{2}\lambda)}{\pi\lambda}. \tag{3.34}$$

A suitable limit where $\delta \to 0$ is reserved for a later stage.

We limit the discussion on \mathbb{E}^l here to formal arguments for good reason. It is well-known that \mathbb{E}^l is uniquely determined by its coherent state matrix elements [15]. Since the reproducing kernel is defined by the very same coherent state matrix elements of \mathbb{E} (3.5) and provides Gaussian smoothing owing to the test function properties of the coherent states, we insist on the coherent state matrix elements of \mathbb{E} as being rigorously-defined mathematical quantities [15].

Knowing that the Projection Operator factorizes according to (3.33), with the definition of (3.5) the reduced reproducing kernel can be given the functional forms of the transverse and longitudinal reduced reproducing kernels,

$$\langle\langle p', h'|p, h\rangle\rangle \equiv \langle\langle \mathbf{p}', \mathbf{h}'|\mathbf{p}, \mathbf{h}\rangle\rangle^{TT} \langle\langle p', h'|p, h\rangle\rangle^{T} \langle\langle p', \mathfrak{h}'|\mathfrak{p}, \mathfrak{h}\rangle\rangle^{L}$$

$$= \langle \mathbf{p}', \mathbf{h}'|\mathbf{p}, \mathbf{h}\rangle^{TT} \frac{\langle p', h'|\mathbb{E}^{T}|p, h\rangle}{\langle 0, 0|\mathbb{E}^{T}|0, 0\rangle} \frac{\langle \mathfrak{p}'^{t}, \mathfrak{h}'^{t}|\mathbb{E}^{t}|\mathfrak{p}^{t}, \mathfrak{h}^{t}\rangle}{\langle \mathbf{0}, \mathbf{0}|\mathbb{E}^{t}|\mathbf{0}, \mathbf{0}\rangle} \lim_{\delta \to 0} \frac{\langle \mathfrak{p}'^{l}, \mathfrak{h}'^{l}|\mathbb{E}^{l}|\mathfrak{p}^{l}, \mathfrak{h}^{l}\rangle}{\langle 0, 0|\mathbb{E}^{l}|0, 0\rangle}$$

$$= \langle \mathbf{p}', \mathbf{h}'|\mathbf{p}, \mathbf{h}\rangle^{TT} \mathcal{K}_{2}[\mathbf{p}', h'; \mathbf{p}, h] \mathcal{K}_{2}[\mathfrak{p}'^{t}, \mathfrak{h}'^{t}; \mathfrak{p}^{t}, \mathfrak{h}^{t}] \mathcal{K}_{1}[\mathfrak{p}'^{l}, \mathfrak{h}'^{l}; \mathfrak{p}^{l}, \mathfrak{h}^{l}]$$

$$(3.35)$$

The functional \mathcal{K}_2 represents the functional for reproducing kernels for second-class constraints in (3.35), while \mathcal{K}_1 represents the reproducing kernels for their first-class counterparts. Each of these reproducing kernels can be shown to be the reproducing kernels of one-dimensional Hilbert spaces.

This fact may not be immediately obvious. Writing down any member of a dense set $\Psi \in D_{\mathfrak{H}_T}$ of the transverse Hilbert space \mathfrak{H}_T leads to

$$\Psi[\mathbf{p}, \mathbf{h}] = \sum_{m} \beta_{m} \langle \mathbf{p}, \mathbf{h} | \mathbf{p}_{m}, \mathbf{h}_{m} \rangle
= \sum_{m} \beta_{m} \mathcal{K}_{2}[\mathbf{p}, \mathbf{h}; \mathbf{p}_{m}, \mathbf{h}_{m}]
= \sum_{m} \beta_{m} \langle \mathbf{p}, \mathbf{h} | O^{T} \rangle \langle O^{T} | \mathbf{p}_{m}, \mathbf{h}_{m} \rangle$$

$$\rightarrow \Psi_{0}[\mathbf{p}, \mathbf{h}] \sum_{m} \beta_{m} e^{-\sum \left[\frac{|\mathbf{p}_{m}|^{2}}{4k} + \frac{k|\mathbf{h}_{m}|^{2}}{4}\right]} \propto \Psi_{0}[\mathbf{p}, \mathbf{h}], \quad (3.37)$$

The Ψ_0 in the above equation is the ground state representative in \mathcal{H}_T :

$$\Psi_0[\mathsf{p},\mathsf{h}] = \langle\!\langle \mathsf{p},\mathsf{h}|0,0\rangle\!\rangle = e^{-\sum \left[\frac{|\mathsf{p}|^2}{4k} + \frac{k|\mathsf{h}|^2}{4}\right]}.$$
 (3.38)

The fact that every vector in the dense set $D_{\mathfrak{H}_T}$ is proportional to the ground state Ψ_0 leads to the conclusion that the transverse reproducing kernel characterizes a one-dimensional Hilbert space. The same analysis also holds for $\Psi \in D_{\mathfrak{H}_t}$ given by expansions of $\mathcal{K}_2[\mathfrak{p}'^t, \mathfrak{h}'^t; \mathfrak{p}^t, \mathfrak{h}^t]$. A similar result follows for the reduction of \mathcal{K}_1 , the specifics of which are included in Appendix B.

Now that it has been demonstrated that the factors \mathcal{K}_2 and \mathcal{K}_1 in (3.35) are extraneous, one may also choose to absorb these factors into a redefinition of the coherent states [10]. With this operation, the reproducing kernel takes on the simplified form of

$$\langle \langle p', h' | p, h \rangle \rangle = \langle \mathbf{p}', \mathbf{h}' | \mathbf{p}, \mathbf{h} \rangle^{TT}.$$
 (3.39)

In representing the reduced reproducing kernel in (3.39), we have suppressed the trivial nature of the labels $\{\mathfrak{p},\mathfrak{h};\mathfrak{p},\mathsf{h}\}.$

The true dynamical degrees of freedom left reside entirely in the transversetraceless components of the metric and momentum fields, as expected. The Hilbert space \mathfrak{H}_{TT} is the physical Hilbert space \mathfrak{H}_{P} , and general vectors in this space are given by expansions of the (3.39) reproducing kernel, producing vectors of the form of (3.6). The calculations that went into finding (3.39) show how this was achieved without choosing any lapses, shifts, or time representations; instead, a projection operator was used to enforce the quantum constraints.

4 Gravitonic states in the physical Hilbert space

The reproducing kernel formalism not only provides a useful vehicle to describe reduction of the original Hilbert space, but for the problem under consideration, it also can be used to build a functional Fock space representation of the physical Hilbert space. To start with, the ground state functional representative in the physical Hilbert space may be written in the continuum limit of $L \to \infty$ and $N \to \infty$ as

$$\Psi_0[\mathbf{p}, \mathbf{h}] \equiv \langle \mathbf{p}, \mathbf{h} | 0 \rangle^{TT}$$

$$= \exp \left[-\frac{1}{2} \int d^3k \frac{1}{\omega(\mathbf{k})} \left(|\mathbf{p}_{ab}|^2 + \frac{\omega(\mathbf{k})^2}{4} |\mathbf{h}_{ab}|^2 \right) \right],$$
(4.1)

where $\omega = |\mathbf{k}|$. Using (3.17), the complex modulus-squared of each label may be expanded in terms of its components as

$$|\mathbf{h}_{ab}|^2 = |h_+(\mathbf{k})|^2 + |h_-(\mathbf{k})|^2, \quad |\mathbf{p}_{ab}|^2 = |p_+(\mathbf{k})|^2 + |p_-(\mathbf{k})|^2,$$
 (4.2)

where the + and - subscripts denote the two different graviton polarizations. This means that the ground state functional can also be seen as an independent functional for each polarization, or

$$\Psi_0[\mathbf{p}, \mathbf{h}] = \Psi_{0+}[p_+, h_+] \Psi_{0-}[p_-, h_-],$$
 (4.3)

where the functional $\Psi_{0\pm}[p_{\pm}, h_{\pm}]$ is defined in the same way as (4.1), but with $(\mathbf{p}_{ab}, \mathbf{h}_{ab})$ replaced by (p_{\pm}, h_{\pm}) .

To populate Fock space with gravitonic states, one needs to know what the creation and annihilation operators are in the appropriate coherent state representation [17]. The most convenient way to do this is to introduce the complex label $\mathbf{z}(\mathbf{k})$, the components of which are given by

$$z_{\pm}(\mathbf{k}) = \frac{\sqrt{\omega}}{2} h_{\pm}(\mathbf{k}) + \frac{i}{\sqrt{\omega}} p_{\pm}(\mathbf{k}), \tag{4.4}$$

for each polarization state. Using this new label, (4.1) can be expressed as

$$\Psi_0[\mathbf{z}] = \exp\left[-\frac{1}{2} \int d^3k \left(|z_+(\mathbf{k})|^2 + |z_-(\mathbf{k})|^2\right)\right].$$
 (4.5)

The annihilation operators for both the + and - polarization state appear then as

$$a_{\pm}(\mathbf{k}) = \frac{z_{\pm}(\mathbf{k})}{2} + \frac{\delta}{\delta z_{+}^{*}(\mathbf{k})}.$$
(4.6)

Likewise, the creation operator for each state is

$$a_{\pm}^{\dagger}(\mathbf{k}) = \frac{z_{\pm}^{*}(\mathbf{k})}{2} - \frac{\delta}{\delta z_{+}(\mathbf{k})}.$$
(4.7)

These operators act on the ground state $\Psi[\mathbf{z}]$ to give

$$a_{\pm}(\mathbf{k})\Psi_0[\mathbf{z}] = 0, \tag{4.8}$$

$$a_{\pm}^{\dagger}(\mathbf{k}_1) \dots a_{\pm}^{\dagger}(\mathbf{k}_l) \Psi_0[\mathbf{z}] = z_{\pm}^*(\mathbf{k}_1) \dots z_{\pm}^*(\mathbf{k}_l) \Psi_0[\mathbf{z}].$$
 (4.9)

5 Conclusion

This work examined the application of the Projection Operator Method to the theory of linearized gravity. In the classical version of the theory, perturbing around a flat background and insisting on gauge independence led to a set of partially second-class, classical constraints. In fact, only one degree of freedom per lattice point was constrained in a first class manner, corresponding to a gauge choice which could have been made. While the emergence of the second-class nature of the constraints at the classical level certainly seems novel and merits further investigation, the main thrust of this work involves the demonstration that the Projection Operator can unambiguously quantize an infinite number of mixed first and second-class constraints.

The core of the quantum analysis was the calculation of the reproducing kernels. These results showed explicitly how the Projection Operator, though it contained a factorizable product of projection operators, reduced all constrained subspaces to independent copies of the one-dimensional Hilbert space of complex numbers ($\mathbf{1}_{\mathbb{C}}$). Symbolically, leaving needed rescaling as implicit, this may be expressed as

$$\mathfrak{H}_{P} = \mathfrak{H}_{TT} \otimes \mathbb{E}^{T} \mathfrak{H}_{T} \otimes (\mathbb{E}^{t} \mathfrak{H}_{t} \otimes \mathbb{E}^{l} \mathfrak{H}_{l})
= \mathfrak{H}_{TT} \otimes \mathbf{1}_{\mathbb{C}} \otimes (\mathbf{1}_{\mathbb{C}} \otimes \mathbf{1}_{\mathbb{C}})
\rightsquigarrow \mathfrak{H}_{TT}.$$
(5.1)

Here, we have introduced the symbol \rightsquigarrow as meaning that a bijection can be found relating the second and third lines, (5.1) and (5.2). This result agrees with the well-known result that the transverse-traceless degrees of freedom are the only degrees of freedom involved in quantum dynamics. This reduction mirrors the result obtained recently by Dittrich and Thiemann using the Master Constraint Programme [6].

To further illustrate the connection to previous results, equation (4.1) may be compared with prior work by Kuchař [18]. In his version, the ground state functional for linearized gravity is given by

$$\Psi_0[h^{TT}] = \mathcal{N} \exp\left[-\frac{1}{4} \int d^3k \ \omega(\mathbf{k}) \ \left|h_{ab}^{TT}\right|^2\right],\tag{5.3}$$

the two physical degrees of freedom in the theory residing within the tensor h_{ab}^{TT} . For us these same degrees of freedom are embedded in $\{\mathbf{p}, \mathbf{h}\}$. Here \mathcal{N} is a formal normalization constant. It is clear that in passing to a representation involving only the $\mathbf{h}_{ab}(\mathbf{k})$, (4.1) results in an expression identical to (5.3).

In essence, we can say that the Projection Operator Method, combined with the reduced reproducing kernel calculations, has shown that the transverse and longitudinal degrees of freedom completely decouple from linearized gravity after quantization. Kuchař discovered the same dynamics as a result of traditional canonical quantum gravity techniques: by using the so-called extrinsic time representation, a gauge choice, and embedding the quantum dynamics into the WdW equation. Our result for the ground state functional is similar to what one would expect from using the Wheeler-DeWitt equation, however we remark again that no gauge has been chosen in our approach.

A Appendix: Simple Toy Model

To help clarify the way in which the Projection Operator reduces the kinematical Hilbert space to the physical Hilbert space for a second-class system of constraints, we have chosen to include an example problem analogous to the situation of gauge-independent linearized gravity. Consider a system in a flat phase space, coordinatized by (p_a, q^a) , a = 1, 2, 3, such that

$$\{q^b, p_a\} = \delta_a^b \tag{A.1}$$

Let this system be described by the constrained action

$$S[p,q] = \int dt \left[p_a \dot{q}^a - H(p,q) - \lambda^A p_A \right], \tag{A.2}$$

where A = 2, 3. The unconstrained Hamiltonian is then given by

$$H(p,q) = \frac{p_1^2}{2} + \frac{q_1^2}{2} + \sum_{J=2}^{3} \left(\frac{p_J^2}{2} - \frac{q_J^2}{2}\right). \tag{A.3}$$

Evolution of the constraints is directly analogous to the reduction involving \mathfrak{H}_t in gauge-independent linearized gravity. The constraint p_A in (A.2) most similar to the constraint piece involving \mathfrak{p}_a^t in (2.16), while the Hamiltonian in (A.3) is of the same basic form as (2.32). Calculating the evolution of p_A , one finds

$$\dot{p}_A = \{p_A, H\} = q_A \equiv 0 ,$$
 (A.4)

meaning that we must consider q_A as a secondary constraint.

Placing primary and secondary constraints back in the action, we get

$$S[p,q] = \int dt \left[p_a \dot{q}^a - H(p,q) - \lambda^A p_A - \mu^B q_B \right], \tag{A.5}$$

where the set of Lagrange multipliers $\{\mu_A, \lambda_A\}$ are determined by the equations of motion. Now (A.5) is used to recursively calculate the evolution of the constraints. After again insisting on the vanishing of the time-derivatives of all constraints, the solution

$$\lambda_A = q_A = 0 = -p_B = \mu_B \tag{A.6}$$

is obtained as an end result.

Foregoing further analysis and physical interpretation of the classical solution, we proceed to the quantized version of the theory. Observing that the classical constraint algebra,

$$\{q^B, p_A\} = \delta_A^B, \tag{A.7}$$

does not vanish on the constraint hypersurface, we come to the conclusion that this is a second-class system of constraints. Since the constraints are just the canonical variables, promotion of the phase-space coordinates to self-adjoint, irreducible operators via

$$\{q^a, p_b\} = \delta_b^a \mapsto -i[Q^a, P_b]/\hbar, \tag{A.8}$$

implies the promotion of the constraints to self-adjoint operators. The fact that the constraint commutator algebra is by definition

$$[Q^B, P_A] = i\hbar, (A.9)$$

means that there is still a second class constraint structure for the quantum analysis.

Construction of the Projection Operator is straightforward. This operator is defined as

$$\mathbb{E} \equiv \mathbb{E}[\Sigma \Phi^2 \le \delta^2] = \mathbb{E}[\Sigma (P_A^2 + Q_A^2) \le 2\hbar]. \tag{A.10}$$

Since $\Sigma\Phi^2$ is clearly the Hamiltonian operator for a unit-frequency, twodimensional harmonic oscillator, δ^2 may be chosen as $2\hbar$ to capture only the ground state and prevent \mathbb{E} from vanishing. Therefore, working in the energy eigenbasis the Projection Operator may be simply written as

$$\mathbb{E}[\Sigma(P_A^2 + Q_A^2) \le 2\hbar] = |O_2, O_3\rangle\langle O_2, O_3|, \tag{A.11}$$

where $|O_2, O_3\rangle\langle O_2, O_3|$ is the ground state for the J=2,3 degrees of freedom. Introducing $|\mathbf{O}\rangle$ as the ground state for all three degrees of freedom, the coherent states over the unconstrained Hilbert space are defined as

$$|\mathbf{p}, \mathbf{q}\rangle = e^{i(p_a Q^a - q^a P_a)} |\mathbf{O}\rangle$$
 (A.12)

$$\equiv |p_1, q_1\rangle \otimes |p_2, q_2\rangle \otimes |p_3, q_3\rangle, \tag{A.13}$$

such that q_a and p_a are the expectations of the Q_a and P_a operators respectively. Using (A.11) with (A.13), the reproducing kernel is given by

$$\langle \langle \mathbf{p}', \mathbf{q}' | \mathbf{p}, \mathbf{q} \rangle \rangle = \langle \mathbf{p}', \mathbf{q}' | O \rangle \langle O | \mathbf{p}, \mathbf{q} \rangle$$
 (A.14)

$$= \mathcal{K}_0[p'_1, q'_1; p_1, q_1] \mathcal{K}_2[p'_A, q'_A; p_A, q_A], \qquad (A.15)$$

where $\mathcal{K}_2[p_A', q_A'; p_A, q_A] = e^{-(p_A'^2 + q_A'^2)/4 - (p_A^2 + q_A^2)/4}$. This equation for \mathcal{K}_2 should be compared with $\mathcal{K}_0[p_1', q_1'; p_1, q_1]$, which is the conventional coherent state overlap

$$\mathcal{K}_{0}[p'_{1}, q'_{1}; p_{1}, q_{1}] = \langle p'_{1}, q'_{1} | p_{1}, q_{1} \rangle
= e^{-(p_{1} - p'_{1})^{2}/4 - (q_{1} - q'_{1})^{2}/4 + i(p_{1}q'_{1} - p'_{1}q_{1})/2}.$$
(A.16)

The physical Hilbert space characterized by the reproducing kernel in (A.15) may be expressed as

$$\mathfrak{H}_P = \mathfrak{H}_1 \otimes \mathbb{E} \left(\mathfrak{H}_2 \otimes \mathfrak{H}_3 \right). \tag{A.17}$$

Vectors $\Psi \in D_{\mathfrak{H}_P}$ for a dense set $D_{\mathfrak{H}_P} \subset \mathfrak{H}_P$ can be written as

$$\Psi[\mathbf{p}, \mathbf{q}] = \sum_{m} \beta_{m} \langle \langle \mathbf{p}, \mathbf{q} | \mathbf{p}_{m}, \mathbf{q}_{m} \rangle \rangle
= \sum_{m} \beta_{m} \mathcal{K}_{0}[p_{1}, q_{1}; p_{1m}, q_{1m}] \mathcal{K}_{2}[p_{A}, q_{A}; p_{Am}, q_{Am}]
= \sum_{m} \left(\beta_{m} e^{-(p_{Am}^{2} + q_{Am}^{2})/4} e^{-(p_{A}^{2} + q_{A}^{2})/4} \right) \mathcal{K}_{0}[p_{1}, q_{1}; p_{1m}, q_{1m}]
= \sum_{m} \beta'_{m} \mathcal{K}_{0}[p_{1}, q_{1}; p_{1m}, q_{1m}]$$
(A.18)

where K_2 has been absorbed into the complex coefficients β'_m . The reduction depicted in (A.18) shows how the structure of \mathcal{H}_P is only dependent on the single (p_1, q_1) degree of freedom.

B Appendix: Theorem on the reduction of a firstclass reproducing kernel using coherent states

To show the collapse of the constrained Hilbert space \mathfrak{H}_l into a one-dimensional Hilbert space, we shall prove the following

Theorem: The Hilbert space \mathfrak{H}_l characterized by $\mathcal{K}_1[\mathfrak{p}'^l,\mathfrak{h}'^l;\mathfrak{p}^l,\mathfrak{h}^l]$ is a one-dimensional Hilbert space.

Proof: The direct proof to follow involves the lattice setup of (3.11) and (3.12) and the properties of coherent states contained in , e.g., [15] and [17]. The longitudinal vector projection operator, extracted from (3.32), is given by

$$\mathbb{E}^l = \mathbb{E}^l[\Sigma|\hat{\varsigma}|^2 \le \delta^2]. \tag{B.1}$$

It pays to examine the argument of this operator more closely and we find that this expression may be re-expressed as

$$\mathcal{V}^{-1} \sum_{\mathbf{n} \in \mathcal{I}} k_{\mathbf{n}}^4 |\hat{\mathbf{p}}^{\mathbf{n}}|^2 \leq \delta^2.$$
 (B.2)

A useful intermediate bound is given by

$$\mathcal{V}^{-1} \sum_{\mathbf{n} \in \mathcal{I}} k_{\mathbf{n}}^{4} |\hat{\mathbf{p}}^{\mathbf{n} l}|^{2} \leq \mathcal{V}^{-1} \sum_{\mathbf{n} \in \mathcal{I}} \delta_{\mathbf{n}}^{2} \leq \delta^{2}.$$
 (B.3)

There are a finite number of small parameters in the sum in (B.3), meaning a convergent sum. This sum is bounded by δ^2 and vanishes as $\delta \to 0$. We can just as well take every $\delta_{\mathbf{n}} \to 0$. In this case, \mathbb{E}^l may be expanded as

$$\mathbb{E}^{l} = \prod_{\mathbf{n} \in \mathcal{I}} \mathbb{E}^{\mathbf{n} l} \left[k_{\mathbf{n}}^{4} |\hat{\mathbf{p}}^{\mathbf{n} l}|^{2} \le \delta_{\mathbf{n}}^{2} \right].$$
 (B.4)

Using (B.4) in (3.35), results in

$$\mathcal{K}_{1} \equiv \lim_{\delta \to 0} \frac{\langle \mathbf{p}^{\prime l}, \mathbf{h}^{\prime l} | \mathbb{E}^{l} \left[\Sigma k_{\mathbf{n}}^{4} | \hat{\mathbf{p}}^{\mathbf{n} \ l} |^{2} \leq \delta^{2} \right] | \mathbf{p}^{l}, \mathbf{h}^{l} \rangle}{\langle 0, 0 | \mathbb{E}^{l} \left[\Sigma k_{\mathbf{n}}^{4} | \hat{\mathbf{p}}^{\mathbf{n} \ l} |^{2} \leq \delta^{2} \right] | 0, 0 \rangle}$$
(B.5)

$$= \prod_{\mathbf{n} \in \mathcal{I}} \lim_{\delta_{\mathbf{n}} \to 0} \frac{\langle \mathbf{p}_{\mathbf{n}}^{\prime l}, \mathbf{p}_{\mathbf{n}}^{\prime l} | \mathbb{E}^{\mathbf{n} l} \left[k_{\mathbf{n}}^{4} | \hat{\mathbf{p}}^{\mathbf{n} l} |^{2} \leq \delta_{\mathbf{n}}^{2} \right] | \mathbf{p}_{\mathbf{n}}^{l}, \mathbf{p}_{\mathbf{n}}^{l} \rangle}{\langle 0, 0 | \mathbb{E}^{\mathbf{n} l} \left[k_{\mathbf{n}}^{4} | \hat{\mathbf{p}}^{\mathbf{n} l} |^{2} \leq \delta_{\mathbf{n}}^{2} \right] | 0, 0 \rangle}$$
(B.6)

It is then sufficient to evaluate the reproducing kernel for some **n** pair $\mathbf{b}, -\mathbf{b} \in \mathcal{I}$ of (3.11). For each point, a resolution of unity may be inserted

into (B.6) to give

$$\prod_{\mathbf{n}=\mathbf{b},-\mathbf{b}} \langle \mathbf{p}'_{\mathbf{n}}, \mathbf{h}'_{\mathbf{n}} | \mathbb{E}^{\mathbf{n} l} \left[k_{\mathbf{n}}^{4} | \hat{\mathbf{p}}'' |^{2} \leq \delta_{\mathbf{n}}^{2} \right] | \mathbf{p}_{\mathbf{n}}, \mathbf{h}_{\mathbf{n}} \rangle
= \mathcal{M} \int_{-\delta_{\mathbf{b}}/k_{\mathbf{b}}^{2}}^{\delta_{\mathbf{b}}/k_{\mathbf{b}}^{2}} d\bar{\mathbf{p}}'' d\mathbf{p}'' \exp \left\{ -4 \mathcal{V}^{-1} k_{\mathbf{b}}^{4} \left[\frac{|\mathbf{p}'_{\mathbf{b}} - \mathbf{p}''|^{2}}{2} + 2i\Re(\bar{\mathbf{p}}'' [\mathbf{h}'_{\mathbf{b}} - \mathbf{h}_{\mathbf{b}}]) + \frac{|\mathbf{p}'' - \mathbf{p}_{\mathbf{b}}|^{2}}{2} \right] \right\},
\times \exp \left\{ -4i \mathcal{V}^{-1} k_{\mathbf{b}}^{4} \Re(\mathbf{p}'_{\mathbf{b}} \mathbf{h}'_{\mathbf{b}} - \mathbf{p}_{\mathbf{b}} \mathbf{h}_{\mathbf{b}}) \right\},$$
(B.7)

where we have suppressed all l superscripts, used $\Re(\cdot)$ to indicate the operation of extracting the real part, and assumed a Gaussian form for the fiducial vector $|\eta^l\rangle$ in (3.23). Also, the finite, constant factor \mathcal{M} in (B.7) need not be determined as it cancels out when (B.7) is plugged back into (B.6). Expression (B.7) allows us to observe the reduction

$$\lim_{\delta_{\mathbf{b}}\to 0} \prod_{\mathbf{n}=\mathbf{b},-\mathbf{b}} \frac{\langle \mathfrak{p}'_{\mathbf{n}},\mathfrak{h}'_{\mathbf{n}}|\mathbb{E}^{\mathbf{n}}\left[k_{\mathbf{n}}^{4}|\hat{\mathfrak{p}}^{\mathbf{n}}|^{2} \leq \delta_{\mathbf{n}}^{2}\right]|\mathfrak{p}_{\mathbf{n}},\mathfrak{h}_{\mathbf{n}}\rangle}{\langle 0,0|\mathbb{E}^{\mathbf{n}}\left[k_{\mathbf{n}}^{4}|\hat{\mathfrak{p}}^{\mathbf{n}}|^{2} \leq \delta_{\mathbf{n}}^{2}\right]|0,0\rangle}$$

$$= \exp\left\{-4\mathcal{V}^{-1}k_{\mathbf{b}}^{4}\left[\frac{|\mathfrak{p}'_{\mathbf{b}}|^{2}}{2} + \frac{|\mathfrak{p}_{\mathbf{b}}|^{2}}{2}\right]\right\} \exp\left\{-4i\mathcal{V}^{-1}k_{\mathbf{b}}^{4}\Re(\mathfrak{p}'_{\mathbf{b}}\mathfrak{h}'_{\mathbf{b}} - \mathfrak{p}_{\mathbf{b}}\mathfrak{h}_{\mathbf{b}})\right\}.$$
(B.8)

It then follows that

$$\mathcal{K}_1[\mathfrak{p}',\mathfrak{h}';\mathfrak{p},\mathfrak{h}] \tag{B.9}$$

$$= \exp \left\{ -\mathcal{V}^{-1} \sum_{\mathbf{n} \in \mathcal{I}} 4k_{\mathbf{n}}^{4} \left[\frac{|\mathfrak{p}_{\mathbf{n}}'|^{2}}{2} + \frac{|\mathfrak{p}_{\mathbf{n}}|^{2}}{2} + i\Re \left(\mathfrak{p}_{\mathbf{n}}'\mathfrak{h}_{\mathbf{n}}' - \mathfrak{p}_{\mathbf{n}}\mathfrak{h}_{\mathbf{n}} \right) \right] \right\}.$$
(B.10)

The fiducial vector representative can be written in terms of the reproducing kernel as

$$\Psi_0[\mathfrak{p}^l,\mathfrak{h}^l] = \mathcal{K}_1[\mathfrak{p}^l,\mathfrak{h}^l;0,0] = \exp\left\{-\mathcal{V}^{-1}\sum_{\mathbf{n}\in\mathcal{I}}4k_{\mathbf{n}}^4 \left[\frac{|\mathfrak{p}_{\mathbf{n}}^l|^2}{2} + i\Re(\mathfrak{p}_{\mathbf{n}}^l\mathfrak{h}_{\mathbf{n}}^l)\right]\right\}.$$
(B.11)

Writing down any member of a dense set $\Psi \in D_{\mathfrak{H}_l}$ of this Hilbert space \mathfrak{H}_l leads to a vector

$$\Psi[\mathfrak{p}^{l},\mathfrak{h}^{l}] = \sum_{m} \beta_{m} \mathcal{K}_{1}[\mathfrak{p}^{l},\mathfrak{h}^{l};\mathfrak{p}_{m}^{l},\mathfrak{h}_{m}^{l}]$$

$$= \sum_{m} \beta_{m} \exp\left\{-\mathcal{V}^{-1} \sum_{\mathbf{n} \in \mathcal{I}} 4k_{\mathbf{n}}^{4} \left[\frac{|\mathfrak{p}_{\mathbf{n}}^{l}|^{2}}{2} + \frac{|\mathfrak{p}_{\mathbf{n}m}^{l}|^{2}}{2} + i\Re\left(\mathfrak{p}_{\mathbf{n}}^{l}\mathfrak{h}_{\mathbf{n}}^{l} - \mathfrak{p}_{\mathbf{n}m}^{l}\mathfrak{h}_{\mathbf{n}m}^{l}\right)\right]\right\}$$

$$= \Psi_{0}[\mathfrak{p}^{l},\mathfrak{h}^{l}] \sum_{m} \beta_{m}^{l} \qquad (B.12)$$

which is proportional to the fiducial vector representative in the same way that we obtained (3.37) earlier. The Hilbert space \mathfrak{H}_l , like \mathfrak{H}_T and \mathfrak{H}_t , is equivalent to the space of complex numbers \mathbb{C} . This completes the proof. \square

C Appendix: Fundamental classical and quantum algebras

C.1 Classical Poisson algebra

The following equations give the classical Poisson bracket algebra for the tensor components of $\{\widetilde{p}_{ab}(\mathbf{k}), \widetilde{h}_{ab}(\mathbf{k})\}$.

Transverse:

$$\left\{\widetilde{\widetilde{\mathsf{h}}}(t,\mathbf{k}),\widetilde{\mathsf{p}}(t,\mathbf{k}')\right\} = 2\delta^3(\mathbf{k} - \mathbf{k}').$$
 (C.1)

Longitudinal:

$$\left\{ \widetilde{\widetilde{\mathfrak{h}}}_{1}^{t}(t,\mathbf{k}), \widetilde{\mathfrak{p}}_{1}^{t}(t,\mathbf{k}') \right\} = \left\{ \widetilde{\widetilde{\mathfrak{h}}}_{2}^{t}(t,\mathbf{k}), \widetilde{\mathfrak{p}}_{2}^{t}(t,\mathbf{k}') \right\} = \frac{\delta^{3}(\mathbf{k} - \mathbf{k}')}{4k^{2}} \quad (C.2)$$

$$\left\{ \widetilde{\widetilde{\mathfrak{h}}}^{l}(t,\mathbf{k}), \widetilde{\mathfrak{p}}^{l}(t,\mathbf{k}') \right\} = \frac{\delta^{3}(\mathbf{k} - \mathbf{k}')}{4k^{4}} \quad (C.3)$$

In the above equation and throughout the paper, we make use of the following abbreviation for the transverse components of \mathfrak{f}_a :

$$\widetilde{\mathfrak{f}}_a = \frac{m_a \widetilde{\mathfrak{f}}_1^t + \bar{m}_a \widetilde{\mathfrak{f}}_2^t}{2}.$$
(C.4)

Transverse-Traceless:

$$\left\{ \overline{\widetilde{\mathbf{h}}}^{+}(t,\mathbf{k}), \widetilde{\mathbf{p}}^{+}(t,\mathbf{k}') \right\} = \left\{ \overline{\widetilde{\mathbf{h}}}^{-}(t,\mathbf{k}), \widetilde{\mathbf{p}}^{-}(t,\mathbf{k}') \right\} = \delta^{3}(\mathbf{k} - \mathbf{k}'). \tag{C.5}$$

The $\widetilde{\mathbf{f}}^+$ and $\widetilde{\mathbf{f}}^-$ come from the expansion

$$\widetilde{\mathbf{f}}_{ab} = \widetilde{\mathbf{f}}^{+} m_a m_b + \widetilde{\mathbf{f}}^{-} \bar{m}_a \bar{m}_b. \tag{C.6}$$

C.2 Quantum commutation relations

The fundamental commutation relations may then be determined for each set of operator components, using the lattice prescription of Section 3.1.

Transverse:

$$\left[\hat{\bar{\mathbf{h}}}^{\mathbf{m}}, \hat{\mathbf{p}}^{\mathbf{n}}\right] = 2i\mathcal{V}\delta^{\mathbf{m},\mathbf{n}}.\tag{C.7}$$

Longitudinal:

$$\left[\hat{\hat{\mathfrak{h}}}_{1}^{\mathbf{m}t}, \hat{\mathfrak{p}}_{1}^{\mathbf{n}t}\right] = \left[\hat{\hat{\mathfrak{h}}}_{2}^{\mathbf{m}t}, \hat{\mathfrak{p}}_{2}^{\mathbf{n}t}\right] = \frac{i\mathcal{V}\delta^{\mathbf{m},\mathbf{n}}}{4k_{\mathbf{m}}^{2}}$$
(C.8)

$$\left[\bar{\hat{\mathfrak{h}}}^{\mathbf{m}l}, \hat{\mathfrak{p}}^{\mathbf{n}l}\right] = \frac{i\mathcal{V}\delta^{\mathbf{m},\mathbf{n}}}{4k_{\mathbf{n}}^{4}}$$
 (C.9)

Transverse-Traceless:

$$\left[\hat{\mathbf{h}}^{\mathbf{m}+}, \hat{\mathbf{p}}^{\mathbf{n}+}\right] = \left[\hat{\mathbf{h}}^{\mathbf{m}-}, \hat{\mathbf{p}}^{\mathbf{m}-}\right] = i\mathcal{V}\delta^{\mathbf{m},\mathbf{n}}.$$
 (C.10)

Again, the $\hat{\mathbf{f}}^+$ and $\hat{\mathbf{f}}^-$ come from the expansion

$$\hat{\mathbf{f}}_{ab} = \hat{\mathbf{f}}^{+} m_a m_b + \hat{\mathbf{f}}^{-} \bar{m}_a \bar{m}_b \tag{C.11}$$

Acknowledgements

The authors would like to thank Abhay Ashtekar, for helpful comments and questions surrounding the $\delta \to 0$ limit; J. Scott Little for help fruitful useful discussions; and Sergei Shabanov on the application of the Projection Operator to field theories.

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