Quantum Theory of Geometry I: Area Operators[*]

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Abstract

A new functional calculus, developed recently for a fully non-perturbative treatment of quantum gravity, is used to begin a systematic construction of a quantum theory of geometry. Regulated operators corresponding to areas of 2-surfaces are introduced and shown to be self-adjoint on the underlying (kinematical) Hilbert space of states. It is shown that their spectra are *purely* discrete indicating that the underlying quantum geometry is far from what the continuum picture might suggest. Indeed, the fundamental excitations of quantum geometry are 1-dimensional, rather like polymers, and the 3-dimensional continuum geometry emerges only on coarse graining. The full Hilbert space admits an orthonormal decomposition into finite dimensional sub-spaces which can be interpreted as the spaces of states of spin systems. Using this property, the complete spectrum of the area operators is evaluated. The general framework constructed here will be used in a subsequent paper to discuss 3-dimensional geometric operators, e.g., the ones corresponding to volumes of regions.

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I. INTRODUCTION

In his celebrated inaugural address, Riemann suggested [3] that geometry of space may be more than just a fiducial, mathematical entity serving as a passive stage for physical phenomena, and may in fact have a direct physical meaning in its own right. General relativity proved this vision to be correct: Einstein's equations put geometry on the same footing as matter. Now, the physics of this century has shown us that matter has constituents and the 3-dimensional objects we perceive as solids in fact have a discrete underlying structure. The continuum description of matter is an approximation which succeeds brilliantly in the macroscopic regime but fails hopelessly at the atomic scale. It is therefore natural to ask if the same is true of geometry. Does geometry also have constituents at the Planck scale? What are its atoms? Its elementary excitations? Is the space-time continuum only a "coarse-grained" approximation? If so, what is the nature of the underlying quantum geometry?

To probe such issues, it is natural to look for hints in the procedures that have been successful in describing matter. Let us begin by asking what we mean by quantization of physical quantities. Let us take a simple example –the hydrogen atom. In this case, the answer is clear: while the basic observables –energy and angular momentum– take on a continuous range of values classically, in quantum mechanics their spectra are discrete. So, we can ask if the same is true of geometry. Classical geometrical observables such as areas of surfaces and volumes of regions can take on continuous values on the phase space of general relativity. Are the spectra of corresponding quantum operators discrete? If so, we would say that geometry is quantized.

Thus, it is rather easy to pose the basic questions in a precise fashion. Indeed, they could have been formulated soon after the advent of quantum mechanics. Answering them, on the other hand, has proved to be surprisingly difficult. The main reason, it seems, is the inadequacy of the standard techniques. More precisely, the traditional approach to quantum field theory has been perturbative, where one *begins* with a continuum, background geometry. It is then difficult to see how discreteness would arise in the spectra of geometric operators. To analyze such issues, one needs a fully non-perturbative approach: geometric operators have to be constructed ab-initio without assuming *any* background geometry. To probe the nature of quantum geometry, we can not begin by *assuming* the validity of the continuum picture. We must let quantum gravity itself decide whether this picture is adequate at the Planck scale; the theory itself should lead us to the correct microscopic picture of geometry.

In this paper, we will use the non-perturbative, canonical approach to quantum gravity based on connections to probe these issues. Over the past three years, this approach has been put on a firm mathematical footing through the development of a new functional calculus on the space of gauge equivalent connections [4-11]. This calculus does not use any background fields (such as a metric) and is therefore well-suited to a fully non-perturbative treatment. The purpose of this paper is to use this framework to explore the nature of quantum geometry.

In section 2, we recall the relevant results from the new functional calculus and out-

line the general strategy. In section 3, we present a regularization of the area operator. Its properties are discussed in section 4; in particular, we exhibit its entire spectrum. Our analysis is carried out in the "connection representation" and the discussion is self-contained. However, at a non-technical level, there is a close similarity between the basic ideas used here and those used in discussions based on the "loop representation" [12,13]. Indeed, the development of the functional calculus which underlies this analysis itself was motivated, in a large measure, by the pioneering work on loop representation by Rovelli and Smolin [14]. The relation between various approaches will discussed in section 5.

The main result of this paper should have ramifications on the statistical mechanical origin of the entropy of black holes along the lines of [15,16]. This issue is being investigated.

II. PRELIMINARIES

This section is divided into three parts. In the first, we will recall [4,5] the basic structure of the quantum configuration space and, in the second, that of the Hilbert space of (kinematic) quantum states [10]. The overall strategy will be summarized in the third part.

A. Quantum configuration space

In general relativity, one can regard the space $\mathcal{A}/\mathcal{G}$ of $SU(2)$ connections modulo gauge transformations on a ("spatial") 3-manifold Σ as the classical configuration space [17–19]. For systems with only a finite number of degrees of freedom, the classical configuration space also serves as the domain space of quantum wave functions, i.e., as the quantum configuration space. For systems with an infinite number of degrees of freedom, on the other hand, this is not true: generically, the quantum configuration space is an enlargement of the classical. In free field theory in Minkowski space (as well as exactly solvable models in low space-time dimensions), for example, while the classical configuration space can be built from suitably smooth fields, the quantum configuration space includes all (tempered) distributions. This is an important point because, typically, the classical configuration spaces are of zero measure; wave functions with support only on smooth configurations have zero norm! The overall situation is the same in general relativity. The quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ is a certain completion of $\mathcal{A}/\mathcal{G}$ [4,5].

The space $\overline{\mathcal{A}/\mathcal{G}}$ inherits the quotient structure of $\mathcal{A}/\mathcal{G}$, i.e., $\overline{\mathcal{A}/\mathcal{G}}$ is the quotient of the space $\overline{\mathcal{A}}$ of generalized connections by the space $\overline{\mathcal{G}}$ of generalized gauge transformations. To see the nature of the generalization involved, recall first that each smooth connection defines a holonomy along paths [1] in Σ: $h_p(A) := \mathcal{P} \exp - \int_p A$. Generalized connections

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[1]For technical reasons, we will assume that all paths are analytic. An extension of the frame-

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capture this notion. That is, each $\bar{A}$ in $\overline{\mathcal{A}}$ can be defined [6,8] as a map which assigns to each oriented path p in Σ an element $\bar{A}(p)$ of $SU(2)$ such that: i) $\bar{A}(p^{-1}) = (\bar{A}(p))^{-1}$; and, ii) $\bar{A}(p_2 \circ p_1) = \bar{A}(p_2) \cdot \bar{A}(p_1)$, where p^{-1} is obtained from p by simply reversing the orientation, $p_2 \circ p_1$ denotes the composition of the two paths (obtained by connecting the end of p_1 with the beginning of p_2) and $\bar{A}(p_2) \cdot \bar{A}(p_1)$ is the composition in $SU(2)$. A generalized gauge transformation is a map g which assigns to each point v of Σ an $SU(2)$ element $g(x)$ (in an arbitrary, possibly discontinuous fashion). It acts on $\bar{A}$ in the expected manner, at the end points of paths: $\bar{A}(p) \rightarrow g(v_+)^{-1} \cdot \bar{A}(p) \cdot g(v_-)$, where v_- and v_+ are respectively the beginning and the end point of p. If $\bar{A}$ happens to be a smooth connections, say A, we have $\bar{A}(p) = h_p(A)$. However, in general, $\bar{A}(p)$ can not be expressed as a path ordered exponential of a smooth 1-form with values in the Lie algebra of $SU(2)$ [5]. Similarly, in general, a generalized gauge transformation can not be represented by a smooth group valued function on Σ.

At first sight the spaces $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ seem too large to be mathematically controllable. However, they admit three characterizations which enables one to introduce differential and integral calculus on them [4,5,7]. We will conclude this sub-section by summarizing the characterization –as suitable limits of the corresponding spaces in lattice gauge theory– which will be most useful for the main body of this paper.

We begin with some definitions. An *edge* is an oriented, 1-dimensional sub-manifold of Σ with two boundary points, called *vertices*, which is analytic everywhere, including the vertices. A *graph* in Σ is a collection of edges such that if two distinct edges meet, they do so only at vertices. In the physics terminology, one can think of a graph as a "floating lattice", i.e., a lattice whose edges are not required to be rectangular. (Indeed, they may even be non-trivially knotted!) Using the standard ideas from lattice gauge theory, we can construct the configuration space associated with the graph γ. Thus, we have the space $\mathcal{A}_\gamma$, each element A_γ of which assigns to every edge in γ an element of $SU(2)$ and the space $\mathcal{G}_\gamma$ each element g_γ of which assigns to each vertex in γ an element of $SU(2)$. (Thus, if N is the number of edges in γ and V the number of vertices, $\mathcal{A}_\gamma$ is isomorphic with $[SU(2)]^N$ and $\mathcal{G}_\gamma$ with $[SU(2)]^V$). $\mathcal{G}_\gamma$ has the obvious action on $\mathcal{A}_\gamma$: $A_\gamma(e) \rightarrow g(v_+)^{-1} \cdot A_\gamma(e) \cdot g(v_-)$. The (gauge invariant) configuration space associated with the floating lattice γ is just $\mathcal{A}_\gamma/\mathcal{G}_\gamma$. The spaces $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ can be obtained as well-defined (projective) limits of the spaces $\mathcal{A}_\gamma$, $\mathcal{G}_\gamma$ and $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ [7,5]. Note however that this limit is *not* the usual "continuum limit" of a lattice gauge theory in which one lets the edge length go to zero. Here, we are already in the continuum and have available to us all possible floating lattices from the beginning. We are just expressing the quantum configuration space of the continuum theory as a suitable limit of the configuration spaces of theories associated with all these lattices.

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work to allow for smooth paths is being carried out [20]. The general expectation is that the main results will admit natural generalizations to the smooth category. In this article, A has the physical dimensions of a connection, $(\text{length})^{-1}$ and is thus related to the configuration variable A_{old} in the literature by $A = GA_{\text{old}}$ where G is Newton's constant.

To summarize, the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ is a specific extension of the classical configuration space $\mathcal{A}/\mathcal{G}$. Quantum states can be expressed as complex-valued, square-integrable functions on $\overline{\mathcal{A}/\mathcal{G}}$, or, equivalently, as $\overline{\mathcal{G}}$-invariant square-integrable functions on $\overline{\mathcal{A}}$. As in Minkowskian field theories, while $\mathcal{A}/\mathcal{G}$ is dense in $\overline{\mathcal{A}/\mathcal{G}}$ topologically, measure theoretically it is generally sparse; typically, $\mathcal{A}/\mathcal{G}$ is contained in a subset set of zero measure of $\overline{\mathcal{A}/\mathcal{G}}$ [7]. Consequently, what matters is the value of wave functions on "genuinely" generalized connections. In contrast with the usual Minkowskian situation, however, $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ are all *compact* spaces in their natural (Gel'fand) topologies [4-8]. This fact simplifies a number of technical issues.

Our construction can be compared with the general framework of 'second quantization' proposed by Kijowski [21] already twenty years ago. He introduced the space of states for a field theory by using the projective limit of spaces of states associated to a family of finite dimensional theories. He also, suggested, as an example, the lattice approach. The common element with the present approach is that in our case the space of measures on $\overline{\mathcal{A}}$ is also the projective limit of the spaces of measures defined on finite dimensional spaces A_γ.

B. Hilbert space

Since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, it admits regular (Borel, normalized) measures and for every such measure we can construct a Hilbert space of square-integrable functions. Thus, to construct the Hilbert space of quantum states, we need to select a specific measure on $\overline{\mathcal{A}/\mathcal{G}}$.

It turns out that $\overline{\mathcal{A}}$ admits a measure μ^o that is preferred by both mathematical and physical considerations [5,6]. Mathematically, the measure μ^o is natural because its definition does not involve introduction of any additional structure: it is induced on $\overline{\mathcal{A}}$ by the Haar measure on $SU(2)$. More precisely, since $\mathcal{A}_\gamma$ is isomorphic to $[SU(2)]^N$, the Haar measure on $SU(2)$ induces on it a measure μ^o_γ in the obvious fashion. As we vary γ, we obtain a family of measures which turn out to be compatible in an appropriate sense and therefore induce a measure μ^o on $\overline{\mathcal{A}}$. This measure has the following attractive properties [5]: i) it is faithful; i.e., for any continuous, non-negative function f on $\overline{\mathcal{A}}$, $\int d\mu^o f \geq 0$, equality holding if and only if f is identically zero; and, ii) it is invariant under the (induced) action of Diff$[\Sigma]$, the diffeomorphism group of Σ. Finally, μ^o induces a natural measure $\tilde{\mu}^o$ on $\overline{\mathcal{A}/\mathcal{G}}$: $\tilde{\mu}^o$ is simply the push-forward of μ^o under the projection map that sends $\overline{\mathcal{A}}$ to $\overline{\mathcal{A}/\mathcal{G}}$. Physically, the measure $\tilde{\mu}^o$ is selected by the so-called "reality conditions". More precisely, the classical phase space admits an (over)complete set of naturally defined configuration and momentum variables which are real, and the requirement that the corresponding operators on the quantum Hilbert space be self-adjoint selects for us the measure $\tilde{\mu}^o$ [10].

Thus, it is natural to use $\tilde{\mathcal{H}}^o := L^2(\overline{\mathcal{A}/\mathcal{G}}, d\tilde{\mu}^o)$ as our Hilbert space. Elements of $\tilde{\mathcal{H}}^o$ are the kinematic states; we are yet to impose quantum constraints. Thus, $\tilde{\mathcal{H}}^o$ is the classical analog of the *full* phase-space of quantum gravity (prior to the introduction of

the constraint sub-manifold). Note that these quantum states can be regarded also as *gauge invariant* functions on $\overline{\mathcal{A}}$. In fact, since the spaces under consideration are compact and measures normalized, we can regard $\tilde{\mathcal{H}}^o$ as the gauge invariant *sub-space* of the Hilbert space $\mathcal{H}^o := L^2(\overline{\mathcal{A}}, d\mu^o)$ of square-integrable functions on $\overline{\mathcal{A}}$ [6,7]. *In what follows, we we will often do so.*

What do "typical" quantum states look like? To provide an intuitive picture, we can proceed as follows. Fix a graph γ with N edges and consider functions Ψ_γ of generalized connections of the form $\Psi_\gamma(\bar{A}) = \psi(\bar{A}(e_1), ..., \bar{A}(e_N))$ for *some* smooth function ψ on $[SU(2)]^N$, where $e_1, ..., e_N$ are the edges of the graph γ. Thus, the functions Ψ_γ know about what the generalized connections do only to those paths which constitute the edges of the graph γ; they are precisely the quantum states of the gauge theory associated with the "floating lattice" γ. This space of states, although infinite dimensional, is quite "small" in the sense that it corresponds to the Hilbert space associated with a system with only a *finite* number of degrees of freedom. However, if we vary γ through all possible graphs, the collection of all states that results is very large. Indeed, one can show that it is *dense* in the Hilbert space $\mathcal{H}^o$. (If we restrict ourselves to Ψ_γ which are gauge invariant, we obtain a dense sub-space in $\tilde{\mathcal{H}}^o$.) Since each of these states depends only on a finite number of variables, borrowing the terminology from the quantum theory of free fields in Minkowski space, they are called *cylindrical functions* and denoted by Cyl. Gauge invariant cylindrical functions represent the "typical" kinematic states. In many ways, Cyl is analogous to the space $C_o^\infty(R^3)$ of smooth functions of compact support on R^3 which is dense in the Hilbert space $L^2(R^3, d^3x)$ of quantum mechanics. Just as one often defines quantum operators – e.g., the position, the momentum and the Hamiltonians– on C_o^∞ first and then extends them to an appropriately larger domain in the Hilbert space $L^2(R^3, d^3x)$, we will define our operators first on Cyl and then extend them appropriately.

Cylindrical functions provide considerable intuition about the nature of quantum states we are led to consider. These states represent 1-dimensional polymer-like excitations of geometry/gravity rather than 3-dimensional wavy undulations on flat space. Just as a polymer, although intrinsically 1-dimensional, exhibits 3-dimensional properties in sufficiently complex and densely packed configurations, the fundamental 1-dimensional excitations of geometry can be packed appropriately to provide a geometry which, when coarse-grained on scales much larger than the Planck length, lead us to continuum geometries [12,22]. Thus, in this description, gravitons can arise only as approximate notions in the low energy regime [23]. At the basic level, states in $\tilde{\mathcal{H}}^o$ are fundamentally different from the Fock states of Minkowskian quantum field theories. The main reason is the underlying diffeomorphism invariance: In absence of a background geometry, it is not possible to introduce the familiar Gaussian measures and associated Fock spaces.

C. Statement of the problem

We can now outline the general strategy that will be followed in sections 4 and 5.

Recall that the classical configuration variable is an $SU(2)$ connection[2] A_a^i on a 3-manifold Σ, where i is the $su(2)$-internal index with respect to a basis τ_i. Its conjugate momentum E_j^b has the geometrical interpretation of an orthonormal triad with density weight one [24,17], the precise Poisson brackets being:

$$\{A_a^i(x), E_j^b(y)\} = G\delta_a^b\delta_j^i\delta^3(x,y)\,, \tag{2.1}$$

where G is Newton's constant. (Recall from footnote 1 that the field A, used here, is related to A_{old} used in the literature [25] via $A = GA_{\text{old}}$.)

Therefore, geometrical observables –functionals of the 3-metric– can be expressed in terms of this field E_i^a. Fix within the 3-manifold Σ any analytic, finite 2-surface S without boundary such that the closure of S in Σ is a compact. The area A_S of S is a well-defined, real-valued function on the *full* phase space of general relativity (which happens to depend only on E_i^a). It is easy to verify that these kinematical observables can be expressed as:

$$A_S := \int_S dx^1 \wedge dx^2 \, [E_i^3 E^{3i}]^{\frac{1}{2}}\,, \tag{2.2}$$

where, for simplicity, we have used adapted coordinates such that S is given by $x^3 = 0$, and x^1, x^2 parameterize S, and where the internal index i is raised by a the inner product we use on $su(2)$, $k(\tau_i, \tau_j) = -2\text{Tr}(\tau_i\tau_j)$.

Our task is to find the corresponding operators on the kinematical Hilbert space $\tilde{\mathcal{H}}^o$ and investigate their properties.

There are several factors that make this task difficult. Intuitively, one would expect that $E_i^a(x)$ to be replaced by the "operator-valued distribution" $-i\hbar G\delta/\delta A_a^i(x)$. Unfortunately, the classical expression of A_S involves *square-roots of products* of E's and hence the formal expression of the corresponding operator is badly divergent. One must introduce a suitable regularization scheme. Unfortunately, we do not have at our disposal the usual machinery of Minkowskian field theories and even the precise rules that are to underlie such a regularization are not apriori clear.

There are however certain basic expectations that we can use as guidelines: i) the resulting operators should be well-defined on a dense sub-space of $\tilde{\mathcal{H}}^o$; ii) their final expressions should be diffeomorphism covariant, and hence, in particular, independent of any background fields that may be used in the intermediate steps of the regularization procedure; and, iii) since the classical observables are real-valued, the operators should be self-adjoint. These expectations seem to be formidable at first. Indeed, these demands are rarely met even in Minkowskian field theories; in presence of interactions, it is extremely difficult to establish rigorously that physically interesting operators are well-defined and

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[2]We assume that the underlying 3-manifold Σ is orientable. Hence, principal $SU(2)$ bundles over Σ are all topologically trivial. Therefore, we can represent the $SU(2)$ connections on the bundle by a $su(2)$-valued 1-form on Σ. The matrices τ_i are anti-Hermitian, given, e.g., by $(-i/2)$-times the Pauli matrices.

self-adjoint. As we will see, the reason why one can succeed in the present case is two-folds. First, the requirement of diffeomorphism covariance is a powerful restriction that severely limits the possibilities. Second, the background independent functional calculus is extremely well-suited for the problem and enables one to circumvent the various road blocks in subtle ways.

Our general strategy will be following. We will define the regulated versions of area operators on the dense sub-space Cyl of cylindrical functions and show that they are essentially self-adjoint (i.e., admit unique self-adjoint extensions to $\tilde{\mathcal{H}}^o$). This task is further simplified because the operators leave each sub-space $\mathcal{H}_\gamma$ spanned by cylindrical functions associated with any one graph γ invariant. This in effect reduces the field theory problem (i.e., one with an infinite number of degrees of freedom) to a quantum mechanics problem (in which there are only a finite number of degrees of freedom). Finally, we will find that the operators in fact leave invariant certain *finite* dimensional sub-space of $\mathcal{H}^o$ (associated with extended spin networks, introduced in Sec. 4.2). This powerful simplification further reduces the task of investigating the properties of these operators; in effect, the quantum mechanical problem (in which the Hilbert space is still infinite dimensional) is further simplified to a problem involving spin systems (where the Hilbert space is finite dimensional). It is because of these simplifications that a complete analysis is possible.

III. REGULARIZATION

Our task is to construct a well-defined operator $\hat{A}_S$ starting from the classical expression (2.2). As is usual in quantum field theory, we will begin with the formal expression obtained by replacing E_i^3 in (2.2) by the corresponding operator valued distribution $\hat{E}_i^3$ and then regulate it to obtain the required $\hat{A}_S$. (For an early discussion of non-perturbative regularization, see, in particular, [26]). Our discussion will be divided in to two parts. In the first, we introduce the basic tools and, in the second, we apply them to obtain a well-defined operator $\hat{A}_S$.

To simplify the presentation, let us first assume that S is covered by a single chart of adapted coordinates. Extension to the general case is straightforward: one mimics the procedure used to define the integral of a differential form over a manifold. That is, one takes advantage of the coordinates invariance of the the resulting 'local' operator and uses a partition of unity.

A. Tools

The regularization procedure involves two main ingredients. We will begin by summarizing them.

The first involves smearing of (the operator analog of) $E_i^3(x)$ and point splitting of the integrand in (2.2). Since in this integrand, the point x lies on the 2-surface S, let

us try to use a 2-dimensional smearing function. Let $f_\epsilon(x, y)$ be a 1-parameter family of fields on S which tend to the $\delta(x, y)$ as ϵ tends to zero; i.e., such that

$$\lim_{\epsilon \to 0} \int_S d^2y\, f_\epsilon(x^1, x^2; y^1, y^2) g(y^1, y^2) = g(x^1, x^2)\,, \tag{3.1}$$

for all smooth densities g of weight 1 and of compact support on S. (Thus, $f_\epsilon(x, y)$ is a density of weight 1 in x and a function in y.) The smeared version of $E_i^3(x)$ will be defined to be:

$$[E_i^3]_f(x) := \int_S d^2y\, f_\epsilon(x, y) E_i^3(y)\,, \tag{3.2}$$

so that, as ϵ tends to zero, $[E_i^3]_f$ tends to $E_i^3(x)$. The point-splitting strategy now provides a 'regularized expression' of area:

$$[A_S]_f := \int_S d^2x\, [\, \int_S d^2y\, f_\epsilon(x, y) E_i^3(y) \int_S d^2z\, f_\epsilon(x, z) E^{3i}(z)\,]^{\frac{1}{2}}$$
$$= \int_S d^2x\, [[E_i^3]_f(x)[E^{3i}]_f(x)]^{\frac{1}{2}}\,, \tag{3.3}$$

which will serve as the point of departure in the next subsection. To simplify technicalities, we will assume that the smearing field $f_\epsilon(x, y)$ has the following additional properties for sufficiently small $\epsilon > 0$: i) for any given y, $f_\epsilon(x, y)$ has compact support in x which shrinks uniformly to y; and, ii) $f_\epsilon(x, y)$ is non-negative. These conditions are very mild and we are thus left with a large class of regulators.[3]

We now introduce the second ingredient. To go over to the quantum theory, we want to replace E_i^3 in (3.3) by $\hat{E}_i^3 = -iG\hbar\delta/\delta A_3^i$. However, it is not apriori clear that, even after smearing, $[\hat{E}_i^3]_f$ is a well-defined operator because: i) our wave functions Ψ are functionals of *generalized* connections $\bar{A}$, whence it is not obvious what the functional derivative means; and, ii) we have smeared the operator only along two dimensions. Let us discuss these points one by one.

First, let us fix a graph γ and consider a cylindrical function Ψ_γ on $\overline{\mathcal{A}}$,

$$\Psi_\gamma(\bar{A}) = \psi(\bar{A}(e_1), .., \bar{A}(e_N))\,, \tag{3.4}$$

where, as before, N is the total number of edges of γ and where ψ is a smooth function on $[SU(2)]^N$. Now, a key fact about generalized connections is that, for any given graph γ, each $\bar{A}$ is equivalent to some smooth connection A [5]: Given any $\bar{A}$, there exists an A such that

$$\bar{A}(e_k) = h_k[A] := \mathcal{P}\exp - \int_{e_k} A\,, \tag{3.5}$$

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[3] For example, $f_\epsilon(x, y)$ can be constructed as follows. Take *any* non-negative function f of compact support on S such that $\int d^2x f(x) = 1$ and set $f_\epsilon(x, y) = (1/\epsilon^2) f((x - y)/\epsilon)$. Here, we have implicitly used the given chart to give $f_\epsilon(x, y)$ a density weight in x.

for all $k = 1, ..., N$. (For any given $\bar{A}$, the smooth connection A is of course not unique. However, this ambiguity does not affect the considerations that follow.) Hence, there is a 1-1 correspondence between the cylindrical function Ψ_γ on $\overline{\mathcal{A}}$ and function $\psi(h_1(A), ...,h_E(A))$ on the space $\mathcal{A}$ of smooth connections and we can apply the operator $[\hat{E}_i^3]_f$ to the latter. The result is:

$$[\hat{E}_i^3]_f(x) \cdot \Psi_\gamma(\bar{A}) = -iG\hbar \sum_{I=1}^{N} \int_S d^2y f_\epsilon(x, y) \left(\frac{\delta h_I}{\delta A_a^i(y)} \right)|_{y^3=0} \left(\frac{\partial \psi}{\partial h_I} \right)(A)$$

$$= i\ell_P^2 \int_S d^2y \, f_\epsilon(x, y) \sum_{I=1}^{N} \left[\int_0^1 dt \, \dot{e}_I^{\,3}(t) \, \delta(y^1, e_I^1(t)) \delta(y^2, e_I^2(t)) \delta(0, e_I^3(t)) \right.$$

$$\left. (h_I(1, t)\tau^i h_I(t, 0))_B^A \right] \frac{\partial \psi}{\partial h_{I_B^A}}(A) \,, \tag{3.6}$$

where, $\ell_P = \sqrt{G\hbar}$ is the Planck length, the index I labels the edges in the graph, $[0, 1] \ni t \mapsto e_I(t)$ is any parameterization of an edge e_I, $h_I(t', t) := \mathcal{P} \exp - \int_t^{t'} A_a(e_I(s))\dot{e}_I^a(s)ds$ is the holonomy of the connection A along the edge e_I from parameter value t to t'. Thus, the functional derivative has a well-defined action on cylindrical functions; the first of the two problems mentioned above has been overcome.

However, because of the presence of the delta distributions, it is still not clear that $[\hat{E}_i^3]_f$ is a genuine operator (rather than a distribution-valued operator). To explicitly see that it is, we need to specify some further details. Given a graph γ, we can just subdivide some of its its edges and thus obtain a graph γ' which occupies the same points in Σ as γ but has (trivially) more vertices and edges. Every function which is cylindrical with respect to the "smaller" graph γ is obviously cylindrical with respect to the "larger" graph γ' as well. The idea is to use this freedom to simplify the discussion by imposing some conditions on our graph γ. We will assume that: i) if an edge e_I contains a segment which lies in S, then it lies entirely in the closure of S; ii) each isolated intersection of γ with the 2-surface S is a vertex of γ; and, iii) each edge e_I of γ intersects S at most once. (The overlapping edges are often called edges 'tangential' to S; they should not be confused with edges which 'cross' S but whose tangent vector at the intersection point is tangent to S). If the given graph does not satisfy one or more of these conditions, we can obtain one which does simply by sub-dividing of some of the edges. Thus these conditions are not restrictive. They are introduced to simplify the "book-keeping" in calculations.

Let us now return to (3.6). If an edge e_I has no point in common with S, it does not contribute to the sum. If it is contained in S, $\dot{e}_I^3$ vanishes identically whence its contribution also vanishes. (For a subtlety, see the remark below Eq (3.11).) We are thus left with edges which intersect S in isolated points. Let us first consider only those edges which are 'outgoing' at the intersection. Then, at the intersection point, the value of the parameter t is zero and, for a given edge e_I, $\dot{e}_I^{\,3}$ is positive (negative) if e_I is directed "upwards" along increasing x^3 ("downwards" along decreasing x^3). Hence, (3.6)

becomes[4]:

$$[\hat{E}_i^3]_f(x) \cdot \Psi_\gamma = \frac{i\ell_P^2}{2} \Big[\sum_{I=1}^{N} \big[\int_S d^2y \kappa_I f_\epsilon(x,y) \delta(y^1, e_I^1(0)) \delta(y^2, e_I^2(0)) (h_I \tau^i)_B^A \big] \frac{\partial \psi}{\partial h_{IB}^A}$$

$$= \frac{i\ell_P^2}{2} \sum_{I=1}^{N} \kappa_I f_\epsilon(x, e_I(0)) \; L_I^i \cdot \psi(\bar{A}(e_1), ..., \bar{A}(e_N)), \tag{3.7}$$

where, the constant κ_I associated with the edge e_I is given by:

$$\kappa_I = \begin{cases} 0, & \text{if } e_I \text{ is tangential to } S \text{ or does not intersect } S, \\ +1, & \text{if } e_I \text{ has an isolated intersection with } S \text{ and lies above } S \\ -1, & \text{if } e_I \text{ has an isolated intersection with } S \text{ and lies below } S \end{cases} \tag{3.8}$$

and where L_I^i is the left invariant vector field in the i-th internal direction on the copy of $SU(2)$ corresponding to the I-th edge

$$L_I^i \cdot \psi(\bar{A}(e_1), ..., \bar{A}(e_N)) = (\bar{A}(e_I)\tau^i)_B^A \frac{\partial \psi}{\partial (\bar{A}(e_I))_B^A}. \tag{3.9}$$

If some of the edges are 'incoming' at the intersection point, then the final expression of $[\hat{E}_i^a]_f(x)$ can be written as:

$$[\hat{E}_i^3]_f(x) \cdot \Psi_\gamma = \frac{i\ell_P^2}{2} \Big[\sum_{I=1}^{N} \kappa_I f_\epsilon(x, v_{\alpha_I}) \; X_I^i \Big] \cdot \psi(\bar{A}(e_1), ..., \bar{A}(e_N)), \tag{3.10}$$

where X_I^i is an operator assigned to a vertex v and an edge e_I intersecting v by the following formula

$$X_I^i \cdot \psi(\bar{A}(e_1), ..., \bar{A}(e_N)) = \begin{cases} (\bar{A}(e_I)\tau^i)_B^A \frac{\partial \psi}{\partial (\bar{A}(e_I))_B^A}, & \text{when } e_I \text{ is outgoing} \\ -(\tau^i \bar{A}(e_I))_B^A \frac{\partial \psi}{\partial (\bar{A}(e_I))_B^A}, & \text{when } e_I \text{ is incoming} \end{cases} \tag{3.11}$$

Remark: Let us briefly return to the edges which are tangential to S. In this case, although $\dot{e}_I^3$ vanishes, we also have a singular term $\delta(0,0)$ (in the x^3 direction) in (3.6). Hence, to recover an unambiguous answer, *for these edges*, we need to smear also in the third direction using an additional regulator, say $g_{\epsilon'}(x^3, y^3)$. When this is done, one finds that the contribution of the tangential edges vanishes even before removing the regulator; as stated earlier, the tangential edges do not contribute. We did not introduce the smearing in the third direction right in the beginning to emphasize the point that this step is unnecessary for the edges whose contributions survive in the end.

The right side again defines a cylindrical function based on the (same) graph γ. Denote by $\mathcal{H}_\gamma^o$ the Hilbert space $L^2(\mathcal{A}_\gamma, d\mu_\gamma^o)$ of square integrable cylindrical functions associated

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[4] In the first step, we have used the regularization $\int_0^\infty dz g(z) \delta(z) = \frac{1}{2}g(0)$ which follows if the $\delta(z)$ is obtained, in the standard fashion, as a limit of functions which are symmetric about 0.

with a fixed graph γ. Since μ_γ^o is the induced Haar measure on $\mathcal{A}_\gamma$ and since the operator is just a sum of right/left invariant vector fields, standard results in analysis imply that, with domain Cyl_γ^1 of all C^1 cylindrical functions based on γ, it is an essentially self-adjoint on $\mathcal{H}_\gamma^o$. Now, it is straightforward to verify that the operators on $\mathcal{H}_\gamma^o$ obtained by varying γ are all compatible[5] in the appropriate sense. Hence, it follows from the general results in [8] that $[\hat{E}_i^3]_f(x)$, with domain Cyl^1 (the space of all C^1 cylindrical functions), is an essentially self-adjoint operator on $\mathcal{H}^o$. For notational simplicity, we will denote its self-adjoint extension also by $[\hat{E}_i^3]_f(x)$. (The context should make it clear whether we are referring to the essentially self-adjoint operator or its extension.)

The fact that this operator is well-defined may seem surprising at first sight since we have used only a 2-dimensional smearing. Recall however that in free field theory in Minkowski space, the action of the momentum operator on cylindrical functions is well-defined in the same sense without any smearing at all. In our case, a 2-dimensional smearing is needed because our states contain one –rather than three- dimensional excitations.

B. Area operators

Let us now turn to the integrand of the smeared area operator (corresponding to (3.3)). Denoting the determinant of the intrinsic metric on S by g_S, we have:

$$
\begin{aligned}
[\hat{g}_S]_f(x) \cdot \Psi_\gamma &:= [E_i^3]_f(x)[E^{3i}]_f(x) \cdot \Psi_\gamma \\
&= -\frac{\ell_P^4}{4}[\sum_{I,J} \kappa(I,J)f_\epsilon(x,v_{\alpha_I})f_\epsilon(x,v_{\alpha_J})\, X_I^i X_J^i] \cdot \Psi_\gamma\,,
\end{aligned}
\tag{3.12}
$$

where the summation goes over all the oriented pairs (I,J); v_{α_I} and v_{α_J} are the vertices at which edges e_I and e_J intersect S; $\kappa(I,J) = \kappa_I\kappa_J$ equals 0 if either of the two edges e_I and e_J fails to intersect S or lies entirely in S, +1 if they lie on the same side of S, and, -1 if they lie on the opposite sides. (For notational simplicity, from now on we shall not keep track of the position of the internal indices i; as noted in Sec. 2.3, they are contracted using the invariant metric on the Lie algebra $su(2)$.) The next step is to consider vertices v_α at which γ intersects S and simply rewrite the above sum by re-grouping terms by vertices. The result simplifies if we choose ϵ sufficiently small so that, $f_\epsilon(x,v_{\alpha_I})f_\epsilon(x,v_{\alpha_J})$ is zero unless $v_{\alpha_I} = v_{\alpha_J}$. We then have:

---

[5]Given two graphs, γ and γ', we say that $\gamma \geq \gamma'$ if and only if every edge of γ' can be written as a composition of edges of γ. Given two such graphs, there is a projection map from $\mathcal{A}_\gamma$ to $\mathcal{A}_{\gamma'}$, which, via pull-back, provides an unitary embedding $U_{\gamma,\gamma'}$ of $\tilde{\mathcal{H}}_{\gamma'}^o$ into $\tilde{\mathcal{H}}_\gamma^o$. A family of operators $\mathcal{O}_\gamma$ on the Hilbert spaces $\mathcal{H}_\gamma^o$ is said to be compatible if $U_{\gamma,\gamma'}\mathcal{O}_{\gamma'} = \mathcal{O}_\gamma U_{\gamma,\gamma'}$ and $U_{\gamma,\gamma'}D_{\gamma'} \subset D_\gamma$ for all $\gamma \geq \gamma'$, where D_γ and $D_{\gamma'}$ are the domains of $\mathcal{O}_\gamma$ and $\mathcal{O}_{g'}$.

$$[\hat{g}_S]_f(x) \cdot \Psi_\gamma = -\frac{\ell_P^4}{4}[\sum_\alpha (f_\epsilon(x, v_\alpha))^2 \sum_{I_\alpha, J_\alpha} \kappa(I_\alpha, J_\alpha) X^i_{I_\alpha} X^i_{J_\alpha}] \cdot \Psi_\gamma, \qquad (3.13)$$

where the index α labels the vertices on S and I_α and J_α label the edges at the vertex α.

The next step is to take the square-root of this expression. The same reasoning that established the self-adjointness of $[\hat{E}^3_i]_f(x)$ now implies that $[\hat{g}_S]_f(x)$ is a non-negative self-adjoint operator and hence has a well-defined square-root which is also a positive definite self-adjoint operator. Since we have chosen ϵ to be sufficiently small, for any given point x in S, $f_\epsilon(x, v_\alpha)$ is non-zero for at most one vertex v_α. We can therefore take the sum over α outside the square-root. One then obtains

$$([\hat{g}_S]_f)^{\frac{1}{2}}(x) \cdot \Psi_\gamma = \frac{\ell_P^2}{2}\sum_\alpha f_\epsilon(x, v_\alpha)[\sum_{I_\alpha, J_\alpha} \kappa(I_\alpha, J_\alpha) X^i_{I_\alpha} X^i_{J_\alpha}]^{\frac{1}{2}} \cdot \Psi_\gamma. \qquad (3.14)$$

Note that the operator is neatly split; the x-dependence all resides in f_ϵ and the operator within the square-root is "internal" in the sense that it acts only on copies of $SU(2)$.

Finally, we can remove the regulator, i.e., take the limit as ϵ tends to zero. By integrating both sides against test functions on S and then taking the limit, we conclude that the following equality holds in the distributional sense:

$$\widehat{\sqrt{g_S}}(x) \cdot \Psi_\gamma = \frac{\ell_P^2}{2}\sum_\alpha \delta^{(2)}(x, v_\alpha)[\sum_{I_\alpha, J_\alpha} \kappa(I_\alpha, J_\alpha) X^i_{I_\alpha} X^i_{J_\alpha}]^{\frac{1}{2}} \cdot \Psi_\gamma. \qquad (3.15)$$

Hence, the regularized area operator is given by:

$$\hat{A}_S \cdot \Psi_\gamma = \frac{\ell_P^2}{2}\sum_\alpha [\sum_{I_\alpha, J_\alpha} \kappa(I_\alpha, J_\alpha) X^i_{I_\alpha} X^i_{J_\alpha}]^{\frac{1}{2}} \cdot \Psi_\gamma. \qquad (3.16)$$

(Here, as before, α labels the vertices at which γ intersects S and I_α labels the edges of γ at the vertex v_α.) With Cyl^2 as its domain, $\hat{A}_S$ is essentially self-adjoint on the Hilbert space $\mathcal{H}^o$.

Let us now remove the assumption that the surface Σ is covered by a single chart of adapted coordinates. If such a global chart does not exist, we can cover Σ with a family $\mathcal{U}$ of neighborhoods such that for each $U \in \mathcal{U}$ there exists a local coordinates system (x^a) adapted to Σ. Let $(\varphi_U)_{U \in \mathcal{U}}$ be a partition of unity associated to $\mathcal{U}$. We just repeat the above regularization for a slightly modified classical surface area functional, namely for

$$A_{S,U} := \int_S dx^1 \wedge dx^2 \, \varphi_U [E^3_i E^{3i}]^{\frac{1}{2}} \qquad (3.17)$$

which has support within a domain U of an adapted chart. Thus, we obtain the operator $\hat{A}_{SU}$. Then we just define

$$\hat{A}_S = \sum_{U \in \mathcal{U}} \hat{A}_{S,U}. \qquad (3.18)$$

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The result is given again by the formula (3.16). The reason why the functions φ_U disappear from the result is that the operator obtained for a single domain of an adapted chart is insensitive on changes of this chart. This concludes our technical discussion.

The classical expression A_S of (2.2) is a rather complicated. It is therefore somewhat surprising that the corresponding quantum operators can be constructed rigorously and have quite manageable expressions. The essential reason is the underlying diffeomorphism invariance which severely restricts the possible operators. Given a surface and a graph, the only diffeomorphism invariant entities are the intersection vertices. Thus, a diffeomorphism covariant operator can only involve structure at these vertices. In our case, it just acts on the copies of $SU(2)$ associated with various edges at these vertices.

We have presented this derivation in considerable detail to spell out all the assumptions, to bring out the generality of the procedure and to illustrate how regularization can be carried out in a fully non-perturbative treatment. While one is free to introduce auxiliary structures such as preferred charts or background fields in the intermediate steps, the final result must respect the underlying diffeomorphism invariance of the theory. These basic ideas will be used repeatedly for other geometric operators in the sequel to this paper.

C. General properties of operators

1. *Discreteness of the spectrum:* By inspection, it follows that the total area operator $\hat{A}_S$ leaves the sub-space of Cyl_γ^2 which is associated with any one graph γ invariant and is a self-adjoint operator on the sub-space $\mathcal{H}_\gamma^o$ of $\mathcal{H}^o$ corresponding to γ. Next, recall that $\mathcal{H}_\gamma^o = L^2(\mathcal{A}_\gamma, d\mu^o)$, where $\mathcal{A}_\gamma$ is a compact manifold, isomorphic with $(SU(2))^N$ where N is the total number of edges in γ. As, we explained below, the restriction of $\hat{A}_S$ to $\mathcal{H}_\gamma^o$ is given by certain commuting elliptic differential operators on this compact manifold. Therefore, all its eigenvalues are discrete. Now suppose that the complete spectrum of $\hat{A}_S$ on $\mathcal{H}^o$ has a continuous part. Denote by P_c the associated projector. Then, given any Ψ in $\mathcal{H}^o$, $P_c \cdot \Psi$ is orthogonal to $\mathcal{H}_\gamma^o$ for any graph γ, and hence to the space Cyl of cylindrical functions. Now, since Cyl^2 is dense in $\mathcal{H}^o$, $P_c \cdot \Psi$ must vanish for all Ψ in $\mathcal{H}^o$. Hence, the spectrum of $\hat{A}_S$ has no continuous part.

Note that this method is rather general: It can be used to show that *any* self-adjoint operator on $\mathcal{H}^o$ which maps (the intersection of its domain with) $\mathcal{H}_\gamma^o$ to $\mathcal{H}_\gamma^o$, and whose action on $\mathcal{H}_\gamma^o$ is given by elliptic differential operators, has a purely discrete spectrum on $\mathcal{H}^o$. Geometrical operators, constructed purely from the triad field tend to satisfy these properties.

2. *Area element:* Note that not only is the total area operator well-defined, but in fact it arises from a local area element, $\widehat{\sqrt{g_S}}$, which is an operator-valued distribution in the usual sense. Thus, if we integrate it against test functions, the operator is densely defined on $\mathcal{H}^o$ (with C^2 cylindrical functions as domain) and the matrix elements

$$\langle \Psi'_{\gamma'}, \, \widehat{\sqrt{g_S}}(x) \cdot \Psi_\gamma \rangle \tag{3.19}$$

are 2-dimensional distributions on S. Furthermore, since we did not have to renormalize the regularized operator (3.14) before removing the regulator, there are *no* free renormalization constants involved. The local operator is completely unambiguous.

3. $[\hat{g}_S]_f$ *versus its square-root:* Although the regulated operator $[\hat{g}_s]_f$ is well-defined, if we let ϵ to go zero, the resulting operator is in fact divergent: roughly, it would lead to the square of the 2-dimensional δ distribution. Thus, the determinant of the 2-metric is not a well-defined in the quantum theory. As we saw, however, the square-root of the determinant *is well* defined: We have to first take the square-root of the *regulated* expression and *then* remove the regulator. This, in effect, is the essence of the regularization procedure.

To get around this divergence of $\hat{g}_S$, as is common in Minkowskian field theories, we could have first rescaled $[\hat{g}_S]_f]$ by an appropriate factor and then taken the limit. Then result can be a well-defined operator, but it will depend on the choice of the regulator, i.e., additional structure introduced in the procedure. Indeed, if the resulting operator is to have the same density character as its classical analog $g_S(x)$ –which is a scalar density of weight two– then the operator can not respect the underlying diffeomorphism invariance.[6] For, there is no metric/chart independent distribution on S of density weight two. Hence, such a 'renormalized' operator is not useful to a fully non-perturbative approach. For the square-root, on the other hand, we need a local density of weight *one*. And, the 2-dimensional Dirac distribution provides this; now is no apriori obstruction for a satisfactory operator corresponding to the area element to exist. This is an illustration of what appears to be typical in non-perturbative approaches to quantum gravity: Either the limit of the operator exists as the regulator is removed without the need of renormalization or it inherits back-ground dependent renormalization fields (rather than constants).

4. *Vertex operators:* As noted already, in the final expressions of the area element and area operators, there is a clean separation between the 'x-dependent' and the 'internal' parts. Given a graph γ, the internal part is a sum of square-roots of the operators

$$\triangle_{S,v_\alpha} := \sum_{I_\alpha, J_\alpha} \kappa(I_\alpha, J_\alpha) X^i_{I_\alpha} X^i_{J_\alpha} \tag{3.20}$$

associated with the surface S and the vertex v_α on it. It is straightforward to check that operators corresponding to different vertices commute. Therefore, to analyze the properties of area operators, we can focus just on one vertex operator at a time.

Furthermore, given the surface S and a point v on it, we can define an operator $\triangle_{S,v}$ on the dense sub-space Cyl^2 on $\mathcal{H}^o$ as follows:

---

[6]If, on the other hand, for some reason, we are willing to allow the limiting operator to have a *different* density character than its classical analog, one can renormalize $[\hat{g}]_f(x)$ in such a way as to obtain a background independent limit. For instance, we may use $f_\epsilon = (1/\epsilon^2)\theta(|x - x'| - \frac{\epsilon}{2})$, and rescale $[\hat{g}]_f$ by ϵ^2 before taking the limit. Then the limit is a well defined, diffeomorphism covariant operator but it is a scalar density of weight *one* rather than two.

$$\triangle_{S,v} \cdot \Psi_\gamma := \begin{cases} \sum_{I,J} \kappa(I,J) X_I^i X_J^i \cdot \Psi_\gamma, & \text{if } \gamma \text{ intersects } S \text{ in } v, \\ 0, & \text{Otherwise} \end{cases} \qquad (3.21)$$

where I and J label the edges of γ which have v as a vertex. (Recall that every cylindrical function is associated with *some* graph γ. As before, if γ intersects S at v but v is not a vertex of γ, one can extend γ just by adding a new vertex v and orienting the edges at v to outgoing.) It is straightforward to verify that this definition is unambiguous: if a cylindrical function can be represented in two different ways, say as Ψ_γ and $\Psi_{\gamma'}$, then $\triangle_{S,v} \cdot \Psi_\gamma$ and $\triangle_{S,v} \cdot \Psi_{\gamma'}$ are two representations of the same function on $\overline{\mathcal{A}}$. There is a precise sense [8] in which $\triangle_{S,v}$ can be regarded as a Laplacian operator on $\mathcal{H}^o$. The area operator is a sum over all the points v of S of square-roots of Laplacians,

$$\hat{A}_S = \frac{\ell_P^2}{2} \sum_{v \in S} \sqrt{-\triangle_{S,v}} . \qquad (3.22)$$

(Here the sum is well defined because, for any cylindrical function, it contains only a finite number of non-zero terms, corresponding to the isolated intersection points of the associated graph with S). We will see in the next subsection that this fact is reflected in its spectrum.

5. *Gauge invariance:* The classical area element $\sqrt{g_S}$ is invariant under the internal rotations of triads E_i^a; its Poisson bracket with the Gauss constraint functional vanishes. This symmetry is preserved in the quantum theory: the quantum operator $\widehat{\sqrt{g_S}}$ commutes with the induced action of $\overline{\mathcal{G}}$ on the Hilbert space $\mathcal{H}^o$. Thus, $\widehat{\sqrt{g_S}}$ and the total area operator $\hat{A}_S$ map the space of gauge invariant states to itself; they project down to the Hilbert space $\tilde{\mathcal{H}}^o$ of kinematic states.

Note, however, that the regulated triad operators $[\hat{E}_i^3]_f$ are *not* gauge invariant; they are defined only on $\mathcal{H}^o$. Nonetheless, they are useful; they feature in an important way in our regularization scheme. In the loop representation, by contrast, one can only introduce gauge invariant operators and hence the regulated triad operators do not exist. Furthermore, even in the definition (3.3) of the regularized area element, one must use holonomies to transport indices between the two points y and z. While this manifest gauge invariance is conceptually pleasing, in practice it often makes the calculations in the loop representation cumbersome; one has to keep track of these holonomy insertions in the intermediate steps although they do not contribute to the final result.

6. *Overall Factors*: The overall numerical factors in the expressions of various operators considered above depend on two conventions. The first is the convention noted in footnote 4 used in the regularization procedure. Could we not have used a different convention, setting $\int_0^\infty dz g(z)\delta(z) = cg(0)$ and $\int_{-\infty}^0 dz g(z)\delta(z) = (1-c)g(0)$ for some constant $c \neq 1/2$? The answer is in the negative. For, in this case, the constant κ_I would take values:

$$\kappa_I = \begin{cases} 0, & \text{if } e_I \text{ is tangential to } S \text{ or does not intersect } S, \\ +2c, & \text{if } e_I \text{ has an isolated intersection with } S \text{ and lies above } S \\ -2(1-c), & \text{if } e_I \text{ has an isolated intersection with } S \text{ and lies below } S \end{cases} \qquad (3.23)$$

It then follows that, unless $c = 1/2$, the action of the area operator $\hat{A}_S$ on a given cylindrical function would change if we simply reverse the orientation on S (keeping the orientation on Σ the same). Since this is physically inadmissible, we must have $c = 1/2$; there is really no freedom in this part of the regularization procedure.

The second convention has to do with the overall numerical factor in the action, which dictates the numerical coefficients in the symplectic structure. Here, we have adopted the convention of [25] (see chapter 9) which makes the Poisson bracket $\{A_a^i(x), E_j^b(y)\} = G\delta_a^b\delta_j^i\delta(x,y)$, enabling us to express $\hat{E}_i^a(x)$ as $-iG\hbar\delta/\delta A_a^i(x)$. (Had we rescaled the action by $1/8\pi$ as is sometimes done, in our expressions, Newton's constant G would be replaced by $8\pi G$.)

IV. EIGENVALUES AND EIGENVECTORS

This section is divided into three parts. In the first, we derive the complete spectrum of the area operators, in the second, we extend the notion of spin networks, and in the third, we use this extension to discuss eigenvectors.

A. The complete spectrum

We are now ready to calculate the complete spectrum of $\hat{A}_S$. Since $\hat{A}_S$ is a sum of square-roots of vertex operators which all commute with one another, the task reduces to that of finding the spectrum of each vertex operator. Furthermore, since vertex operators map (C^2) cylindrical functions associated with any one graph to (C^0) cylindrical functions associated with the *same* graph, we can begin with an arbitrary but fixed graph γ. Consider then a vertex operator $\triangle_{S,v}$ and focus on the edges of γ which intersect S at v. Let us divide the edges in to three categories: let $e_1, ..., e_d$ lie 'below' S ('down'), $e_{d+1}, ..., e_u$ lie 'above' S ('up') and let $e_{u+1}, ..., e_t$ be tangential to S. (As before, the labels 'down' and 'up' do not have an invariant significance; the orientation of S and of Σ enable us to divide the non-tangential edges in to two parts and we just label one as 'down' and the other as 'up'.) Let us set:

$$
\begin{aligned}
J_{S,v}^{(d)\,i}{}_\gamma &:= -i\,(X_1^i + ... + X_d^i), \quad & J_{S,v}^{(u)\,i}{}_\gamma &:= -i\,(X_{d+1}^i + ... + X_u^i), \\
J_{S,v}^{(t)\,i}{}_\gamma &:= -i\,(X_{u+1}^i + ... + X_t^i), \quad & J_{S,v}^{(d+u)\,i}{}_\gamma &:= J_{S,v}^{(d)\,i} + J_{S,v}^{(u)\,i}
\end{aligned}
\tag{4.1}
$$

where X_I^i is the operator defined in (3.11) assigned to the point v and an edge e_I at v. This notation is suggestive. We can associate with each edge e a particle with only a spin degree of freedom. Then, the operators $-iX_e^i$ can be thought of as the ith component of angular momentum operators associated with that particle and $J_{S,v}^{(d)\,i}, J_{S,v}^{(u)\,i}$ and $J_{S,v}^{(t)\,i}$ as the total 'down', 'up' and 'tangential' angular momentum operators at the vertex v.

By varying the graph, we thus obtain a family of operators. It is easy to check that they satisfy the compatibility conditions and thus define operators $J_{S,v}^{(u)\,i}, J_{S,v}^{(d)\,i}.J_{S,v}^{(t)\,i}$ and

$J_{S,v}^{(d+u)^i}$ on Cyl. It is also easy to verify that they all commute with one another. Hence, one can express the vertex operator $\triangle_{S,v}$ simply as:

$$- \triangle_{S,v} = (J_{S,v}^{(d)^i} - J_{S,v}^{(u)^i})(J_{S,v}^{(d)^i} - J_{S,v}^{(u)^i}) \, ; \tag{4.2}$$

because of the factor $\kappa(I, J)$ in (3.21), the edges which are tangential do not feature in this expression.

The evaluation of possible eigenvalues is now straightforward. It is simplest to express $\triangle_{S,v}$ as

$$- \triangle_{S,v} = 2(J_{S,v}^{(d)})^2 + 2(J_{S,v}^{(u)})^2 - (J_{S,v}^{(d+u)})^2 \, . \tag{4.3}$$

and, as in elementary text-books, go to the representation in which the operators $(J_{S,v}^{(d)})^2, (J_{S,v}^{(u)})^2$ and $(J_{S,v}^{(d+u)})^2$ are diagonal. If we restrict now the operators to Cyl_γ associated to a fixed graph, it is obvious that the possible eigenvalues λ of $\triangle_{S,v}$ are given by:

$$\lambda_{S,v} = 2j^{(d)}(j^{(d)} + 1) + 2j^{(u)}(j^{(u)} + 1) - j^{(d+u)}(j^{(d+u)} + 1) \tag{4.4}$$

where $j^{(d)}, j^{(u)}$ and $j^{(d+u)}$ are half integers subject to the usual condition:

$$j^{(d+u)} \in \{|j^{(d)} - j^{(u)}|, |j^{(d)} - j^{(u)}| + 1, \dots, j^{(d)} + j^{(u)}\}. \tag{4.5}$$

Returning to the total area operator, we note that the vertex operators associated with distinct vertices commute. Although the sum (3.22) is not finite, restricted to any graph γ and Cyl_γ it becomes finite. Therefore, the eigenvalues a_S of $\hat{A}_S$ are given by

$$a_S = \frac{\ell_P^2}{2} \sum_\alpha \left[2j_\alpha^{(d)}(j_\alpha^{(d)} + 1) + 2j_\alpha^{(u)}(j_\alpha^{(u)} + 1) - j_\alpha^{(d+u)}(j_\alpha^{(d+u)} + 1) \right]^{\frac{1}{2}} \tag{4.6}$$

where α labels a finite set of points in S and the non-negative half-integers assigned to each α are subject to the inequality (4.5). The question now is if all these eigenvalues are actually attained, i.e., if, given *any* a_S of the form (4.6), there are eigenvectors in $\mathcal{H}^o$ with that eigenvalue. In Sec. 4.3, will show that the full spectrum is indeed realized on $\mathcal{H}^o$.

The area operators map the subspace $\tilde{\mathcal{H}}^o$ of *gauge invariant* elements of $\mathcal{H}^o$ to itself. Hence we can ask for their spectrum on $\tilde{\mathcal{H}}^o$. We will see in Sec. 4.3 that further restrictions can now arise depending on the topology of the surface S. There are three cases:

- i) The case when S is an open surface whose closure is contained in Σ. An example is provided by the disk $z = 0$, $x^2 + y^2 < r_o$ in R^3. In this case, there is no additional condition; all a_S of (4.6) subject to (4.5) are realized.

- ii) The case when the surface S is closed ($\partial S = \emptyset$) and divides Σ into disjoint open sets Σ_1 and Σ_2 (i.e., $\Sigma = \Sigma_1 \cup S \cup \Sigma_2$ with $\Sigma_1 \cap \Sigma_2 = \emptyset$.) An example is given by:

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$\Sigma = R^3$ and $S = S^2$. In this case, there is an a condition on the half integers $j_\alpha^{(d)}$ and $j_\alpha^{(u)}$ that appear in (4.6) in addition to (4.5):

$$\sum_\alpha j_\alpha^{(d)} = N, \quad \text{and} \quad \sum_\alpha j_\alpha^{(u)} = N' \tag{4.7}$$

for some integers N and N'.

- iii) The case when S is closed but not of type ii). An example is given by; $\Sigma = S^1 \times S^1 \times S^1$ and $S = S^1 \times S^1$. In this case, the additional condition is milder:

$$\sum_\alpha (j_\alpha^{(d)} + j_\alpha^{(u)}) = N \tag{4.8}$$

for some interger N.

Next, let us note some properties of this spectrum of $\hat{A}_S$. By inspection, it is clear that the smallest eigenvalue is 0 and that the spectrum is unbounded from above. One can ask for the 'area gap' i.e., the value of the smallest non-zero eigenvalue. On the full Hilbert space $\mathcal{H}^o$, it is given by:

$$a_S^o = \frac{\sqrt{3}}{4} \ell_P^2. \tag{4.9}$$

This is a special case of the situation when there is only one term in the sum in (4.6) with $j^{(d)} = 0$, $j^{(u)} = j^{(d+u)} = j$. Then

$$a_S = \frac{\ell_P^2}{2} \sqrt{j(j+1)}, \tag{4.10}$$

and, if we choose $j = \frac{1}{2}$, we obtain the eigenvalue a_S^o. On the Hilbert space $\tilde{\mathcal{H}}^o$ of gauge invariant states, on the other hand, because of the constraints on the spectrum discussed above, the area gap is sensitive to the topology of S:

$$a_S^o = \frac{\sqrt{3}}{4} \ell_P^2 \text{ if } S \text{ is of type i)}$$
$$a_S^o = \frac{2\sqrt{2}}{4} \ell_P^2 \text{ if } S \text{ is of type ii)}$$
$$a_S^o = \frac{2}{4} \ell_P^2 \text{ if } S \text{ is of type iii).} \tag{4.11}$$

Another important feature of the spectrum is its behavior for large a_S. As noted above, the spectrum is discrete. However, an interesting question is if it approaches continuum and, if so, in what manner. We will now show that as $a_S \to \infty$, the difference Δa_S between a_S and its closest eigenvalue satisfies the inequality

$$\Delta a_S \leq (\ell_P^2/2)(\ell_P/\sqrt{a_S}) + O((\ell_P^2/a_S))\ell_P^2 \tag{4.12}$$

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and hence tends to zero (irrespective of the topology of S). Specifically, given (odd) integers M and N satisfying $1 \le M \le 2\sqrt{N}$, we will obtain an eigenvalue $a_{S,N,M}$ of $\hat{A}_S$ such that for sufficiently large N, the bound (4.12) is explicitly realized. [7] Let us label representations of $SU(2)$ by their dimension, $n_\alpha = 2j_\alpha + 1$. Let n_α, $\alpha = 1, ..., M$ be (odd) integers such that $\sum_{\alpha=1}^M n_\alpha = N$, and $|n_\alpha - \frac{N}{M}| < 2$ Then, for each M, we have from (4.10) an eigenvalue $a_{S,N,M}$

$$
\begin{aligned}
a_{S,N,M} &= \frac{\ell_P^2}{2} \sum_{\alpha=1}^M \sqrt{j_\alpha(j_a+1)} \\
&= \frac{\ell_P^2}{4} \sum_{\alpha=1}^M (n_\alpha - \frac{1}{2n_\alpha}) + O(\frac{1}{N})) \\
&= \frac{\ell_P^2}{4}(N - \frac{M^2}{2N} + \frac{kM^2}{N^2} + O(\frac{1}{N}))
\end{aligned}
\tag{4.13}
$$

for some integer $k \in [1, M/2]$. As M varies between 1 and $2\sqrt{N}$, $a_{S,N,M}$ varies between $(\ell_P^2/4)N$ and $(\ell_P^2/4)(N-2)+4k/N \le (\ell_P^2/4)((N-2)+4/\sqrt{N})$. Hence, given a sufficiently large a_S, there exist integers N, M satisfying the conditions given above such that $\Delta a_S := |a_S - a_{n,m}|$ satisfies the inequality (4.12).

We will conclude this discussion of the spectrum by providing an alternative form of the expression (4.4) which holds for gauge invariant states. This form will be useful in comparing our result with those obtained in the loop representation (where, from the beginning, one restricts oneself to gauge invariant states.) Let Ψ_γ be a gauge invariant cylindrical function on $\overline{\mathcal{A}}$. Then, the Gauss constraint implies that, at every vertex v of γ, the following condition must hold:

$$
\sum_I X_I^i \cdot \Psi_\gamma = 0,
\tag{4.14}
$$

where I labels the edges of γ at the vertex v and X_I^i is assigned to the point v and vertex e_I (see (3.11). Therefore,

$$
J_{S,v}^{(d)\,i} + J_{S,v}^{(u)\,i} + J_{S,v}^{(t)\,i} = 0.
\tag{4.15}
$$

Hence, one can now express the operator (4.3) in an alternate form,

$$
- \triangle_{S,v} = 2(J_{S,v}^{(d)})^2 + 2(J_{S,v}^{(u)})^2 - (J_{S,v}^{(t)})^2 .
\tag{4.16}
$$

---

[7]This calculation was motivated by the results of Bekenstein and Mukhanov [15] and our estimate has an interesting implication on whether the Hawking spectrum is significantly altered due to quantum gravity effects. Because the "level spacing" Δa_S goes to zero as a_S goes to infinity, the considerations of [15] do not apply to *large* black holes in our approach and there is no reason to expect deviations from Hawking's semi-classical results. On the other hand, for small black-holes, –i.e., the final stages of evaporation– the estimate does not apply and one expects transitions between area eigenstates to show significant deviations.

Furthermore, if it happens that γ has no edges which are tangential to S at v, (4.14) implies:

$$-\triangle_{S,v} = 4(J_{S,v}^{(d)})^2 = 4(J_{S,v}^{(u)})^2, \tag{4.17}$$

whence the corresponding *restricted* eigenvalues of $\hat{A}_S$ are given by $\sum \ell_P^2 \sqrt{j(j+1)}$, where j are half-integers.

B. Extended spin networks

As a prelude to the discussion on eigenvectors, in this sub-section we will generalize the constructions and results obtained in [9,10,27] on spin networks and spin network states. The previous work showed that the spin network states provide us with a natural orthogonal decomposition of the Hilbert space $\tilde{\mathcal{H}}^o$ of gauge invariant states in to *finite* dimensional sub-spaces. Here, we will extend those results to the space $\mathcal{H}^o$.

We begin by fixing some terminology. Given N irreducible representations $\pi_1, ..., \pi_N$ of $SU(2)$, an associated *invariant tensor* $c^{m_{k+1}...m_N}{}_{m_1...m_k}$ is a multi-linear map from $\otimes_{I=1}^k \pi_I$ to $\otimes_{I=k+1}^N \Pi_I$ such that:

$$\pi_{k+1}(g)^{n_{k+1}}_{m_{k+1}} ... \pi_N(g)^{n_N}_{m_N} c^{m_{k+1}...m_N}{}_{m_1...m_k} \pi_1(g^{-1})^{m_1}_{n_1} ... \pi_N(g^{-1})^{n_k}_{m_k} = c^{n_{k+1}...n_N}{}_{n_1...n_k}, \tag{4.18}$$

for arbitrary $g \in SU(2)$, where $\pi_I(g)$ is the matrix representing g in the representation π_I. An invariant tensor $c_{m_1...m_k}{}^{m_{k+1}...m_N}$ is also called an *intertwining tensor* from the representations $\pi_1, ..., \pi_k$ into $\pi_{k+1}, ..., \pi_N$. All the invariant tensors are given by the standard Clebsch-Gordon theory.

An *extended spin network* is an quintuplet $(\gamma, \vec{\pi}, \vec{c}, \vec{\rho}, \vec{M})$ consisting of:

i) A graph γ;

ii) A labeling $\vec{\pi} := (\pi_1, .., \pi_N)$ of the edges $e_1, ..., e_N$ of that graph γ with irreducible and non-trivial representations of $SU(2)$;

iii) A labeling $\vec{\rho} := (\rho_1, ..., \rho_V)$ of the vertices $v_1, ..., v_V$ of γ with irreducible representations of $SU(2)$, the constraint being that for every vertex v_α the representation ρ_α emerges in the decomposition of the tensor product of representations assigned by $\vec{\pi}$ to the edges intersecting v_α;

iv) A labeling $\vec{c} = (c_1, ..., c_V)$ of the vertices $v_1, ..., v_V$ of γ with certain invariant tensors, namely, assigned to a vertex v_α is an intertwining tensor c_α from the representations assigned to the edges coming to v_α and ρ_α to the representations assigned to the outgoing edges at v_α; and,

v) A labeling $\vec{M} := (M_\alpha)_{\alpha=1,...,V} = (M_1, ..., M_V)$ of the vertices $v_1, ..., v_V$ of γ which assigns to every vertex v_α a vector M_α in the representation ρ_α;

It should be emphasized that every π_I is necessarily non-trivial whereas ρ_α may be trivial (i.e., 1-dimensional). In the gauge invariant context [9,10], ρ_α are all trivial whence the items iii) and v) are unnecessary. The details of these conditions may seem somewhat complicated but they are necessary to achieve the orthogonal decomposition (4.22).

From spin networks, we can construct states in $\mathcal{H}^o$. An *extended spin network state* $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ is simply a C^∞ cylindrical function on $\overline{\mathcal{A}}$ constructed from an extended spin network $(\gamma, \vec{\pi}, \vec{\rho}, \vec{c}, \vec{M})$,

$$\mathcal{N}_{\gamma,\vec{c},\vec{M}}(\bar{A}) := [\bigotimes_{I=1}^{N} \pi_I(\bar{A}(e_I)) \otimes \bigotimes_{\alpha=1}^{V} M_\alpha] \cdot [\otimes_{\alpha=1}^{V} c_\alpha], \tag{4.19}$$

for all $\bar{A} \in \overline{\mathcal{A}}$, where, as before, $\bar{A}(e_I)$ is an element of G associated with an edge e_I and '·' stands for contracting, at each vertex v_α of γ, the upper indices of the matrices corresponding to all the incoming edges, the lower indices of the matrices assigned to all the outgoing edges and the upper index of the vector M_α with all the corresponding indices of c_α. (We skip $\vec{\pi}$ and $\vec{\rho}$ in the symbol for the extended spin network function because the intertwiners c contain this information.) Thus, for example, in the simple case when the network has only two vertices, and all edges originate at the first vertex and end at the second, $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ can be written out explicitly as:

$$\mathcal{N}_{\gamma,\vec{c},\vec{M}} = \pi_1(\bar{A}(e_1))_{m_1}^{n_1} \ldots \pi_N(\bar{A}(e_N))_{m_N}^{n_N} \, M_1^{m'_1} M_2^{m'_2} \, c_1^{m_1\ldots m_N}{}_{m'_1} \, c_{2n_1\ldots n_N m'_2}, \tag{4.20}$$

where indices m_I, n_I range over $1, ..., 2j_I + 1$ and m'_α ranges over $1, ..., 2j_{\alpha+1}$. Given any spin network, (4.19) provides a function on $\overline{\mathcal{A}}$ which is square-integrable with respect to the measure μ^o. Given an extended spin network function on $\overline{\mathcal{A}}$, the the range $R(\gamma)$ of the associated graph γ is completely determined. Thus, two spin networks can define the same function on $\overline{\mathcal{A}}$ if one can be obtained from the other by subdividing edges and changing arbitrarily the orientations.

It turns out that the spin network states provide a decomposition of the full Hilbert space $\mathcal{H}^o$ into *finite* dimensional orthogonal sub-spaces (compare with [9,10]). Given a triplet $(\gamma, \vec{\pi}, \vec{\rho})$ defined by $(i - iii)$ above, consider the vector space $\mathcal{H}_{\gamma,\vec{\pi},\vec{\rho}}$ spanned by the spin network functions $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ given by all the possible choices for $\vec{c}, \vec{M}$ compatible with fixed labelings $\vec{\pi}, \vec{\rho}$. Note that, according to the representation theory of compact groups, every $\mathcal{H}_{\gamma,\vec{\pi},\mathcal{G}}$ is a finite dimensional irreducible representation of $\overline{\mathcal{G}}$ in Cyl. The group acts there via

$$\mathcal{N}_{\gamma,\vec{c},\vec{M}}(g^{-1}\bar{A}g) = \mathcal{N}_{\gamma,\vec{c},\vec{M}'}(\bar{A}), \quad M'_\alpha = \rho_\alpha(g(v_\alpha))M_\alpha. \tag{4.21}$$

Modulo the obvious completions, we have the following orthogonal decomposition,

$$\mathcal{H}^o = \bigoplus_{R(\gamma),\vec{\pi},\vec{\rho}} \mathcal{H}_{\gamma,\vec{\pi},\vec{\rho}} \tag{4.22}$$

where, given a graph γ, the labelings $\vec{\pi}$ and $\vec{\rho}$ range over all the data defined above by (i-iii) whereas for γ in the sum we take exactly one representative from every range of an

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analytic graph in Σ. When $\vec{\rho}$ is trivial we skip ρ in $\mathcal{H}_{\gamma,\vec{\pi},\vec{\rho}}$. On $\mathcal{H}_{\gamma,\vec{\pi}}$, the action of the gauge transformations group $\overline{\mathcal{G}}$ is trivial and we have the following orthogonal decomposition of the Hilbert space of gauge invariant cylindrical functions,

$$\tilde{\mathcal{H}}^o = \bigoplus_{R(\gamma),\vec{\pi}} \mathcal{H}_{\gamma,\vec{\pi}}, \qquad (4.23)$$

where we used the same conventions as in (4.22). Thus, we recover the result on spin network states obtained in [9,10].

We conclude this sub-section with a general comment on spin network states. Consider *tri-valent* graphs, i.e. graphs γ each vertex of which has three (or less) edges. In this case, the standard Clebsch-Gordon theory implies that the the number of associated gauge invariant spin network functions is severely limited: the corresponding sub-space of $\tilde{\mathcal{H}}^o$ is one dimensional. Hence, on the sub-space Cyl of $\tilde{\mathcal{H}}^o$ corresponding only to tri-valent graphs, the (normalized) spin network states provide a natural orthonormal basis. What is remarkable is that these spin networks were first introduced by Penrose [28] already twenty five years ago to probe the microscopic structure of geometry, although in a different context. Because of the simplicity (and other attractive properties) of these Penrose spin network states it is tempting to hope that they might suffice also in the present approach to quantum gravity. Indeed, there were conjectures that the higher valent graphs are physically redundant. However, it turns out that detailed physical considerations rule out this possibility; quantum gravity seems to need graphs with unlimited complexity.

C. Eigenvectors

We are now ready to exhibit eigenvectors of the operators $\triangle_{S,v}$ and $\hat{A}_S$ for any of the potential eigenvalues found in section 4.1. We will begin with the full, non-gauge invariant Hilbert space $\mathcal{H}^o$ and consider an arbitrary surface S. Since $\mathcal{H}^o$ serves as the (gravitational part) of the kinematical Hilbert space in theories in which gravity is coupled to spinor fields, our construction is relevant to that case. In the second part of this sub-section, we will turn to the gauge invariant Hilbert space $\tilde{\mathcal{H}}^o$ and exhibit eigenvectors for the restricted range of eigenvalues presented in section IV A.

Fix a point v in the surface S. We will investigate the action of the operators $(J_{S,v}^{(d)})^2$, $(J_{S,v}^{(u)})^2$, $(J_{S,v}^{(d+u)})^2$ and $\triangle_{S,v}$ on extended spin network states. Without loss of generality we can restrict ourselves to graphs which are adapted to S and contain v as a vertex, say $v = v_1$. Given a graph γ and labeling $\vec{\pi}$ and $\vec{\rho}$ of its edges and vertices by representations of $SU(2)$, we shall denote by $\mathcal{C}_v$ the linear space of the intertwining tensors which are compatible with $\vec{\pi}$ and $\vec{\rho}$ at v in the sense of section IV B. Let $(\gamma,\vec{\pi},\vec{\rho},\vec{c},\vec{M})$ be an extended spin network and $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ be the corresponding state. As one can see from Eqs. (4.1, 3.21), each of the four operators above is given by a linear combination (with constant coefficients) of gauge invariant terms of the form $b_{i_1...i_E} X_{I_1}^{i_1}...X_{I_E}^{i_E}$ where $b_{i_1...i_E}$ is a constant tensor and all the Xs are associated with the point v and the edges which meet there.

On $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ the action of any operator of this type reduces to a linear operator o_v acting in $\mathcal{C}_v$. More precisely, if $\mathcal{O}$ is any of the above operators, we have

$$\mathcal{O}\mathcal{N}_{\gamma,\vec{c},\vec{M}} = \mathcal{N}_{\gamma,\vec{c}',\vec{M}} \tag{4.24}$$

where $\mathcal{N}_{\gamma,\vec{c}',\vec{M}}$ is again an extended spin network state and the network $(\gamma,\vec{\pi},\vec{\rho},\vec{c}',\vec{M})$ differs from the first one only in one entry of the labeling $\vec{c}$ corresponding to the vertex v; $c'_\alpha = c_\alpha$ for all the vertices $v_\alpha \neq v$ and $c'_1 = o_v c_1$. Consequently, the problem of diagonalizing these operators reduces to that of diagonalizing a *finite* symmetric matrix of o_v. Note that a constant vector M assigned to v does not play any role in this action and hence will just make eigenvectors degenerate.

In the case of operators $(J^{(d)}_{S,v})^2$, $(J^{(u)}_{S,v})^2$ and $(J^{(d+u)}_{S,v})^2$, the (simultaneous) eigenstates are given by the group representation theory. We can now spell out the general construction.

Let us fix a graph γ and arrange the edges that meet at v into three classes as before; $e_1,...,e_d$; $e_{d+1},....,e_u$; $e_{u+1},...,e_t$. Let us also fix a labeling $\pi_1,...,\pi_t$ of these edges by irreducible, non-trivial representations of $SU(2)$ and an irreducible (possibly trivial) representation ρ which emerges in the decomposition of $\pi_1 \otimes ... \otimes \pi_t$. Consider now the following ingredients:

a) Irreducible representations $\mu_{(d)}$, $\mu_{(u)}$ and $\mu_{(d+u)}$;

b) Invariant tensors $c_{(d)}{}^{m_1...m_d m'}$, $c_{(u)}{}^{m_{d+1}...m_u m''}$ and $c_{(u+d)\,m'm''}{}^m$ associated, respectively, to the representations $\pi_1,...,\pi_d,\mu_{(d)}$, and to $\pi_{d+1},...,\pi_u,\mu_{(u)}$ and finally to $\mu_{(d)},\mu_{(u)},\mu_{(d+u)}$; and,

c) Invariant tensor $c_{(t)\,n}{}^{m_{u+1}...m_t m}$ associated to $\mu_{(d+u)},\pi_{u+1},...,\pi_t,\rho$.

From this structure, construct the following invariant tensor,

$$c^{m_1...m_t n} := c_{(d)}{}^{m_1...m_d m'} c_{(u)}{}^{m_{d+1}...m_u m''} c_{(d+u)\,m'm''}{}^n c_{(t)\,n}{}^{m_{u+1}...m_t m}, \tag{4.25}$$

associated with the representations $\pi_1,...,\pi_t,\rho$. To obtain a non-trivial result in the end, we need all the tensors to be non-zero. The existence of such tensors is equivalent to the following two conditions on the data (a-c):

d) The representations $\mu_{(d)}$ and $\mu_{(u)}$ emerge respectively, in $\pi_1\otimes...\otimes\pi_d$ and $\pi_{d+1}\otimes...\otimes\pi_u$; and,

e) the representation $\mu_{(d+u)}$ emerges both in $\mu_{(d)} \otimes \mu_{(u)}$ and $\pi_{u+1} \otimes ... \otimes \pi_t \otimes \rho$.

Finally, introduce an extended spin network $(\gamma,\vec{\pi},\vec{\rho},\vec{c},\vec{M})$ such that

$$\vec{\pi} = (\pi_1,...,\pi_t,...,\pi_N), \quad \vec{\rho} = (\rho,\rho_2,...,\rho_V), \quad \vec{c} = (c,c_2,...,c_V), \tag{4.26}$$

the remaining entries being arbitrary. Then, the corresponding state $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ is an eigenvector of the operators $(J^{(d)}_{S,v})^2$, $(J^{(u)}_{S,v})^2$, and $(J^{(d+u)}_{S,v})^2$ with the eigenvalues $j^{(d)}(j^{(d)}+1)$,

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$j^{(u)}(j^{(u)} + 1)$ and $j^{(d+u)}(j^{(d+u)} + 1)$, respectively, where the half integers $j^{(d)}, j^{(u)}$ and $j^{(d+u)}$ correspond to the representations $\mu_{(d)}, \mu_{(u)}$ and $\mu_{(d+u)}$. Hence, this $\mathcal{N}_{\gamma,\vec{c},\vec{M}}$ is also an eigenvector of $\triangle_{S,v}$ with the eigenvalue (4.4,4.5). It is obvious, that for any triple of representations $\mu_{(d)}, \mu_{(u)}$ and $\mu_{(d+u)}$ satisfying the constraint (4.5) there exists an extended spin network (4.26).

This construction provides *all* eigenvectors of $\triangle_{S,v}$. The key reason behind this completeness is that, given any choice of $\pi_1, ..., \pi_d, ..., \pi_u, ..., \pi_t$ and ρ as above, the invariant tensors which can be written in the form (4.25) with any $\mu_{(d)}, \mu_{(u)}$, and $\mu_{(d+u)}$ span the entire space $\mathcal{C}_v$ of invariant tensors at v compatible with that data. Since the defining formula for a spin network function (4.19) is linear with respect to every component of $\vec{c}$, given any spin network $(\gamma, \vec{\pi}, \vec{\rho}, \vec{c}, \vec{M})$ it suffices to decompose the component c_1 of $\vec{c}$ at $v_1 = v$ into invariant tensors of the form (4.25) in any manner to obtain a decomposition of the corresponding spin network function into a linear combination of extended spin network functions given by (4.25,4.26). The desired result now follows from the orthogonal decomposition of $\mathcal{H}^o$ in to the extended spin network subspaces.

Let us now turn to the operator $\hat{A}_S$. A basis of eigen vectors can be obtained in the following way. Since the area operator can be expressed in terms of and commutes with $(J_{S,v}^{(d)})^2, (J_{S,v}^{(u)})^2$, and $(J_{S,v}^{(d+u)})^2$ at any point v in S, we can simultaneously diagonalize all these operators. Because for every graph the area operator preserves the subspace of spin-network states associated with that graph and for two different graphs the spin network spaces spaces are orthogonal, it is enough to look for eigen vectors for an arbitrary graph γ. Given a graph γ, labelings $\vec{\pi}$, $\vec{\rho}$ and $\vec{M}$ as in section 4.2, at every vertex v contained in the surface S choose a basis in the space $\mathcal{C}_v$ consisting of invariant tensors of the form (4.25). The set of the spin network functions (4.19) constructed by varying $\gamma, \vec{\pi}, \vec{\rho}, \vec{M}$ and picking at each vertex v an element of the basis in $\mathcal{C}_v$ constitutes a basis in $\mathcal{H}^o$. (If we restrict the labelings to ρ consisting only of the trivial representations, then the resulting set of spin network states provide a basis for the space $\tilde{\mathcal{H}}^o$ of gauge invariant functions.) Each of such states is automatically an eigenvector of $\hat{A}_S$ with eigenvalue (4.6).

We conclude the first part of this sub-section with a simple example of an eigen vector of the area operator with eigenvalue a_S, where a_S is any real number satisfying (4.6,4.6).

- Example. Suppose $(j_\alpha^{(d)}, j_\alpha^{(u)}, j_\alpha^{(d+u)})$, $\alpha = 1, ..., W$, is a finite set of triples of half integers which for every α satisfy (4.5)). Rather than repeating the construction a)-e) above step by step, we will specify only the simplest of the resulting (extended) spin-networks. In S choose W distinct points v_α, $\alpha = 1, ..., W$. To every point v_α assign two finite analytic curves $e_{d,\alpha}$ and $e_{u,\alpha}$ starting at v_α, not intersecting S otherwise, and going in opposite directions of S. For a graph γ take the graph $\{e_{d,1}, e_{u,1}, ..., e_{d,W}, e_{u,W}\}$, the vertices being the intersection points v_α and the ends of the edges $e_{d,\alpha}$ and $e_{u,\alpha}$ (the curves being chosen such that the points v_α are the only intersections). Label each edge $e_{d,\alpha}$ with the irreducible representation $\pi_{d,\alpha}$ corresponding to a given $j_\alpha^{(d)}$ and every edge $e_{u,\alpha}$ with the irreducible representation $\pi_{u,\alpha}$ defined by $j_\alpha^{(u)}$. That defines a labeling $\vec{\pi}$ of γ. (The absence of edges $e_{t,\alpha}$ is equivalent to introducing these edges in any manner and assigning to them the trivial representations.) To define a labeling $\vec{\rho}$ at the vertices v_α, assign to every vertex

v_α a representation ρ_α defined by a given $j_\alpha^{(d+u)}$. Next, to each vertex v_α assign an invariant tensor $c_\alpha^{m_d m_u m}$ associated to the triple of representations $(\pi_{d,\alpha}, \pi_{u,\alpha}, \rho_\alpha)$ introduced above. The construction of a spin-network is completed by: (i) labeling that end point of each $e_{d,\alpha}$ and, respectively, of $e_{u,\alpha}$ which is not contained in S, with the representation $\rho_{d,\alpha} := \pi_{d,\alpha}$ and, respectively, $\rho_{u,\alpha} := \pi_{u,\alpha}$; (ii) labeling of these ends of the edges with the unique invariants corresponding to the representations $\mu_{(d),\alpha}, \rho_{(d),\alpha}$ or, respectively, to $\mu_{(u),\alpha}, \rho_{(u),\alpha}$; (iii) defining a labeling $\vec{M}$ of vertices which can be chosen arbitrarily, provided at a vertex v_α the associated vector M_α belongs to the representation $\rho_{(d+u),\alpha}$ and at an endpoint of either of the edges $e_{d/u,\alpha}$ the associated $M_{d/u,\alpha}$ belongs to $\rho_{(d/u),\alpha}$.

As we noted in section 2, the Hilbert space $\mathcal{H}^o$ is the quantum analog of the full phase space. Now, in the classical theory, the imposition of the Gauss constraint on the phase space does not restrict the allowed values of the functional A_S of (2.2). It is therefore of interest to see if this feature persists in the quantum theory: Is the spectrum of $\hat{A}_S$ on the full $\mathcal{H}^o$ the same as that on its gauge invariant sub-space $\tilde{\mathcal{H}}^o$? As was indicated in Sec. 4.1, the answer is in the affirmative only if the surface is open. If S is closed, there are restrictions on the spectrum which depend on topological properties of S embedded in Σ. The second part of this section is devoted to this issue. As indicated in Sec. 4.1, we need to consider three separate cases.

Case i): $\partial S \neq \emptyset$ *(and* $\partial S \subset \Sigma$*).*
We will modify the spin-network of the above Example in such a way as to obtain a gauge-invariant eigenstate *without* changing the eigen value of the area operator. Let γ and the labeling $\vec{\pi}$ be the ones defined in the Example. To each vertex v_α assign one more edge $e_{t,\alpha}$ beginning in v_α and *contained* in S. Label it by the representation $\pi_{t,\alpha}$ corresponding to a given $j_\alpha^{(d+u)}$ at that point. The labeling $\vec{\rho}$ is now taken to be trivial. To every point v_α assign, as in the Example, an invariant tensor c_α associated now to the representations $(\pi_{d,\alpha}, \pi_{u,\alpha}, \pi_{t,\alpha})$. Every extension of this data to a spin-network will define a spin-network state which is gauge invariant at each of the points v_α. Now, we need to define a closed spin-network which contains all the edges $e_{d,\alpha}, e_{u,\alpha}, e_{t,\alpha}$ and provides an extension for the labelings already introduced. For this, we use a key property of the area operator associated to a surface with boundary: vertices which lie on ∂S do not contribute to the action of the operator. Therefore, we can simply extend every edge $e_{t,\alpha}$ within S to the boundary of S. Denote the intersection point with ∂S by $v_{t,\alpha}$. Next, for every α we extend (in a piecewise analytic way) the edges $e_{d,\alpha}$ and $e_{u,\alpha}$ such that they end at $v_{t,\alpha}$. The extended edges form a graph $\gamma' = \{e'_{d,1}, e'_{u,1}, e'_{t,1}, ..., e'_{d,W}, e'_{u,W}, e'_{t,W}\}$. Let us label each primed edge by the irreducible representation assigned before to the edge it is an extension of. This defines a labeling $\vec{\pi'}$ of γ'. Finally, assign to each new vertex $v_{t,\alpha}$ the non-zero invariant tensor $c'_{t,\alpha\, m_u, m_d, m_t}$ (which is unique up to rescaling) associated to the triplet of representations $(\pi_{d,\alpha}, \pi_{u,\alpha}, \pi_{t,\alpha})$. This completes the construction of a gauge invariant extension of a spin-network state constructed in the Example. Thus, for an open surface, the spectrum of the area operator $\hat{A}_S$ on $\tilde{\mathcal{H}}^o$ is the same as that on $\mathcal{H}^o$.

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Case ii): $\partial S = \emptyset$ and S splits Σ in to two open sets.

In this case we can not repeat the above construction: Since S has no boundary, if additional vertices are needed to close the open spin-network, they must now lie in S and can make unwanted contributions to the action of the area operator. Consequently, there are further restrictions on the possible eigenvalues of the operators $(J_{S,v}^{(d)})^2$, $(J_{S,v}^{(u)})^2$ and $(J_{S,v}^{(d+u)})^2$. To see this explicitly, consider arbitrary spin-network state $(\gamma, \vec{\pi}, \vec{c})$ given by the construction *a)-e)* of Sec 4.3. Let $\{v_1, ..., v_W\}$ be a set of the vertices of γ contained in the surface S. Graph γ can be split into three graphs: γ_t which is contained in S, γ_u which is contained in a one side of S in Σ and γ_d contained in the other side of S in Σ. The only intersection between the two parts is the set $\{v_1, ..., v_W\}$ of vertices of γ which are contained in S. Let γ_r be one of the parts of γ (i.e. $r = d$ or $r = u$ or $r = t$). According to the construction *a)-e)*, the labelings $\vec{\pi}$ and $\vec{c}$ define naturally on γ_r an extended spin-network. The labeling of the edges of γ_r by irreducible representations is defined just by the restriction of $\vec{\pi}$ to γ_r. The labeling of the vertices by irreducible representations and invariant tensors is defined in the following way. For the vertices of γ_r which are not contained in S, the labelings are taken to be again the restriction of $\vec{\rho}$ (which are all trivial) and $\vec{c}$. To a vertex v_α contained in S we assign the representation corresponding to a given $j_{r,\alpha}$ and the invariant tensor c_r defined in *b)* (for $r = d, u$) and *c)* (for $r = t$) of the construction *a) - e)*. Finally, we complete it by arbitrary nonzero labeling $\vec{M}$ of the vertices with vectors in appropriate representations. The construction *a) - e)* guarantees that a resulting extended spin-network state is not zero. Now, for an extended spin-network $(\gamma', \vec{\pi}', \vec{\rho}', \vec{c}', \vec{M}')$ we have the following "fermion conservation law":

$$\sum_v j_{\rho'(v)} = N \tag{4.27}$$

for some *integer* N, where v runs through the vertices of a graph γ' and each $j_{\rho(v)}$ is an half-integer corresponding to an representation assigned to v by $\vec{\rho}'$. In our case we therefore obtain the restriction:

$$\sum_\alpha j_{r,\alpha} = N_r \tag{4.28}$$

for $r = d, u, d + u$ which gives the conditions (4.7) listed in Sec 4.1. (In fact either two of the above conditions imply the third one).

The conditions (4.28) are also sufficient for an eigen vector to exist. Suppose we are given a set of half integers as in the Example above, which satisfy the restriction (4.28). A statement "converse to the fermion conservation law" is that for any set $\{v_1, ..., v_W\}$ of points in S and any assignment $v_\alpha \mapsto j_\alpha$ where j_α are non-negative half integer satisfying (4.28), there exists an extended spin-network $(\gamma', \vec{\pi}', \vec{\rho}', \vec{c}', \vec{M}')$ such that every v_α is its index, j_α corresponds to the representation assigned to v_α by $\vec{\rho}'$, and for every vertex $v \neq v_\alpha$, $\alpha = 1, ..., W$, of γ', the representation assigned by $\vec{\rho}'$ is trivial. From extended spin-networks provided by the above statement it is easy to construct an eigen vector of the corresponding eigen values.

Case iii): $\partial S = \emptyset$ but S does not split Σ.

The only difference between this case and the previous one is that now a graph γ representing an eigen vector is cut by S into two components: γ_t contained in S and γ_{d+u} which corresponds to the rest of γ. Since γ_{d+u} can be now connected by the same arguments as above, we prove that a necessary and sufficient condition for an eigen vector to exists is (4.28) imposed only on the half integers $j_\alpha^{(d+u)}$.

V. DISCUSSION

In section 1, we began by formulating what we mean by quantization of geometry: Are there geometrical observables which assume continuous values on the classical phase space but whose quantum analogs have discrete spectra? In the last two sections, we answered this question in the affirmative in the case of area operators. In the next paper in this series we will show that the same is true of other ('3-dimensional') operators. The discreteness came about because, at the microscopic level, geometry has a distributional character with 1-dimensional excitations. This is the case even in semi-classical states which approximate classical geometries macroscopically [12,22].

We will conclude this paper by examining our results on the area operators from various angles.

1. Inputs: The picture of quantum geometry that has emerged here is strikingly different from the one in perturbative, Fock quantization. Let us begin by recalling the essential ingredients that led us to the new picture.

This task is made simpler by the fact that the new functional calculus provides the degree of control necessary to distill the key assumptions. There are only two essential inputs. The first assumption is that the Wilson loop variables, $T_\alpha = \text{Tr}\,\mathcal{P} - \exp\int_\alpha A$, should serve as the configuration variables of the theory, i.e., that the Hilbert space of (kinematic) quantum states should carry a representation of the $C^\star$-algebra generated by the Wilson loop functionals on the classical configuration space $\mathcal{A}/\mathcal{G}$. The second assumption singles out the measure $\tilde{\mu}^o$. In essence, if we assume that $\hat{E}_i^a$ be represented by $-i\hbar\delta/\delta A_a^i$, the 'reality conditions' lead us to the measure $\tilde{\mu}^o$ [10]. Both these assumptions seem natural from a mathematical physics perspective. However, a deeper understanding of their *physical* meaning is still needed for a better understanding of the overall situation [8].

Compactness of $SU(2)$ plays a key role in all our considerations. Let us therefore briefly recall how this group arose. As explained in [17,19], one can begin with the ADM

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[8]In particular, in the standard spin-2 Fock representation, one uses quite a different algebra of configuration variables and uses the flat background metric to represent it. It then turns out that the Wilson loops are *not* represented by well-defined operators; our first assumption is violated. One can argue that in a fully non-perturbative context, one can not mimic the Fock space strategy. Further work is needed, however, to make this argument water-tight.

phase-space in the triad formulation, i.e., with the fields (E_i^a, K_a^i) on Σ as the canonical variables, and then make a canonical transformation to a new pair $(A_a^i := (\Gamma_a^i + K_a^i), E_i^a)$, where K_a^i is the extrinsic curvature and Γ_a^i, the spin-connection of E_i^a. Then, A_a^i is an $SU(2)$ connection, the configuration variable with which we began our discussion in section 2. It is true that, in the Lorentzian signature, it is not straightforward to express the Hamiltonian constraint in these variables; one has to introduce an additional step, e.g., a generalized Wick transform [18]. However, this point is not directly relevant in the discussion of geometric operators which arise at the *kinematical* level. (See, however, below). Finally, we could have followed the well-known strategy [25] of simplifying constraints by using a complex connection ${}^{\mathbb{C}}A_a^i := (\Gamma_a^i - iK_a^i)$ in place of the real A_a^i. The internal group would then have been complexified $SU(2)$. However, for *real* (Lorentzian) general relativity, the kinematic states would then have been *holomorphic* functionals of ${}^{\mathbb{C}}A_a^i$. To construct this representation rigorously, certain technical issues still need to be overcome. However, as argued in [18], in broad terms, it is clear that the results will be equivalent to the ones obtained here with real connections.

2. Kinematics versus Dynamics: As was emphasized in the main text, in the classical theory, geometrical observables are defined as functionals on the *full* phase space; these are kinematical quantities whose definitions are quite insensitive to the precise nature of dynamics, presence of matter fields, etc. Thus, in the connection dynamics description, all one needs is the presence of a canonically conjugate pair consisting of a connection and a (density weighted) triad. Therefore, one would expect the result on the area operator presented here to be quite robust. In particular, they should continue to hold if we bring in matter fields or extend the theory to super-gravity.

There is, however, a subtle caveat: In field theory, one can not completely separate kinematics and dynamics. For instance, in Minkowskian field theories, the kinematic field algebra typically admits an infinite number of *inequivalent* representations and a given Hamiltonian may not be meaningful on a given representation. Therefore, whether the kinematical results obtained in any one representation actually hold in the physical theory depends on whether that representation supports the Hamiltonian of the model. In the present case, therefore, a key question is whether the quantum constraints of the theory can be imposed meaningfully on $\tilde{\mathcal{H}}^o$.[9] Results to date indicate (but do not yet conclusively prove) that this is likely to be the case for general relativity. The general expectation is that this would be the case also for a class of theories such as super-gravity, which are 'near' general relativity. The results obtained here would continue to be applicable for this class of theories.

3. Dirac Observable: Note that $\hat{A}_S$ has been defined for *any* surface S. Therefore, these operators will not commute with constraints; they are not Dirac observables. To obtain a Dirac observable, one would have to specify S *intrinsically*, using, for example,

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[9]Note that this issue arises in *any* representation once a sufficient degree of precision is reached. In geometrodynamics, this issue is not discussed simply because generally the discussion is rather formal.

matter fields. In view of the Hamiltonian constraint, the problem of providing an explicit specification is extremely difficult. However, this is true already in the classical theory. In spite of this, *in practice* we do manage to specify surfaces and furthermore compute their areas using the standard formula from Riemannian geometry which is quite insensitive to the details of how the surface was actually defined. Similarly, in the quantum theory, *if* we could specify a surface S intrinsically, we could compute the spectrum of $\hat{A}_S$ using results obtained in this paper.

4. Comparison: Let us compare our methods and results with those available in the literature. Area operators were first examined in the loop representation. The first attempt [12] was largely exploratory. Thus, although the key ideas were recognized, the very simplest of loop states were considered and the simplest eigenvalues were looked at; there was no claim of completeness. In the present language, this corresponds to restricting oneself to bi-valent graphs. In this case, apart from an overall numerical factor (which does, however, have some conceptual significance) our results reduce to that of [12].

A more complete treatment, also in the framework of the loop representation, was given in [13]. It may appear that our results are in contradiction with those in [13] on two points. First, the final result there was that the spectrum of the area operator is given by $\ell_P^2 \sum \sqrt{j_l(j_l + 1)}$, where j_l are half-integers, rather than by (4.6). However, the reason behind this discrepancy is rather simple: the possibility that some of the edges at any given vertex can be tangential to the surface was ignored in [13]. It follows from our remark at the end of section 4.2 that, given a surface S, if one restricts oneself to only to graphs in which none of the edges are tangential, our result reduces to that of [13]. Thus, the eigenvalues reported in [13] do occur in our spectrum. It is just that the spectrum reported in [13] is incomplete. Second, it is suggested in [13] that, as a direct consequence of the diffeomorphism covariance of the theory, local operators corresponding to volume (and, by implication, area) *elements* would be necessarily ill-defined (which makes it necessary to by-pass the introduction of volume (and area) elements in the regularization procedure). This assertion appears to contradict our finding that the area element $\widetilde{\sqrt{g_S}}$ *is* a well-defined operator-valued distribution which can be used to construct the total area operator $\hat{A}_S$ in the obvious fashion. We understand [29], however, that the intention of the remark in [13] was only to emphasize that the volume (and area) elements are 'genuine' operator-valued distributions; thus there is no real contradiction.

The difference in the methodology is perhaps deeper. First, as far as we can tell, in [13] only states corresponding to trivalent graphs are considered in actual calculations. Thus, even the final expression (Eq (48) in [13]) of the area operator after the removal of the regulator is given only on tri-valent graphs. similarly, their observation that every spin network is an eigenvector of the area operator holds only in the tri-valent case. Second, for the limiting procedure which removes the regulator to be well-defined, there is an implicit assumption on the continuity properties of loop states (spelled out in detail in [30]). A careful examination shows that this assumption is *not* satisfied by the states of interest and hence an alternative limiting procedure, analogous to that discussed in section 3.1, is needed. Work is now progress to fill this gap [29]. Finally, not only is the level of precision

achieved in the present paper significantly higher but the approach adopted is also more systematic. In particular, in contrast to [13], in the present approach, the Hilbert space structure is known *prior* to the introduction of operators. Hence, we can be confident that we did not just omit the continuous part of the spectrum by excising by fiat the corresponding sub-space of the Hilbert space.

Finally, the main steps in the derivation presented in this paper were sketched in the Appendix D of [10]. The present discussion is more detailed and complete.

5. Manifold versus Geometry: In this paper, we began with an orientable, analytic, 3-manifold Σ and this structure survives in the final description. As noted in footnote 1, we believe that the assumption of analyticity can be weakened without changing the qualitative results. Nonetheless, a smoothness structure of the underlying manifold will persist. What is quantized is 'geometry' and not smoothness. Now, in 2+1 dimensions, using the loop representation one can recast the final description in a purely combinatorial fashion (at least in the so-called 'time-like sector' of the theory). In this description, at a fundamental level, one can avoid all references to the underlying manifold and work with certain abstract groups which, later on, turn out to be the homotopy groups of the 'reconstructed/derived' 2-manifold (see, e.g., section 3 in [31]). One might imagine that, if and when our understanding of knot theory becomes sufficiently mature, one would also be able to get rid of the underlying manifold in the 3+1 theory and introduce it later as a secondary/derived concept. At present, however, we are quite far from achieving this.

In the context of geometry, however, a detailed combinatorial picture *is* emerging. Geometrical quantities are being computed by counting; integrals for areas and volumes are being reduced to genuine sums. (However, the sums are *not* the 'obvious' ones, often used in approaches that *begin* by postulating underlying discrete structures. In the computation of area, for example, one does not just count the number of intersections; there are precise and rather intricate algebraic factors that depend on the representations of $SU(2)$ associated with the edges at each intersection.) It is striking to note that, in the same address [3] in which Riemann first raised the possibility that geometry of space may be a physical entity, he also introduced ideas on discrete geometry. The current program comes surprisingly close to providing us with a concrete realization of these ideas.

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