Twistors, charge structure, and BMS symmetries

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Abstract

Corresponding to the Bondi-Metzner-Sachs (BMS) symmetry algebra of asymptotically-flat spacetimes are a set of BMS charges. These are formally constructed via the symplectic formalism of Wald and Zoupas, but the same charge expression may be arrived at by the simpler twistorial procedure of Dray and Streubel. Here, we formalize the connection between twistors and asymptotic symmetries which underlies the Dray-Streubel charge by demonstrating an isomorphism between twistors in flat spacetime and twistors on radiation-free sections of \mathscr{I}^+ . In the corresponding formalism, the Dray-Streubel charge finds a natural reinterpretation as exactly the part of Penrose's twistorial charge which is invariant with respect to a certain gauge transformation. Furthermore, we argue that the twistorial picture of the radiative phase space, properly formalized, provides a tool alongside the symplectic formalism or shear structure for analyzing radiative data on \mathscr{I}^+ .

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I. INTRODUCTION

In a generic curved spacetime, in the absence of a preferred direction such as a Killing vector field or a fluid 4-velocity, there is no obvious way to define charges like energy and angular momentum. Charges tend to be defined with respect to symmetries of the background and outside of certain exact solutions, general relativity admits no such symmetries. As a consequence, there can be no grounds for comparison between any two spacetimes without additional structure since there is no phase space of observables that is shared by all spacetimes.

The advancement that would recover the ability to compare spacetimes came out of the attempt to covariantly describe gravitational radiation beginning with Bondi and culminating in Penrose's procedure of conformal compactification [1–3]. The abstract manifold \mathscr{I} emerges as the place at which gravitational radiation can be unambiguously defined. It forms a universal structure shared by all spacetimes which admit a smooth conformal boundary. As shown by Ashtekar and Streubel [4–6], this universal structure is a symplectic structure and so it can be thought of as a phase space shared by all asymptotically-flat spacetimes.

We would expect that a charge structure should emerge with respect to symmetries of this universal structure and so it is significant that the symmetry group of the universal structure is the infinite-dimensional BMS group, corresponding to the Poincaré group with the translations replaced by the infinite-dimensional supertranslations [7].

Taking the position that the infinite-dimensional enhancement of the symmetry group in the infrared is something to be taken seriously, a set of adjoint *BMS charges* may be defined [8–11]. Two of these prescriptions are formally equivalent and have all of the properties that one would reasonably expect. The most conceptually satisfactory of these is the Wald-Zoupas prescription which defines a charge in terms of the explicit symplectic structure at null infinity, taking into account the leaking of symplectic current through the boundary [10, 12].

The second is the Dray-Streubel prescription which exploits the spin-lowering properties of the twistor equation to define a charge in analogy with Maxwell electromagnetism [8, 13, 14]. It is formally equivalent to the Wald-Zoupas charge in asymptotically-flat general relativity, yet little has been said about the reason for this connection. The form of the charge arose through careful flux considerations given previous unsuccessful attempts and

the BMS algebra appears only via a component-wise correspondence between the symmetry algebra and solutions to the twistor equation.

This article formalizes this correspondence between twistors and the BMS algebra by first considering a twistor space on a section of null infinity for which no gravitational radiation is arriving. We begin with a review of the previous use of twistors in charge construction in both flat and curved spacetime (§II). We show that most previous applications of twistors to charge structure at \mathscr{I}^+ failed to account for a certain gauge ambiguity in the twistor which stands in for an element of the symmetry algebra (§III). With a proper formalism in place, we reinterpret the Dray-Streubel charge procedure as a more general procedure for regulating the gauge freedom in the twistor which stands in for an element of the underlying symmetry algebra so that the resulting charge is adjoint not just to some twistor space but to the symmetry algebra proper (§IV). We conclude with a discussion of the status of the twistorial description of the radiative phase space and make an argument for its value as an alternative description of the radiative observables described by the shear structure or the symplectic formalism at \mathscr{I}^+ (§V).

A. Notation and conventions

Following Penrose's procedure of conformal compactification [3], we consider a spacetime up to a conformal rescaling

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}.\tag{1}$$

A spacetime may be called asymptotically-flat if it admits a conformal boundary defined as the level set

$$\mathscr{I} = \{ x \in \mathcal{M} : \Omega(x) = 0 \}$$
 (2)

where \mathcal{M} is the rescaled manifold. It may be shown that \mathscr{I} is a null hypersurface with null normal $N_a = -\nabla_a \Omega$. Furthermore, we assume \mathscr{I} is made up of two connected components with topology $\mathbb{R} \times S^2$ called future and past null infinity. We will work primarily with future null infinity \mathscr{I}^+ , but it is straightforward to demonstrate that all results hold for past null infinity \mathscr{I}^- .

1. Tetrad on \mathscr{I}^+

We define a null tetrad on \mathscr{I} via a spin frame. Choose a spinor ι^A such that its flagpole $n^a \equiv \iota^A \bar{\iota}^{A'}$ is a representative from the one parameter set of vectors which kill the degenerate metric on \mathscr{I} , i.e.,

$$n_a = -A\nabla_a\Omega \tag{3}$$

for some A. Note that we use the abstract index notation of Penrose and so the spinor and tensor indices need not be thought of as taking on values and instead denote virtual copies of a given object [15].

Next, choose a spinor o^A , so that

$$o_A \iota^A = 1 \tag{4}$$

and its flagpole $l^a = o^A \bar{o}^{A'}$, points off \mathscr{I} . The complex null vector $m^a = o^A \bar{\iota}^{A'}$ and its complex conjugate are taken to span *cuts* of null infinity.

We will regulate the scaling and rotation freedom in the null tetrad using the GHP formalism [16] on top of the Newman-Penrose formalism [17]. In the literature, two special gauges are employed. The cylinder gauge corresponds to the vanishing of ρ' so that each cut shares a unit sphere metric. The Bondi gauge corresponds to this condition plus the vanishing of τ so that the null lapse is constant across a cut. We will employ a further modified GHP formalism that is covariant with respect to these gauge transformations.

2. Conformal GHP formalism

The conformal GHP (cGHP) formalism acts on quantities which transform covariantly with respect to conformal transformations of the metric [18, 19]. In that way, it is useful for dealing with quantities on the *universal radiative structure* at \mathscr{I}^+ [20].

We say that a scalar η has conformal weight w if under the conformal map $\Omega \to \omega \Omega$ it scales according to

$$\eta \to \omega^w \eta.$$
 (5)

A quantity which is also spin- and boost-weighted will have a weight denoted [w; p, q]. There is the freedom to choose how the null tetrad scales under such a transformation. We choose

$$l^a \to \omega^{-2} l^a, \qquad m^a \to \omega^{-1} m^a, \qquad n^a \to n^a.$$
 (6)

As a consequence, the scalar A has weight [-1; 1, 1].

One notices that the action of GHP weighted derivatives does not transform covariantly under (5) and so we introduce new conformally-covariant derivatives,

$$b'_c \eta = b' \eta + (w + p + q) \rho' \eta, \tag{7}$$

$$\eth_c \eta = \eth \eta + (w+q)\tau \eta, \tag{8}$$

$$\bar{\eth}_c \eta = \eth' \eta + (w+p)\bar{\tau}\eta,\tag{9}$$

with corresponding weights

$$b'_c \eta : [w; p - 1, q - 1], \tag{10}$$

$$\eth_c \eta : [w-1; p+1, q-1],$$
 (11)

$$\bar{\eth}_c \eta : [w-1; p-1, q+1].$$
 (12)

These derivatives do not commute in general. Their non-commutation is given by,

$$[b'_c, \eth_c]\eta = (w+q)\mathcal{P}\eta, \tag{13}$$

$$[b'_c, \bar{\eth}_c]\eta = (w+p)\bar{\mathcal{P}}\eta,$$
 (14)

$$[\eth, \eth']\eta = -(p-q)\mathcal{Q}\eta, \tag{15}$$

where $\mathcal{P} = \beta' \tau - \eth \rho'$ and $\mathcal{Q} = \Phi_{11} + \Lambda - \rho \rho' - \eth' \tau$ is related to the Gauß curvature of the cut.

The components of the rescaled Weyl tensor are properly weighted and their propagation equations along the generators of \mathscr{I}^+ are

$$b'_c \psi_k - \eth_c \psi_{k+1} = (3-k)\sigma \psi_{k+2}, \quad k = 0:3.$$
(16)

The case k=2 will be important in later sections.

In a Bondi gauge where cuts are taken to be metric spheres, the Bondi news is the derivative of σ along the generators, $\dot{\sigma} = -\bar{N}$, but the quantity which appears in the Bondi energy in this less restricted gauge is $\mathcal{N} = N + \bar{R}$, where R is the unique solution to

$$\bar{\eth}_c R + \eth_c \mathcal{Q} = 0 \tag{17}$$

[8, 19].

One can show that \mathcal{N} satisfies

$$p_c' \mathcal{N} = A\psi_4, \quad \eth_c \mathcal{N} = A\psi_3 \tag{18}$$

so that $\mathcal{N}=N+\bar{R}$ should be taken as the definition of the Bondi news in this gauge [8, 19, 20].

3. BMS algebra

We can write down concrete expressions for elements of the BMS algebra by considering the Lie dragging of Geroch's universal structure tensor along the integral curves of a vector field $X^a = \eta n^a + \bar{\xi} m^a + \xi \bar{m}^a$ on \mathscr{I} :

$$\mathcal{L}_X \Gamma^{ab}{}_{cd} = 0. \tag{19}$$

By taking X^a to be invariant with respect to conformal, spin, and boost transformations, the weights corresponding to its components are, $\eta:[0;1,1]$ and $\xi:[1;1,-1]$. One finds that elements of the BMS algebra are those vector fields X^a such that

$$\eth_c \xi = 0,$$
 $\flat'_c \xi = 0,$ $\flat'_c \eta = \frac{1}{2} \left(\bar{\eth}_c \xi + \bar{\eth}_c \bar{\xi} \right).$ (20)

The value of η on some initial cut is completely unconstrained so that the BMS algebra is infinite-dimensional. This presentation of the BMS algebra is that of Frauendiener and Stevens [19].

It is known that the Bondi-Sachs energy-momentum offers an unambiguous supertranslationfree definition of energy-momentum on a cut of \mathscr{I}^+ [2]. The saving grace which allows for this construction is the existence of an ideal of translations within the supertranslations [7].

In cGHP notation, one finds that the translations are solutions to (20) with the further condition,

$$\eth_c^2 V = RV \tag{21}$$

where $\eta = AV$. It makes sense to talk of a *supertranslation-free* translation, but the semidirect product structure of the BMS group means that there is no obvious way to write down a *supertranslation-free* Lorentz sub-algebra of the BMS algebra. We must make a choice of supertranslation.

II. SPIN-LOWERING AND TWISTORIAL CHARGES

First, we review the conventional use of twistors in charge construction. In flat spacetime, twistors provide a very natural description of energy, momentum, and angular momentum. The charge structure of a system in Minkowski space is entirely encoded in a symmetric kinematic twistor $A_{\alpha\beta}$ [18].

One may access the individual charges by contracting with a necessarily symmetric dual twistor $S^{\alpha\beta}$. That is,

$$A(S^{\alpha\beta}) = A_{\alpha\beta}S^{\alpha\beta},\tag{22}$$

so that the choices for $S^{\alpha\beta}$ will span the charge structure in the same way that the set of Killing vectors in flat spacetime spans the charge structure.

Penrose's construction of a quasi-local twistorial charge makes use of the spin-lowering property of solutions to the twistor equation [13]. If we take a spinor solution to the valence- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ symmetric twistor equation,

$$\nabla_{A'}^{(A} \gamma^{BC)} = 0, \tag{23}$$

then contracting with a spin-s zero rest-mass field yields a zero rest-mass field of spin s-1. If we contract a spin-2 graviton field ψ_{ABCD} , the result is a spin-1 Maxwell field,

$$\phi_{AB} := \psi_{ABCD} \gamma^{CD}, \tag{24}$$

for which there is a well-motivated charge construction: choose a relevant 2-surface, contract the area 2-form with the Maxwell field and integrate over the surface. The independent solutions to (23) span the discrepancy between the 1 complex charge in electromagnetism and the 10 charges we expect in analogy with the linearized gravitational theory.

Furthermore, this procedure is conformally-invariant and so we may construct such a charge in conformally-flat spacetimes if we take the Weyl spinor as a spin-2 zero rest-mass field with field equations given by the non-linear Bianchi identities.

A. Conventional use of twistors on \mathscr{I}^+

The work of Dray and Streubel applies this construction to \mathscr{I}^+ [8]. Therefore a prescription for a twistor space on \mathscr{I}^+ is needed. A key observation is that when one applies the

twistor equation to \mathscr{I}^+ only those components of the twistor equation which are intrinsic to a cut of \mathscr{I}^+ are integrable.

Let us consider the valence- $\begin{bmatrix} 1\\0 \end{bmatrix}$ twistor equation,

$$\nabla_{A'}^{(A}\omega^{B)} = 0, \tag{25}$$

and apply it to \mathscr{I}^+ taking components with respect to the adapted spin-frame of Section IA1. In terms of cGHP operators,

$$\bar{\eth}_c \omega^0 = 0, \quad \mathbf{p}_c' \omega^0 = 0,
\eth_c \omega^1 = \sigma \omega^0, \quad \mathbf{p}_c' \omega^1 = \eth_c \omega^0,$$
(26)

for $\omega^A = \omega^0 o^A + \omega^1 \iota^A$, where the components carry weights $\omega^0 : [1; -1, 0]$ and $\omega^1 : [0; 1, 0]$ if we choose ω^A to have zero weight.

The constraint equation for ω^1 on a cut does not commute with its evolution equation along the generators of \mathscr{I}^+ unless a stringent integrability constraint is met.

If we would like to make use of the spin-lowering property of solutions to the twistor equation then we need integrability only on a 2-surface of interest. Therefore, most authors have taken only those components of the twistor equation intrinsic to a cut of null infinity, avoiding the integrability condition [8, 13, 21].

It is here that one finds the well-known correspondence between twistors and the BMS algebra of symmetries.

B. Twistor components and the BMS algebra

According to the procedure of Dray and Streubel, one can apply Penrose's quasi-local charge definition to cuts of \mathscr{I}^+ by defining twistors on these cuts as described above. Then the valence- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ twistor equation lowers the spin of zero rest-mass fields on this cut by 1. The Dray-Streubel charge may claim the title of BMS charge because one finds that the components of this equation on a cut of \mathscr{I}^+ match those of an element of the BMS algebra.

If we define the components of a solution to (23) on a cut by,

$$\gamma^{AB} = Uo^A o^B + Vo^{(A}\iota^{B)}, \tag{27}$$

then the single component of (23) on a cut is,

$$\bar{\eth}_c U = 0. \tag{28}$$

Because U and V are propagated uniquely along the generators of \mathscr{I}^+ once their values are specified on a cut, the identification,

$$\xi \equiv \bar{U}, \qquad \qquad \eta \equiv V \tag{29}$$

is made between components of γ^{AB} and components of an element of the BMS algebra X^a which satisfy (20).

One can see that the component V is unconstrained by (23) and η is similarly unconstrained since it encodes the *supertranslation* degrees of freedom of the BMS algebra.

This is the identification made by Dray and Streubel and made explicit in a recent work [21]. Therefore, solutions to the twistor equation on a cut are often labeled *BMS twistors*.

This would appear to be an odd identification to make. Although it is true that both sets of equations span the same degrees of freedom, they describe two very different objects and the identification only holds at the level of components. An element of the BMS algebra is represented by a vector X^a while a solution to (23) is a valence-2 spinor. Another objection may be that the restriction to a single cut is not a well-motivated procedure and is equivalent to a self-evidently non-covariant selection of preferred components of (23).

Despite these points, the Dray-Streubel charge, which is constructed upon this identification, exactly aligns with the charge resulting from the Wald-Zoupas procedure due to entirely different considerations [10]. Therefore, it would be advantageous to provide a physical interpretation for this identification between the BMS algebra and twistors, leading to a physical connection between the twistorial analysis and the symplectic procedure of Wald and Zoupas.

III. FLAT-SPACE TWISTORS AND QUIESCENT TWISTORS

The physical interpretation we seek is to be found in an isomorphism between twistors in flat-space and twistors on 3-dimensional *sections* of \mathscr{I}^+ for which the Bondi news vanishes, i.e., sections for which no gravitational radiation is arriving at \mathscr{I}^+ .

The vanishing of the Bondi news for any finite period is unphysical for a generic dynamical spacetime. It is for this reason that we must take the infinite-dimensional enhancement from Poincaré to BMS seriously. Gravitational radiation induces an angle-dependent *shunt-ing* or *supertranslation* of the frame in which corresponding observables may be measured.

Therefore, it is useful to consider the stationary case when thinking about constructing a coherent frame, however we should be clear that this condition is entirely unphysical and is used only as a way to probe the radiative phase space.

Let us consider the twistor equation (25) on such a section. We have the four equations (26) intrinsic to \mathscr{I}^+ . Implicitly, there is also a condition due to the non-commutativity of the constraint and evolution equations for ω^1 . We can construct it explicitly by attempting to propagate the constraint along the generators. One finds,

$$\eth_c^2 \omega^0 + \bar{N}\omega^0 = 0. \tag{30}$$

To see how this condition compares with the explicit constraint $\bar{\eth}_c \omega^0 = 0$ we can act with $\bar{\eth}_c^2$ and commute derivatives to find,

$$\eth_c^2 \omega^0 - R\omega^0 = 0. \tag{31}$$

where R is the *co-curvature* (17) related to the Gaussian curvature of each cut. Therefore, for the integrability condition to be satisfied, one must have,

$$\mathcal{N} = N + \bar{R} = 0. \tag{32}$$

That is, the *Bondi news* as it is defined in this gauge must uniformly vanish.

This is a well known result and is often given as motivation to consider the twistor equation only on individual cuts [8, 13, 21]. However, if we do take the Bondi news to vanish, at least on some portion of null infinity, we are still left with the constraint (31). Then, the twistor equation (25) on a radiation-free section of null infinity satisfies,

$$\mathfrak{F}_c^2 \omega^0 - R \omega^0 = 0, \quad \bar{\mathfrak{F}}_c \omega^0 = 0, \quad \mathbf{p}_c' \omega^0 = 0,
\mathfrak{F}_c \omega^1 - \sigma \omega^0 = 0, \qquad \qquad \mathbf{p}_c' \omega^1 = \mathfrak{F}_c \omega^0.$$
(33)

By considering expansions of ω^0 and ω^1 in spin-weighted spherical harmonics [22, 23] one can deduce that the solution space of (33) is 4-complex-dimensional, matching the degrees of freedom of the twistor equation (25) in flat spacetime.

Formally, we may identify the twistor space on a radiation-free section of \mathscr{I}^+ with the twistor space on the *Minkowski space of origins* corresponding to this section [24]. The same cannot be said for a twistor space on a cut of null infinity since there is no such unique corresponding Minkowski space of origins. This is a natural consequence of the fact that

 \mathscr{I}^+ is a conformal structure shared by *all* asymptotically-flat spacetimes, and therefore there is no covariant way to reconstruct the spacetime points in the bulk of any individual asymptotically-flat spacetime from the conformal structure on \mathscr{I}^+ .

The condition of stationarity (at least for some finite retarded time) picks out a specific class of spacetimes and admits a shared Minkowski space of origins between them. This allows for an isomorphism between a twistor space on a stationary section and flat-space twistors.

We are particularly interested in valence- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ twistors due to their utility in charge construction as spin-lowering operators, so let us consider the solution space on a stationary section of \mathscr{I}^+ .

The same analysis applies to the valence- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ twistor equation (23) as for the valence- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ case. If the components with respect to the spin-frame on \mathscr{I}^+ are labeled,

$$\gamma^{AB} = U \ o^A o^B + V \ o^{(A} \iota^{B)} + W \ \iota^A \iota^B, \tag{34}$$

then the 5 components of (23) with respect to the spin-frame are,

$$\bar{\eth}_c U = 0, \qquad \beta'_c U = 0,$$

$$\beta'_c V = \bar{\eth}_c U, \qquad (35)$$

$$\bar{\eth}_c W = V \sigma, \quad \beta'_c W = \bar{\eth}_c V - 2\sigma U.$$

where the components of γ^{AB} have weights, U:[2;-2,0], V:[1;0,0], and W:[0;2,0] if we take γ^{AB} to have zero weight.

Again, there is an implicit constraint due to integrability along the generators of \mathscr{I}^+ , this time due to the non-commutativity of the constraint and evolution equations for the component W. Again we find that the Bondi news \mathcal{N} must vanish and there is the implicit constraint,

$$\eth_c^2 V - 2U\eth_c \sigma - 3\sigma\eth_c U - RV = 0. \tag{36}$$

This constraint bears an informal resemblance with the condition which picks out the four-dimensional ideal of translations from the supertranslations (21).

The constraint (36) may be regarded as a deformation of this condition which we will discuss in more detail in Section IIIB, but for now it suffices to point out that the solution space degrees of freedom of (35) and (36) match those of the valence- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ twistor equation

in flat spacetime. This can be confirmed formally via the Atiyah-Singer index theorem or informally using an expansion in spin-weighted spherical harmonics.

What we have is a true *isomorphism* between twistors in Minkowski spacetime and twistors on *radiation-free* sections of \mathscr{I}^+ and not just a selection of components of the twistor equation. This will be the key observation towards formalizing the connection between twistors and asymptotic symmetries because a certain class of twistors in Minkowski spacetime encode the Killing vectors of *Minkowski spacetime*. We will see that this connection may be translated to the isomorphic twistor spaces on \mathscr{I}^+ .

A. Twistors and symmetries

We typically regard a charge as adjoint to some group of symmetries. In the quasi-local charge construction of Penrose, no such symmetry group appears. The degrees of freedom are spanned by twistorial degrees of freedom.

In fact, there is a well-defined connection between twistors and symmetries in flat spacetime. The link is not direct but passes through a gauge freedom in the twistor which represents a given element of the Poincaré algebra [18]. We will see that this link can be translated to the isomorphic twistor spaces on stationary sections of \mathscr{I}^+ .

The natural twistorial charge structure in flat spacetime (22) is related to the algebra of Killing vectors of Minkowski space. If the spinor components of $S^{\alpha\beta}$ are denoted

$$S^{\alpha\beta} = \begin{bmatrix} \sigma^{AB} & \rho^{A}_{B'} \\ \tau_{A'}{}^{B} & \kappa_{A'B'} \end{bmatrix}, \tag{37}$$

then the component $\rho^{AB'}$ is a *complex* Killing vector which has a real part that satisfies Killing's equation. $\rho^{AB'}$ is related to the primary component of $S^{\alpha\beta}$ by

$$\nabla_{CC'}\gamma^{AB} = -2i\epsilon_C{}^{(A}\rho^{B)}{}_{C'}.$$
(38)

It is quite crucial that this twistor does not determine a Killing vector directly but through a complex Killing vector. Since more than one twistor may define a $\rho^{AB'}$ whose real part is a given Killing vector, the space of symmetric twistors $S^{\alpha\beta}$ is strictly larger than the space of Killing vectors. Explicitly, a given Killing vector may be represented by a twistor $S^{\alpha\beta}$ not uniquely but up to a transformation,

$$S^{\alpha\beta} \to S^{\alpha\beta} + 2iG^{(\alpha}{}_{\gamma}I^{\beta)\gamma},$$
 (39)

for an arbitrary Hermitian twistor $G^{\alpha}{}_{\beta} = \bar{G}_{\beta}{}^{\alpha}$ [18].

This relationship between twistors and symmetries may be carried over to the isomorphic twistor space on stationary sections of \mathscr{I}^+ meaning that the solution space of (35) and (36) spans the degrees of freedom of the Poincaré algebra up to a gauge transformation (39).

If we denote components of $\rho^{AB'}$ with respect to the spin-frame by

$$\rho^a = \eta n^a + \bar{\xi} m^a + \xi \bar{m}^a, \tag{40}$$

then at the level of components the link between twistors and symmetries takes the form of an identification,

$$\bar{\xi} = AU, \qquad \eta = AV, \tag{41}$$

where U and V are the components of γ^{AB} .

The ambiguity (39) is reflected in the condition that only the *real* part of ρ^a corresponds to the symmetry algebra degrees of freedom. At the level of components, Im(V) should not appear in any twistorial quantity which describes a property related to a symmetry.

Penrose's quasi-local charge prescription defines a set of charges adjoint to the space of valence- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ twistors on a 2-dimensional cut by exploiting the spin-lowering property of twistors. If such a charge purports to be adjoint to this underlying symmetry group also, we must ensure that it is invariant under the actions of the gauge transformations (39). We will see that the Dray-Streubel charge may be reinterpreted as a procedure for regulating this gauge freedom to define a charge adjoint to the symmetry group proper.

B. Quiescent twistors and flat spacetime

Let us say more about translating the indirect connection between twistors and symmetries described in Section III A to twistor spaces defined on stationary or *quiescent* sections of \mathscr{I}^+ .

To proceed, consider a spacetime for which the Bondi news vanishes initially for some finite retarded time before a period of radiation arrives at \mathscr{I}^+ . In this initial quiescent period \mathcal{S}_0 , we can define a quiescent twistor space isomorphic to flat-space twistors. In \mathcal{S}_0 , we have $\sigma = 0$ and the constraint (36) becomes

$$\eth_c^2 V - RV = 0. (42)$$

This is exactly the condition which picks out the ideal of translations from the supertranslations to distinguish the Poincaré algebra from the BMS algebra. In fact, under the identification (41), this condition becomes exactly the condition (21).

Now, if a burst of gravitational radiation were to pass, displacing σ and delimiting the start of a new radiation-free region, it would disrupt the integrability of the twistor equation, but if we impose (23) on the new radiation-free region we have the constraint (36) with $\sigma \neq 0$ in general.

If we consider a series of these quiescent regions S_0 , S_1 , S_2 , ... interspersed with bursts of radiation, each one will have an independent twistor space T_0 , T_1 , T_2 , ... which is each exactly isomorphic to the twistor space of flat spacetime. However, there is no canonical map between each of these T_n so that we do not have a shared twistor space which may be used to define a supertranslation-free angular momentum or something similar on either side of a burst of radiation. The T_n are relatively supertranslated.

The reason one should consider these spaces despite the unphysical nature of the absence of gravitational radiation is that through the isomorphism between \mathcal{T}_n and flat spacetime twistors, we can pull the connection between flat spacetime twistors and Killing vectors of Minkowski spacetime to null infinity so that each \mathcal{T}_n carries its own set of Killing vectors of Minkowski spacetime subject to the gauge discrepancy (39).

To summarize, each quiescent region S_n has a corresponding quiescent twistor space T_n to which there is a corresponding copy of the Poincaré algebra P_n through the connection described in Section III. However, because there is no canonical map between each of the T_n , there can be no canonical map between the Poincaré subalgebras P_n .

In general, the \mathcal{P}_n are relatively supertranslated Poincaré subalgebras of the BMS algebra. Each subalgebra shares only the *translations* with the others which can be seen by considering the pure supertranslations which meet the constraint (36). This situation has been described by Penrose and Rindler but here we arrive at it twistorially [18].

In each case the local integrability of the twistor equation in the absence of radiation gives rise to an isomorphism between the \mathcal{T}_n and flat-space twistors which in turn gives access to a global property, a Killing vector. Symmetry-adjoint charges are part of the ontology of a theory constructed on top of the global structure of Minkowski spacetime. The above interplay between the local and global gives us some insight into why we are able to define a charge in curved spacetime.

C. BMS twistors

The above analysis demonstrates the connection between a twistor space on a section of \mathscr{I}^+ for which there is no arriving radiation and the Poincaré algebra. What is the connection to the Dray-Streubel charge which is adjoint to the infinite-dimensional BMS algebra of symmetries?

We may recover the BMS algebra from this construction by restricting to a cut and observing the corresponding effect on an element of the Poincaré algebra under the map (38). This removes the integrability condition on the twistor equation which imposed an absence of radiation and picked out the translations. We are left with the unconstrained supertranslations.

Twistors on a cut of \mathscr{I}^+ have been called BMS twistors since the solution space degrees of freedom of the twistor equation on a cut match those degrees of freedom of the BMS algebra [21]. There is now a convincing formalization of the connection between twistors and the asymptotic symmetry algebra in the contraction of a twistor space on all of \mathscr{I}^+ to a cut and the effect of this contraction on the spinor component of the twistor which represents a complex Killing vector of Minkowski space.

Importantly, the same gauge freedom (39) holds for a *BMS twistor* which purports to stand in for an element of the BMS algebra. We will now see that the Dray-Streubel charge may be reinterpreted as exactly the necessary procedure to regulate this gauge freedom in a twistorial charge.

IV. THE DRAY-STREUBEL CHARGE PRESCRIPTION

We now demonstrate that the charge of Dray and Streubel [8] may be thought of as a more general procedure to define a charge which is adjoint to a space of twistors at null infinity but which regulates the gauge freedom in the choice of twistor which stands in for an element of the symmetry algebra. This regulation is done by the introduction of a vanishing term with a non-vanishing flux and the resulting charge may be identified as adjoint to the symmetry algebra proper since these gauge degrees of freedom are constrained.

The details of the procedure are as follows. One may define a natural gravitational charge on a 2-surface by first lowering the spin-2 Weyl spinor to a spin-1 Maxwell field ϕ_{AB} using

a symmetric valence- $\begin{bmatrix} 2\\0 \end{bmatrix}$ twistor with primary spinor component denoted γ^{AB} . Then,

$$\phi_{AB} := \Psi_{ABCD} \gamma^{CD} \tag{43}$$

is a Maxwell field since it satisfies the zero rest-mass field equations [13].

There is a natural charge definition for Maxwell fields given by contraction with the area 2-form of the surface and integrating. This is Penrose's quasi-local charge prescription. Since the degrees of freedom will be entirely contained in the choice of twistor, we say that the resulting charge is a charge adjoint to the space of twistors γ^{AB} . Explicitly, we have

$$A(\gamma^{AB}) = \oint_{\Sigma} \Psi_{ABCD} \gamma^{CD} o^{A} \iota^{B} dS, \tag{44}$$

where Σ is the 2-surface on which the charge is evaluated and $o^A \iota^B dS$ is the natural area 2-form on Σ .

On \mathscr{I}^+ with respect to the natural spin-frame, (44) takes the form,

$$A(\gamma^{AB}) = \oint_{\Sigma} U\psi_1 + V\psi_2 + W\psi_3 \ dS$$
$$= \oint_{\Sigma} U\psi_1 + V\left(\psi_2 - A^{-1}\sigma\mathcal{N}\right) \ dS \tag{45}$$

where U, V, and W are components of γ^{AB} with respect to the spin-frame and we have integrated the third term by parts to arrive at the second equality.

When the space of twistors is defined on a single cut, this is Penrose's elegant definition of a twistorial charge. The space of symmetric valence- $\begin{bmatrix} 2\\0 \end{bmatrix}$ twistors spans its degrees of freedom. However, (45) is not invariant with respect to map (39) and so it cannot yet be thought of as a charge adjoint to the corresponding symmetry algebra.

To define a symmetry-adjoint charge, we should take the component of Penrose's twistorial charge which is invariant with respect to (39), the gauge freedom in the twistor which stands in for an element of the symmetry algebra. At the level of components this corresponds to the condition that Im(V) does not appear in the charge expression. One can see that taking the real part of the charge (45) does not immediately guarantee that Im(V) will not appear since the coefficient of V in (45) is not necessarily real.

The key insight is the existence of the integral

$$\oint_{\Sigma} V\left[\bar{\eth}_c \sigma - \bar{R}\sigma\right] dS = 0. \tag{46}$$

The vanishing of (46) may be shown by integrating by parts and applying the constraint on V.

We may add any multiple q of (46) to the charge expression (45) so that,

$$A(\gamma^{AB}) = \oint_{\Sigma} U\psi_1 + V \left[\psi_2 - A^{-1}\sigma \mathcal{N} - q \left(\bar{\eth}_c \sigma - \bar{R}\sigma \right) \right] dS.$$
 (47)

then the real part of $A(\gamma^{AB})$ contains no reference to Im(V) so long as

$$\operatorname{Im}\left(\psi_2 - A^{-1}\sigma\mathcal{N} - q\left(\bar{\eth}_c\sigma - \bar{R}\sigma\right)\right) = 0. \tag{48}$$

From the propagation equations for Ricci components (16), this condition is satisfied if q = 1. Then $Re(A(\gamma^{AB}))$ with the inclusion of the above flux term is invariant with respect to (39) and is therefore adjoint to the symmetry algebra proper.

Dray and Streubel identify a second vanishing term of non-vanishing flux,

$$\oint_{\Sigma} \left[U \left(2\sigma \eth_c \bar{\sigma} + \eth_c (\sigma \bar{\sigma}) \right) + V \left(\eth_c^2 \bar{\sigma} - R \bar{\sigma} \right) \right] dS = 0, \tag{49}$$

which has the ability to shift flux between the *supermomentum* and angular momentum components of the charge, naively identified. One may instead add a linear combination of (46) and (49) to the charge expression constrained by p + q = 1. The identification by Dray and Streubel of this ability to shift flux is what resolved the nefarious *factor of two* discrepancy suffered by earlier charge prescriptions.

Including this second flux term, one is left with the Dray-Streubel charge when the twistor space is taken to be twistors defined on a cut of null infinity. However, for Dray and Streubel, the motivation for including the flux term (46) in their charge expression was that it gave a real supermomentum flux for real supertranslations. Their final charge was constructed by splitting the angular momentum and supermomentum components of the twistorial charge and specifying

$$Q_{\rm DS} = Q_V + Q_U + \bar{Q}_U, \tag{50}$$

where subscripts denote the term in the charge with the corresponding coefficient. This form was chosen to align with previous attempts at charge constructions.

The above procedure shows that in fact the flux term (46) plays the crucial role in the Dray-Streubel construction since it is what allows for the charge to be adjoint to the symmetry algebra. Furthermore, we are led to consider the real part of the twistorial charge in service of this same condition. The formal connection between twistors and the asymptotic symmetry algebra resolves the essential ad hoc elements of the Dray-Streubel procedure.

When the twistor space is the space of twistors on a cut of \mathscr{I}^+ then $\operatorname{Re}(A(\gamma^{AB}))$ is the Dray-Streubel BMS charge, but we can now consider other twistor spaces on \mathscr{I}^+ , in particular, the quiescent twistors defined on radiation-free sections of \mathscr{I}^+ . It is clear that the corresponding Dray-Streubel charge will be adjoint to the Poincaré algebra by the discussion of Section III A.

With a formalization of the connection between twistors and the asymptotic symmetry algebra, the task of defining a BMS charge follows naturally from the known relationship between twistors and symmetries in flat spacetime. It is surprising that the charge which one arrives at by this simple procedure matches the Wald-Zoupas charge derived from the explicit symplectic structure of the radiative phase space [10]. Perhaps the twistor formalism can function more generally as a tool for working with the symplectic structure at \mathscr{I}^+ .

V. DISCUSSION

The construction of *quiescent* twistors requires a finite region for which there is no gravitational radiation arriving so that the frame in which the charge is measured may be determined. One might imagine that for $u \to -\infty$ and $u \to \infty$, such regions may exist, perhaps in a limiting sense, for a certain class of radiating spacetimes. Likely, this is still too stringent a constraint to place on physically reasonable spacetimes.

The utility of studying such unphysical spacetimes is that they turn the problem of strong supertranslation ambiguity where radiation imparts a continuous angle-dependent shunting of the frame into a number of discrete supertranslations induced by bursts of radiation between a number of quiescent regimes for which no radiation arrives. From this point of view, it is clearer to see the the relationship between arriving radiation, shear structure, and the appearance of the global structure of Minkowski spacetime at \mathscr{I}^+ .

A charge structure is necessarily a global structure because it relies on a covariant phase space from which to select values. In flat spacetime this structure is represented by the existence of a set of Killing vectors to which one may define a set of adjoint charges. Twistorial structure is fundamentally connected to the global structure of Minkowski space and so on \mathscr{I}^+ , local twistor integrability carries global properties of Minkowski space like its

set of Killing vectors to the radiative phase space. Furthermore, the breaking of integrability along \mathscr{I}^+ exactly corresponds to the widening of Poincaré to BMS. The Dray-Streubel charge exploits this interplay between the local and global.

We would like to suggest that twistors do more than just provide a useful trick for defining a charge in asymptotically-flat spacetimes. Since they have the ability to reproduce the BMS charge expression of the symplectic formalism and their integrability along the generators of \mathscr{I}^+ directly mirrors shear structure, they may provide a framework for analyzing the phase space of radiative observables on par with these standard descriptions of radiative data at \mathscr{I}^+ . Previously, the use of twistors here was limited by the ad hoc connections between twistors and the asymptotic symmetry group. With a proper formalization of this connection, the value of the twistorial description of the radiative phase space is clear.

Future work will explore the ability for this twistorial description to reproduce constructions in the symplectic formalism in the vein of the correspondence between the Wald-Zoupas and Dray-Streubel charges. Perhaps it is possible to call upon the considerable suite of tools from twistor theory to answer questions that arise in the symplectic description of the radiative phase space.

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