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Double-Copying Self-Dual Yang–Mills Theory to Self-Dual Gravity on Twistor Space

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Abstract

We construct a simple Lorentz-invariant action for maximally supersymmetric self-dual Yang–Mills theory that manifests colour–kinematics duality. We also show that this action double copies to a known action for maximally supersymmetric self-dual gravity. Both actions live on twistor space and illustrate nicely the homotopy algebraic perspective on the double copy presented in arXiv:2307.02563. This example is particularly interesting as the involved Hopf algebra controlling the momentum dependence is non-commutative and suggests a generalisation to gauged maximally supersymmetric self-dual gravity.

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1. Introduction and results

It is a remarkable discovery of recent decades that a large class of natural gauge theories feature a hidden symmetry known as color–kinematics (CK) duality [1–3], which implies a surprising relation to gravitational theories, known as the double copy [1–3] (for reviews, see [4–8]). When confronted with such a fundamental new feature of quantum field theory, it is natural to examine the simplest non-trivial example of the phenomenon that nevertheless exhibits all interesting features. A strong contender for this title is (supersymmetric) self-dual Yang–Mills theory (SDYM; see e.g. [9] for more details) — which is even simpler than full $\mathcal{N} = 4$ supersymmetric Yang–Mills theory — and its double copy, which is (supersymmetric) self-dual gravity (SDG; see e.g. [10] for a review).

We present a remarkably simple twistor action for maximally supersymmetric SDYM theory that is reminiscent of the Leznov–Mukhtarov–Parkes action [11–13] on space-time. Yet the action is Lorentz-invariant, manifests a kinematic Lie algebra and CK duality, and is based on a straightforward gauge-fixing from twistor space. It directly double-copies to another simple and known Lorentz-invariant twistor action for self-dual gravity [14].

Our discussion of CK duality and the double copy makes use of the algebraic framework developed in [15, 16] and based on ideas by [17] (see also [18, 19] as well as [20] for related work). Nevertheless, we have minimised the amount of mathematical prerequisites and omitted a review of the formalism; for a detailed discussion of the general constructions, the reader should consult [15]. Below, we shall only expect some familiarity with the facts that a theory of fields taking values in a Lie algebra $\mathfrak{g}$ and with exclusively cubic interaction terms is encoded in a metric differential graded Lie algebra[1] that factorises into a tensor

[1] 'Graded' here always means $\mathbb{Z}$-graded.

product of $\mathfrak{g}$ and a metric differential graded commutative algebra. If the field theory comes with a kinematic Lie algebra, the latter can be promoted to a BV$^{■}$-algebra by identifying a particular second-order differential operator b that defines the kinematic Lie bracket as a grade-shifted derived bracket.

We note that SDYM theory has been studied extensively in the context of CK duality and the double copy. In particular, the tree-level currents of SDYM theory in light-cone gauge were shown to exhibit CK duality in [21] and to double copy to those of SDG. In the same paper, the kinematic Lie algebra of SDYM theory in light-cone gauge was identified with the area-preserving diffeomorphisms on $\mathbb{C}^2$. More recently, we showed that holomorphic Chern–Simons theory on twistor space for (supersymmetric) SDYM theory manifests CK duality and the action implies CK duality for loop amplitudes in the maximally supersymmetric case [22]. The corresponding full (un-gauge-fixed) kinematic Lie algebra is given by the Schouten–Nijenhuis-type Lie algebra of bosonic holomorphic multivector fields on twistor space, which reduces to the area-preserving diffeomorphisms on $\mathbb{C}^2$, identified in [21], upon reducing to space-time and imposing light-cone gauge. Very recently [23], a kinematic homotopy Lie algebra up to trilinear maps (encoding violations of the Jacobi identity up to homotopy) was derived directly from a gauge-invariant off-shell formulation of SDYM theory on space-time and put to a test in a double-copy construction of SDG in light-cone gauge. Again, by going to light-cone gauge, the kinematic Lie algebra of area-preserving diffeomorphisms on $\mathbb{C}^2$ was also recovered. For further related work, see also [24–34].

2. Twistors, self-dual Yang–Mills theory, and self-dual gravity

2.1. Self-dual Yang–Mills theory

Supersymmetric self-dual Yang–Mills theory. Let $\mathfrak{g}$ be a metric Lie algebra with basis e_a, structure constants $f_{ab}{}^c$, and metric g_{ab}. We set $f_{abc} := g_{cd} f_{ab}{}^d$. The classical solutions to SDYM theory on Euclidean $\mathbb{R}^4$ are $\mathfrak{g}$-valued gauge potentials $A_\mu = A_\mu^a \mathsf{e}_a$ with self-dual field strength

$$F_{\mu\nu}^a \;=\; \tfrac{1}{2}\varepsilon_{\mu\nu}{}^{\kappa\lambda} F_{\kappa\lambda}^a \quad \text{with} \quad F_{\mu\nu}^a \;:=\; \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}{}^a A_\mu^b A_\nu^c\ , \tag{2.1}$$

where we have coordinatised $\mathbb{R}^4$ by x^μ with $\mu, \nu, \ldots = 1, \ldots, 4$, $\partial_\mu := \frac{\partial}{\partial x^\mu}$, and $\varepsilon_{\mu\nu\kappa\lambda}$ is the Levi-Civita symbol. An action for these configurations was given in [35], and there are supersymmetric extensions of both the equations of motion and the action from $\mathcal{N} = 1$ to $\mathcal{N} = 4$ [36].

For the twistorial description of these solutions, it is convenient to switch to spinor notation. That is, we use the well known fact that the defining representation $\mathbf{4}$ of $\mathsf{Spin}(4) \cong \mathsf{SU}(2)_{\mathrm{L}} \times \mathsf{SU}(2)_{\mathrm{R}}$ decomposes as $\mathbf{4} \cong \mathbf{2}_{\mathrm{L}} \otimes \mathbf{2}_{\mathrm{R}}$, and we may set $x^{\alpha\dot\alpha} := \sigma_\mu^{\alpha\dot\alpha} x^\mu$ where $\sigma_\mu^{\alpha\dot\alpha}$ are the sigma matrices with $\alpha, \beta, \ldots = 1, 2$ the chiral spinor indices and $\dot\alpha, \dot\beta, \ldots = \dot 1, \dot 2$ the anti-chiral ones. Then the SDYM equation (2.1) translates to

$$\varepsilon^{\alpha\beta}(\partial_{\alpha\dot\alpha} A^a_{\beta\dot\beta} - \partial_{\beta\dot\beta} A^a_{\alpha\dot\alpha} + f_{bc}{}^a A^b_{\alpha\dot\alpha} A^c_{\beta\dot\beta}) \;=\; 0\;, \tag{2.2}$$

where $\partial_{\alpha\dot\alpha} := \frac{\partial}{\partial x^{\alpha\dot\alpha}}$ with $\varepsilon_{\alpha\beta} = \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon^{\alpha\beta} = -\varepsilon^{\dot\alpha\dot\beta}$, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, and $\varepsilon_{12} = +1$, which implies $\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha^\gamma$.

CK duality of SDYM theory is most easily identified in the Leznov–Mukhtarov–Parkes form of the action [11–13], which is a result of adopting Leznov gauge, in which

$$A_{\alpha\dot 1} \;=\; \tfrac{1}{4}\partial_{\alpha\dot 2}\phi \quad\text{and}\quad A_{\alpha\dot 2} \;=\; 0 \tag{2.3}$$

for ϕ some $\mathfrak{g}$-valued function on $\mathbb{R}^4$ also called the prepotential. In this gauge, (2.2) reduces to

$$\Box\,\phi^a + \tfrac{1}{2}\varepsilon^{\alpha\beta} f_{bc}{}^a (\partial_{\alpha\dot 2}\phi^b)(\partial_{\beta\dot 2}\phi^c) \;=\; 0 \tag{2.4}$$

with $\partial_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha} = \frac{1}{2}\partial_\mu\partial^\mu = \frac{1}{2}\Box$. This equation follows variationally from the action

$$S^{\mathrm{LMP}} \;:=\; \int \mathrm{d}^4 x\,\Big\{\tfrac{1}{2} g_{ab}\phi^a\,\Box\,\phi^b + \tfrac{1}{3!} f_{abc}\varepsilon^{\alpha\beta}\phi^c(\partial_{\alpha\dot 2}\phi^a)(\partial_{\beta\dot 2}\phi^b)\Big\}\;. \tag{2.5}$$

However, the equation of motion and the action are not Lorentz-covariant and not Lorentz-invariant, respectively.

We can generalise this action to $\mathcal{N}$-extended supersymmetric SDYM theory by extending $\mathbb{R}^4$ to $\mathbb{R}^{4|2\mathcal{N}}$ by supplementing fermionic coordinates $\eta_i^{\dot\alpha}$ with $i, j, \ldots = 1, \ldots, \mathcal{N}$. The extended action reads as [36]

$$S^{\mathrm{LMP}} \;:=\; \int \mathrm{d}^4 x\,\mathrm{d}\eta_1^{\dot 2}\cdots\mathrm{d}\eta_{\mathcal{N}}^{\dot 2}\,\Big\{\tfrac{1}{2} g_{ab}\phi^a\,\Box\,\phi^b + \tfrac{1}{3!} f_{abc}\varepsilon^{\alpha\beta}\phi^c(\partial_{\alpha\dot 2}\phi^a)(\partial_{\beta\dot 2}\phi^b)\Big\}\;, \tag{2.6}$$

in which ϕ is a superfield on $\mathbb{R}^{4|2\mathcal{N}}$ independent of $\eta_i^{\dot 1}$.

Both actions exhibit CK duality of SDYM theory, as they feature a kinematic Lie algebra $\mathfrak{K}$ with Lie bracket

$$[\phi_1, \phi_2]_{\mathfrak{K}} \;:=\; \varepsilon^{\alpha\beta}(\partial_{\alpha\dot 2}\phi_1)(\partial_{\beta\dot 2}\phi_2)\;. \tag{2.7}$$

Twistor basics. As is well-known, $\mathcal{N}$-extended supersymmetric SDYM theory has a twistorial reformulation in terms of holomorphic Chern–Simons theory [37–40]; see e.g. [41] for a review. In the following, we summarise the underlying geometry.

The twistor space Z is the total space of the holomorphic vector bundle $\mathcal{O}(1)\otimes\mathbb{C}^{2|\mathcal{N}}\to\mathbb{C}P^1$. Geometrically, it parametrises all orthogonal almost-complex structures on $\mathbb{R}^4$. We write $(z^A) = (z^\alpha,\eta_i)$ for the fibre coordinates and $\pi_{\dot\alpha}$ for the (homogeneous) base coordinates, where each of the indices $A,B,\ldots$ combines an α index and an i index. Let us henceforth assume that $\mathcal{N}$ is even; then Z admits an anti-holomorphic involution $\tau:(z^A,\pi_{\dot\alpha})\mapsto(\hat z^A,\hat\pi_{\dot\alpha})$ with

$$\hat z^A \;:=\; \overline{z^B}C_B{}^A \quad\text{and}\quad \hat\pi_{\dot\alpha} \;:=\; C_{\dot\alpha}{}^{\dot\beta}\overline{\pi_{\dot\beta}}\ , \tag{2.8a}$$

where

$$(C_A{}^B) \;:=\; \mathrm{diag}(C_\alpha{}^\beta, C_i{}^j)\ ,\quad (C_\alpha{}^\beta) \;:=\; \varepsilon\ ,\quad (C_i{}^j) \;:=\; \mathbb{1}_{\frac{\mathcal{N}}{2}}\otimes\varepsilon\ ,\quad\text{and}\quad (C_{\dot\alpha}{}^{\dot\beta}) \;=\; -\varepsilon\ . \tag{2.8b}$$

In the following, it will be useful to introduce the notation

$$|\pi|^2 \;:=\; \varepsilon^{\dot\alpha\dot\beta}\pi_{\dot\alpha}\hat\pi_{\dot\beta} \;=\; \pi_{\dot\alpha}\hat\pi^{\dot\alpha}\ . \tag{2.9}$$

The anti-holomorphic exterior derivative $\bar\partial$ on Z can now be written as

$$\bar\partial \;=\; \mathrm{d}\hat z^A\frac{\partial}{\partial\hat z^A} + \hat e^\pi\hat E_\pi\ , \tag{2.10a}$$

where

$$\hat E_\pi \;:=\; |\pi|^2\pi_{\dot\alpha}\frac{\partial}{\partial\hat\pi_{\dot\alpha}} \quad\text{and}\quad \hat e^\pi \;:=\; \frac{\hat\pi^{\dot\alpha}\mathrm{d}\hat\pi_{\dot\alpha}}{|\pi|^4}\ . \tag{2.10b}$$

There is a diffeomorphism between Z and $\mathbb{R}^{4|2\mathcal{N}}\times\mathbb{C}P^1$. If we coordinatise the latter by $(x^{A\dot\alpha},\lambda_{\dot\alpha}) = (x^{\alpha\dot\alpha},\eta_i^{\dot\alpha},\lambda_{\dot\alpha})$ with $\lambda_{\dot\alpha}$ homogeneous coordinates on $\mathbb{C}P^1$ and

$$\hat x^{A\dot\alpha} \;:=\; \tau(x^{A\dot\alpha}) \;=\; x^{A\dot\alpha} \quad\Leftrightarrow\quad x^{A\dot\alpha} \;=\; \overline{x^{B\dot\beta}}C_B{}^AC_{\dot\beta}{}^{\dot\alpha}\ , \tag{2.11}$$

then the diffeomorphism $Z\cong\mathbb{R}^{4|2\mathcal{N}}\times\mathbb{C}P^1$ is given by

$$(z^A,\pi_{\dot\alpha}) \;=\; (x^{A\dot\alpha}\lambda_{\dot\alpha},\lambda_{\dot\alpha}) \quad\text{and}\quad (x^{A\dot\alpha},\lambda_{\dot\alpha}) \;=\; \left(\frac{z^A\hat\pi^{\dot\alpha}-\hat z^A\pi^{\dot\alpha}}{|\pi|^2},\pi_{\dot\alpha}\right). \tag{2.12}$$

Under this diffeomorphism, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial\hat z^A},\hat E_\pi\right) \;&=\; \left(-\frac{1}{|\lambda|^2}\hat E_A,\hat E_\lambda + x^{A\dot\alpha}\lambda_{\dot\alpha}\hat E_A\right),\\ (\mathrm{d}\hat z^A,\hat e^\pi) \;&=\; \left(-|\lambda|^2\hat e^A + |\lambda|^2x^{A\dot\alpha}\lambda_{\dot\alpha}\hat e^\lambda,\hat e^\lambda\right)\end{aligned} \tag{2.13a}$$

with

$$\begin{aligned}(\hat E_A,\hat E_\lambda) \;&:=\; \left(\lambda^{\dot\alpha}\frac{\partial}{\partial x^{A\dot\alpha}},|\lambda|^2\lambda_{\dot\alpha}\frac{\partial}{\partial\hat\lambda_{\dot\alpha}}\right),\\ (\hat e^A,\hat e^\lambda) \;&:=\; \left(-\frac{\hat\lambda_{\dot\alpha}\mathrm{d}x^{A\dot\alpha}}{|\lambda|^2},\frac{\hat\lambda^{\dot\alpha}\mathrm{d}\hat\lambda_{\dot\alpha}}{|\lambda|^4}\right).\end{aligned} \tag{2.13b}$$

We also set

$$E_A \;:=\; \frac{\partial}{\partial z^A} \;=\; \frac{1}{|\lambda|^2}\hat\lambda^{\dot\alpha}\frac{\partial}{\partial x^{A\dot\alpha}}\ . \tag{2.13c}$$

Holomorphic Chern–Simons theory. Let $E \to Z$ be a complex vector bundle over Z with vanishing first Chern class. Furthermore, let $\bar{\nabla} = \bar{\partial} + A$ be a $(0,1)$-connection on E where A is a $\mathfrak{g}$-valued $(0,1)$-form on Z. We also assume that there is a gauge in which A has no anti-holomorphic fermionic directions $\mathrm{d}\hat{\eta}_i$ and depends holomorphically on the fermionic coordinates η_i; this is sometimes called Witten gauge [38]. Then, the $\mathcal{N}$-extended supersymmetric SDYM equation on $\mathbb{R}^4$ is equivalent to the holomorphic Chern–Simons equation

$$\bar{\partial} A^a + f_{bc}{}^a A^b \wedge A^c \;=\; 0 \tag{2.14}$$

on Z [37–40]; see e.g. [41] for a review. In the case of maximal ($\mathcal{N} = 4$) supersymmetry, Z is a Calabi–Yau supermanifold, and (2.14) follows from varying a holomorphic Chern–Simons action on Z [38]. See also [42] for a similar Chern–Simons-type action in harmonic superspace in this case.

This holomorphic Chern–Simons formulation manifests a gauge-invariant, off-shell kinematic Lie algebra and CK duality directly at the level of the action and further extends CK duality to the loop level as explained in [22]. For $\mathcal{N} < 4$, the holomorphic Chern–Simons form of the equation of motion still implies CK duality for the tree-level currents.

Twistorial prepotential action. Given that we have both the Siegel action [36] as well as the prepotential action (2.5) for SDYM theory on space-time, it is natural to ask if, besides the holomorphic Chern–Simons action, there is also a twistorial prepotential action.

To derive such an action, we write the holomorphic Chern–Simons equation (2.14) for $A = \mathrm{d}\hat{z}^\alpha A_\alpha + \hat{e}^\pi A_\pi$ as

$$\begin{aligned} \frac{\partial}{\partial \hat{z}^\alpha} A^a_\beta - \frac{\partial}{\partial \hat{z}^\beta} A^a_\alpha + f_{bc}{}^a A^b_\alpha A^c_\beta \;&=\; 0 \;, \\ \hat{E}_\pi A^a_\alpha - \frac{\partial}{\partial \hat{z}^\alpha} A^a_\pi + f_{bc}{}^a A^b_\pi A^c_\alpha \;&=\; 0 \;. \end{aligned} \tag{2.15}$$

Since we have assumed that E has vanishing first Chern class, we may work in the axial gauge

$$A^a_\pi \;=\; 0 \;. \tag{2.16a}$$

In this gauge, the gauge potential has prepotentials ϕ^a and ψ^a, which are $\mathfrak{g}$-valued functions of weight 2 and 0 on Z, respectively. In particular,

$$A^a_\alpha \;=\; \frac{1}{4|\lambda|^2}(E_\alpha \phi^a + \hat{E}_\alpha \psi^a) \tag{2.16b}$$

with E_α and $\hat{E}_\alpha$ defined in (2.13). This can be seen by regarding this equation as a vector-valued differential equation. The determinant of the differential operator $(E_\alpha \; \hat{E}_\alpha)$ is $\hat{E}_\alpha E^\alpha = -\frac{1}{4}\partial_\mu \partial^\mu = -\frac{1}{4}\square$. After restricting to functions that do not blow up at infinity, the kernel of $\Box$ consists of the constant functions, which are irrelevant in A^a_α. Hence, the differential equation always has a solution.

Let us further restrict to solutions to the holomorphic Chern–Simons equations (2.15). These are holomorphic in $\pi_{\dot\alpha}$, and we can therefore impose Lorenz gauge along the fibres,

$$E^\alpha A^a_\alpha \;=\; 0 \; . \tag{2.16c}$$

This further restricts the prepotential to $\psi^a \in \ker(\Box)$, and hence we can put $\psi^a = 0$. Moreover, the fact that A^a_α is holomorphic in $\pi_{\dot\alpha}$ allows us to demand that ϕ^a is holomorphic in $\pi_{\dot\alpha}$.

Altogether, we see that the solutions to the holomorphic Chern–Simons equations are captured by a $\mathfrak{g}$-valued function of weight 2 on Z that depends holomorphically on the fermionic coordinates η_i as well as $\pi_{\dot\alpha}$. Substituting[1]

$$A^a_\alpha \;=\; \frac{1}{4|\lambda|^2} E_\alpha \phi^a \tag{2.16d}$$

into (2.15), we obtain the remaining equation of motion

$$\Box\,\phi^a + \tfrac{1}{2}\varepsilon^{\alpha\beta} f_{bc}{}^a (E_\alpha\phi^b)(E_\beta\phi^c) \;=\; 0 \; . \tag{2.17}$$

For maximal ($\mathcal{N} = 4$) supersymmetry, this equation follows from the variation of the action

$$S^{\rm SDYM} \;:=\; \int \mathrm{vol}_{\rm SDYM} \left\{ \tfrac{1}{2} g_{ab}\phi^a\,\Box\,\phi^b + \tfrac{1}{3!} f_{abc}\varepsilon^{\alpha\beta}\phi^c (E_\alpha\phi^a)(E_\beta\phi^b) \right\} , \tag{2.18a}$$

where

$$\mathrm{vol}_{\rm SDYM} \;:=\; \mathrm{d}^4 x\, \frac{\lambda^{\dot\alpha}\mathrm{d}\lambda_{\dot\alpha}\,\hat\lambda^{\dot\alpha}\mathrm{d}\hat\lambda_{\dot\alpha}}{|\lambda|^4}\, \mathrm{d}\eta_1\cdots\mathrm{d}\eta_4 \; . \tag{2.18b}$$

This twistor action resembles the space-time action (2.6), but it is manifestly Lorentz-invariant. As mentioned in the introduction, this twistor action appears to be new; we have not found a description of a similar action in the literature, not even for harmonic superspace. It could have been found from the single copy of the corresponding self-dual supergravity action, which we describe in Section 2.2.

Relation to space-time. The superfield expansion of ϕ^a reads as

$$\phi^a \;=\; A^a + \eta_i\,\chi^{ia} + \tfrac{1}{2}\eta_i\eta_j\,W^{ija} + \tfrac{1}{3!}\varepsilon^{ijkl}\eta_i\eta_j\eta_k\,\tilde\chi^a_l + \eta_1\eta_2\eta_3\eta_4\,\tilde A^a \; , \tag{2.19a}$$

and after performing the Penrose–Ward transform for the gauge potential $A^a = \frac{1}{4|\lambda|^2}\mathrm{d}\hat z^\alpha E_\alpha\phi^a$, we recover the expected degrees of freedom on $\mathbb{R}^4$ as displayed in Table 2.1.

[1] Note that this expression is very reminiscent of the Woodhouse representative, cf. [43].

Field	A^a	χ^{ia}	W^{ija}	$\tilde{\chi}^a_i$	$\tilde{A}^a$
Helicity	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
Multiplicity	1	4	6	4	1

Table 2.1: Space-time SDYM fields and their helicities and multiplicities.

To relate the twistor action (2.18) to space-time, one can Kaluza–Klein-expand the scalar field in terms of spherical harmonics on $\mathbb{C}P^1$ and then integrate over the sphere. As is often the case for CK-dual actions, there are infinitely many auxiliary fields in this expansion that enforce the equations of motion. Holomorphy in $\pi_{\dot{\alpha}}$ amounts to the equation $(\hat{E}_\lambda + x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}\hat{E}_\alpha)\phi^a = 0$, which relates different terms in the Kaluza–Klein expansion. The latter is of the form

$$\phi^a \;=\; \lambda^{\dot{\alpha}}\lambda^{\dot{\beta}}\phi^a_{\dot{\alpha}\dot{\beta}} + \lambda^{\dot{\alpha}}\lambda^{\dot{\beta}}\frac{\lambda^{\dot{\gamma}}\hat{\lambda}^{\dot{\delta}}}{|\lambda|^2}\phi^a_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + \cdots \;, \tag{2.20}$$

and setting $\phi^a_{\dot{1}\dot{1}} = \phi^a_{\text{space-time}}$, $\phi^a_{\dot{1}\dot{2}} = \phi^a_{\dot{2}\dot{2}} = 0$, e.g., we recover equation (2.4) from (2.17).

2.2. Self-dual gravity

There is an analogous picture for self-dual gravity, which we describe in the following.

Supersymmetric self-dual gravity. Let (M, g) be a four-dimensional oriented Riemannian manifold with metric g. The self-dual gravity equation is an equation on the curvature for the Levi-Civita connection. In particular, for vanishing cosmological constant,[1] the SDG equation on a local patch $U \cong \mathbb{R}^4$ of M reads as

$$R_{\mu\nu\kappa}{}^\lambda \;=\; \frac{\sqrt{\det(g)}}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma\kappa}{}^\lambda \;, \tag{2.21a}$$

where

$$R_{\mu\nu\kappa}{}^\lambda \;:=\; \partial_\mu\Gamma_{\nu\kappa}{}^\lambda - \partial_\nu\Gamma_{\mu\kappa}{}^\lambda + \Gamma_{\mu\kappa}{}^\sigma\Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\kappa}{}^\sigma\Gamma_{\mu\sigma}{}^\lambda \tag{2.21b}$$

is the Riemann curvature tensor, and

$$\Gamma_{\mu\nu}{}^\kappa \;:=\; \tfrac{1}{2}g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\kappa g_{\mu\nu}) \tag{2.21c}$$

are the Christoffel symbols. Suppose now that M also admits a spin structure. Then we can pick a vierbein $e^{\alpha\dot{\alpha}} = \sigma_a^{\alpha\dot{\beta}}e^a$ on U such that

$$g \;=\; \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}e^{\alpha\dot{\alpha}} \otimes e^{\beta\dot{\beta}} \;. \tag{2.22}$$

[1] In the case of non-zero cosmological constant, one may modify the self-duality condition [34].

The Riemann curvature tensor decomposes as

$$
\begin{aligned}
R_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma}{}^{\delta\dot\delta} \ &= \ R_{\alpha\dot\alpha\beta\dot\beta\gamma}{}^{\delta}\delta_{\dot\gamma}{}^{\dot\delta} + R_{\alpha\dot\alpha\beta\dot\beta\dot\gamma}{}^{\dot\delta}\delta_{\gamma}{}^{\delta}\ , \\
R_{\alpha\dot\alpha\beta\dot\beta\gamma}{}^{\delta} \ = \ \varepsilon_{\alpha\beta}R_{\dot\alpha\dot\beta\gamma}{}^{\delta} + \varepsilon_{\dot\alpha\dot\beta}R_{\alpha\beta\gamma}{}^{\delta}\ , &\quad R_{\alpha\dot\alpha\beta\dot\beta\dot\gamma}{}^{\dot\delta} \ = \ \varepsilon_{\alpha\beta}R_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} + \varepsilon_{\dot\alpha\dot\beta}R_{\alpha\beta\dot\gamma}{}^{\dot\delta} \\
R_{\alpha\beta\gamma}{}^{\delta} \ = \ C_{\alpha\beta\gamma}{}^{\delta} + \Lambda\varepsilon_{\gamma(\alpha}\delta_{\beta)}{}^{\delta}\ , &\quad R_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} \ = \ C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} + \Lambda\varepsilon_{\dot\gamma(\dot\alpha}\delta_{\dot\beta)}{}^{\dot\delta}
\end{aligned}
\tag{2.23a}
$$

with

$$
\begin{aligned}
&R_{\alpha\beta\dot\gamma\dot\delta} \ = \ R_{\dot\gamma\dot\delta\alpha\beta}\ , \quad R_{\alpha\beta\dot\gamma\dot\delta} \ = \ R_{(\alpha\beta)(\dot\gamma\dot\delta)}\ , \\
&C_{\alpha\beta\gamma}{}^{\delta} \ = \ C_{(\alpha\beta\gamma)}{}^{\delta}\ , \quad C_{\alpha\beta\gamma}{}^{\gamma} \ = \ 0\ , \quad C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} \ = \ C_{(\dot\alpha\dot\beta\dot\gamma)}{}^{\dot\delta}\ , \quad C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\gamma} \ = \ 0\ .
\end{aligned}
\tag{2.23b}
$$

The components $C_{\alpha\beta\gamma}{}^{\delta}$ and $C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta}$ constitute the self-dual and anti-self-dual parts of the Weyl tensor, and Λ is the cosmological constant. The Ricci tensor is

$$
R_{\alpha\dot\alpha\beta\dot\beta} \ := \ R_{\gamma\dot\gamma\alpha\dot\alpha\beta\dot\beta}{}^{\gamma\dot\gamma} \ = \ -2R_{\alpha\beta\dot\alpha\dot\beta} + 3\Lambda\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\ , \tag{2.24}
$$

and the curvature scalar is then given by

$$
R \ := \ 2R_{\alpha\dot\alpha}{}^{\alpha\dot\alpha} \ = \ 24\Lambda\ . \tag{2.25}
$$

The SDG equation (2.21) is now equivalent to requiring

$$
\Big\{\ R_{\alpha\dot\alpha\beta\dot\beta\dot\gamma}{}^{\dot\delta} \ = \ 0\ \Big\} \quad \Leftrightarrow \quad \Big\{\ R_{\alpha\beta\dot\gamma}{}^{\dot\delta} \ = \ 0\ , \quad C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} \ = \ 0\ , \quad \text{and} \quad \Lambda \ = \ 0\ \Big\}, \tag{2.26a}
$$

and hence,

$$
R_{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma}{}^{\delta\dot\delta} \ = \ \varepsilon_{\dot\alpha\dot\beta}C_{\alpha\beta\gamma}{}^{\delta}\delta_{\dot\gamma}{}^{\dot\delta}\ . \tag{2.26b}
$$

It was shown in [44] that (2.26) is equivalent to the existence of volume-preserving vector fields $E_a = \sigma_a^{\alpha\dot\alpha}E_{\alpha\dot\alpha}$ whose Lie brackets satisfy

$$
\big[E_{\alpha(\dot\alpha}, E_{\beta\dot\beta)}\big] \ = \ 0\ . \tag{2.27}
$$

By Frobenius' theorem, we may now choose local coordinates in which

$$
E_{\alpha\dot2} \ = \ \partial_{\alpha\dot2}\ , \tag{2.28a}
$$

and we may further fix a gauge such that

$$
E_{\alpha\dot1} \ = \ \partial_{\alpha\dot1} + \tfrac14\varepsilon^{\beta\gamma}(\partial_{\alpha\dot2}\partial_{\beta\dot2}\phi)\partial_{\gamma\dot2} \tag{2.28b}
$$

for ϕ a real-valued function on U. Then, the SDG equation (2.27) reduces to

$$
\Box\,\phi + \tfrac12\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}(\partial_{\alpha\dot2}\partial_{\gamma\dot2}\phi)(\partial_{\beta\dot2}\partial_{\delta\dot2}\phi) \ = \ 0\ . \tag{2.29}
$$

This is Plebański's second heavenly equation [45], and it resembles the SDYM equation (2.4) in Leznov gauge.

The equation (2.29) generalises to $\mathcal{N}$-extended supersymmetric SDG [46],

$$\Box\,\phi + \tfrac{1}{2}\varepsilon^{\alpha\beta}(-1)^{|A|}\Pi^{AB}(\partial_{\alpha\dot{2}}\partial_{A\dot{2}}\phi)(\partial_{\beta\dot{2}}\partial_{B\dot{2}}\phi) \;=\; 0\ , \tag{2.30}$$

where ϕ now is a superfield on $\mathbb{R}^{4|2\mathcal{N}}$ independent of $\eta_i^{\dot{1}}$. In addition, $(\Pi^{AB}) := \mathrm{diag}(\varepsilon^{\alpha\beta},\Pi^{ij})$ with the rank of Π^{ij} depending on how much of the R-symmetry group $\mathsf{SO}(\mathcal{N},\mathbb{C})$ is gauged: in the ungauged case, $\Pi^{ij} = 0$. The equation (2.30) is variational, and follows from the action [46]

$$S^{\mathrm{SP}} \;:=\; \int \mathrm{d}^4x\,\mathrm{d}\eta_1^{\dot{2}}\cdots\mathrm{d}\eta_{\mathcal{N}}^{\dot{2}}\left\{\tfrac{1}{2}\phi\,\Box\,\phi + \tfrac{1}{3!}\varepsilon^{\alpha\beta}(-1)^{|A|}\Pi^{AB}(\partial_{\alpha\dot{2}}\partial_{A\dot{2}}\phi)(\partial_{\beta\dot{2}}\partial_{B\dot{2}}\phi)\right\}\ . \tag{2.31}$$

Note that (2.27) also generalises to $\mathcal{N}$-extended supersymmetric SDG [47], and so does then the above derivation of (2.30).

Twistor description. Like $\mathcal{N}$-extended supersymmetric SDYM theory, also $\mathcal{N}$-extended supersymmetric SDG enjoys a twistorial reformulation via Penrose's non-linear graviton construction [48–54, 47, 14]. In the following, we follow the treatment in [47, 14].

By studying finite complex structure deformations on the twistor space Z, it was shown in [14] that the $\mathcal{N}$-extended supersymmetric SDG equation can be reformulated on Z as a holomorphic Chern–Simons equation with the (infinite-dimensional) gauge group given by the holomorphic Poisson transformations. Concretely, we introduce the holomorphic Poisson structure

$$[f,g] \;:=\; (-1)^{|A|(|f|+1)}\Pi^{AB}\frac{\partial f}{\partial z^A}\frac{\partial g}{\partial z^B} \tag{2.32}$$

on Z, where Π^{AB} is the tensor that already appeared in (2.30). The $\mathcal{N}$-extended supersymmetric SDG equation is then equivalent to [14]

$$\bar{\partial}h + \tfrac{1}{2}[h,h] \;=\; 0\ , \tag{2.33}$$

where h is a $(0,1)$-form on Z of weight 2. Just as the holomorphic gauge potential in the SDYM setting, also h is assumed to have no anti-holomorphic fermionic directions $\mathrm{d}\hat{\eta}_i$ and to depend holomorphically on the fermionic coordinates η_i. Note that [14] also discusses the more general case of non-vanishing cosmological constant. For maximal ($\mathcal{N} = 8$) supersymmetry, equation (2.33) follows from the variation of a holomorphic Chern–Simons action [14]. It should be noted that in this case, however, the twistor space Z is not a Calabi–Yau supermanifold; nevertheless, the weights of h cancel appropriately so as to render the action well defined.

Twistorial prepotential action. In order to connect (2.33) to (2.30), we follow the closely our discussion of SDYM theory and write the holomorphic Chern–Simons equation (2.33) as

$$\begin{aligned}\frac{\partial}{\partial \hat{z}^\alpha} h_\beta - \frac{\partial}{\partial \hat{z}^\beta} h_\alpha + [h_\alpha, h_\beta] \ &= \ 0 \ , \\ \hat{E}_\pi h_\alpha - \frac{\partial}{\partial \hat{z}^\alpha} h_\pi + [h_\pi, h_\alpha] \ &= \ 0 \ .\end{aligned} \tag{2.34}$$

Considering the case of vanishing cosmological constant, we may impose the gauge

$$h_\pi \ = \ 0 \ , \tag{2.35a}$$

just as for SDYM theory. We obtain prepotentials that, for solutions to (2.34), we can further constrain by imposing Lorenz gauge along the fibres, $E^\alpha h_\alpha = 0$, so that we arrive at

$$h_\alpha \ = \ \frac{1}{4|\lambda|^2} E_\alpha \phi \tag{2.35b}$$

for ϕ now a function of weight 4 on Z that depends holomorphically on the fermionic coordinates η_i as well as on $\pi_{\dot\alpha}$. Hence, (2.34) reduces to

$$\Box \phi + \tfrac{1}{2}(-1)^{|A|}\Pi^{AB}\varepsilon^{\alpha\beta}(E_A E_\alpha \phi)(E_B E_\beta \phi) \ = \ 0 \ , \tag{2.36}$$

where we have again used (2.13) as well as $\hat{E}_\alpha E^\alpha = -\frac{1}{4}\Box$. For maximal ($\mathcal{N} = 8$) supersymmetry, this equation arises from variation of the action [14]

$$S^{\rm SDG} \ := \ \int \mathrm{vol}_{\rm SDG} \left\{ \tfrac{1}{2}\phi \Box \phi + \tfrac{1}{3!}(-1)^{|A|}\Pi^{AB}\varepsilon^{\alpha\beta}(E_A E_\alpha \phi)(E_B E_\beta \phi)\right\} , \tag{2.37a}$$

where now

$$\mathrm{vol}_{\rm SDG} \ := \ \mathrm{d}^4 x \, \frac{\lambda^{\dot\alpha}\mathrm{d}\lambda_{\dot\alpha} \, \hat\lambda^{\dot\alpha}\mathrm{d}\hat\lambda_{\dot\alpha}}{|\lambda|^4} \, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_8 \ . \tag{2.37b}$$

This twistor action resembles the space-time action (2.31); however, again, it should be noted that (2.37) is manifestly Lorentz invariant. A similar action (and the corresponding equation of motion) exists also on harmonic superspace [55].

Relation to space-time. The superfield expansion of ϕ reads as

$$\phi \ = \ g + \eta_i \psi^i + \eta_{ij} A^{ij} + \eta_{ijk}\chi^{ijk} + \eta_{ijkl} W^{ijkl} + \eta^{ijk}\tilde\chi_{ijk} + \eta^{ij}\tilde A_{ij} + \eta^i \tilde\psi_i + \eta \tilde g \ , \tag{2.38a}$$

where

$$\eta_{i_1\cdots i_k} \ := \ \tfrac{1}{k!}\eta_{i_1}\cdots\eta_{i_k} \quad \text{and} \quad \eta^{i_1\cdots i_{8-k}} \ := \ \tfrac{1}{k!}\varepsilon^{i_1\cdots i_8}\eta_{i_{9-k}}\cdots\eta_{i_8} \ . \tag{2.38b}$$

The Penrose–Ward transform of the field h with this expansion substituted in then yields the correspondence between the various components and the SDG fields on $\mathbb{R}^4$ displayed in Table 2.2.

Field	g	ψ^i	A^{ij}	χ^{ijk}	W^{ijkl}	$\tilde{\chi}_{ijk}$	$\tilde{A}_{ij}$	$\tilde{\psi}_i$	$\tilde{g}$
Helicity	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
Multiplicity	1	8	28	56	70	56	28	8	1

Table 2.2: Space-time SDG fields and their helicities and multiplicities.

3. Colour–kinematics duality and the double copy

Let us now show that both actions feature CK duality, and the action (2.29) is indeed the double copy of (2.4) according to our formalism [15].

Differential graded Lie algebra. Let $\mathscr{S}(m)$ denote the space of smooth functions of weight m on Z that are holomorphic in the fermionic coordinates η_i as well as the bosonic coordinates $\pi_{\dot{\alpha}}$ and are bounded on Z. Recall that cubic actions correspond to metric differential graded Lie algebras, see e.g. [56, 57]. In the case of the action (2.18), we have the differential graded Lie algebra $\mathfrak{L}^{\rm SDYM} \cong \mathfrak{L}_1^{\rm SDYM} \oplus \mathfrak{L}_2^{\rm SDYM}$ concentrated in degrees 1 and 2 with the underlying cochain complex

$$\mathsf{Ch}(\mathfrak{L}^{\rm SDYM}) \;:=\; \Big(* \longrightarrow \underbrace{\mathfrak{g} \otimes \mathscr{S}(2)}_{:=\,\mathfrak{L}_1^{\rm SDYM}} \xrightarrow{\mathrm{id}_{\mathfrak{g}} \otimes \Box} \underbrace{\mathfrak{g} \otimes \mathscr{S}(2)}_{:=\,\mathfrak{L}_2^{\rm SDYM}} \longrightarrow * \Big) \tag{3.1}$$

and differential $\mu_1|_{\mathfrak{L}_1^{\rm SDYM}} := \mathrm{id}_{\mathfrak{g}} \otimes \Box$. It comes equipped with an invariant inner product, whose components vanish except between degrees 1 and 2, for which

$$\langle \phi, \chi \rangle \;:=\; \int \mathrm{vol}_{\rm SDYM}\, g_{ab} \phi^a \chi^b \tag{3.2}$$

for all $\phi \in \mathfrak{L}_1^{\rm SDYM}$ and $\chi \in \mathfrak{L}_2^{\rm SDYM}$. The interactions are encoded in the Lie bracket $\mu_2 : \mathfrak{L}^{\rm SDYM} \times \mathfrak{L}^{\rm SDYM} \to \mathfrak{L}^{\rm SDYM}$, which vanishes except between two elements of degree 1, for which

$$\mu_2(\phi_1, \phi_2) \;:=\; \mathsf{e}_c \otimes f_{ab}{}^c \varepsilon^{\alpha\beta} (E_\alpha \phi_1^a)(E_\beta \phi_2^b) \tag{3.3}$$

for all $\phi_{1,2} \in \mathfrak{L}_1^{\rm SDYM}$.

BV$^{\blacksquare}$-algebra and colour–kinematics duality. The differential graded Lie algebra $\mathfrak{L}^{\rm SDYM}$ naturally factorises as

$$\mathfrak{L}^{\rm SDYM} \;\cong\; \mathfrak{g} \otimes \mathfrak{C}^{\rm SDYM} , \tag{3.4}$$

where $\mathfrak{g}$ is the gauge Lie algebra and $\mathfrak{C}^{\mathrm{SDYM}} = (\mathfrak{C}^{\mathrm{SDYM}}, \mathsf{d}, \mathsf{m}_2)$ is a differential graded commutative algebra; see [58] for a generic description of this procedure, which is referred to as colour-stripping in the physics literature. The cochain complex underlying $\mathfrak{C}^{\mathrm{SDYM}}$ is concentrated in degrees 1 and 2,

$$\mathsf{Ch}(\mathfrak{C}^{\mathrm{SDYM}}) := \Big(* \longrightarrow \underbrace{\mathscr{S}(2)}_{:= \mathfrak{C}_1^{\mathrm{SDYM}}} \xrightarrow{\ \square\ } \underbrace{\mathscr{S}(2)}_{:= \mathfrak{C}_2^{\mathrm{SDYM}}} \longrightarrow * \Big) \tag{3.5a}$$

with $\mathsf{d}|_{\mathfrak{C}_1^{\mathrm{SDYM}}} := \square$. Its associative graded-commutative product m_2 and inner product $\langle -, - \rangle$ are

$$\mathsf{m}_2(\phi_1, \phi_2) := \varepsilon^{\alpha\beta}(E_\alpha \phi_1)(E_\beta \phi_2) \quad \text{and} \quad \langle \phi_1, \chi \rangle := \int \mathrm{vol}_{\mathrm{SDYM}}\, \phi_1 \chi \tag{3.5b}$$

for all $\phi_{1,2} \in \mathfrak{C}_1^{\mathrm{SDYM}}$ and $\chi \in \mathfrak{C}_2^{\mathrm{SDYM}}$.

The propagator in this theory is $P = \mathsf{id}_{\mathfrak{g}} \otimes \frac{\mathsf{b}}{\square}$, where b is the shift isomorphism

$$\mathsf{b} \,:\, \mathfrak{C}_2^{\mathrm{SDYM}} \xrightarrow{\ \cong\ } \mathfrak{C}_1^{\mathrm{SDYM}} . \tag{3.6}$$

This operator satisfies

$$[\mathsf{d}, \mathsf{b}] = \mathsf{d}\mathsf{b} + \mathsf{b}\mathsf{d} = \square , \tag{3.7}$$

and it is a second-order differential operator in the sense of [59], cf. [60, 15]. Hence, $(\mathfrak{C}^{\mathrm{SDYM}}, \mathsf{d}, \mathsf{m}_2, \mathsf{b})$ forms a BV$^{\blacksquare}$-algebra [17], see also [59, 15].

The extension of $\mathfrak{C}^{\mathrm{SDYM}}$ from a differential graded commutative algebra to a BV$^{\blacksquare}$-algebra implies the existence of a kinematic Lie algebra $\mathfrak{K}$ with Lie bracket given by

$$\{\Phi_1, \Phi_2\} := \mathsf{b}\{\Phi_1, \Phi_2\} - \{\mathsf{b}\Phi_1, \Phi_2\} - (-1)^{|\Phi_1|}\{\Phi_1, \mathsf{b}\Phi_2\} \tag{3.8}$$

for all $\Phi_{1,2} \in \mathfrak{C}^{\mathrm{SDYM}}$ [17, 22, 61]. Explicitly, we have here

$$\begin{aligned} \{\phi_1, \phi_2\} &= \mathsf{b}(\mathsf{m}_2(\phi_1, \phi_2)) = \varepsilon^{\alpha\beta}(E_\alpha \phi_1)(E_\beta \phi_2) \in \mathfrak{C}_1^{\mathrm{SDYM}} , \\ \{\phi_1, \phi_2^+\} &= \mathsf{m}_2(\phi_1, \mathsf{b}\phi_2^+) = \varepsilon^{\alpha\beta}(E_\alpha \phi_1)(E_\beta \phi_2^+) = \{\phi_2^+, \phi_1\} \in \mathfrak{C}_2^{\mathrm{SDYM}} \end{aligned} \tag{3.9}$$

for all $\phi_{1,2} \in \mathfrak{C}_1^{\mathrm{SDYM}}$ and $\phi_2^+ \in \mathfrak{C}_2^{\mathrm{SDYM}}$. The existence of this kinematic Lie algebra leads to a Feynman diagram expansion of the currents and the amplitudes of SDYM theory with interaction vertices given by the Lie brackets $[-,-]_{\mathfrak{g}} \otimes \{-,-\}$ and propagator $\frac{1}{\square}$, as explained in [61], rendering CK duality manifest.

Altogether, we conclude that the tree-level currents of SDYM theory with an arbitrary amount of supersymmetry exhibit CK duality.[1]

[1] The tree-level amplitudes therefore do so as well, but these are trivial.

Double copy: Hopf algebra. It is now apparent, even on a cursory examination, that the ungauged version of the $\mathcal{N}=8$ supersymmetric SDG twistor action (2.37) is the double copy of the $\mathcal{N}=4$ supersymmetric SDYM twistor action action (2.18). In the following, we make this connection algebraically rigorous to provide an explicit and easy-to-follow example for the formalism developed in [15].

An interesting new feature of this example is the appearance of a non-commutative Hopf algebra. To control the momentum dependence on twistor space $Z \cong \mathbb{R}^{4|2\mathcal{N}} \times \mathbb{C}P^1$, we can use the usual bosonic momentum operators ∂_μ on $\mathbb{R}^4$ as well as a generator of $\mathfrak{su}(2)$ together with the quadratic Casimir operator of $\mathfrak{su}(2)$ to characterise the spherical harmonics on $\mathbb{C}P^1$. The smallest Hopf algebra $\mathfrak{H}_Z$ that contains these is the vector space of constant coefficient differential operators on $\mathbb{R}^4$ tensored with the universal enveloping algebra of $\mathfrak{su}(2)$. Note that, contrary to the examples discussed before, e.g. in [17] or [15], this Hopf algebra is non-commutative.[1]

Free fields. The construction of the double copied differential graded Lie algebra described in [15] starts from the restricted tensor product

$$\hat{\mathfrak{C}} \;:=\; \mathfrak{C}^{\mathrm{SDYM}} \otimes^{\mathfrak{H}_Z} \mathfrak{C}^{\mathrm{SDYM}}\;, \tag{3.10}$$

which has underlying chain complex

$$\mathsf{Ch}(\hat{\mathfrak{C}}) \;:=\; \Big(* \longrightarrow \underbrace{\mathscr{S}(4)}_{:=\hat{\mathfrak{C}}_2} \longrightarrow \underbrace{\mathscr{S}(4) \;\oplus\; \mathscr{S}(4)}_{:=\hat{\mathfrak{C}}_3} \longrightarrow \underbrace{\mathscr{S}(4)}_{:=\hat{\mathfrak{C}}_4} \longrightarrow * \Big)\,. \tag{3.11}$$

This tensor product is then degree-shifted and restricted to the kernel of the operator

$$\hat{\mathsf{b}}_- \;:=\; \mathsf{b} \otimes \mathsf{id} - \mathsf{id} \otimes \mathsf{b} \tag{3.12}$$

with the operator b defined in (3.6), which results in the chain complex

$$\mathsf{Ch}(\mathfrak{L}^{\mathrm{SDG}}) \;:=\; \Big(* \longrightarrow \underbrace{\mathscr{S}(4)}_{:=\mathfrak{L}_1^{\mathrm{SDG}}} \xrightarrow{\;\square\;} \underbrace{\mathscr{S}(4)}_{:=\mathfrak{L}_2^{\mathrm{SDG}}} \longrightarrow * \Big)\,. \tag{3.13}$$

Interactions. The interaction terms in the equation of motion are given by the Lie bracket of the kinematic Lie algebra of $\hat{\mathfrak{C}}$, restricted to $\mathfrak{L}^{\mathrm{SDG}}$. We have

$$[\hat{\phi}_1[1], \hat{\phi}_2[1]] \;=\; \hat{\mathsf{b}}_+ \hat{\mathsf{m}}_2(\hat{\phi}_1, \hat{\phi}_2) \;=\; \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}(E_\alpha E_\gamma \hat{\phi}_1)(E_\beta E_\delta \hat{\phi}_2)\;, \tag{3.14}$$

for all $\hat{\phi}_{1,2}[1] \in \mathfrak{L}^{\mathrm{SDG}}$. Hence, we obtain the interactions of SDG with none of the R-symmetry gauged.

[1] The paper [17] discusses the possibility of using the universal enveloping algebra, but does not actually use it in the main example.

Action. An ingredient mostly ignored e.g. in [17] and [20],[1] but crucial for the discussion of an action principle and scattering amplitudes, is the metric on $\mathfrak{L}^{\mathrm{SDG}}$. According to the prescription of [15], we define

$$\langle\hat{\phi}_1[1],\hat{\phi}_2[1]\rangle_{\mathfrak{L}^{\mathrm{SDG}}} \;\coloneqq\; (-1)^{|\hat{\phi}_1|}\langle\Box^{-1}(\Box\otimes\mathsf{id}-\mathsf{id}\otimes\Box)\hat{\phi}_1,\hat{\phi}_2\rangle_{\hat{\mathfrak{C}}} \tag{3.15}$$

for all $\hat{\phi}_{1,2}[1]\in\mathfrak{L}^{\mathrm{SDG}}$. The operators in this inner product cancel to the shift isomorphism

$$\Box^{-1}(\Box\otimes\mathsf{id}-\mathsf{id}\otimes\Box) \;=\; [-1]\otimes\mathsf{id}-\mathsf{id}\otimes[-1]\;, \tag{3.16}$$

which involves the shift isomorphism $[-1]:\mathfrak{C}_1^{\mathrm{SDYM}}\xrightarrow{\cong}\mathfrak{C}_2^{\mathrm{SDYM}}$. For further details, see the discussion of biadjoint scalar field theory in [15]. As also discussed in that paper, there is an infinite volume factor from the integral over the doubled bosonic directions that needs to be removed, so that we have schematically

$$\int \mathrm{vol}_{\mathrm{SDYM}}\otimes\mathrm{vol}_{\mathrm{SDYM}} \quad\longrightarrow\quad \mathrm{vol}(\mathbb{R}^4\times\mathbb{C}P^1)\int\mathrm{vol}_{\mathrm{SDG}}\;. \tag{3.17}$$

Altogether, we are left with a metric uniquely characterised by

$$\langle\hat{\phi}_1,\hat{\phi}_2^+\rangle_{\mathfrak{L}^{\mathrm{SDG}}} \;=\; \int\mathrm{vol}_{\mathrm{SDG}}\,\hat{\phi}_1\hat{\phi}_2^+\;, \tag{3.18}$$

and together with the differential (3.13) and the Lie bracket (3.14), we recover the metric differential graded Lie algebra whose corresponding action is the scalar SDG action on twistor space (2.37).

Remarks. We note that the gauged version of the self-dual gravity twistor action (2.37) may also be obtained as the double copy of self-dual Yang–Mills theory with another theory whose action is

$$\tilde{S}^{\mathrm{SDYM}} \;\coloneqq\; \int\mathrm{vol}_{\mathrm{SDYM}}\left\{\tfrac{1}{2}g_{ab}\phi^a\,\Box\,\phi^b+\tfrac{1}{3!}f_{abc}(-1)^{|A|}\Pi^{AB}\phi^c(E_A\phi^a)(E_B\phi^b)\right\}\;. \tag{3.19}$$

Choosing appropriate Π^{AB}, we can obtain an action in which an arbitrary amount of R-symmetry is gauged.

If we are content with CK duality and double copy at the level of currents, we can also consider the corresponding non-maximally supersymmetric theories. All our constructions bar that of the metric go through as described above. If one wishes to work with an action for these and related theories, one can achieve this by replacing twistor space with a fattened complex manifold, as in [62], or with a weighted projective space, as in [63].

[1]The paper [17] mentions the need for the metric in the loop case but does not develop it further.

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