Chapter 11
Standard measure spaces

There are many examples of measure spaces that present various pathologies and on which some of the deeper theorems of measure theory fail. Elements of the class of standard measure spaces, to be defined shortly, do not display any of these pathologies and, in addition, can be classified up to a natural notion of isomorphism. The results we present here use little in addition to the observation that a separable metric space can be written as a countable union of closed subsets of arbitrarily small diameter (for instance, the closed balls of a fixed radius centered at the points of a countable dense set). They were developed initially in order to resolve questions about the measurability of the projection onto a one-dimensional subspace of a Borel subset of $\mathbb{R}^2$. We develop enough of the theory of standard spaces to display some of the desirable features of measure theory on such spaces. We start with the idea of isomorphism between measurable spaces.

Definition 11.1. Two measurable spaces $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are said to be *isomorphic* if there is a bijection $f : X \to Y$ such that, for $E \subset X$, we have $E \in \mathbf{S}$ if and only if $f(E) \in \mathbf{T}$. We say that $(X, \mathbf{S})$ *embeds* in $(Y, \mathbf{S})$ if there exists $F \in \mathbf{T}$ such that $(X, \mathbf{S})$ is isomorphic to $(F, \mathbf{T}_F)$.

Another way to state the first condition on the function f above is to say that it is measurable $[\mathbf{S}, \mathbf{T}]$ and the inverse map f^{-1} is measurable $[\mathbf{T}, \mathbf{S}]$. The following result is an analog of the Cantor-Bernstein theorem of set theory (see [**I**, Theorem 4.1]).

Theorem 11.2. *Assume that the measurable spaces $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are such that $(X, \mathbf{S})$ embeds in $(Y, \mathbf{T})$ and $(Y, \mathbf{T})$ embeds in $(X, \mathbf{S})$. Then $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are isomorphic.*

Proof. Let $E \in \mathbf{S}$ and $F \in \mathbf{T}$ be such that $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are isomorphic to $(F, \mathbf{T}_F)$ and $(E, \mathbf{S}_E)$, respectively, and let $f : X \to F$ and $g : Y \to E$ be bijections realizing these isomorphisms. It suffices to find sets $A \in \mathbf{S}$ and

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$B \in \mathbf{T}$ such that $f(A) = Y \setminus B$ and $g(B) = X \setminus A$. Indeed, once such sets are found, an isomorphism between $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ is given by the bijection $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & x \in A, \\ g^{-1}(x), & x \in X \setminus A. \end{cases}$$

To find the set A, consider the map $u : \mathbf{S} \to \mathbf{S}$ defined by

$$u(D) = X \setminus g(Y \setminus f(D)), \quad D \in \mathbf{S}.$$

The map u is monotone increasing (that is, $D_1 \subset D_2$ implies $u(D_1) \subset u(D_2)$), and it carries countable unions to countable unions. It follows that the set $A \in \mathbf{S}$ defined by

$$A = \bigcup_{n=1}^{\infty} \underbrace{u(u(\cdots u(}_{n \text{ times}} \varnothing) \cdots))$$

satisfies the equation $A = u(A)$. To conclude the proof, set $B = Y \setminus f(A)$ and verify that these sets satisfy the desired conditions. □

Definition 11.3. A measurable space $(X, \mathbf{S})$ is said to be *standard* if there exists a complete, separable metric space Y such that $(X, \mathbf{S})$ is isomorphic (as a measurable space) to $(Y, \mathbf{B}_Y)$. A measure space $(X, \mathbf{S}, \mu)$ is said to be *standard* if $(X, \mathbf{S})$ is a standard measurable space.

Another way to state this definition is to say that $(X, \mathbf{S})$ is standard if there is a complete metric d on X such (X, d) is separable and $\mathbf{S} = \mathbf{B}_X$. Observe that the definition of the Borel σ-algebra $\mathbf{B}_X$ only requires the topology of the metric space X, not its metric. There are usually many different topologies on X which generate the same Borel σ-algebra. See, for instance, Theorem 11.9 below.

Example 11.4. The space $((0, 1), \mathbf{B}_{(0,1)})$, where $(0, 1)$ is given its usual topology as a subset of $\mathbb{R}$, is standard. Indeed, $(0, 1)$ is homeomorphic to $\mathbb{R}$ so $((0, 1), \mathbf{B}_{(0,1)})$ is isomorphic to $(\mathbb{R}, \mathbf{B}_{\mathbb{R}})$. Similarly, $[0, 1)$ is homeomorphic to $[0, +\infty)$, and therefore $([0, 1), \mathbf{B}_{[0,1)})$ is standard.

Definition 11.5. A topological space (X, τ) is called a *Polish space* if it is homeomorphic to a complete, separable metric space. The topology τ is said to be *Polish* if (X, τ) is a Polish space.

Remark 11.6. Example 11.4 shows that a subspace of a separable, complete metric space (X, d) may be Polish even though it is not complete in the metric d. Such subspaces are precisely the G_δ sets in X. See Problem D.

Example 11.7. If $\{(X_n, \tau_n)\}_{n=1}^{\infty}$ is a sequence of Polish spaces, then the space $X = \prod_{n=1}^{\infty} X_n$ endowed with the product topology is also a Polish space. Indeed, it is easy to see that X is separable. Moreover, if τ_n is defined by a complete metric d_n on X_n, the metric

$$d((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, d_n(x_n, y_n)\}, \quad (x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in X,$$

is complete and this metric topology is the product topology. Two important particular cases are the space $\mathcal{S} = \mathbb{N}^{\mathbb{N}}$ of all sequences of positive integers and the *Cantor space* $\mathcal{C} = \{0,1\}^{\mathbb{N}}$. Both of these are Polish spaces when endowed with the product topology, where every factor is given the discrete topology. The reason for this terminology is that $\mathcal{C}$ is homeomorphic to the standard ternary Cantor set $C \subset [0,1]$ via the continuous bijection $f : \mathcal{C} \to C$ defined by

$$f((x_n)_{n=1}^\infty) = \sum_{n=1}^{\infty} \frac{2x_n}{3^n}, \quad (x_n)_{n=1}^\infty \in \mathcal{C}.$$

(Recall that the inverse of a continuous bijection between compact Hausdorff spaces is necessarily continuous.) The space $\mathcal{S}$ is also homeomorphic to a set of real numbers, namely the set $[0,1] \setminus \mathbb{Q}$. The identification $g : \mathcal{S} \to [0,1] \setminus \mathbb{Q}$ is obtained by use of continued fractions:

$$g((x_n)_{n=1}^\infty) = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots}}}, \quad (x_n)_{n=1}^\infty \in \mathcal{S}.$$

Of course, $\mathcal{C}$ is a compact subset of $\mathcal{S}$. The space $\mathcal{S}$ is also homeomorphic to a subspace of $\mathcal{C}$ via the homeomorphism

$$h((x_n)_{n=1}^\infty) = (\underbrace{0, \ldots, 0}_{x_1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{x_2 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{x_3 \text{ times}}, 1, \ldots).$$

Thus, via Theorem 11.2, $\mathcal{C}$ and $\mathcal{S}$ are isomorphic as measurable spaces.

Another useful Polish space is the *Hilbert cube* $\mathcal{Q} = [0,1]^{\mathbb{N}}$ with the product topology, where $[0,1]$ is given its usual topology as a subset of $\mathbb{R}$.

Theorem 11.8. *Let (X, τ) be a Polish space. Then*

(1) *X is homeomorphic to a subspace of $\mathcal{Q}$.*
(2) *If X is uncountable, then X contains a subspace homeomorphic to $\mathcal{C}$.*
(3) *There exists a continuous surjective map $h : \mathcal{S} \to X$.*

In particular, if X is not countable then it has cardinality $\mathfrak{c} = 2^{\aleph_0}$.

Proof. Fix a complete metric d defining the topology τ and a dense sequence $\{x_n\}_{n=1}^\infty$ in X. The map $f : X \to \mathcal{Q}$ defined by

$$f(x) = (\max\{1, d(x, x_n)\})_{n=1}^\infty$$

is easily seen to be a homeomorphism of X onto $f(X)$. To prove (2), assume that X is uncountable. We claim that there exist closed, disjoint, uncountable subsets $X_0, X_1 \subset X$ each of which has diameter at most 1. Indeed, assume that such sets cannot be found, and let ε be a positive number. Denote by $B_{n,\varepsilon}$ the closed ball of radius ε centered at x_n. Then any two uncountable sets of the form $B_{n,\varepsilon}$ must intersect. Therefore the union of all uncountable sets $B_{n,\varepsilon}$ is contained in a closed ball of the form $B_{m_\varepsilon,3\varepsilon}$. In other words, $X \setminus B_{m_\varepsilon,3\varepsilon}$ is countable for some m. This argument, applied to $\varepsilon_k = 4^{-k}$, yields a ball B_k of radius $3\varepsilon_k$ such that $X \setminus B_k$ is at most countable, so $X \setminus \left[\bigcap_{k\in\mathbb{N}} B_k\right]$ is at most countable as well. The intersection $\bigcap_{k\in\mathbb{N}} B_k$ consists of at most one point, thus leading to the conclusion that X itself is countable, contrary to the hypothesis. We conclude that there exist closed, disjoint uncountable sets $X_0, X_1 \subset X$ of diameter at most 1. We can now construct, by induction, for each $N \in \mathbb{N}$, closed, uncountable, pairwise disjoint sets $X_{n_1,n_2,\dots,n_N}$ for $(n_1, \dots, n_N) \in \{0,1\}^N$ such that $X_{n_1,n_2,\dots,n_N} \subset X_{n_1,n_2,\dots,n_{N-1}}$ and the diameter of $X_{n_1,n_2,\dots,n_N}$ is less than 2^{-N}. Indeed, we just apply the preceding observation to the uncountable Polish space $X_{n_1,n_2,\dots,n_{N-1}}$. Once this is done, a function $g : \mathcal{C} \to X$ such that

$$\bigcap_{N=1}^{\infty} X_{n_1,n_2,\dots,n_N} = \{g(t)\}, \quad t = (n_j)_{j=1}^{\infty} \in \mathcal{C},$$

exists because d is a complete metric. The map g is a homeomorphism from $\mathcal{C}$ to $g(\mathcal{C})$.

The proof of (3) is similar to that of (2). We construct, for every $k \in \mathbb{N}$ and every $(n_1, \dots, n_k) \in \mathbb{N}^k$, a closed subset $A_{n_1,\dots,n_k} \neq \varnothing$ with diameter at most 2^{-k} such that $X = \bigcup_{n_1\in\mathbb{N}} A_{n_1}$ and

$$A_{n_1,\dots,n_k} = \bigcup_{n\in\mathbb{N}} A_{n_1,\dots,n_k,n}, \quad k, n_1, \dots, n_k \in \mathbb{N}.$$

The function $h : \mathcal{S} \to X$ is defined by the requirement that

$$\{h(\mathbf{n})\} = \bigcap_{k=1}^{\infty} A_{n_1,\dots,n_k}$$

for every sequence $\mathbf{n} = (n_1, n_2, \dots) \in \mathcal{S}$. □

It is convenient in what follows to write $\mathbf{B}_\tau$ for the Borel σ-algebra of a topological space (X, τ). Thus $(X, \mathbf{B}_\tau)$ is a standard measurable space if (X, τ) is a Polish space. There are usually different Polish topologies τ, τ' on a set X such that $\mathbf{B}_\tau = \mathbf{B}_{\tau'}$, and which one is used is immaterial for questions of measure theory.

Theorem 11.9. *Let $(X, \mathbf{S})$ be a standard measurable space, and fix a set $F \in \mathbf{S}$. There exists a Polish topology τ on X such that $\mathbf{S} = \mathbf{B}_\tau$ and F is both open and closed in (X, τ).*

Proof. Let d be a metric on X that defines a Polish topology σ on X such that $\mathbf{S} = \mathbf{B}_\sigma$. Denote by $\mathbf{T}$ the collection of those sets $E \in \mathbf{S}$ for which there exists a Polish topology τ on X finer than σ such that $\mathbf{S} = \mathbf{B}_\tau$ and E is both open and closed in τ. We prove the theorem by showing that $\mathbf{T} = \mathbf{S}$. To do this it suffices to show that $\mathbf{T}$ contains all the open sets in σ and that it is closed under the formation of finite intersections and countable unions.

Assume first that E is an arbitrary open set in σ. The subspace $X \setminus E \subset X$, endowed with the restriction of d, is again a complete metric space. We now show that E is also a Polish space in the relative topology on E induced by σ. A complete metric d' on E is obtained by selecting a dense sequence $\{x_n\}_{n=1}^\infty$ in $X \setminus E$ and setting

$$d'(x,y) = d(x,y) + \sum_{n=1}^\infty 2^{-n} |h_n(x) - h_n(y)|, \quad x, y \in E,$$

where

$$h_n(x) = \min\left\{1, \frac{1}{d(x,x_n)}\right\}, \quad n \in \mathbb{N}.$$

We define a metric d_E on X by setting

$$d_E(x,y) = \begin{cases} d(x,y), & x,y \in X \setminus E, \\ d'(x,y), & x,y \in E, \\ \max\{1, d(x,y)\}, & x \in E, y \in X \setminus E \text{ or } x \in X \setminus E, y \in E. \end{cases}$$

The topology τ_E on X defined by this metric is Polish, $\mathbf{S} = \mathbf{B}_{\tau_E}$, and E is both open and closed in τ_E. (See Problem 11C.) This argument shows that $\mathbf{T}$ contains all open sets in σ.

Suppose next that $\{E_n\}_{n=1}^\infty \subset \mathbf{T}$, and d_n is a metric on X defining a Polish topology τ_n finer than σ such that E_n and $X \setminus E_n$ are open in τ_n and $\mathbf{B}_{\tau_n} = \mathbf{S}$ for all $n \in \mathbb{N}$. Define a new metric $\widetilde{d}$ on X by setting

$$\widetilde{d}(x,y) = \sum_{n=1}^\infty 2^{-n} \min\{1, d_n(x,y)\}, \quad x, y \in X.$$

The inequality $\widetilde{d} \geq 2^{-n} \min\{1, d_n\}$ shows that the topology $\widetilde{\tau}$ generated by $\widetilde{d}$ is finer than τ_n for every n. In particular, E_n and $X \setminus E_n$ are open in $\widetilde{\tau}$ for all $n \in \mathbb{N}$. A sequence $\{x_k\}_{k=1}^\infty$ is Cauchy in the metric $\widetilde{d}$ if and only if it is Cauchy in each d_n. Since each d_n is complete, there exists $a_n \in X$ such that $\lim_{k\to\infty} d_n(x_k, a_n) = 0$. Moreover, since τ_n is finer than σ, $\lim_{k\to\infty} d(x_k, a_n) = 0$ as well. We conclude that there exists $a \in X$ such that $a_n = a$ for all $n \in \mathbb{N}$ and $\lim_{k\to\infty} \widetilde{d}(x_k, a) = 0$. Thus $\widetilde{d}$ is a complete metric.

We show next that the space $(X, \widetilde{\tau})$ is second countable. Assume that $\{B_{n,j}\}_{j=1}^\infty$ is a base of open sets for τ_n and note that each $B_{n,j} \in \mathbf{S}$. We show that the finite intersections of sets in $\{B_{n,j}\}_{n,j=1}^{\infty}$ form a base for $\widetilde{\tau}$. Let $G \subset X$ be open in $\widetilde{\tau}$. For every $x \in G$, we have

$$G \supset \{y \in X : \widetilde{d}(x,y) < \varepsilon\}$$

for some $\varepsilon > 0$, and

$$\{y \in X : \widetilde{d}(x,y) < \varepsilon\} \supset \bigcap_{n=1}^{N} \{y \in X : d_n(x,y) < \varepsilon/2\}$$

provided that $2^{-N} < \varepsilon/2$. It follows that $\{y \in X : \widetilde{d}(x,y) < \varepsilon\}$ contains a finite intersection W_x of sets in $\{B_{n,j}\}_{n,j=1}^{\infty}$ such that $x \in W_x$ and thus $G = \bigcup_{x \in G} W_x$. Since $B_{n,j} \in \mathbf{S}$ for all $n, j \in \mathbb{N}$, we conclude that $\mathbf{B}_{\widetilde{\tau}} = \mathbf{S}$. Moreover, E_n is both open and closed in $\widetilde{\tau}$ for $n \in \mathbb{N}$. We see immediately that $E_1 \cap E_2$ is both open and closed in $\widetilde{\tau}$, and this establishes the fact that $\mathbf{T}$ is closed under the formation of finite intersections. It also follows that the union $E = \bigcup_{n \in \mathbb{N}} E_n$ is open in $\widetilde{\tau}$. The first part of the proof provides a Polish topology τ on X finer than $\widetilde{\tau}$ such that $\mathbf{B}_{\tau} = \mathbf{B}_{\widetilde{\tau}} = \mathbf{S}$ and E is closed as well as open in τ. This shows that $\mathbf{T}$ is closed under the formation of countable unions and thus concludes the proof of the theorem. $\square$

The last part of the proof of Theorem 11.9 yields the following strengthening of the theorem.

Corollary 11.10. *Let $(X, \mathbf{S})$ be a standard measurable space, and fix a sequence $\{F_n\}_{n \in \mathbb{N}} \subset \mathbf{S}$. There exists a Polish topology τ on X such that $\mathbf{S} = \mathbf{B}_{\tau}$ and each F_n is both open and closed in (X, τ), $n \in \mathbb{N}$.*

Corollary 11.11. *Let $(X, \mathbf{S})$ be a standard measurable space, and let $E \in \mathbf{S}$. Then the measurable space $(E, \mathbf{S}_E)$ is standard.*

Proof. Let τ be a Polish topology on X such that $\mathbf{S} = \mathbf{B}_{\tau}$. By Theorem 11.9 we may assume that E is closed in X, and therefore is a Polish space in the relative topology. The conclusion follows now because $\mathbf{S}_E$ is the Borel σ-algebra of the topology induced by τ on E. $\square$

Corollary 11.12. *Let $(X, \mathbf{S})$ be a standard measurable space. There exists an injective map $f : X \to \mathcal{C}$ which is measurable $[\mathbf{S}, \mathbf{B}_{\mathcal{C}}]$.*

Proof. Let τ be a Polish topology on X such that $\mathbf{B}_{\tau} = \mathbf{S}$, and let $\{G_n\}_{n=1}^{\infty}$ be a base of open sets for τ, so $\mathbf{S}$ is the σ-algebra generated by $\{G_n\}_{n=1}^{\infty}$. By Corollary 11.10 there is a Polish topology τ' on X finer than τ such that $\mathbf{B}_{\tau'} = \mathbf{S}$ and such that the sets G_n are also closed in τ'. If $x, y \in X$ and $x \neq y$, there exists $n \in \mathbb{N}$ such that $x \in G_n$ and $y \notin G_n$. Thus the map $f : X \to \mathcal{C}$ defined by

$$f(x) = (\chi_{G_n}(x))_{n=1}^{\infty}$$

is injective and continuous on (X, τ'), and hence measurable $[\mathbf{S}, \mathbf{B}_{\mathcal{C}}]$. $\square$

We have seen in Theorem 11.8 that the Cantor space $\mathcal{C}$ is homeomorphic to a closed subset of any uncountable Polish space. This implies that $(\mathcal{C}, \mathbf{B}_{\mathcal{C}})$ embeds into any standard space $(X, \mathbf{S})$ such that X is uncountable.

Proposition 11.13. *Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be standard measurable spaces and let $f : X \to Y$ be measurable $[\mathbf{S}, \mathbf{T}]$. Then there exist Polish topologies σ and τ on X and Y, respectively, such that $\mathbf{S} = \mathbf{B}_\sigma$, $\mathbf{T} = \mathbf{B}_\tau$, and $f : (X, \sigma) \to (Y, \tau)$ is continuous.*

Proof. Let τ be a Polish topology on Y such that $\mathbf{T} = \mathbf{B}_\tau$, and let $\{B_n\}_{n\in\mathbb{N}}$ be a base of open sets for this topology. By Corollary 11.10 there exists a Polish topology σ on X such that $\mathbf{S} = \mathbf{B}_\sigma$ and $f^{-1}(B_n)$ is open for every $n \in \mathbb{N}$. The map $f : (X, \sigma) \to (Y, \tau)$ is obviously continuous. $\square$

Theorem 11.14. *Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be two standard measurable spaces, and let $f : X \to Y$ be a measurable function. Assume that $A, B \in \mathbf{S}$ and satisfy $f(A) \cap f(B) = \varnothing$. Then there exist disjoint sets $C, D \in \mathbf{T}$ such that $f(A) \subset C$ and $f(B) \subset D$, that is, $f(A)$ and $f(B)$ are* separated by measurable sets*.*

Proof. Let d_1 and d_2 be complete metrics on X and Y that define Polish topologies σ and τ on X and Y, respectively, such that $\mathbf{B}_\sigma = \mathbf{S}$, $\mathbf{B}_\tau = \mathbf{T}$, and $f : (X, \sigma) \to (Y, \tau)$ is continuous. This is possible by Proposition 11.13. Suppose, to get a contradiction, that $f(A)$ and $f(B)$ cannot be separated by measurable sets. We claim that there exist decreasing sequences

$$A = A^{(0)} \supset A^{(1)} \supset \cdots \supset A^{(n)} \supset \cdots$$

and

$$B = B^{(0)} \supset B^{(1)} \supset \cdots \supset B^{(n)} \supset \cdots$$

of closed sets such that for each $n \in \mathbb{N}$, $A^{(n)}$ and $B^{(n)}$ have diameter less than $1/n$ and $f(A^{(n)})$ cannot be separated from $f(B^{(n)})$ by measurable sets. These sequences of sets are constructed inductively. Suppose that $A^{(k)}$ and $B^{(k)}$ have been constructed for $k = 0, \ldots, n-1$, and write $A^{(n-1)} = \bigcup_{j\in\mathbb{N}} A_j$ and $B^{(n-1)} = \bigcup_{j\in\mathbb{N}} B_j$ as unions of closed measurable sets A_j and B_j with diameter less than $1/n$. If, for all $j, k \in \mathbb{N}$, $f(A_j)$ and $f(B_k)$ can be separated by measurable sets $C_{j,k} \supset f(A_j)$ and $D_{j,k} \supset f(B_k)$, then the sets

$$C = \bigcup_{j=1}^{\infty} \left[\bigcap_{k=1}^{\infty} C_{j,k} \right], \quad D = Y \setminus C,$$

separate $f(A^{(n-1)})$ and $f(B^{(n-1)})$, contrary to the inductive hypothesis. Thus there exist $j, k \in \mathbb{N}$ such that $f(A_j)$ and $f(B_k)$ cannot be separated by measurable sets. We complete the inductive process by setting $A^{(n)} = A_j$ and $B^{(n)} = B_k$.

Since d_1 is a complete metric, there exist two unique points

$$x_1 \in \bigcap_{n=1}^{\infty} A^{(n)},\ x_2 \in \bigcap_{n=1}^{\infty} B^{(n)}.$$

We have $f(x_1) \neq f(x_2)$ because $x_1 \in A$, $x_2 \in B$, and $f(A) \cap f(B) = \varnothing$. Therefore $f(x_1)$ and $f(x_2)$ have disjoint neighborhoods. In other words, there exist $p, q \in \mathbb{N}$ so $G_p \cap G_q = \varnothing$, $f(x_1) \in G_p$, and $f(x_2) \in G_q$. Using the continuity of f at x_1 and x_2, we see that $f(A^{(n)}) \subset G_p$ and $f(B^{(n)}) \subset G_q$ for sufficiently large n, showing that $f(A^{(n)})$ and $f(B^{(n)})$ are separated by measurable sets. This contradiction concludes the proof. $\square$

Repeated application of Theorem 11.14 shows that it is possible to separate more than two sets.

Corollary 11.15. *Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be two standard measurable spaces, and let $f : X \to Y$ be a measurable function. Assume that the sets $\{E_n\}_{n=1}^{\infty} \subset \mathbf{S}$ are such that the sets $\{f(E_n)\}_{n=1}^{\infty}$ are pairwise disjoint. Then there exist pairwise disjoint sets $\{F_n\}_{n=1}^{\infty} \subset \mathbf{T}$ such that $f(E_n) \subset F_n$ for all $n \in \mathbb{N}$.*

Proof. By Theorem 11.14, we can select sets $C_n \in \mathbf{T}$ such that $f(E_n) \subset C_n$ and $f(\bigcup_{m \neq n} E_m) \subset Y \setminus C_n$, for $n \in \mathbb{N}$, and set

$$F_n = C_n \cap \bigcap_{m \neq n} (Y \setminus C_m).$$

The sequence $\{F_n\}_{n=1}^{\infty}$ has the two desired properties. $\square$

Theorem 11.16. *Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be two standard measurable spaces, and let $f : X \to Y$ be an injective measurable function. Then for every $E \in \mathbf{S}$ we have $f(E) \in \mathbf{T}$.*

Proof. For every $E \in \mathbf{S}$, the space $(E, \mathbf{S}_E)$ is standard, and thus we may, and do, assume with no loss of generality that $E = X$. By Proposition 11.13, there exist Polish topologies σ and τ on X and Y such that $\mathbf{B}_\sigma = \mathbf{S}$, $\mathbf{B}_\tau = \mathbf{T}$ and f is continuous from (X, σ) to (Y, τ). Assume that σ is defined by the complete metric d on X. Construct a partition $X = \bigcup_{n=1}^{\infty} A_n$ into Borel sets of diameter less than $1/2$. Moreover, for each $n_1 \in \mathbb{N}$, A_{n_1} can be partitioned as $A_{n_1} = \bigcup_{m=1}^{\infty} A_{n_1,m}$ where each $A_{n_1,m}$ is a Borel set of diameter less than $1/4$, and continuing by induction, we construct, for each finite sequence $n_1, n_2, \ldots, n_k$ of natural numbers, a (possibly empty) Borel set $A_{n_1,n_2,\ldots,n_k}$ of diameter less than 2^{-k} such that the nonempty sets in the collection $\{A_{n_1,n_2,\ldots,n_{k-1},n} : n \in \mathbb{N}\}$ form a partition of $A_{n_1,n_2,\ldots,n_{k-1}}$. The sets $\{f(A_{n_1,n_2,\ldots,n_k}) : n_1, n_2, \ldots, n_k \in \mathbb{N}\}$ are pairwise disjoint, and Corollary 11.15 yields pairwise disjoint sets $\{B_{n_1,n_2,\ldots,n_k} : n_1, n_2, \ldots, n_k \in \mathbb{N}\} \subset \mathbf{T}$ such that

$$A_{n_1,n_2,\ldots,n_k} \subset B_{n_1,n_2,\ldots,n_k}, \quad n_1, n_2, \ldots, n_k \in \mathbb{N}.$$

Replacing $B_{n_1,\dots,n_k}$ by $B_{n_1} \cap B_{n_1,n_2} \cap \cdots \cap B_{n_1,\dots,n_k}$ we may also assume that $B_{n_1,\dots,n_k} \subset B_{n_1,\dots,n_{k-1}}$ for all $n_1, n_2, \dots, n_k \in \mathbb{N}$ and for all $k \in \mathbb{N} \setminus \{1\}$. The theorem will follow from the equality

$$f(X) = \bigcap_{k=1}^{\infty} \left[\bigcup_{n_1,n_2,\dots,n_k \in \mathbb{N}} (B_{n_1,n_2,\dots,n_k} \cap f(A_{n_1,n_2,\dots,n_k})^-) \right],$$

because the right-hand side of the equation is clearly a Borel set. Here we denote by $f(A_{n_1,n_2,\dots,n_k})^-$ the closure of $f(A_{n_1,n_2,\dots,n_k})$ in the topology τ. To prove this equality, it suffices to show that each point y in the intersection above is in $f(X)$. Indeed, given such a point y, there exists (see Problem 11I) a sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that

$$y \in B_{n_1,n_2,\dots,n_k} \cap \overline{f(A_{n_1,n_2,\dots,n_k})}, \quad k \in \mathbb{N}.$$

In particular, the sets $A_{n_1,n_2,\dots,n_k}$ corresponding to these indices are not empty. In addition the diameter of $A_{n_1,n_2,\dots,n_k}$ tends to zero as $k \to \infty$. Since d is a complete metric, there is a unique point $x \in \bigcap_{k=1}^{\infty} \overline{A_{n_1,n_2,\dots,n_k}}$. Continuity of f at x implies that $f(x) \in \bigcap_{k=1}^{\infty} \overline{f(A_{n_1,n_2,\dots,n_k})}$ and also that the diameters of the sets in this intersection tend to zero. We conclude that $f(x) = y$. □

A consequence of the above results is that there are very few equivalence classes of standard measurable spaces with respect to the equivalence relation of isomorphism.

Theorem 11.17. *Two standard measurable spaces are isomorphic if and only if they have the same cardinality. All uncountable standard measurable spaces are isomorphic to* $(\mathcal{C}, \mathbf{B}_{\mathcal{C}})$.

Proof. It suffices to verify the last statement. Let $(X, \mathbf{S})$ be an uncountable standard measurable space. By Theorem 11.8, $(\mathcal{C}, \mathbf{B}_{\mathcal{C}})$ embeds into $(X, \mathbf{S})$, while Theorems 11.12 and 11.16 show that $(X, \mathbf{S})$ embeds into $(\mathcal{C}, \mathbf{B}_{\mathcal{C}})$. Theorem 11.2 then applies to yield the desired conclusion. □

It was shown in Problem 1Z that analytic sets in a complete metric space X (see Definition 1.24) are precisely the images of continuous functions defined on some other complete metric space Y. (As noted in that problem, the space Y can be taken to be the space $\mathcal{S}$.) The following result allows us to define the concept of an analytic set in an arbitrary standard measurable space.

Proposition 11.18. *Let* $(X, \mathbf{S})$ *be a standard measurable space and let* $A \neq \varnothing$ *be a subset of* X. *The following assertions are equivalent.*

(1) *There exists a Polish topology* τ *on* X *such that* $\mathbf{S} = \mathbf{B}_\tau$ *and* A *is analytic in* (X, τ).
(2) *There exist a standard measurable space* $(Y, \mathbf{T})$ *and a measurable function* $f : Y \to X$ *such that* $f(Y) = A$.

(3) *There exists a measurable function* $f : \mathcal{S} \to X$ *such that* $f(\mathcal{S}) = A$.
(4) *There exist sets* $\{F_{n_1,\dots,n_k} : k, n_1, \dots, n_k\}$ *in* $\mathbf{S}$ *such that*

$$A = \bigcup_{(n_1,n_2,\dots)\in\mathbb{N}^{\mathbb{N}}} (F_{n_1} \cap F_{n_1,n_2} \cap \cdots \cap F_{n_1,\dots,n_k} \cap \cdots).$$

Sets A *that satisfy these equivalent conditions are said to be* analytic *in* $(X, \mathbf{S})$.

Proof. The equivalence of (1) and (4) follows immediately from Definition 1.24 and Corollary 11.10. Indeed the sets $\{F_{n_1,\dots,n_k} : k, n_1, \dots, n_k\}$ from condition (4) are closed in some Polish topology τ such that $\mathbf{S} = \mathbf{B}_\tau$. Similarly, the equivalence between (2) and (3) follows from Problem 1Z and Proposition 11.13 because the map f in (2) is continuous when X and Y are endowed with appropriate topologies. Finally, the equivalence of (2) and (3) follows from Theorem 11.8(3). □

For our next result about standard spaces it is useful to consider the *lexicographical order* on the space $\mathcal{S} = \mathbb{N}^{\mathbb{N}}$. Given $\mathbf{m} = (m_1, m_2, \dots)$ and $\mathbf{n} = (n_1, n_2, \dots)$ in $\mathcal{S}$, we write $\mathbf{m} < \mathbf{n}$ if $\mathbf{m} \neq \mathbf{n}$ and for the first index j such that $m_j \neq n_j$ we have $m_j < n_j$. We also write $\mathbf{m} \leq \mathbf{n}$ if either $\mathbf{m} = \mathbf{n}$ or $\mathbf{m} < \mathbf{n}$. The set $\mathcal{S}$ is totally ordered by this relation, but it is not well ordered as illustrated by the decreasing sequence $\{\mathbf{n}_k\}_{k\in\mathbb{N}}$, where

$$\mathbf{n}_k = (\underbrace{0, \dots, 0,}_{k \text{ times}} 1, 1, \dots), \quad k \in \mathbb{N}.$$

Proposition 11.19. *Every closed subset* $A \neq \varnothing$ *of* $\mathcal{S}$ *contains a least element.*

Proof. Define $m_1 \in \mathbb{N}$ to be the smallest integer with the property that there is some sequence in A with first entry equal to m_1. Then define inductively $m_k \in \mathbb{N}$ for $k \in \mathbb{N} \setminus \{1\}$ to be the smallest integer with the property that there exists a sequence in A starting with $m_1, \dots, m_{k-1}, m_k$. We show that $\mathbf{m} = (m_1, m_2, \dots)$ is an element of A. Indeed, the definition of this sequence ensures the existence of elements $\mathbf{n}_k \in A$ starting with $(m_1, \dots, m_k)$. Clearly $\mathbf{m} = \lim_{k\to\infty} \mathbf{n}_k$. The inequality $\mathbf{m} \leq \mathbf{n}$, $\mathbf{n} \in A$, follows immediately from the definition of $\mathbf{m}$. □

Theorem 11.16 implies that a measurable bijection between standard measure spaces is automatically an isomorphism. The following result is a substitute of this fact for noninjective maps.

Theorem 11.20. *Let* $(X, \mathbf{S})$ *and* $(Y, \mathbf{T})$ *be standard measurable spaces, and let* $f : X \to Y$ *be a measurable map. There exists a function* $g : f(X) \to Y$ *such that* $f(g(y)) = y$ *for all* $y \in f(X)$ *and, for every* $A \in \mathbf{T}$, $g^{-1}(A)$ *belongs to the* σ*-algebra generated by the analytic sets in* Y.

Proof. We may, and do, assume that $X \neq \varnothing$. Let σ and τ be Polish topologies on X and Y, respectively, such that $\mathbf{S} = \mathbf{B}_\sigma$, $\mathbf{T} = \mathbf{B}_\tau$, and $f : (X, \sigma) \to (Y, \tau)$ is continuous. We show first that it suffices to prove the theorem in the special case in which X equals the sequence space $\mathcal{S}$. To do this, choose a surjective continuous function $h : \mathcal{S} \to X$. The existence of such a function is guaranteed by Theorem 11.8(3). Then $f \circ h : \mathcal{S} \to Y$ is surjective. If the theorem is true for $X = \mathcal{S}$, then there exists a map $u : f(X) \to \mathcal{S}$, measurable when $f(X)$ is endowed with the σ-algebra generated by the analytic sets, such that $f(h(u(y))) = y$ for every $y \in f(X)$. Therefore the map $g = h \circ u$ satisfies the conditions in the statement of the theorem.

For the remainder of the proof, we assume that $X = \mathcal{S}$. For each $y \in f(X)$, the set $f^{-1}(\{y\})$ is closed in $\mathcal{S}$ and therefore it has a smallest element $g(y)$. To conclude the proof, it suffices to show that $g^{-1}(F)$ is analytic in Y for F in a collection of sets that generates the σ-algebra $\mathbf{B}_{\mathcal{S}}$. Such a collection is provided by the open sets

$$F_{\mathbf{m}} = \{\mathbf{n} \in \mathcal{S} : \mathbf{n} < \mathbf{m}\}, \quad \mathbf{m} \in \mathcal{S}.$$

We show that $g^{-1}(F_{\mathbf{m}}) = f(F_{\mathbf{m}})$, and these sets are analytic by Theorem 11.18. The inclusion $g^{-1}(F_{\mathbf{m}}) \subset f(F_{\mathbf{m}})$ follows from the equality $f(g(y)) = y$, $y \in g^{-1}(F_{\mathbf{m}})$. On the other hand, if $y = f(\mathbf{n})$ for some $\mathbf{n} \in F_{\mathbf{m}}$, then $g(y) \leq \mathbf{n} < \mathbf{m}$ so $g(y) \in F_{\mathbf{m}}$ The fact that the sets $F_{\mathbf{m}}$ generate $\mathbf{B}_{\mathcal{S}}$ is left as an exercise (see Problem 11O). □

Standard measure spaces are also easy to classify. The appropriate concept of isomorphism is as follows.

Definition 11.21. Two measure spaces $(X, \mathbf{S}, \mu)$ and $(Y, \mathbf{T}, \nu)$ are said to be *isomorphic* if there exist sets $X_0 \in \mathbf{S}$, $Y_0 \in \mathbf{T}$, and a bijection $f : X_0 \to Y_0$ such that:

(1) $\mu(X \setminus X_0) = \nu(Y \setminus Y_0) = 0$,
(2) f and f^{-1} are measurable relative to the σ-algebras $\mathbf{S}_{X_0}$ and $\mathbf{T}_{Y_0}$, and
(3) $\mu(f^{-1}(E)) = \nu(E)$ for every $E \in \mathbf{T}_{Y_0}$.

A measurable map $f : X \to Y$ (injective or not) satisfying condition (3) is said to be *measure preserving*.

Example 11.22. Endow the Cantor space $(\mathcal{C}, \mathbf{B}_{\mathcal{C}})$, where $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$, with the product measure $\mu = \rho \times \rho \times \cdots$, where $\rho(\{0\}) = \rho(\{1\}) = 1/2$. The map $f : \mathcal{C} \to [0, 1]$ defined by

$$f((t_n)_{n=1}^\infty) = \sum_{n=1}^\infty \frac{t_n}{2^n}, \quad (t_n)_{n=1}^\infty \in \mathcal{C},$$

is a bijection with range $[0, 1]$ when restricted to the complement of a (countable) set of measure zero $[\mu]$. (The countable set in question consists of those sequences $(t_n)_{n=1}^{\infty}$ such that $t_n = 1$ for all sufficiently large n.) This map realizes an isomorphism between $(\mathcal{C}, \mathbf{B}_{\mathcal{C}}, \mu)$ and $([0,1], \mathbf{B}_{[0,1]}, \lambda_1|\mathbf{B}_{[0,1]})$.

Example 11.23. The map $h : [0,1) \to [0,1)$ defined by

$$h(t) = \begin{cases} 2t, & 0 \le t < \frac{1}{2}, \\ 2t-1, & \frac{1}{2} \le t < 1, \end{cases}$$

is measure preserving on $([0,1), \mathbf{B}_{[0,1)}, \lambda_1|\mathbf{B}_{[0,1)})$, but it is not an isomorphism because it is two-to-one.

Proposition 11.24. *Let $(X, \mathbf{S}, \mu)$ be a standard measure space and let $A \in \mathbf{S}$ be an atom for μ. Then there exists $x \in E$ such that $\mu(\{x\}) = \mu(A)$.*

Proof. Fix a Polish topology τ on X such that $\mathbf{S} = \mathbf{B}_\tau$, and assume that τ is defined by a complete metric d. We show that for each $n \in \mathbb{N}$ we can find a set $A_n \in \mathbf{S}$ such that $A_n \subset A_{n-1}$, $\mu(A_n) = \mu(A)$, and the diameter of A_n is less than $1/n$. This follows by induction because, given $n \in \mathbb{N}$ and supposing that A_{n-1} with the desired properties has been constructed, A_{n-1} can be written as a union of countably many pairwise disjoint Borel sets of diameter less than $1/n$. Only one of these sets can have positive measure (equal to $\mu(A)$) and we define A_n to be that set. Then $B = \bigcup_{n=1}^{\infty}(A \setminus A_n)$ satisfies $\mu(B) = 0$ and $A \setminus B = \bigcap_{n=1}^{\infty} A_n$ has diameter zero, hence it consists of a single point x with $\mu(\{x\}) = \mu(A)$. □

Theorem 11.25. *Let $(X, \mathbf{S}, \mu)$ be a σ-finite standard measure space of positive measure such that $\mu(\{x\}) = 0$ for every $x \in X$ (that is, such that μ is atom-free). Then $(X, \mathbf{S}, \mu)$ is isomorphic to*

$$((0, \mu(X)), \mathbf{B}_{(0,\mu(X))}, \lambda_1|\mathbf{B}_{(0,\mu(X))}).$$

Proof. Since a σ-finite measure space is isomorphic to a countable direct sum of finite measure spaces, it suffices to prove the theorem for finite measure spaces. Assume for simplicity that $\mu(X) = 1$. The space X cannot be countable, and by Theorem 11.17 we can assume that $X = \mathbb{R}$ and $\mathbf{S} = \mathbf{B}_{\mathbb{R}}$. The function $f : \mathbb{R} \to [0,1]$ defined by $f(t) = \mu((-\infty, t))$ is easily seen to be monotone increasing and continuous, $\lim_{t\to-\infty} f(t) = 0$, and $\lim_{t\to+\infty} f(t) = 1$. Thus the range of f contains $(0,1)$. Denote by $G \subset \mathbb{R}$ the open set consisting of those points $t \in \mathbb{R}$ for which $\mu((t-\varepsilon, t+\varepsilon)) = 0$ for some $\varepsilon > 0$. We can then write G as the disjoint union of its components

$$G = \bigcup_n (\alpha_n, \beta_n),$$

each of which satisfies $\mu((\alpha_n,\beta_n])=0$. Setting

$$X_0=\mathbb{R}\setminus\bigcup_n(\alpha_n,\beta_n],$$

we see that $\mu(X\setminus X_0)=0$ and f is a bijection from $X\setminus X_0$ to $(0,1)$. Clearly, $f|(X\setminus X_0)$ and its inverse are Borel functions, and the equality $\mu(f^{-1}(E))=\lambda_1(E)$ is easily verified for intervals $E\subset[0,1]$. The theorem follows from the regularity of λ_1 (see Proposition 7.10). □

Isomorphic models for arbitrary standard σ-finite measures spaces are easily deduced from the preceding result. Indeed, if $(X,\mathbf{S},\mu)$ is such a space, the set $A=\{x:\mu(\{x\})>0\}$ is countable, and the space $(X\setminus A,\mathbf{S}_{X\setminus A},\mu|(X\setminus A))$ is a standard atom free space, so Theorem 11.25 can be applied. See Problem 11N for the precise statement.

We turn now to the study of conditional expectations in the context of standard spaces. Assume that $(X,\mathbf{S},\mu)$ is a probability space and $\mathbf{T}\subset\mathbf{S}$ is a σ-algebra. Fix, for every $E\in\mathbf{S}$, a conditional expectation f_E for the function χ_E relative to $\mathbf{T}$, that is, f_E is measurable $\mathbf{T}$ and

$$\int_F f_E\,d\mu=\mu(F\cap E),\quad F\in\mathbf{T}.$$

Given pairwise disjoint sets $E,F\in\mathbf{S}$, the function f_E+f_F is a conditional expectation for $\chi_{E\cup F}$ relative to $\mathbf{T}$, and therefore the equality

$$f_{E\cup F}(x)=f_E(x)+f_F(x)$$

holds for $[\mu]$-almost every $x\in X$. This equality may not hold for every $x\in X$, so in general we cannot conclude that the map $E\mapsto f_E(x)$ is finitely additive for any given $x\in X$. This issue can be overcome for standard probability spaces.

Theorem 11.26. *Let $(X,\mathbf{S},\mu)$ be a standard probability space, and let $\mathbf{T}\subset\mathbf{S}$ be a σ-algebra. There exists a map $\nu:X\times\mathbf{S}\to[0,1]$ with the following properties:*

(1) *For each $x\in X$, the map $\nu_x(E)=\nu(x,E)$, $E\in\mathbf{S}$, is a probability measure on $\mathbf{S}$.*
(2) *For each $E\in\mathbf{S}$, the map $x\mapsto\nu(x,E)$ is a conditional expectation of χ_E relative to $\mathbf{T}$.*

If $\nu':X\times\mathbf{S}\to[0,1]$ is another map with these properties, then $\nu'_x=\nu_x$ almost everywhere $[\mu]$.

Proof. The result is easily verified when X is countable, so we consider only the uncountable case, and we assume without loss of generality that $X=\mathbb{R}$ and $\mathbf{S}=\mathbf{B}_{\mathbb{R}}$. For every $r\in\mathbb{Q}$, fix a Borel function $f_r:\mathbb{R}\to[0,1]$ which is a conditional expectation of $\chi_{(-\infty,r]}$ relative to $\mathbf{T}$, that is, f_r is measurable $[\mathbf{T}]$ and

$$\int_E f_r \, d\mu = \mu(E \cap (-\infty, r]), \quad E \in \mathbf{T}.$$

For two rational numbers $r \leq s$, the inequality $f_r(x) \leq f_s(x)$ must hold for x outside some $[\mu]$-null set $E_{r,s} \in \mathbf{T}$. The functions

$$f_{-\infty}(x) = \inf_{r \in \mathbb{Q}} f_r(x), \;\; f_{+\infty}(x) = \sup_{r \in \mathbb{Q}} f_r(x)$$

are conditional expectations for the constant functions 0 and 1, and therefore $f_{-\infty}(x) = 1 - f_{+\infty}(x) = 0$ for x outside some $[\mu]$-null set $F \in \mathbf{T}$. Redefine the functions f_r on the $[\mu]$-null set $F \cup \left[\bigcup_{r \leq s} E_{r,s}\right]$ by setting, for example,

$$f_r(x) = \begin{cases} 0, & r < 0, \\ 1, & r \geq 0, \end{cases}$$

and set

$$g_t(x) = \inf\{f_r(x) : r \in \mathbb{Q} \cap (t, +\infty)\}, \quad t, x \in \mathbb{R}.$$

Then g_t is a conditional expectation of $\chi_{(-\infty,t]}$ relative to $\mathbf{T}$ for every $t \in \mathbb{R}$. In addition, for fixed x, the map $t \mapsto g_t(x)$ is monotone increasing, right continuous, and has limits 0 and 1 at $-\infty$ and $+\infty$, respectively. Therefore there exists a Borel probability measure ν_x on $\mathbb{R}$ satisfying

$$\nu_x((-\infty, t]) = g_t(x), \quad t, x \in \mathbb{R}.$$

The function $\nu(x, E) = \nu_x(E)$ obviously satisfies condition (1) in the statement of the theorem. We show now that it also satisfies condition (2). Indeed, the collection $\mathbf{C}$ of those sets $E \in \mathbf{S}$ which satisfy (2) is a σ-algebra containing all intervals of the form $(-\infty, t]$, and therefore $\mathbf{C} = \mathbf{S}$.

Assume finally that ν' is another map satisfying properties (1) and (2) (with ν' in place of ν). Given a rational number r, we must have $\nu'_x((-\infty, r]) = f_r(x)$ for x outside a $[\mu]$-null set E_r. It follows that $\nu'_x = \nu_x$ for x outside the $[\mu]$-null set $\bigcup_{r \in \mathbb{Q}} E_r$. $\square$

The function ν constructed above is called a *system of conditional measures* or a *disintegration* of the measure μ relative to $\mathbf{T}$.

Example 11.27. Consider standard measurable spaces $(X, \mathbf{S})$ and $(Y, \mathbf{T})$, and let μ be a probability measure on the product space $(X \times Y, \mathbf{S} \times \mathbf{T})$. The collection

$$\mathbf{C} = \{E \times Y : E \in \mathbf{S}\}$$

is a σ-algebra contained in $\mathbf{S} \times \mathbf{T}$, and therefore Theorem 11.26 produces numbers $\nu_{x,y}(G) \in [0, 1]$, $x \in X$, $y \in Y$, $G \in \mathbf{S} \times \mathbf{T}$. Since the map $(x, y) \mapsto \nu_{x,y}(G)$ is measurable $[\mathbf{C}]$, it follows that $\nu_{x,y}(G)$ is independent of $y \in Y$. We can thus define $\nu_x(G) = \nu_{x,y}(G)$. The fact that $x \mapsto \nu_x(G)$ is a conditional expectation relative to $\mathbf{C}$ means that

$$\mu(G \cap (E \times Y)) = \int_{E\times Y} \nu_x(G)\, d\mu(x,y), \quad E \times Y \in \mathbf{C}, G \in \mathbf{S} \times \mathbf{T}.$$

The integral above can be written in terms of the *marginal* ρ on $(X, \mathbf{S})$ defined by

$$\rho(E) = \mu(E \times Y), \quad E \in \mathbf{S}.$$

Thus,

$$\mu(G \cap (E \times Y)) = \int_E \nu_x(G)\, d\rho(x), \quad E \in \mathbf{S}, G \in \mathbf{S} \times \mathbf{T}.$$

Consider now the marginal measures $\sigma_x : \mathbf{T} \to [0,1]$ defined by

$$\sigma_x(F) = \nu_x(X \times F), \quad x \in X, F \in \mathbf{T}.$$

Given $E \in \mathbf{S}$ and $F \in \mathbf{T}$, we have

$$\mu(E \times F) = \mu((X \times F) \cap (E \times Y)) = \int_E \nu_x(X \times F)\, d\rho(x) = \int_E \sigma_x(F)\, d\rho(x).$$

This extends easily to

$$\mu(G) = \int_X \sigma_x(G_x)\, d\rho(x), \quad G \in \mathbf{S} \times \mathbf{T},$$

so μ can be viewed as a generalized product measure.

The following result is used in the construction of probability measures, other than product measures, on infinite product spaces. The result, due to Kolmogorov, does not hold for arbitrary measure spaces.

Theorem 11.28. *Let $(X, \mathbf{S}) = \prod_{n=1}^{\infty}(X_n, \mathbf{S}_n)$ be a product of standard measurable spaces, and let $\mathbf{T}_N \subset \mathbf{S}$, $N \in \mathbb{N}$, be the σ-algebra consisting of sets of the form*

$$E \times \prod_{n=N+1}^{\infty} X_n, \quad E \in \prod_{n=1}^{N} \mathbf{S}_n.$$

Assume that $\mu : \mathbf{T} = \bigcup_{N\in\mathbb{N}} \mathbf{T}_N \to [0,1]$ is a set function such that $\mu|\mathbf{T}_N$ is countably additive for all $N \in \mathbb{N}$. Then μ is countably additive on $\mathbf{T}$.

Proof. We may assume that each X_n is a complete metric space and $\mathbf{S}_n = \mathbf{B}_{X_n}$. Suppose to the contrary that there exist a set $E \in \mathbf{S}$ and a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathbf{S}$ such that $E = \bigcup_{n=1}^{\infty} E_n$ but

$$\alpha = \mu(E) - \sum_{n=1}^{\infty} \mu(E_n) > 0.$$

The sets $A_n = \bigcup_{m=n}^{\infty} E_m = E \setminus \bigcup_{m=1}^{n-1} E_m$ belong to $\mathbf{T}$, $A_n \supset A_{n+1}$ for $n \in \mathbb{N}$, and $\mu(A_n) \geq \alpha$. We obtain a contradiction by showing that these conditions imply that $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$. Indeed, Theorem 7.3 applied to the measure $\mu|\mathbf{T}_N$ implies the existence of compact sets $K_n \subset A_n$ such that $\mu(A_n \setminus K_n) < 2^{-n}\alpha$ for all $n \in \mathbb{N}$. Note that

$$A_n \setminus (K_1 \cap \cdots \cap K_n) \subset (A_1 \setminus K_1) \cup \cdots \cup (A_n \setminus K_n),$$

and thus

$$\mu(A_n \setminus (K_1 \cap \cdots \cap K_n)) \leq \sum_{k=1}^{n} \frac{\alpha}{2^n} < \alpha.$$

It follows that the intersections $K_1 \cap \cdots \cap K_n$ are not empty and therefore $\bigcap_{n=1}^{\infty} A_n \supset \bigcap_{n=1}^{\infty} K_n \neq \varnothing$ since the K_j have the finite intersection property. □

Theorem 11.20 has a more convenient form in the context of standard measure spaces.

Theorem 11.29. *Let $(X, \mathbf{S})$ be a standard measurable space, let $(Y, \mathbf{T}, \nu)$ be a σ-finite standard measure space, and let $f : X \to Y$ be a surjective measurable map. There exist a set $E \in \mathbf{T}$ and a measurable function $h : Y \setminus E \to X$ such that $\nu(E) = 0$ and $f(h(y)) = y$ for every $y \in Y \setminus E$.*

Proof. There is a partition $Y = \bigcup_{n\in\mathbb{N}} Y_n$ such that for each $n \in \mathbb{N}$, $Y_n \in \mathbf{T}$, $\nu(Y_n) < +\infty$, and Y_n is either a finite atom or $\nu|Y_n$ is atom free. It suffices to prove the theorem for each of the maps $f_n = f|f^{-1}(Y_n)$. Equivalently, we can restrict ourselves to the case in which $\nu(Y) < +\infty$ and either Y is a finite atom or ν_Y is atom free. If Y is a finite atom, simply pick $y \in Y$ such that $\nu(\{y\}) = \nu(Y)$, set $E = Y \setminus \{y\}$, and define $h(y)$ to be any point in $f^{-1}(y)$. Suppose therefore that $0 < \nu(Y) < +\infty$ and ν is atom free. By Theorem 11.25, we may assume that $Y = (0, \nu(Y))$, $\mathbf{T} = \mathbf{B}_Y$, and $\nu = \lambda_1|\mathbf{B}_Y$. Let $g : Y \to X$ be a function that satisfies the conditions in Theorem 11.20, let $\{A_n\}_{n\in\mathbb{N}} \subset \mathbf{S}$ be a sequence of sets which generates the σ-algebra $\mathbf{S}$ and set $B_n = f(A_n)$, $n \in \mathbb{N}$. It follows from Proposition 5.33 that each set B_n is measurable $[\lambda_1]$, so there exist Borel sets C_n and D_n such that $C_n \subset B_n \subset D_n \subset Y$ and $\nu(D_n \setminus C_n) = 0$. To conclude the proof, we set $E = \bigcup_{n\in\mathbb{N}}(D_n \setminus C_n)$ and $h = g|(Y \setminus E)$. The measurability of h follows because

$$h^{-1}(A_n) = B_n \cap (Y \setminus E) = C_n \cap (Y \setminus E)$$

is a Borel set for every $n \in \mathbb{N}$. □

Problems

11A. Show that the metric d defined on a product space in Example 11.7 is complete. Also verify that the functions f, g, h of that example are homeomorphisms onto their ranges.

11B. Show that the map g defined in the proof of Theorem 11.8 is indeed a homeomorphism.

11C. Verify that the metric d' defined on the open set E in the first part of the proof of Theorem 11.9 is complete, and show that it induces the same topology as the original metric d.

11D. Let (X_1, d_1) and (X_2, d_2) be metric spaces with X_2 complete, let $A \subset X_1$ be an arbitrary set, and let $f : X_1 \to X_2$ be a continuous map. Denote by A_1 the set of those points $x \in A^-$ with the following property: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_2(f(y), f(z)) < \varepsilon$ for any points y, z in X_1 satisfying $d_1(y, x) < \delta$ and $d_1(z, x) < \delta$.

(i) Show that A_1 is a G_δ set in X_1.
(ii) Show that f extends to a continuous function $f_1 : A_1 \to X_2$.
(iii) Suppose that A is a Polish space with the induced topology. Show that A is a G_δ set in X_1. (Hint: consider the identity map on A.)
(iv) Conversely, show that every G_δ set in a Polish space is a Polish space in the induced topology.

11E. Recall that $\mathcal{S} = \mathbb{N}^{\mathbb{N}}$ is endowed with a Polish topology which is the product topology obtained when each factor $\mathbb{N}$ is given the discrete topology.

(i) Show that every compact subset $K \subset \mathcal{S}$ has empty interior in $\mathcal{S}$.
(ii) Let $X \neq \varnothing$ be a Polish space such that every compact subset $K \subset X$ has empty interior in X. Assume, in addition that X is *totally disconnected*, that is, the topology of X has a base consisting of sets which are closed as well as open. Show that X is homeomorphic to $\mathcal{S}$.
(iii) Let $X \neq \varnothing$ be a compact, totally disconnected Polish space without isolated points. Show that X is homeomorphic to $\mathcal{C}$.

11F. (Cantor-Bendixson) Given a Polish space X, denote by $A \subset X$ the set consisting of those points $x \in X$ which have a countable open neighborhood. Show that A is open and at most countable. Show that the set $X \setminus A$ has no isolated points (and hence is a perfect set), and it is equal to the intersection $\bigcap_{\mathrm{card}(\alpha) \le \aleph_0} X^\alpha$, where X^α is defined inductively for countable ordinals α as follows: $X^0 = X$, and if X^α has been defined, then $X^{\alpha+1}$ is the set of those $x \in X^\alpha$ which are not isolated in X^α. Moreover, if α is a limit ordinal, then X^α is defined as $\bigcap_{\beta<\alpha} X^\beta$.

11G. Verify that the maps f and g defined in the proof of Theorem 11.8 are indeed homeomorphisms.

11H. In the proof of Theorem 11.14, verify that the sets C and D separate $f(A)$ and $f(B)$. Similarly, verify that the sets F_n in Corollary 11.15 are pairwise disjoint and $f(E_n) \subset F_n$, $n \in \mathbb{N}$.

11.I With the notation as in the proof of Theorem 11.16, consider a point

$$y \in \bigcap_{k=1}^{\infty} \left[\bigcup_{n_1, n_2, \ldots, n_k \in \mathbb{N}} (B_{n_1, n_2, \ldots, n_k} \cap f(A_{n_1, n_2, \ldots, n_k})^-) \right],$$

so for every $k \in \mathbb{N}$, $y \in B_{n_1(k), n_2(k), \ldots, n_k(k)} \cap f(A_{n_1, n_2, \ldots, n_k})^-$ for some integers $n_1(k), \ldots, n_k(k) \in \mathbb{N}$. Show that $n_j(k) = n_j(k+1)$ when $j \le k$.

11J. Verify that the function g_t defined in the proof of Theorem 11.26 is indeed a conditional expectation of the function $\chi_{(-\infty,t]}$ relative to $\mathbf{T}$.

11K. Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be standard measurable spaces, and let $f : X \to Y$ be a measurable bijection of X onto Y. Show that f^{-1} is also measurable, and thus that the two measurable spaces are isomorphic.

11L. Let τ and τ' be two Polish topologies on a set X such that $\mathbf{B}_\tau \subset \mathbf{B}_{\tau'}$. Show that $\mathbf{B}_\tau = \mathbf{B}_{\tau'}$.

11M. Let $(X.\mathbf{S})$ be a standard measurable space, and let $E \in \mathbf{S}$ be an uncountable set. Show that the cardinality of E is $\mathfrak{c} = 2^{\aleph_0}$.

11N. Suppose that $(X, \mathbf{S}, \mu)$ is a σ-finite standard measure space. Show that there exist measure spaces $(X_1, \mathbf{S}_1, \mu_1)$ and $(X_2, \mathbf{S}_2, \mu_2)$ such that

(i) $(X, \mathbf{S}, \mu)$ is isomorphic to the direct sum of $(X_1, \mathbf{S}_1, \mu_1)$ and $(X_2, \mathbf{S}_2, \mu_2)$,
(ii) μ_1 is atom free and μ_2 is purely atomic,
(iii) X_1 is a (possibly empty) open interval $(0, \alpha)$ in $\mathbb{R}$, $\mathbf{S}_1 = \mathbf{B}_{X_1}$, and $\mu_1 = \lambda_1|\mathbf{S}_1$,
(iv) X_2 is an initial segment of $\mathbb{N}$ (possibly empty) and $\mu_2(\{n\}) > 0$ for every $n \in X_2$.

11O. Given $k, a_1, \ldots, a_k \in \mathbb{N}$, show that the set $V_{a_1,\ldots,a_k} = \{\mathbf{n} = (n_1, \ldots) \in \mathcal{S} : n_1 = a_1, \ldots, n_k = a_k\}$ belongs to the algebra generated by the sets $F_{\mathbf{m}}$ used in the proof of Theorem 11.20, with $\mathbf{m}$ of the form $(m_1, \ldots, m_p, 1, 1, \ldots)$ for some $p, m_1, \ldots, m_p \in \mathbb{N}$. (The sets $V_{a_1,\ldots,a_k}$ form a base for the topology of $\mathcal{S}$.)

11P. Let $(X, \mathbf{S})$ be a standard measurable space, and let $A \subset X$ be an analytic set such that $X \setminus A$ is also analytic. Show that $A \in \mathbf{S}$.

11Q. Let X be a separable metric space and let $\{U_n\}_{n\in\mathbb{N}}$ be a base of open sets in X.

(i) Show that the set
$F = \{(x, \mathbf{n}) \in X \times \mathcal{S} : x \in \bigcap_{k=1}^{\infty}(X \backslash U_k), \mathbf{n} = (n_1, n_2 \ldots)\}$
is closed in $X \times \mathcal{S}$ and that every closed subset of X is equal to some $\mathcal{S}$-section $F^{\mathbf{n}}$ of F. (The $\mathcal{S}$-sections are defined before the statement of Theorem 8.3.)
(ii) Let $F \subset \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ be a closed subset such that the collection $\{F^{\mathbf{n}} : \mathbf{n} \in \mathcal{S}\}$ contains all the closed subsets of $\mathcal{S} \times \mathcal{S}$. Define a set $A \subset \mathcal{S} \times \mathcal{S}$ by
$A = \{(\mathbf{p}, \mathbf{n}) :$ there exists $\mathbf{p} \in \mathcal{S}$ such that $(\mathbf{p}, \mathbf{q}, \mathbf{n}) \in F\}$.
Show that A is analytic in $\mathcal{S} \times \mathcal{S}$ and that every analytic subset of $\mathcal{S}$ is equal to some $\mathcal{S}$-section $A^{\mathbf{n}}$ of A.

11R. Suppose that $(X, \mathbf{S})$ is an uncountable standard measure space. Show that there exists an analytic set $A \in \mathbf{S} \times \mathbf{S}$ such that every analytic subset of X is equal to some section A^y of A.

(i) Deduce that the set

$$B = \{x \in X : (x, x) \in A\}$$

is analytic in X but its complement $X \setminus B$ is not analytic. (Hint: By Problem 11P, $B \notin \mathbf{S}$.)
(ii) Show that the cardinality of the collection of analytic sets in X equals $\mathfrak{c} = 2^{\aleph_0}$.

11S. Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be standard measurable spaces, let A be an arbitrary subset of X, and let $f : A \to Y$ be measurable $[\mathbf{S}_A, \mathbf{T}]$. Show that there exists a measurable function $g : X \to Y$ such that $g|A = f$. (Hint: Consider first the case of a function with countable range. Assume that Y is a separable complete metric space and approximate f uniformly by countably valued functions f_n. The set where the corresponding functions g_n have a limit as $n \to \infty$ is measurable.)

11T. Let $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ be standard measurable spaces, let A be an arbitrary subset of X, let B be an arbitrary subset of Y, and let $f : A \to B$ be an isomorphism of $(A, \mathbf{S}_A)$ to $(B, \mathbf{T}_B)$. Show that f can be extended to an isomorphism from $(A_1, \mathbf{S}_{A_1})$ to $(B_1, \mathbf{T}_{B_1})$, where $A \subset A_1 \in \mathbf{S}$ and $B \subset B_1 \in \mathbf{T}$. (Hint: Apply Problem 11S to f and f^{-1} to obtain extensions g_1, g_2 and consider the set $\{x \in X : g_1(g_2(x)) = x\}$.)

11U. Suppose that a function $f : [0,1] \times [0,1] \to \mathbb{R}$ is such that the map $x \mapsto f(x,y)$ is Borel measurable for every $y \in [0,1]$ and the map $y \mapsto f(x,y)$ is continuous for every $x \in [0,1]$. Show that f is Borel measurable. (Hint: $f(x,y) = \lim_{n\to\infty} f(x, 2^{-n}[2^n y])$, where $[2^n y]$denotes the integer part of $2^n y$. Here we use $[x]$ to denote the integer part of x, that is, $[x]$ is an integer satisfying $[x] \le x < [x]+1$.)

11V. Denote by X the space $[0,1]^{[0,1]}$ of all functions $g : [0,1] \to [0,1]$ endowed with the topology of pointwise convergence, so X is a compact Hausdorff space. Define $f_n : [0,1] \to X$, $n \in \mathbb{N}$, by

$$(f_n(x))(y) = \max\{0, 1 - n|x-y|\}, \quad x, y \in [0,1].$$

Show that each f_n is a continuous function but the pointwise limit f of $\{f_n\}_{n\in\mathbb{N}}$ is not Borel measurable.

11W. Let $(X, \mathbf{S})$ be a measurable space.

(i) Suppose that there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathbf{S}$ that generates $\mathbf{S}$ and separates the points of X, that is, for every $x, y \in X$ such that $x \ne y$ there is $n \in \mathbb{N}$ such that $x \in A_n$ and $y \notin A_n$. Show that $(X, \mathbf{S})$ embeds into $([0,1], \mathbf{B}_{[0,1]})$. (Hint: $f = \sum_{n\in\mathbb{N}} 3^{-n}\chi_{A_n}$.)

(ii) Suppose that $(X, \mathbf{S})$ is standard and $\{A_n\}_{n\in\mathbb{N}} \subset \mathbf{S}$ is a sequence that separates the points of X. Show that $\{A_n\}_{n\in\mathbb{N}}$ generates $\mathbf{S}$.

11X. Show that the σ-algebra generated by the sets $\{x\}$, $x \in [0,1]$, is not countably generated. (Thus a sub-σ-algebra of a countably generated σ-algebra (namely, $\mathbf{B}_{[0,1]}$) need not be countably generated.)

11Y. (Dieudonné) Consider the standard measure space $([0,1], \mathbf{T}, \lambda_1|\mathbf{T})$ where $\mathbf{T} = \mathbf{B}|[0,1]$, and let $A \subset [0,1]$ be a set that is not Lebesgue measurable (Example 5.27).

(i) Show that there exist sets $A_+, A_- \in \mathbf{T}$ such that $A_- \subset A \subset A_+$, $\lambda_1(A_+ \setminus A_-) > 0$, and for every $B_+, B_- \in \mathbf{T}$ satisfying $B_- \subset A \subset B_+$ we have $\lambda_1(A_+ \setminus B_+) = \lambda_1(B_- \setminus A_-) = 0$. The set A_+ can be chosen to be a G_δ set and A_- can be chosen to be an F_σ set.

(ii) Fix sets A_+ and A_- as in (i) and denote by $\mathbf{S}$ the σ-algebra generated by $\mathbf{T} \cup \{A\}$. Show that $\mathbf{S}$ consists of all sets of the form $E \cup (F \cap A) \cup G$ where $E, F, G \in \mathbf{T}$ satisfy $E \subset A_-$, $F \subset A_+ \setminus A_-$, and $G \subset [0,1] \setminus A_-$.

(iii) Define $\mu : \mathbf{S} \to [0,1]$ by setting $\mu(E \cup (F \cap A) \cup G) = \lambda_1(E) + \lambda_1(F) + \lambda_1(G)$, where E, F, G are as in (ii). Show that μ is a probability measure. (The measure μ is a restriction of the outer measure λ_1^* used to define Lebesgue measure.)

(iv) Show that there does not exist a disintegration of μ relative to $\mathbf{T}$. (Hint: Suppose such a disintegration $\nu : [0,1] \times \mathbf{S} \to [0,1]$ exists. Show that there exists a set $E \in \mathbf{T}$ such that $\lambda_1(E) = 0$ and $\nu(x, F) = \delta_x(F)$ for every $x \in [0,1] \setminus E$ and $F \in \mathbf{S}$. Derive a contradiction when $F = A$. As usual, δ_x denotes the unit point mass at x.)

11Z. Show that there exist subsets $\{A_n\}_{n\in\mathbb{N}}$ of $[0,1]$ with the following properties.

(i) A_n is not Lebesgue measurable, $n \in \mathbb{N}$.
(ii) If $E \subset [0,1]\backslash\bigcup_{k=1}^{n} A_k$ is a Borel set, then $\lambda_1(E) = 0$.
(iii) $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$. (Hint: The equivalence relation $\sim$ on $[0,1]$ defined by the requirement that $s \sim t$ precisely when $t - s \in \mathbb{Q}$ has countably infinite equivalence classes. Write each equivalence class E as $E = \{t_{1E}, t_{2E}, \dots\}$ in such a way that $t_{nE} - t_{nF}$ is constant modulo 1 for any two equivalence classes E, F. Define $B_n = \{t_{nE} : E \text{ an equivalence class}\}$ and set $A_n = [0,1]\backslash B_n$.)

11AA. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets satisfying properties (a-c) of Problem 11Z. For each $n \in \mathbb{N}$, set $X_n = [0,1]$ and let $\mathbf{S}_n$ be the σ-algebra generated by $\mathbf{B}_{[0,1]}$ and A_n. Using the notation of Theorem 11.28, define a set function $\mu : \mathbf{T} \to [0,1]$ as follows. For every set $E \in \mathbf{T}$, define

$$D_E = \{t \in [0,1] : (t, t, \dots) \in E\}$$

and $\mu(E) = \lambda_1^*(D_E) = \inf\{\lambda_1(F) : F \in \mathbf{B}_{[0,1]}, F \supset D_E\}$. Show that $\mu|\mathbf{T}_n$ is countably additive for each $n \in \mathbb{N}$ but μ is not countably additive on $\mathbf{T}$. (Hint: The sets $E_n = A_1 \times \cdots \times A_n \times X \times X \times \cdots$ satisfy $E_n \supset E_{n+1}$, $\mu(E_n) = 1$, and $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$.)

11BB. Show that there exists a Borel set $E \subset [0,1] \times [0,1]$ such that the set $\{y \in [0,1] :$ there exists $x \in [0,1]$ such that $(x,y) \in E\}$ is not Borel. (Hint: Replace $[0,1]$ by $\mathcal{S}$ and consider the graph of a continuous function $f : \mathcal{S} \to \mathcal{S}$ whose range is not a Borel set.)