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FINITE AXIOMATIZABILITY USING ADDITIONAL PREDICATES

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Introduction.[1] By a *theory* we shall always mean one of first order, having finitely many non-logical constants. Then for theories with identity (as a logical constant, the theory being closed under deduction in first-order logic with identity), and also likewise for theories without identity, one may distinguish the following three notions of axiomatizability. First, a theory may be recursively axiomatizable, or, as we shall say, simply, *axiomatizable.* Second, a theory may be *finitely axiomatizable using additional predicates* (*f. a.*$^+$), in the syntactical sense introduced by Kleene [9]. Finally, the italicized phrase may also be interpreted semantically. The resulting notion will be called *s. f. a.*$^+$. It is closely related to the modeltheoretic notion **PC** introduced by Tarski [16], or rather, more strictly speaking, to $\mathbf{PC} \cap \mathbf{AC}_\delta$.

For arbitrary theories with or without identity, it is easily seen that *s. f. a.*$^+$ implies *f. a.*$^+$ and it is known that *f. a.*$^+$ implies axiomatizability. Thus it is natural to ask under what conditions the converse implications hold, since then the notions concerned coincide and one can pass from one to the other.

Kleene [9] has shown: (1) For arbitrary theories without identity, axiomatizability implies *f. a.*$^+$. It also follows from his work that: (2) For theories with identity which have only infinite models, axiomatizability implies *f. a.*$^+$.

The three main results of this paper, which supplement those of Kleene, are the following. Modifying the proof of (2), we obtain the stronger result that: (3) For theories with or without identity which have only infinite models, axiomatizability implies *s. f. a.*$^+$, so that all three notions coincide for such theories (§ 2). This result is then used to prove: (4) For arbitrary theories with identity, *f. a.*$^+$ implies *s. f. a.*$^+$ (§ 3). In contrast to (1)—(4), and disclosing a rather unexpected difference between theories with and theories without identity, we also show: (5) For arbitrary theories without identity, *f. a.*$^+$ and *s. f. a.*$^+$ do not always coincide, while for arbitrary theories with identity, axiomatizability and *f. a.*$^+$ do not always coincide (§ 4).

In § 5, we note that, as a consequence of a recent result of Tarski, the new additional predicates required can be reduced to one binary predicate in (1), (2), and (3).

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[1] Summaries of the results of 3.5 to 3.7 of this paper were presented to the American Mathematical Society in [4], and of all other results in [20]. (Professor Tarski has informed us that the phrase in line 11 of [20] referring to him is not correct.)

For convenience, only the logic with identity is considered in § 2—§ 5, and in § 1, where the notions we have been discussing are defined precisely. In § 6, the various results concerning logic without identity (including a proof of (1)) are collected. An idea involved in the proof of (3) leads to some remarks about second-order definability and a problem of Łoś [10], which constitute the final § 7.

§ 1. Preliminaries. We suppose given a universal first-order language Λ. The language Λ has the distinct *logical constants* / (the "neither-nor" symbol), $\wedge$ (the universal quantifier), and $=$ (the binary identity predicate). Λ has also *non-logical constants*, each classified as a *predicate* of a certain *rank* (number of places) $n = 1, 2, 3, \ldots$, or as an *operation symbol* of a certain *rank* $n = 0, 1, 2, \ldots$. It is assumed that Λ has a denumerable infinity of predicates of each rank $n \geqq 1$, and a denumerable infinity of operation symbols of each rank $n \geqq 0$. The *symbols* of Λ are the constants together with two additional symbols $\mathbf{v}$ and $'$. The set of *expressions* of Λ is a set containing the symbols and closed under a binary, associative operation called *concatenation* (denoted by juxtaposition). It is assumed that any expression can be represented in one and only one way as a (non-empty) finite concatenation of symbols.[2]

The expressions $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{v}''$, ... (denoted, respectively, by $v_0, v_1, v_2, \ldots$) are called *variables*. Certain sets of expressions, called the sets of *atomic terms*, *terms*, *atomic formulas*, *formulas*, and *sentences*, are distinguished in the usual way, according to the so-called parenthesis-free notation.[3] The notion of a sentence being *logically valid*, and that of a sentence being (logically) *derivable* from a set of sentences, are both understood in the sense of the first-order logic with identity.

A *theory* T is determined by giving a finite list $C_0, \ldots, C_{p-1}$, of non-logical constants, and a set V. The *expressions of* T are all expressions which are concatenations of the symbols $/, \wedge, =, \mathbf{v}, ', C_0, \ldots, C_{p-1}$. By a *term* (*formula*, *sentence*, etc.) *of* T we mean a *term* (*formula*, *sentence*, etc.) which is an expression of T. It is assumed that V is a set of sentences of T (called the *valid* sentences of T) and that any sentence of T derivable from V belongs to V.

It is well known that the set of expressions of any given theory T can be mapped in a one-to-one way onto the set ω of natural numbers in such a way that the operation corresponding to concatenation under the mapping is recursive. A set or relation or operation among expressions of T is called *recursive* or *recursively enumerable* provided its image under such a mapping

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[2] These assumptions characterize the language Λ up to isomorphism.

[3] Sentences are formulas with no free variables; the notions of a free or bound occurrence of a variable in a formula are to be understood, also, in the ordinary way.

is recursive or recursively enumerable. It is easily seen that these notions do not depend on the particular mapping chosen. A theory T such that the set of expressions of T actually coincides with ω and the concatenation operation of T is recursive (in the number-theoretical sense) is called an *arithmetized* theory.

A theory is called *decidable* if its set of valid sentences is recursive. A set W of sentences valid in T is called a *set of axioms* for the theory T if any valid sentence of T is derivable from W. T is called *axiomatizable* if it has a recursive set of axioms, and *finitely axiomatizable* (*f. a.*) if it has a finite set of axioms.

We shall employ in the metalanguage, when naming expressions, all the usual logical connectives, as well as parentheses.[4] By the closure of a formula φ is meant the sentence $\Lambda v_{k_0} \ldots \Lambda v_{k_{n-1}} \varphi$, where $k_0 < k_1 < \ldots < k_{n-1}$, and $v_{k_0}, \ldots, v_{k_{n-1}}$ are the distinct variables free in φ. We write $\dashv_T \varphi$ to mean that the closure of φ is valid in T. To increase the readability of formulas, we assume, throughout, that x, y, z, u, w, x′, y′, z′, u′, and w′, are, in order, the variables $v_0, v_1, \ldots, v_9$. (Roughly speaking, particular variables and non-logical constants will be denoted by non-italicized letters.) Given arbitrary terms $\tau_0, \ldots, \tau_{n-1}$ in which no variables occur, and a formula φ whose free variables are $v_0, \ldots, v_{n-1}$, we mean by $\varphi(\tau_0, \ldots, \tau_{n-1})$ the formula obtained from φ by substituting, simultaneously, τ_i for all free occurrences of v_i, $i = 0, \ldots, n-1$.

By a *realization* of a theory T based on $C_0, \ldots, C_{p-1}$, and V, is meant an *algebraic system* $\mathfrak{A} = \langle A, X_0, \ldots, X_{p-1} \rangle$, where A is a non-empty set, and, for $i < p$, X_i is an n-ary relation or operation among the elements of A according as C_i is a predicate or operation symbol of the same rank n.[5] The class of all realizations of a theory is called a *similarity class*. By an *assignment* of values in the set A to the variables is meant a function f, whose domain is the set Vb of all variables, whose values are members of A, and such that for some finite set U of variables, $f(\xi) = f(\eta)$ whenever ξ and η are variables not belonging to U. We denote by $A^{(Vb)}$ the set of all assignments with values in A. (The last condition in the definition of an assignment is added in order that A and $A^{(Vb)}$ have the same power, when A is infinite). We assume it is understood under what conditions a member f of $A^{(Vb)}$ is said to *satisfy* a formula φ of a theory T in a realization $\mathfrak{A} = \langle A, \ldots \rangle$ of T.[6] For example, if ξ and η are variables, f satisfies the

[4] As is well known, this can be justified by regarding Λ, for example, as a binary operation such that, for any formulas φ and ψ, $\varphi \land \psi = //\varphi\varphi/\psi\psi$.

[5] An n-ary relation is considered to be a set of ordered n-tuples $\langle x_0, \ldots, x_{n-1} \rangle$. However, ordinarily we shall identify $\langle x \rangle$ and x; thus, a singulary relation is simply an arbitrary set. By the same identification, a 0-ary operation (among the elements of A) is considered to be simply an object (belonging to A).

[6] For a definition of this notion, cf. [13].

formula $\xi = \eta$ in $\mathfrak{A}$ if and only if $f(\xi) = f(\eta)$. A sentence σ is said to be *true* in $\mathfrak{A} = \langle A, \ldots \rangle$, and $\mathfrak{A}$ is called a model of σ, if any $f \in A^{(Vb)}$ satisfies σ in $\mathfrak{A}$. $\mathfrak{A}$ is called a *model* of T if it is a model of all valid sentences of T. By the power of $\mathfrak{A} = \langle A, \ldots \rangle$ we mean the power of A. We shall employ frequently, without explicit mention, the fact that the same sentences are true in isomorphic algebraic systems (cf. [15], Theorem 22).

With minor modifications, the above terminology is that of Tarski in [18]. We turn now to the specific notions to be discussed in this article.

DEFINITION 1.1. *A theory T is called finitely axiomatizable using additional predicates (abbreviated, f. a.$^+$), if there is a theory T' such that:*

(*) *the non-logical constants of T' are those of T together with some additional predicates, T' is finitely axiomatizable, and an arbitrary sentence of T is valid in T if and only if it is valid in T'.*

DEFINITION 1.2. *A theory T is called finitely axiomatizable using additional predicates, in the semantical sense (abbreviated, s. f. a.$^+$), if there is a theory T' such that:*

(*) *the non-logical constants of T' are those of T together with p' additional predicates, T' is finitely axiomatizable, and an arbitrary realization $\mathfrak{A} = \langle A, X_0, \ldots, X_{p-1} \rangle$ of T is a model of T if and only if there exist relations $R_0', \ldots, R'_{p'-1}$ such that $\langle A, X_0, \ldots, X_{p-1}, R_0', \ldots, R'_{p'-1} \rangle$ is a model of T'.*

It may be remarked that, if the notions of second-order formula and satisfaction of second-order formulas, in the standard sense (cf. [2]), are employed, one can say: T is *s. f. a.*$^+$ if and only if there is a second-order formula of the form $\sigma' = \bigvee V_0 \ldots \bigvee V_{p'-1}\sigma$, where σ is a first-order sentence, such that T and σ' have the same models. Using the notions of Tarski [16] concerning classes $\boldsymbol{K}$ of relational systems, one also sees at once that: $\boldsymbol{K}$ is the class of all models of an *s. f. a.*$^+$ theory if and only if $\boldsymbol{K} \in \mathbf{PC} \cap \mathbf{AC}_\delta$.

Using Gödel's completeness theorem ([6]), one establishes easily:

COROLLARY 1.3. *If T is s. f. a.$^+$, then T is f. a.$^+$.*

Noting the result of Craig [3] that any theory whose set of valid sentences is recursively enumerable is axiomatizable, one obtains:

COROLLARY 1.4. *If T is f. a.$^+$, then T is axiomatizable.*

§ 2. Theories having only infinite models.

THEOREM 2.1. *If a theory T is axiomatizable and has only infinite models, then T is s. f. a.$^+$.*

As remarked in the Introduction, 2.1 with *s. f. a.*$^+$ replaced by *f. a.*$^+$ follows immediately from Kleene's results in [9], though Kleene deals explicitly only with theories without identity. In fact from Kleene's work it follows at once that, under the hypothesis of 2.1, a theory T' can be found such that Condition 1.2 (*) of the definition of *s. f. a.*$^+$ holds for any denumerable system $\mathfrak{A}$.

The present stronger result, whose proof will be given in this section, will be obtained by a modification of Kleene's "short proof" of (1) of the Introduction.[7] The rough idea of Kleene's "short proof" is this: New predicates and axioms are introduced in T' so that T' contains a fragment of number theory (and hence a fragment of arithmetized syntax). In addition a new binary predicate St (the "satisfaction predicate") and some axioms concerning it are introduced. Since the notion of satisfaction can now be expressed, so can that of truth, and we can replace the infinite set of axioms of T by a single statement which "says" that all axioms of T are true.

Now satisfaction is a binary relation holding between a number (i.e., a formula) and a sequence (finite, if desired) of members of the entire domain of discourse. Thus, in order to have a first-order concept, the notion of finite sequence here must be replaced by an equivalent notion of lower order. In Kleene's argument, the fragment of number theory in T' is used for this purpose (as well as that of representing syntactical notions) by employing one of the devices by which every number can be considered to determine a finite sequence of numbers. As a result, for Kleene's theory T' the intuitive interpretation would be that all members of the domain of discourse are natural numbers. (Cf., however, Footnote 14 below.) And thus we see (at least directly) only that 1.2* holds for denumerable systems $\mathfrak{A}$.

In our modification of the argument, T' will still contain a fragment of number theory, for representing syntactical notions, and a binary predicate St; but, in addition, T' will contain a fragment of set theory (Cf. Footnote 15 below) or of "finite sequence theory". This fragment will be expressed by the formulas As(x) ("x is an assignment") and $\mathrm{Vl}(x, u) = z$ ("the value of the assignment x at the variable u is z"). Then a notion to replace that of a finite sequence of arbitrary elements of the domain of discourse will be available, even when the domain of discourse is non-denumerably infinite.

(Actually, Kleene's method diverges from ours a bit further than the above discussion might indicate. One might say, very roughly speaking, that by means of certain devices, Kleene succeeds in getting along without ever introducing a full equivalent of the notion of finite sequence. Cf. [9] for details.)

The remainder of this section will be devoted to the detailed proof of 2.1. We assume (for the duration of this section) that T is a fixed, axiomatizable theory. By means of the well-known method which reduces, for most

[7] Kleene [9] gave two proofs of (1) — a "short proof" involving set-theoretical devices, and a "finitary" or "metamathematical" proof. Our proof of 2.1, from which we derive (1) as a corollary (6.2 below), is by no means finitary.

purposes, the discussion of general theories to those whose non-logical constants are predicates (cf., for example, [8], pp. 405–420), it is easily seen that we may, without loss of generality, assume that the non-logical constants of T are all predicates. Suppose, then, that T is based on the predicates $\mathrm{P}_0, \ldots, \mathrm{P}_{p-1}$, of ranks $\rho_0, \ldots, \rho_{p-1}$, respectively. As a notational convenience we shall sometimes write P_p instead of $=$, and we put $\rho_p = 2$. There is clearly no loss of generality in assuming, as we do now, that T is an arithmetized theory, i.e., that the set of expressions of T coincides with ω, and the concatenation operation among the expressions of T, denoted by Cn, is a recursive number-theoretical function of two variables. As is known (cf. [18], p. 13, Footnote 12), it follows that the set Fm of formulas of T and the set Vb of variables are recursive. By hypothesis, T possesses a recursive set Ax of axioms.

The theory $Q^{(\mathrm{N})}$ of [18] will play an auxiliary role in the construction of the desired theory T'. $Q^{(\mathrm{N})}$ is a fragmentary theory of natural numbers based on a predicate N of rank 1 (assumed distinct from $\mathrm{P}_0, \ldots, \mathrm{P}_{p-1}$), and operation symbols $\mathbf{0}$, S, $+$, and $\cdot$, of ranks 0, 1, 2, and 2, respectively, and on a certain finite set of axioms.[8] Roughly speaking the intended interpretations of N, $\mathbf{0}$, S, $+$, and $\cdot$, respectively, are ω, 0, and any extensions of the ordinary number-theoretical S, $+$, and $\cdot$ (to the whole domain of individuals).

As in [18], we define recursively a sequence $\Delta_0, \Delta_1, \ldots$ of terms of $Q^{(\mathrm{N})}$ by the stipulation that $\Delta_0 = \mathbf{0}$ and, for any n, $\Delta_{n+1} = \mathrm{S}(\Delta_n)$.

It is easily seen (cf. [18], pp. 53–54) that:

Lemma 2.2. *If $m, n \in \omega$, then* (i) $\dashv_{Q^{(\mathrm{N})}} \mathrm{N}(\Delta_n)$ *and* (ii) $\dashv_{Q^{(\mathrm{N})}} \Delta_m \neq \Delta_n$ *if $m \neq n$.*

It is proved in [18] (pp. 24–25, 39–74) and in [8] (pp. 295–297) that arbitrary recursive functions and relations are "numeralwise representable" or "definable" in $Q^{(\mathrm{N})}$, from which we infer, in particular:

Lemma 2.3. *There are formulas θ_1, θ_2, and θ_3 of $Q^{(\mathrm{N})}$, each having* x *as its only free variable, and a formula θ_4 of $Q^{(\mathrm{N})}$, whose free variables are* x, y, *and* z, *such that for any natural numbers m, n, and p:*

(i) $\dashv_{Q^{(\mathrm{N})}} \theta_1(\Delta_m)$ *or* $\dashv_{Q^{(\mathrm{N})}} \sim\theta_1(\Delta_m)$

according as $m \in Fm$ *or* $m \notin Fm$,

(ii) $\dashv_{Q^{(\mathrm{N})}} \theta_2(\Delta_m)$ *or* $\dashv_{Q^{(\mathrm{N})}} \sim\theta_2(\Delta_m)$

according as $m \in Vb$ *or* $m \notin Vb$,

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[8] The axioms of $Q^{(\mathrm{N})}$ are the closures of the formulas $\mathrm{N}(\mathbf{0})$ and $\mathrm{N}(x) \wedge \mathrm{N}(y) \rightarrow \{[\mathrm{N}(\mathrm{S}(x)) \wedge \mathrm{N}(x+y) \wedge \mathrm{N}(x\cdot y)] \wedge [\mathrm{S}(x) = \mathrm{S}(y) \rightarrow x = y] \wedge [0 \neq \mathrm{S}(y)] \wedge [x \neq 0 \rightarrow \vee z(x = \mathrm{S}(z))] \wedge [x+0 = x] \wedge [x + \mathrm{S}(y) = \mathrm{S}(x+y)] \wedge [x\cdot 0 = 0] \wedge [x\cdot \mathrm{S}y = (x\cdot y)+x]\}$.

(iii) $\dashv_{Q^{(N)}} \theta_3(\Delta_m)$ *or* $\dashv_{Q^{(N)}} \sim\theta_3(\Delta_m)$

according as $m \in Ax$ *or* $m \notin Ax$,

(iv) $\dashv_{Q^{(N)}} \theta_4(\Delta_m, \Delta_n, \Delta_p)$ *or* $\dashv_{Q^{(N)}} \sim\theta_4(\Delta_m, \Delta_n, \Delta_p)$

according as $Cn(m, n) = p$ *or* $Cn(m, n) \neq p$.

We now construct a theory T_1, from which the desired theory T' will be obtained by the standard process for replacing operation symbols and individual constants by predicates. (We form T_1 first, rather than T' directly, in order to increase the readability of the formulas.) The non-logical constants of T_1 are the distinct symbols:

$\mathrm{P}_0, \ldots, \mathrm{P}_{p-1}$; N, $\mathbf{0}$, S, $+$, $\cdot$;

Fm, Vb, Ax, Cn; As, Vl, E, and St.

Fm, Vb, Ax, As, E, and St are predicates of ranks 1, 1, 1, 1, 4, and 2, respectively, while Cn and Vl are operation symbols each of rank 2. Roughly speaking, the intended interpretations of Fm, Vb, Ax, and Cn, respectively, are *Fm* (the set of formulas of T), *Vb* (the set of variables of T), *Ax* (the set of axioms of T), and *Cn* (the concatenation operation of T). As regards the last four new constants,

(i) As(x) may be read "x *is an assignment*",
(ii) Vl(x, u) = z may be read "*the value of* x *at the variable* u *is* z",
(iii) E(x, x′, u, z) may be read "*assignments* x *and* x′ *have equal values at all variables except at most the variable* u, *and the value of* x′ *at* u *is* z",
and (iv) St(x, w) may be read "x *satisfies* w".

T_1 is based on the axioms which are the closures of the formulas listed in Groups I–V below. Constants Fm, Vb, Ax, Cn, and E and the axioms of Group II are introduced only to avoid dealing with the problem of definitions of new constants in a theory, or to shorten the formulas. The axioms of Group IV are a formalization of the inductive conditions which determine the satisfaction relation (cf. [13]), while that of Group V states that "all axioms are true". Axiom III.2 insures the validity of those basic properties of As and Vl, and the axioms of Group I (on the basis of 2.2 and 2.3) the validity of those basic properties of the syntactical notions Fm, Vb, and Cn, which are needed, together with IV, to demonstrate Lemma 2.4, below, the "adequacy condition" of Tarski [13]. The axioms of Group I also ensure the validity of the formulas described in 2.3 (iii), needed to derive from 2.4 (and III.1) the validity in T_1 of the axioms of T.

I The axioms of $Q^{(N)}$.

II .1 $\mathrm{Fm}(x) \leftrightarrow \theta_1 \wedge \mathrm{N}(x)$.

.2 $\mathrm{Vb}(x) \leftrightarrow \theta_2 \wedge \mathrm{N}(x)$.

.3 $\mathrm{Ax}(x) \leftrightarrow \theta_3 \wedge \mathrm{N}(x)$.

.4 $\mathrm{N}(x) \wedge \mathrm{N}(y) \wedge \mathrm{N}(z) \rightarrow [\mathrm{Cn}(x, y)=z \leftrightarrow \theta_4]$.

.5 $\mathrm{E}(x, x', u, z) \leftrightarrow \mathrm{As}(x) \wedge \mathrm{As}(x') \wedge \mathrm{Vb}(u) \wedge \mathrm{Vl}(x', u)=z$

$\wedge \bigwedge u'[\mathrm{Vb}(u') \wedge u' \neq u \rightarrow \mathrm{Vl}(x', u')=\mathrm{Vl}(x, u')]$.

III.1 $\bigvee x\, \mathrm{As}(x)$.

.2 $\mathrm{As}(x) \wedge \mathrm{Vb}(u) \rightarrow \bigvee x' \mathrm{E}(x, x', u, z)$.

Before stating the remaining axioms, we introduce inductively a new notation, by requiring that if $\tau_0, \ldots, \tau_n$ are terms of T_1, then $\tau_0{}^\frown \ldots {}^\frown\tau_n = \tau_0$ if $n = 0$, while $\tau_0{}^\frown \ldots {}^\frown\tau_n = \mathrm{Cn}(\tau_0{}^\frown \ldots {}^\frown\tau_m, \tau_{m+1})$ if $n = m+1$.

IV .1 $\mathrm{As}(x) \wedge \mathrm{Vb}(v_0) \wedge \ldots \wedge \mathrm{Vb}(v_{\rho_k-1}) \rightarrow [\mathrm{St}(x, \Delta_{\mathrm{P}_k}{}^\frown v_0{}^\frown \ldots {}^\frown v_{\rho_k-1})$

$\leftrightarrow \mathrm{P}_k(\mathrm{Vl}(x, v_0), \ldots, \mathrm{Vl}(x, v_{\rho_k-1})] \qquad (k = 0, \ldots, p)$.

.2 $\mathrm{As}(x) \wedge \mathrm{Fm}(w) \wedge \mathrm{Fm}(w') \rightarrow [\mathrm{St}(x, \Delta_/{}^\frown w^\frown w') \leftrightarrow \mathrm{St}(x, w)/\mathrm{St}(x, w')]$.

.3 $\mathrm{As}(x) \wedge \mathrm{Fm}(w) \wedge \mathrm{Vb}(u) \rightarrow \{\mathrm{St}(x, \Delta_\bigwedge{}^\frown u^\frown w) \leftrightarrow$

$\bigwedge z \bigwedge x'[\mathrm{E}(x, x', u, z) \rightarrow \mathrm{St}(x', w)]\}$.

V $\mathrm{As}(x) \wedge \mathrm{Ax}(w) \rightarrow \mathrm{St}(x, w)$.

LEMMA 2.4. *Suppose φ is a formula of T, ξ is a variable not occurring in φ, and $\varphi^{(\xi)}$ is the formula obtained from φ by substituting, simultaneously, the term $\mathrm{Vl}(\xi, \Delta_\zeta)$ for each free occurrence of every variable ζ in φ. Then*

(1) $$\dashv_{T_1} \mathrm{As}(\xi) \rightarrow [\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \varphi^{(\xi)}].$$

PROOF. (Let us agree to write "$\dashv_1$" instead of "$\dashv_{T_1}$".) Lemma 2.4 will be proved by "induction" on the formula φ.

Suppose φ is $\mathrm{P}_k(\eta_0, \ldots, \eta_t)$ where $k \leqq p$ and $t = \rho_k-1$, and $\eta_0, \ldots, \eta_t \in Vb$.
By II.2, 2.3 (ii), and 2.2 (i),

(2) $$\dashv_1 \mathrm{Vb}(\Delta_{\eta_0}) \wedge \ldots \wedge \mathrm{Vb}(\Delta_{\eta_t});$$

while, by II.4, 2.3 (iv), and 2.2 (i),

(3) $$\dashv_1 \Delta_{\mathrm{P}_k}{}^\frown\Delta_{\eta_0}{}^\frown \ldots {}^\frown\Delta_{\eta_t}=\Delta_\varphi.$$

From (2), (3), and IV.1, (1) follows immediately.

Now suppose 2.4 holds for φ_1 and φ_2, and φ is φ_1 / φ_2. Since ξ does not occur in φ, it occurs neither in φ_1 nor in φ_2, and so, by hypothesis,

(4) $$\dashv_1 \mathrm{As}(\xi) \to [\mathrm{St}(\xi, \Delta_{\varphi_i}) \leftrightarrow \varphi_i^{(\xi)}] \qquad (i = 1, 2).$$

From II.1, 2.2 (i), and 2.3 (i), and from II.4, 2.2 (i), and 2.3 (iv), it follows, respectively, that

(5) $$\dashv_1 \mathrm{Fm}(\Delta_{\varphi_1}) \wedge \mathrm{Fm}(\Delta_{\varphi_2}) \text{ and } \dashv_1 \Delta_\varphi = \Delta_/ {}^\frown \Delta_{\varphi_1} {}^\frown \Delta_{\varphi_2}.$$

By (5) and IV.2,

(6) $$\dashv_1 \mathrm{As}(\xi) \to [\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \mathrm{St}(\xi, \Delta_{\varphi_1}) \,/\, \mathrm{St}(\xi, \Delta_{\varphi_2})].$$

From (4) and (6) follows

(7) $$\dashv_1 \mathrm{As}(\xi) \to [\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \varphi_1^{(\xi)} \,/\, \varphi_2^{(\xi)}];$$

and the desired conclusion (1) is an immediate consequence of (7) since $\varphi_1^{(\xi)} \,/\, \varphi_2^{(\xi)}$ is $\varphi^{(\xi)}$.

Finally, suppose φ is the formula $\wedge\eta\varphi_1$, where 2.4 holds for φ_1, and η is any variable; we suppose, of course, that ξ is a variable not occurring in φ. Let γ be a variable also not occurring in φ, and distinct from ξ. By the inductive hypothesis,

(8) $$\dashv_1 \mathrm{As}(\gamma) \to [\mathrm{St}(\gamma, \Delta_{\varphi_1}) \leftrightarrow \varphi_1^{(\gamma)}].$$

By II, 2.2 and 2.3,

(9) $$\dashv_1 \mathrm{Fm}(\Delta_{\varphi_1}),\ \dashv_1 \mathrm{Vb}(\Delta_\eta), \text{ and } \dashv_1 \Delta_\varphi = \Delta_\wedge {}^\frown \Delta_\eta {}^\frown \Delta_{\varphi_1}.$$

From (9) and IV.3, noting that ξ, η, and γ are distinct, we infer that

(10) $$\dashv_1 \mathrm{As}(\xi) \to \{\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \wedge\eta\wedge\gamma[\mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \to \mathrm{St}(\gamma, \Delta_{\varphi_1})]\}.$$

Noting that $\dashv_1 \mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \to \mathrm{As}(\gamma)$, by II.5, we obtain from (10) and (8)

(11) $$\dashv_1 \mathrm{As}(\xi) \to \{\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \wedge\eta\wedge\gamma[\mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \to \varphi_1^{(\gamma)}]\}.$$

Now if ζ is any variable distinct from η, then, by II, 2.2 and 2.3, $\dashv_1 \mathrm{Vb}(\Delta_\zeta)$, and, by 2.2 (ii), $\vdash_1 \Delta_\zeta \neq \Delta_\eta$. Hence, from II.5 it follows that

(12) $$\dashv_1 \mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \to \mathrm{Vl}(\gamma, \Delta_\zeta) = \mathrm{Vl}(\xi, \Delta_\zeta), \text{ if } \zeta \neq \eta,$$

(13) $$\dashv_1 \mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \to \mathrm{Vl}(\gamma, \Delta_\zeta) = \eta, \text{ if } \zeta = \eta.$$

Let $\varphi_1^{(\xi)\prime}$ be the formula obtained from φ_1 by substituting (simultaneously) $\mathrm{Vl}(\xi, \Delta_\zeta)$ for each free occurrence of every variable $\zeta \neq \eta$. Then $\varphi_1^{(\xi)\prime}$ is also the formula obtained from $\varphi_1^{(\gamma)}$ by substituting $\mathrm{Vl}(\xi, \Delta_\zeta)$ for all occurrences of $\mathrm{Vl}(\gamma, \Delta_\zeta)$, where ζ is any variable $\neq \eta$, and η for each occurrence of $\mathrm{Vl}(\gamma, \Delta_\eta)$. Note that ξ and γ never occur bound in either $\varphi_1^{(\gamma)}$ or $\varphi_1^{(\xi)\prime}$, and that the last described substitution of η always occurs at a location where η occurred free in φ_1 and therefore at a location not within the scope of a quantifier expression $\wedge\eta$ in either $\varphi_1^{(\gamma)}$ or $\varphi_1^{(\xi)\prime}$. Thus, all conditions of the general rule of substitution for identity are met,[9] so that from (12) and (13) we may infer that

$$(14) \qquad \dashv_1 \mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \rightarrow [\varphi_1{}^{(\gamma)} \leftrightarrow \varphi_1{}^{(\xi)\prime}].$$

Substituting, on the basis of (14), $\varphi_1{}^{(\xi)\prime}$ for $\varphi_1{}^{(\gamma)}$ in (11), and noting that γ does not occur in $\varphi_1{}^{(\xi)\prime}$, we obtain

$$(15) \qquad \dashv_1 \mathrm{As}(\xi) \rightarrow \{\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \bigwedge\eta[\bigvee\gamma\mathrm{E}(\xi, \gamma, \Delta_\eta, \eta) \rightarrow \varphi_1{}^{(\xi)\prime}]\}.$$

From (15), (9), and III.2 follows

$$(16) \qquad \dashv_1 \mathrm{As}(\xi) \rightarrow \{\mathrm{St}(\xi, \Delta_\varphi) \leftrightarrow \bigwedge\eta\varphi_1{}^{(\xi)\prime}\}.$$

(16) establishes the conclusion (1) to be proved, since $\bigwedge\eta\varphi_1{}^{(\xi)\prime}$ and $\varphi^{(\xi)}$ are, in virtue of their definitions, identical. Thus the proof of 2.4 is completed.

LEMMA 2.5. *Any sentence σ valid in T is valid in T_1.*

PROOF. Indeed, suppose σ is an axiom, i.e., $\sigma \in Ax$, and ξ is a variable not occurring in σ. Then, by 2.4,

$$(1) \qquad \dashv_1 \mathrm{As}(\xi) \rightarrow [\mathrm{St}(\xi, \Delta_\sigma) \leftrightarrow \sigma].$$

Now, by II.3, 2.2 (i), and 2.3 (iii),

$$(2) \qquad \dashv_1 \mathrm{Ax}(\Delta_\sigma);$$

While, by V and (2),

$$(3) \qquad \dashv_1 \mathrm{As}(\xi) \rightarrow \mathrm{St}(\xi, \Delta_\sigma).$$

From (1), (3), and III.1 it follows that σ is valid in T_1. Thus, all axioms of T, and hence, all sentences of T derivable from them, i.e., all sentences valid in T, are valid in T_1.

An immediate consequence of 2.5 is:

LEMMA 2.6. *If there exist $Y_0, \ldots, Y_{12}$ such that $\langle A, R_0, \ldots, R_{p-1}, Y_0, \ldots, Y_{12}\rangle$ is a model of T_1, then $\langle A, R_0, \ldots, R_{p-1}\rangle$ is a model of T.*

We now establish a partial converse of 2.6.

LEMMA 2.7. *Suppose that $\mathfrak{A} = \langle A, R_0, \ldots, R_{p-1}\rangle$ is an infinite model of T. Then there exist $Y_0, \ldots, Y_{12}$ such that $\langle A, R_0, \ldots, R_{p-1}, Y_0, \ldots, Y_{12}\rangle$ is a model of T_1.*

PROOF. Let $B_0 = A \cup \omega$, and for $n = 1, 2, \ldots$, let $B_{n+1} = B_n \cup B_n{}^{(Vb)}$. Since A is infinite, it is easily seen that $A^* = B_0 \cup B_1 \cup \ldots \cup B_n \cup \ldots$ has the same power as A. (It should be remarked, however, that the proof of this fact involves the axiom of choice.) Moreover, $\omega \subseteq A^*$, and any

[9] Cf. [2] or [8].

assignment with values in A^* belongs, itself, to A^*, i.e., $A^{*(Vb)} \subseteq A^*$. Let f be a one-to-one function on A^* onto A, and let

$$R_k^* = \{\langle x_0, \ldots, x_{p_k-1}\rangle \mid \langle f(x_0), \ldots, f(x_{p_k-1})\rangle \in R_k\} \qquad (k = 0, \ldots, p-1).$$

Thus $\mathfrak{A}^* = \langle A^*, R_0^*, \ldots, R_{p-1}^*\rangle$ is isomorphic to $\mathfrak{A}$. It will clearly suffice to establish the conclusion of our lemma for $\mathfrak{A}^*$ instead of for $\mathfrak{A}$.

For $Y_0, \ldots, Y_8$ (compare the list of non-logical constants of T_1 following Lemma 2.3) we take, respectively, $\omega, 0, S', +', \cdot', Fm, Vb, Ax$ and Cn', where S', $+'$, $\cdot'$, and Cn' are any extensions of S, $+$, $\cdot$, and Cn, respectively, to all of A^*. For Y_9 we take the set $As = A^{*(Vb)}$. For Y_{10}, we take any binary operation Vl over A^* such that $Vl(x, u) = x(u)$ whenever $x \in As$ and $u \in Vb$. For Y_{11}, we take the set E of all quadruples $\langle x, x', u, z\rangle$ such that $x, x' \in As$, $u \in Vb$, $x'(u) = z$, and $x(u') = x'(u')$ for every variable $u' \neq u$. Finally, for Y_{12}, we take the set St of all couples $\langle x, w\rangle$ such that $x \in As$, $w \in Fm$, and x satisfies w in $\mathfrak{A}^*$. Let $\mathfrak{A}_1^* = \langle A^*, R_0^*, \ldots, R_{p-1}^*, Y_0, \ldots, Y_{12}\rangle$.

Since the axioms of $Q^{(N)}$ are simple truths of number theory, the axioms of Group I are all true in $\mathfrak{A}_1^*$. Using 2.3, one sees easily that the axioms II.1–II.4 are true in $\mathfrak{A}_1^*$. From the definition of $Y_{11} = E$ it follows at once that II.5 holds in $\mathfrak{A}_1^*$; and the axioms III are clearly also true in $\mathfrak{A}_1^*$. Using 2.3, one sees easily that the truth of the axioms IV in $\mathfrak{A}_1^*$ amounts exactly to the well-known condition of Tarski characterizing the satisfaction relation, and hence, by the definition of $Y_{12} = St$, these axioms are true in $\mathfrak{A}_1^*$. The truth of Axiom V in $\mathfrak{A}_1^*$ is an immediate consequence of the fact that $\mathfrak{A}$, and hence its isomorph $\mathfrak{A}^*$, are models of T. Thus $\mathfrak{A}_1^*$ is a model of T_1, and the proof of 2.7 is completed.

Assume now that T has only infinite models. Lemmas 2.6 and 2.7 establish that T and T_1 fulfill Condition 1.2 (*) of the definition of *s. f. a.*$^+$, except that T_1 involves additional non-logical constants other than predicates. A theory T' fulfilling (with T) all the requirements of 1.2 (*) may be obtained from T_1 by the standard process for replacing each operation symbol of rank n not in T by a predicate of rank $n+1$ (cf. [8], pp. 405–420). We omit the details of this step, by which the proof of 2.1 is completed.

Using the relationship between *s. f. a.*$^+$ and $\mathbf{PC} \cap \mathbf{AC}_\delta$ described after 1.2, Theorem 2.1 may be restated as follows.

COROLLARY 2.2. *Suppose* $\boldsymbol{K}$ *is a class of infinite relational systems (having finitely many relations). Then* $\boldsymbol{K}$ *is the class of models of an axiomatizable theory if and only if* $\boldsymbol{K} \in \mathbf{PC} \cap \mathbf{AC}_\delta$.

Within any fixed similarity class of relational systems, the family $\mathbf{AC}_\delta^\infty$ of all classes $\boldsymbol{K} \in \mathbf{AC}_\delta$ consisting entirely of infinite relational systems forms a lattice under union and intersection. A sublattice of $\mathbf{AC}_\delta^\infty$ is formed by the family of all classes $\boldsymbol{K} \in \mathbf{AC}_\delta^\infty$ such that $\boldsymbol{K}$ is the class of models of a (recursively) axiomatizable theory. Corollary 2.2 shows that this sublattice may also be described in a different and simple way in terms of elementary notions concerning classes of models, with no reference being made to the notion of a recursive function.

§ 3. Equivalence of *f. a.*$^+$ **and** *s. f. a.*$^+$. We turn now to the consideration of theories which may have finite models. Any finite realization of a theory T is, clearly, isomorphic to a *finite numerical realization of* T, by which we mean a realization of the form $\langle n, X_0, \ldots, X_{p-1}\rangle$, where $n = \{0, 1, \ldots, n-1\}$.

DEFINITION 3.1. *Let T be a theory based on predicates* $\mathrm{P}_0, \ldots, \mathrm{P}_{p-1}$ *of ranks* $\rho_0, \ldots, \rho_{p-1}$. *With each finite numerical realization* $\mathfrak{A} = \langle n, R_0, \ldots, R_{p-1}\rangle$ *of T we correlate the following sentence* $\sigma_{\mathfrak{A}}$ of T:

$$\mathrm{V}\mathrm{v}_0 \ldots \mathrm{V}\mathrm{v}_{n-1}\{\prod_{k<m<n}(\mathrm{v}_k \neq \mathrm{v}_m) \wedge \Lambda \mathrm{v}_n \sum_{k<n}(\mathrm{v}_n = \mathrm{v}_k)$$

$$\wedge \prod_{i<p}[\prod_{r \in R_i} \mathrm{P}_i(\mathrm{v}_{r_0}, \ldots, \mathrm{v}_{r_{\rho_i-1}}) \wedge \prod_{r' \in R_i'} \sim \mathrm{P}_i(\mathrm{v}_{r_0'}, \ldots, \mathrm{v}_{r'_{\rho_i-1}})]\},$$

where, for $i < p$, R_i' is the complement of the relation R_i, and it is understood that an empty conjunction is to be omitted. If T has non-logical constants other than predicates, $\sigma_{\mathfrak{A}}$ may be defined analogously. Sentences of the form $\sigma_{\mathfrak{A}}$ are called finite model descriptions.

COROLLARY 3.2. *If $\mathfrak{A}$ is a finite numerical realization of a theory T, then an arbitrary realization of T is a model of $\sigma_{\mathfrak{A}}$ if and only if it is isomorphic to $\mathfrak{A}$.*[10]

DEFINITION 3.3. *Given an arbitrary theory T, we write T^∞ for the theory, having the same non-logical constants, whose valid sentences are all sentences of T derivable from the set of valid sentences of T augmented by the sentences* $\sim\sigma_n$, $n = 1, 2, \ldots$. (We write briefly σ_n for $\sigma_{\langle n\rangle}$ as defined by 3.1 with $p = 0$.)

The models of T^∞ are simply the infinite models of T. Note that if T is axiomatizable, then so is T^∞.

For the sake of completeness, we remark at this point that, by means of 3.1, 3.2, and 3.3, Theorem 2.1 may easily be slightly generalized. Indeed, the hypothesis that T has no finite models may be replaced by the hypothesis that T has a finite number of non-isomorphic finite models.

LEMMA 3.4. *Suppose T is f. a.$^+$, and T' is a theory with the property demanded in* 1.1 (*). *Then, given any finite model* $\mathfrak{A} = \langle A, X_0, \ldots, X_{p-1}\rangle$

[10] The sentences $\sigma_{\mathfrak{A}}$ have been used by several authors. The term "finite model description" was coined by Henkin [7].

of T, *there exist* $R_0', \ldots, R'_{p'-1}$, *such that* $\langle A, X_0, \ldots, X_{p-1}, R_0', \ldots, R'_{p'-1}\rangle$ *is a model of* T'.

PROOF. We may suppose in addition, without loss of generality, that $\mathfrak{A}$ is a numerical model of T. Since $\mathfrak{A}$ is a model of $\sigma_{\mathfrak{A}}$, $\sim\sigma_{\mathfrak{A}}$ is not valid in T, and hence, by 1.1 (*), $\sim\sigma_{\mathfrak{A}}$ is not valid in T'. Thus, $\sigma_{\mathfrak{A}}$ is consistent with T', and, by Gödel's completeness theorem, there is an algebraic system $\mathfrak{B}' = \langle B, Y_0, \ldots, Y_{p-1}, S_0', \ldots, S'_{p'-1}\rangle$ which is a model of T' and of $\sigma_{\mathfrak{A}}$. Since $\mathfrak{B}'$ is a model of $\sigma_{\mathfrak{A}}$, $\mathfrak{B} = \langle B, Y_0, \ldots, Y_{p-1}\rangle$ is isomorphic to $\mathfrak{A}$. Since $\mathfrak{B}'$ is a model of T', the conclusion of the lemma holds for $\mathfrak{B}$ and hence, clearly, for its isomorph $\mathfrak{A}$.[11]

THEOREM 3.5. *Suppose that theories* T_1 *and* T_2 *have the same non-logical constants* $\mathrm{C}_0, \ldots, \mathrm{C}_{p-1}$, T_1^∞ *is axiomatizable*, T_2 *is f. a.*$^+$, T_1 *and* T_2 *have the same finite models, and any infinite model of* T_2 *is a model of* T_1. *Then* T_1 *is s. f. a.*$^+$.

PROOF. T_1^∞ is axiomatizable, and hence, by 2.1, is *s. f. a.*$^+$. Therefore we may find a theory $T_1^{\infty\prime}$, based on $\mathrm{C}_0, \ldots, \mathrm{C}_{p-1}$ and additional predicates $\mathrm{P}_0', \ldots, \mathrm{P}'_{p'-1}$ and on a single axiom σ^1, such that T_1^∞ and $T_1^{\infty\prime}$ satisfy 1.2 (*). By hypothesis, T_2 is *f. a.*$^+$, so there is a theory T_2' such that 1.1 (*) holds for T_2 and T_2'; we may assume T_2' has the additional predicates $\mathrm{P}_0'', \ldots, \mathrm{P}''_{p''-1}$, and the single axiom σ^2. Let T_1' be the theory with non-logical constants $C_0, \ldots, C_{p-1}, \mathrm{P}_0', \ldots, \mathrm{P}'_{p'-1}, \mathrm{P}_0'', \ldots, \mathrm{P}''_{p''-1}$ and the axiom $\sigma^1 \vee \sigma^2$. We shall show that T_1 and T_1' fulfill 1.2 (*).

Suppose $\mathfrak{A}^* = \langle A, X_0, \ldots, X_{p-1}, R_0', \ldots, R'_{p'-1}, R_0'', \ldots, R''_{p''-1}\rangle$ is a model of T_1', i.e., of either σ^1 or σ^2. If $\mathfrak{A}^*$ is a model of σ^1, then by 1.2, $\mathfrak{A} = \langle A, X_0, \ldots, X_{p-1}\rangle$ is a model of T_1^∞, and hence of T_1; while, if $\mathfrak{A}^*$ is a model of σ^2, then it follows immediately from 1.1 that $\mathfrak{A}$ is a model of T_2, and hence, by the hypothesis of 3.5, $\mathfrak{A}$ is again a model of T_1.

On the other hand, suppose $\mathfrak{A} = \langle A, X_0, \ldots, X_{p-1}\rangle$ is an arbitrary model of T_1. If $\mathfrak{A}$ is finite, then, by hypothesis, $\mathfrak{A}$ is a model of T_2; applying 3.4 to T_2 and T_2', we see that there exist $R_0'', \ldots, R''_{p''-1}$ such that $\langle A, X_0, \ldots, X_{p-1}, R_0'', \ldots, R''_{p-1}\rangle$ is a model of σ^2, and hence, for arbitrary relations $R_0', \ldots, R'_{p'-1}$ of the correct ranks, $\langle A, X_0, \ldots, X_{p-1}, R_0', \ldots, R'_{p'-1}, R_0'', \ldots, R''_{p''-1}\rangle$ is a model of T_1'. If $\mathfrak{A}$ is infinite, then $\mathfrak{A}$ is a model of T_1^∞; hence, by 1.2, there exist $R_0', \ldots, R'_{p'-1}$ such that $\langle A, X_0, \ldots, X_{p-1}, R_0', \ldots, R'_{p'-1}\rangle$ is a model of σ^1, so that for arbitrary relations $R_0'', \ldots, R''_{p''-1}$ of the correct ranks, $\langle A, X_0, \ldots, X_{p-1}, R_0, \ldots, R'_{p'-1}, R_0'', \ldots, R''_{p''-1}\rangle$ is a model of T_1'. Thus the proof that T_1 is *s. f. a.*$^+$ is complete.

Part of the import of 3.5 will be discussed later. For the present we are only interested in the special case of 3.5 in which $T_1 = T_2$, which together with 1.4 and 1.3 yields at once

[11] The use of Gödel's theorem can be avoided by a somewhat longer argument.

THEOREM 3.6. *The notions f. a.$^+$ and s. f. a.$^+$ coincide.*

Like 2.1, Theorem 3.6 may be stated in terms of Tarski's theory of classes of models. Working within any one fixed similarity class of relational systems, and given any class $\boldsymbol{K}$ of such systems, Tarski [16] defines:

$$\boldsymbol{C}(\boldsymbol{K}) = \bigcap \{\boldsymbol{L} \mid \boldsymbol{K} \subseteq \boldsymbol{L} \in \mathbf{AC}\}.$$

Using this notion, it is not hard to show, with the aid of Gödel's completeness theorem [6], that 3.6 is equivalent to the following.

COROLLARY 3.7. *If $\boldsymbol{K} \in \mathbf{PC}$, then $\boldsymbol{C}(\boldsymbol{K}) \in \mathbf{PC}$.*

§ 4. A counterexample. DEFINITION 4.1. *We write $F(T)$ for the set of all finite numerical models of a theory T. A set $\boldsymbol{K}$ of finite numerical realizations of a theory is called recursive provided the corresponding set $\{\sigma_{\mathfrak{A}} \mid \mathfrak{A} \in \boldsymbol{K}\}$ of finite model descriptions is recursive.*

In particular, if T has no non-logical constants, then, as one easily sees, the new concept of a recursive set agrees with the usual one.

It is well known that, if a theory T is finitely axiomatizable, then $F(T)$ is recursive (cf. [7]). One sees easily, by applying 3.4 (or 3.6), that this important property of these theories is shared by all theories which are *f. a.*$^+$:

THEOREM 4.2. *If T is f. a.$^+$, then $F(T)$ is recursive.*

Using 4.2, we shall establish:

THEOREM 4.3. *There are axiomatizable theories which are not f. a.$^+$.*

PROOF. An example is furnished by the theory T_2 constructed by Ehrenfeucht [5]. T_2 is a theory with no non-logical constants and the axiom set $\{\sim\sigma_n \mid n \in J\}$, where J is a recursively enumerable, but not recursive, set of positive integers. T_2 clearly has a recursively enumerable set of valid sentences, and hence, by a theorem of Craig [3], is axiomatizable. On the other hand, $F(T_2)$ is the complement (with respect to the positive integers) of J, and hence is not recursive (or even recursively enumerable). Therefore, by 4.2, T_2 is not *f. a.*$^+$.

Scholz [12] calls a set K of positive integers the *spectrum* of an arbitrary sentence σ (of Λ) if K is the set of positive integers n such that σ has a model of power n, and raises the problem of characterizing those sets of positive integers which are spectra. Applying 3.4 (or 3.6), we see that the function F establishes a one-to-one correspondence between the theories, with no non-logical constants, which are *f. a.*$^+$, on the one hand, and the spectra, on the other. Hence, from the results of Asser [1] and Mostowski [11] concerning spectra, follows at once the conclusion that, if a theory T, with no non-logical constants, is *f. a.*$^+$, then $F(T)$ is not only recursive, but is elementary (in the sense of Kalmár), and in fact, that not even all elementary sets of positive integers are so obtainable.[12]

This discussion would seem to suggest that the general question "What theories are *f. a.*$^+$?" could be regarded as a kind of generalization of the Scholz problem. In all the examples we have been able to construct of axiomatizable theories which are not *f. a.*$^+$, it is solely the properties of $F(T)$ which prevent T from being *f. a.*$^+$. However, the following problem is open.

4.4 Problem. *If T and T' are axiomatizable, $F(T) = F(T')$, and T' is f. a.$^+$, must T be f. a.$^+$? (In particular, if T is axiomatizable and has all finite realizations as models, is T f. a.$^+$?)*

Theorem 3.5 represents a step toward a positive solution of 4.4; it states that, if in addition any infinite model of T' is a model of T, then the answer to 4.4 is affirmative. From this it follows easily that 4.4 can be reduced to the case where all infinite models of T are models of T'.

It may be interesting to mention here the following result of Henkin [7] as an example of the importance of the property shown by 4.2 to be possessed by all *f. a.*$^+$ theories:

4.5. *If T is axiomatizable, $F(T)$ is recursive, and T^∞ is decidable, then T is decidable.*

Although the statement 4.5 is slightly more general than given by Henkin, the proof in [7] constitutes a demonstration of 4.5.[13] As Henkin remarked, the first two hypotheses of 4.5 are satisfied by any finitely axiomatizable theory. By 1.4 and 4.2, the same applies to arbitrary *f. a.*$^+$ theories.

§ 5. Reduction in the number of additional predicates. The theory T' constructed in the proof of § 2 is by no means the only possible one.[14]

[12] By an obvious modification of the proof of 4.3, these results give rise to examples showing that there are even decidable theories which are not *f. a.*$^+$.

It is clearly true, in general, that some analogous results, much stronger than 4.2, apply to theories having non-logical constants; we shall not attempt, however, to define some analogue of the notion 'elementary' in order thus to strengthen 4.2.

[13] In [7], the hypothesis "T^∞ is decidable" is replaced by the hypothesis "T is $\mathfrak{a}$-categorical for some infinite power $\mathfrak{a}$". That, for an axiomatizable theory, the latter implies the former was shown in [10] and [19]. More specifically, it was proved in both [10] and [19] that, if T^∞ is $\mathfrak{a}$-categorical for some infinite power $\mathfrak{a}$, then T^∞ is complete. In [19], it was remarked that consequently, by a well-known theorem (cf., e.g., [18], p. 14, Theorem 1), if T^∞ is also axiomatizable, then T^∞ is decidable. This remark was due to Tarski.

[14] Indeed, it may be interesting to note that essentially the theory constructed by Kleene ([9], pp. 32–43) in his proof of (1) of the Introduction, modified only by the use of only one equality predicate, can be shown to have the desired properties. For this theory, versions of 2.4, 2.5, and 2.6 were established by Kleene; but 2.7 can also be proved, though the proof is much longer and less natural than that of our 2.7. We have not carefully verified all the details of the proof.

For example, one can construct a theory with the desired properties having as its additional predicates only U, ϵ, and St, where U(x) is to be read "x is a set", and $x \epsilon y$ is to be interpreted as usual.[15] However, in § 2 we made no effort to attain economy in the number of additional predicates, in view of the fact that a recent result of Tarski [17] allows the T' of § 2 to be converted easily into a theory with only one additional, binary predicate. The result of Tarski to which we refer involves the following notion.

DEFINITION 5.1. *A formula* φ *(whose free variables are* x *and* y*) of a theory* T *is called a universal pair formula for* T *if the formula*

$$\Lambda z \Lambda z' \mathrm{V} x \Lambda y(\varphi \leftrightarrow y=z \vee y=z')$$

is valid in T.

Roughly speaking, the result of Tarski is that *within any theory* T *possessing a universal pair formula, there may be introduced by definition a binary predicate in terms of which all the original non-logical constants of* T *may be defined.* (For a more precise formulation and a clarification of the notion involved, see [17].) Now it is a simple matter to see that the theory T' (or T_1) constructed in § 2 possesses a universal pair formula. On the basis of this remark, Tarski's result is easily applied to give the following theorem; we shall omit any further indication of the proof.

THEOREM 5.2. *In Theorem* 2.1, *one additional binary predicate suffices.*

Using a theory T with no non-logical constants, it is not difficult to construct a counterexample showing that 5.2 cannot be strengthened (i.e., that no finite number of additional singulary predicates will suffice in 2.1). It should be noted that 5.2 concerns only the case of theories having no finite models.

§ 6. Logic without identity. The notion of a *theory* (formalized in the first-order logic) *without identity* may be obtained by simply dropping the logical constant $=$ from the language Λ and construing derivability in the sense of first-order logic without identity. The other notions of § 1 can then be taken over with little or no change. In particular, the notions *f. a.*$^+$ and *s. f. a.*$^+$ are obtained by construing in 1.1 or 1.2, respectively, T as a theory without identity and T' as a theory with identity. (The definition of *f. a.*$^+$ in Kleene [9] construes also T' as a theory without identity. The two definitions are easily shown to be equivalent.)

Either by carrying out a (slightly modified) analogue of the previous proofs, or by a fairly simple direct argument from the previous theorems,

[15] Such a theory was constructed by us before that of § 2. The present theory T' (or T_1) of § 2 is only inessentially different, but allows a somewhat shorter notation and proof.

analogues of some of the preceding theorems may be obtained. Thus, from 2.1, it is seen that:

THEOREM 6.1. *If a theory T without identity is axiomatizable and has no finite models, then T is s. f. a.*$^+$.

The logic without identity has, of course, the property that any consistent set of its sentences has an infinite model. Therefore, if a theory T without identity is extended first to a theory $T(=)$ by adding $=$ and axioms for $=$ and then to $T(=)^\infty$, then all sentences of T which are valid in $T(=)^\infty$ are also valid in T. Now suppose that T, and hence $T(=)^\infty$, is axiomatizable. Then by 2.1, there is a T' such that T' and $T(=)^\infty$ satisfy Condition 1.2 (*) of *s. f. a.*$^+$, and hence also 1.1 (*) of *f. a.*$^+$. (Hence T' and T satisfy 1.2 (*) in so far as only infinite realizations are considered.) It follows that T' and T satisfy 1.1 (*). This yields the basic result of Kleene [9]:

6.2 *Every axiomatizable theory without identity is f. a.*$^+$.

The example of 4.3 is converted into an axiomatizable theory without identity by dropping the logical axioms for $=$ and then replacing $=$ everywhere, the non-logical axioms $\sim\sigma_n$ included, by a binary non-logical predicate P. Then $\langle n, =\rangle$ is a model of the resulting theory T_{P} if and only if $n \notin J$, where J is recursively enumerable but not recursive. Hence $F(T_{\mathrm{P}})$ is not recursive (so that 4.2 is false for T_{P}). On the other hand, if we replace *f. a.*$^+$ by *s. f. a.*$^+$ in 4.2, then the resulting assertion also holds for theories without identity. This proves:

THEOREM 6.3. *Not all axiomatizable theories without identity are s. f. a.*$^+$.

This suggests for theories without identity a problem analogous to 4.4, with *s. f. a.*$^+$ in place of *f. a.*$^+$. For the special case where any infinite model of T' is a model of T, the answer to this analogue is again affirmative, by the analogue for theories without identity of 3.5 (with *s. f. a.*$^+$ in place of *f. a.*$^+$).

Finally, the argument in § 5, based on Tarski's reduction theorem, can be easily modified to apply to theories without identity, and in this way one establishes:

THEOREM 6.4. *In both* 6.1 *and* 6.2, *one additional binary predicate suffices.*[16]

§ 7. Second-order definability. In this closing section, we want to make some brief remarks which concern second-order formulas, i.e., for-

[16] As regards the new version of 6.2 obtained in this way, the question may be asked whether a proof using less powerful devices may be constructed, analogous to Kleene's "finitary proof" of 6.2. (The answer is not immediate, as the theory Kleene employs in the role of T' may not have a universal pair formula.)

mulas in which quantification of predicates as well as individual variables is permitted (cf., e.g., [2]).

By a *second-order sentence,* we understand a second-order formula with no free variables of either kind. If σ is a second- (or higher-) order sentence, we denote by $\boldsymbol{P}(\sigma)$ the class of all non-zero powers $\mathfrak{a}$ such that a set of power $\mathfrak{a}$ is a model of σ. In [10], Łoś considers the class $\boldsymbol{Ct}(T)$ of all infinite powers $\mathfrak{a}$ for which a given theory T is $\mathfrak{a}$-categorical (i.e., all models of T of power $\mathfrak{a}$ are isomorphic). He remarks that, if T is *f. a.*, then $\boldsymbol{Ct}(T)$ is, clearly, of the form $\boldsymbol{P}(\sigma)$, where σ is a second-order sentence; and asks (cf. [10], Problem (iii)) what is the case for an arbitrary theory T.

Now, from the usual definition of truth (cf. [13]), it is clear that $\boldsymbol{Ct}(T)$ can be obtained in the form $\boldsymbol{P}(\sigma)$, with a *third*-order sentence σ, provided the set of valid sentences of T (which we can assume to be an arithmetized theory) is definable in third-order number theory.[17] On the other hand, Łoś' remark concerning theories which are *f. a.* clearly applies as well to those which are *s. f. a.*$^+$. Applying 2.1, we see that for any axiomatizable theory T, $\boldsymbol{Ct}(T) = \boldsymbol{Ct}(T^\infty)$ is of the form $\boldsymbol{P}(\sigma)$, with σ a second-order sentence. Indeed, by looking more closely at the proof of 2.1, one can obtain the stronger result: *$\boldsymbol{Ct}(T)$ is of the form $\boldsymbol{P}(\sigma)$, where σ is a second-order sentence, provided that the set of valid sentences of T is definable in second-order number theory.*[17] With regard to arbitrary theories, the question (iii) of Łoś remains open.

To obtain this stronger result, the only essential modification of the devices of § 2 which is necessary is that the axioms of Q^{N} must be replaced by a second-order formula characterizing the number-theoretical notions N and $S, +, \cdot, 0$ (or, strictly speaking, the corresponding relations) up to isomorphism. However, if desired, one could also, for the sake of added simplicity, introduce second-order formulas characterizing As, Vl, and St up to isomorphism (and thus avoid the proof of 2.4).

In any case, what is involved in this argument is simply the observation that the devices of § 2 (or alternative ones) are of importance for questions concerning expressibility alone — whereas in § 2 we are concerned with provability as well. Indeed, as a part of the argument for the italicized statement above, we have really already arrived at the following fact, implied by these devices.

The notion of the truth of sentences of a first-order theory in models is expressible in the second-order language. More precisely, given an arithmetized first-order theory T based on predicates R_i of ranks ρ_i ($i = 0, \ldots, p-1$), there

[17] Let Sc be the successor relation among natural numbers. A set $M \subseteq \omega$ is called *definable* in the n-th-order number theory ($n \geqq 2$) if there is an n-th-order formula φ, whose only free variables (or non-logical constants) are the individual variable x and the binary predicate variable X, such that x (for x) and Sc (for X) satisfy φ in ω if and only if $x \in M$.

exists a second-order formula φ such that: (i) the only free variables of φ are a singulary predicate B, the predicates $R_0, \ldots, R_{p-1}$, the singulary predicate N, the binary predicate X and the individual variable x; and (ii) if A is any *infinite* set, and $B, R_0, \ldots, R_{p-1}, N, X$ are relations (of the appropriate ranks) among the elements of A, and $x \in N$, then $B, R_0, \ldots, R_{p-1}, N, X$, and x satisfy φ in A if and only if there is a function f such that f maps $\langle N, X\rangle$ isomorphically onto $\langle \omega, Sc\rangle$ and $f(x)$ is a sentence of T true in $\langle B, R_0, \ldots, R_{p-1}\rangle$.[17]

The same result applies to languages of the n-th and $(n+1)$*-th orders, for each positive integer n.* Specifically, the full statement given above remains correct, if the theory T is replaced by the arithmetized n-th-order language augmented by n-th-order-predicates $R_0, \ldots, R_{p-1}$, the second-order formula φ is replaced by an $(n+1)$-th-order formula, and the relations $B, R_0, \ldots, R_{p-1}$ by entities of the appropriate kind. To establish this more general result, another device is needed (because of the presence of infinitely many kinds of variables), but it is a known one. Roughly speaking, it is this: The notions As and Vl should still be introduced to represent the idea of a finite sequence of individuals, but then the notion, for example, of a class of m-tuples of sets of individuals should be represented by that of a class of sets of individuals, each such set C determining the m-tuple of sets $\{Vl(x, 0) \mid x \in C\}, \{Vl(x, 1) \mid x \in C\}, \ldots, \{Vl(x, m-1) \mid x \in C\}$. For higher orders, this method must be iterated. Of course, the notion of St must be treated now in a more involved way, also. We omit any details.

The fact that the semantical notions for the n-th-order language can be introduced in the $(n+1)$-th-order language was observed by Tarski in [14]. However, the present result is a slight improvement over what is stated explicitly in [14], since there (the number-theoretical predicates are considered as constants in such a way that) the only interpretations considered have a denumerable set of individuals, while we allow the set of individuals to be of any infinite power.[18]

[18] This means that, roughly speaking, the same devices we are using (or some equivalents) were also necessary for Tarski's result, with the possible exception of the remark that finite sequences of individuals can be introduced in the general case, as well as in the denumerable case, and that, as result, the general case can also be handled.

Church ([2], p. 176, Footnote 314) refers, at least indirectly, to the question that Tarski's observations and ours concern. Our use of the term "n-th-order language" is to be understood exactly as in [2]. It may be remarked that the strict interpretation which we have given to the italicized statements above appears to be the strongest possible, since if the domain of individuals is allowed to be finite, then, of course, not even the syntactical notions can be introduced in a language of any finite order.

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