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Derivation of Einstein–Cartan theory from general relativity

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Abstract. General relativity cannot describe exchange of orbital angular momentum and intrinsic angular momentum. In 1922 É. Cartan proposed extending general relativity by including affine torsion which, apparently unknown to Cartan, resolves this problem. In 1986 the author published a derivation of Einstein–Cartan theory from general relativity with classical intrinsic angular momentum, with no additional assumptions [Petti 1986]. This paper adds simpler explanations, correction of a factor of 2, more computational details, a summary of the evidence in support of Einstein–Cartan theory, a discussion of relevance to cosmic inflation, a conjecture that quantized torsion is the 'inflaton' field of inflation theory, and a discussion of limitations of the derivation.

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Contents

1. Introduction

As the master theory of classical physics, general relativity has one outstanding flaw: it cannot describe exchange of orbital angular momentum and intrinsic angular momentum. This process, known as *spin-orbit coupling*, occurs in quantum mechanics and in classical spin fluids. The simplest manifestation of this flaw is that the Einstein tensor in Riemannian geometry is symmetric, hence the momentum tensor must be symmetric. However, classical continuum mechanics shows that the momentum tensor is non-symmetric during exchange of orbital and intrinsic angular momentum.

1.1. Early History of Einstein–Cartan theory

In 1922 E. Cartan proposed extending general relativity by including affine torsion in the most natural way: do not force torsion to zero where it naturally appears in covariant derivatives and curvatures [Cartan 1922, Cartan 1923]. The resulting theory, known as Einstein–Cartan theory, is the minimal extension of general relativity that resolves the problem of exchange of orbital and intrinsic angular momenta. The theory is minimal in that (a) when we drop the assumption that torsion vanishes, general relativity spontaneously becomes Einstein–Cartan theory, and (b) torsion is zero except inside regions that contain spin density.

Cartan tried to explain torsion to Einstein several times, after which Einstein confessed to Cartan in 1929 that he “didn’t at all understand the explanations you gave me [about the role of torsion in geometry]; still less was it clear to me how they might be made useful for physical theory” [Debever 1979]. From their correspondence, it appears that Cartan presented torsion to Einstein as a natural mathematical extension of Riemannian geometry, that he did not have a physical interpretation of torsion in mind, and he hoped that Einstein might find a physical application for it. It appears Cartan did not know that torsion corresponds to intrinsic angular momentum in spacetime physics, or that the resulting geometry enables the law of conservation of momentum to include classical intrinsic angular momentum. Einstein spent the better part of his last two decades trying unsuccessfully to extend general relativity; his main attempts included affine torsion, but he did not adopt Einstein–Cartan theory.

1.2. Mathematical arguments for validating Einstein–Cartan theory

Three types of mathematical arguments have been proposed to validate Einstein–Cartan theory.

The first argument is that Einstein–Cartan theory extends the symmetry group of gravitation from the Lorentz rotation group to the Poincaré group [Kibble 1962]. This is an intuitively appealing argument; however it requires a new assumption, a new symmetry group of the theory. The current view is that classical gravitational theory is based on Cartan connections [Kobayashi 1972], in which the symmetry group is a semidirect product of the Lorentz group and spacetime translations; the splitting of the group into rotations and translations is fixed (the translational gauge is fixed). Full Poincaré symmetry includes translational gauge symmetry [Petti 2006].

The second argument for Einstein–Cartan theory is that linearized models of rotating matter in general relativity have the same first order terms as Einstein–Cartan theory. Adamowicz showed that general relativity plus a linearized classical model of matter with spin yields the same linearized equations for the time-time and space-space components of the metric as linearized Einstein–Cartan theory [Adamowicz 1975]. Adamowicz does not treat the time-space components of the metric, exchange of orbital and intrinsic angular momentum, the non-symmetric momentum tensor, or the geometry of torsion. He does not show that Einstein–Cartan theory follows from general relativity plus classical spin. Indeed, he writes, “It is possible ‘a priori’ to solve this problem [of dust with intrinsic angular momentum] exactly in the formalism of general relativity but in the general situation we have no practical approach because of mathematical difficulties.” Adamowicz’s conclusion is at best incomplete: it is not possible to solve the full problem exactly in general relativity, including exchange of orbital and intrinsic angular momentum, without adopting the larger framework of Einstein–Cartan theory.

The third argument for Einstein–Cartan theory is that general relativistic models of rotating matter generate translational holonomy and, in the continuum limit of a spin fluid, affine torsion. In 1986 this author published a mathematical derivation of Einstein–Cartan theory from general relativity with classical spin [Petti 1986]. The derivation uses no mathematics beyond classical differential geometry and classical continuum mechanics. It requires no additional assumptions except some commonplace limit arguments used in continuum mechanics. This paper contains simpler explanations of the derivation and a more complete account of computations than the author’s 1986 paper. This paper also corrects a factor of 2 in the original derivation.

1.3. Relevance of Einstein–Cartan theory to gravitational research

The traditional view of Einstein–Cartan theory is that it requires a separate assumption beyond general relativity, that it is unsupported by empirical evidence, and that it solves no significant problems. Since the original publication of this derivation over a quarter century ago, another generation of physicists has been trained in the 90-year-old consensus that Einstein–Cartan theory is one of many unproven speculations surrounding general relativity. The review article on Einstein–Cartan theory in *The Encyclopedia of Physics* [Trautman 2006] treats the theory as an unproven speculation and does not mention the derivation.

> “The Einstein–Cartan theory is a viable theory of gravitation ... It is possible that the Einstein–Cartan theory will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion.”

The most prominent exception to this view is the late Yuval Ne’eman [Ne’eman 1990].

If Einstein–Cartan theory is correct, then a complete quantum theory of gravitation must include affine torsion or a quantum precursor of torsion, and the classical limit of the theory is Einstein–Cartan theory.

Poplawski has shown that closed Friedman cosmologies based on Einstein–Cartan theory generate inflationary expansion from classical geometry [Poplawski 2010; Poplawski 2012]. Section 7 discusses the possible role of torsion in cosmic inflation.

This paper is divided into these main sections: Section 2, a brief summary of the derivation; Section 3, mathematical tools; Section 4, translational holonomy around a Kerr rotating mass; Section 5, transition from discrete to continuous matter with spin; Section 6, Relationship between spin and torsion; Section 7, Applications; Section 8, Conclusion.

Appendix C provides previously unpublished details of a rigorous proof of Theorem 1. Appendix E discusses some comments about the derivation that have come to the author's attention. For a quick non-technical summary of this paper, read sections 1, 2, 8.

2. Summary of the derivation

The derivation of the field equations of Einstein–Cartan theory from general relativity is similar to derivation of a differential field equation from an integral law in fluid mechanics or electrodynamics. The computation consists of three steps.

(a) Starting with a rotating mass whose exterior solution is the Kerr-Newman solution. Compute the translational holonomy around a spacelike equatorial loop and a spacelike polar loop. (Section 4)

(Classical fluid analog: Start with a model of a rotating classical particle with which to build a model of a spin fluid.)

(b) Form a discrete distribution of many small rotating masses with correlated rotations. Take the continuum limit of the distribution while holding mass and spin densities constant. (Section 5)

(Classical fluid analog: Approximate a spin fluid by distribution of many classical particles with correlated rotations and take the continuum limit while holding mass and spin densities constant.)

(c) In the continuum limit, translational holonomy becomes affine torsion, the distribution of rotating masses becomes a spin fluid with a nonzero spin density, and the relation between them is exactly that in the field equations of Einstein–Cartan theory.

(Classical fluid analog: The continuum limit yields equations of conservation of momentum and of orbital plus intrinsic angular momentum for a classical spin fluid.)

All mathematics used in the computation is widely known to relativity physicists and differential geometers, except for affine torsion. Affine torsion is most familiar as the antisymmetric part of the connection coefficients, or as the antisymmetrized covariant derivative [Kobayashi and Nomizu 1963; Bishop and Crittenden 1964; Milnor 1963]. Affine torsion is translational curvature, and it is analogous in many respects to Riemannian (rotational) curvature.

- Rotational holonomy measures the net rotation of a linear frame when the frame is parallel translated around a closed loop. Rotational curvature is the rotational holonomy per unit area surrounded by the loop, in the limit of very small loops.
- Translational holonomy measures the "failure-to-close vector" when a closed loop in the manifold with torsion is mimicked (developed) in a space with no torsion, using the same pattern of accelerations as the original loop. Translational curvature is the translational holonomy per unit area surrounded by the loop, in the limit of very small loops.

For more discussion and drawings of curvature and torsion as affine defects, see [Petti 2001].

3. Mathematical tools

For a summary of notational conventions, see Appendix A.

3.1. Development of curves

The motivation for defining the development of curves is quite simple. We want to isolate the influence of local manifold geometry from the influence of acceleration upon the shape of a curve. Given a curve C in a Riemannian manifold Ξ, what would a curve look like which had the same pattern of accelerations as C, if it were drawn on a flat manifold Ξ'? The answer is the development of the curve C into the manifold Ξ'.

Define:

- Ξ is a smooth (pseudo-) Riemannian manifold of dimension n with local coordinates ξ^μ, metric $g_{\mu\nu}$, and metric connection coefficients $\Gamma_{\lambda\nu}{}^{\mu}(s)$ whose covariant differentiation is denoted by D. Denote a point in Ξ by ξ.
- $C: [0,1] \to \Xi$ is a smooth curve in Ξ, with $C(0) = \xi_0$ and tangent vector field u.
- $A(s): T_{C(0)}\Xi \to T_{C(s)}\Xi$ is an isometry of the tangent space at $C(0)$ with the tangent space at $C(s)$, by parallel translation along the curve C.

Lemma 1: The mapping $A(s)$ satisfies the differential system

$$\frac{dA^{\mu}{}_{\nu}}{ds}(s) + u^{\lambda}(s)\ \Gamma_{\lambda\sigma}{}^{\mu}(s)\ A^{\sigma}{}_{\nu}(s) = 0 \tag{1}$$

with initial condition $A^{\mu}{}_{\nu}(0) = \text{kronecker\_delta}^{\mu}{}_{\nu}$.

The index μ refers to a vector basis at $C(s)$, while the index ν refers to a vector basis at $C(0)$. The proof of Lemma 1 consists of applying the formulas for parallel translation along C of the vector from $C(0)$ to $C(s)$.

Define similar structures for another manifold Ξ':

- Ξ' is a flat smooth (pseudo-) Riemannian manifold of the same dimension as Ξ with local coordinates ξ'^{μ}, metric $g'_{\mu\nu}$, and metric connection coefficients $\Gamma'_{\lambda\nu}{}^{\mu}(s)$ and whose covariant differentiation is denoted by D'. Denote a point in Ξ' by ξ'.
- $C': [0,1] \to \Xi'$ a smooth curve in Ξ', with $C'(0) = \xi'_0$ and tangent vector field u'.
- $A'(s): T_{C'(0)}\Xi' \to T_{C'(s)}\Xi'$ is an isometry of the tangent space at $C'(0)$ with the tangent space at $C'(s)$, by parallel translation along the curve C'.

Applying Lemma 1 to the curve on manifold Ξ', the mapping $A'(s)$ satisfies the differential system

$$\frac{dA'^{\mu}{}_{\nu}}{ds}(s) + u'^{\lambda}(s)\ \Gamma\ni_{\lambda\sigma}{}^{\mu}(s)\ A'^{\sigma}{}_{\nu}(s) = 0 \tag{1'}$$

with initial condition $A'^{\mu}{}_{\nu}(0) = \text{kronecker\_delta}^{\mu}{}_{\nu}$.

We are ready to define the development of a curve, after we select an isometry of the tangent spaces at the initial points of the two curves. Choose

$L: T_{\xi}\Xi \to T_{\xi'}\Xi'$ is an arbitrary isometry between tangent spaces at $\xi \in \Xi$ and $\xi' \in \Xi'$.

Definition: The development of the curve C in Ξ into the manifold Ξ' is the curve $C': [0,1] \to \Xi'$ defined by the differential system:

$$D'_{u'}\ u' = A'(s)\ L\ [A(s)]^{-1}\ D_u\ u \tag{2}$$

with $C'(0) = \xi'_0$ and $u'(0) = L\ u(0)$.

Development of curve C maps the tangent vector $u(s)$ back to the point ξ_0, then to a tangent vector at $\xi'_0 \in \Xi'$, then parallel translates the vector from $C'(0)$ to $C'(s)$. Equivalently, development of a curve maps the tangent vector $u(s)$ at $C(s)$ directly to the vector $u'(s)$ at $C(s)$ in Ξ' by parallel translating the mapping L along C from $C(0)$ to $C(s)$ (which yields the mapping $A'(s)\ L\ [A(s)]^{-1}$).

3.2. Relation between development, curvature, and torsion for linear connections

Choose a point $\xi \in \Xi$ and two vectors V and W at ξ. Construct a one-parameter family of closed loops $C_t(s)$ through ξ which are tangent at ξ to the plane of $V \wedge W$, and which converge to ξ as $t \to 0$. Develop the curves C_t into a flat (pseudo-) Euclidean n manifold Ξ'.

3.2.1. Definition of curvature using rotational holonomy

If the developed curve does not close, close it with a short straight line. (This line segment is of sufficiently high order relative to the size of the loop that it does not affect the computations below.) Parallel translate a vector basis around the curves C_t. use the same pattern of rotations as in C_t to translate the vector basis along C'_t.

Rotational holonomy is defined as the linear transformation g_t that maps the starting vector basis onto the ending vector basis that results from parallel translation around the loop C'_t.

Conventional (rotational) curvature is defined as the rotational holonomy per unit area enclosed by the loop, in the limit as the area of the loop approaches zero.

(3) $$- R(V \wedge W) = \lim_{t\to 0} \frac{g_t}{\text{area}(t)}$$

3.2.2. Definition of affine torsion using translational holonomy

The developed curves C'_t described above generally do not close. Translational holonomy is defined as the failure-to-close vector needed to close the developed curve, $C'_t(1) - C'_t(0)$.

Affine torsion is defined as the translational holonomy per unit area enclosed by the loop, in the limit as the area of the loop approaches zero.

(4) $$- T(V \wedge W) = \lim_{t\to 0} \frac{(C'_t(1) - C'_t(0))}{\text{area}(t)}$$

We now have the mathematical tools we need to create torsion out of general relativity, with no additional assumptions. We can generate rotational and translational holonomy around loops in a spacetime Ξ, and interpret them as integral surrogates for (rotational) curvature and torsion. By taking limits as the loops shrink to a point, we have a well-defined procedure for translating the integral concepts of holonomy into the differential concepts of curvature and torsion.

3.3. Rotational holonomy in the Schwarzschild solution

As an introduction to the technique, let us calculate the rotational holonomy of a spacelike circular orbit in the exterior Schwarzschild solution. Use coordinates so that the metric takes the form

(5) $$ds^2 = -(1-2\ m/r)\ dt^2 + \frac{dr^2}{1-2\ m/r}\quad r^2\ (d\theta^2 + \sin^2(\theta)\ d\varphi^2)$$

For a spacelike circular orbit of coordinate radius r around the equator ($\theta=\pi/2$), the rotational holonomy is spacelike and lies in the equatorial plane [the (r, φ) plane]. It is given by

(6) $$2\pi\ [1 - (1 - 2\ m/r)^{½}]$$

In the limit where $M \ll r$, the holonomy per unit area enclosed by the loop (using coordinate area, where the coordinate r directly reflects the area of the sphere r=constant) becomes approximately

(7) $$2\ \frac{m}{r^3} = \frac{8}{3}\ \pi\ \frac{m}{(4/3)\ \pi\ r^3}$$

This expression is an integral surrogate for the sectional curvature $R(r, \varphi)$. Three of these terms provide a surrogate for the scalar curvature

$$R = R_{ij}{}^{ji} \qquad \text{(note that } R_{0ij0} = 0\text{)} \tag{8}$$

With $m / (4/3\ \pi\ r^3)$ approximating the mass density, we have recovered the source term for the field equations of general relativity: the field equations assign the value 8π RHO (where RHO is the energy density in a fluid model of matter) to the scalar curvature.

The measure used for area of the closed loop is derived from the image of the loop seen by an observer at spatial infinity. Since the line integral along the closed loop also avoids the Schwarzschild singularity, the entire construction does not involve an integral over the singularity.

3.4. Geometric definition of affine torsion is the key to the derivation

The conventional algebraic definition torsion is the antisymmetric part of the connection coefficients. A more fundamental definition of torsion is the covariant derivative of the fundamental one-form (or solder form) of an affine connection [Kobayashi and Nomizu 1963; Bishop and Crittenden 1964]. See Appendix B for a discussion of the relationships among connections, curvature and torsion on fiber bundles.

The geometric interpretation of affine torsion is less widely known; yet it provides the key insight that motivates the derivation. The geometric interpretation of torsion has two equivalent forms:

a) Torsion is the continuum limit of translational holonomy per unit area, where translational holonomy is the “failure to close vector” of the development of a closed loop.

b) Torsion is the continuum limit of dislocation density in a discrete affine space.

I suspect that the failure of communication about torsion between Cartan and Einstein [Debever 1979] was partly due to the fact that Cartan communicated in terms of symbolic algebra and Einstein did his most important creative thinking in terms of visualizations and thought experiments.

Without the geometric interpretation of torsion, the derivation herein might be mistakenly interpreted as merely a plausibility argument.

4. Translational holonomy around a charged Kerr rotating mass

4.1. Objective and summary

The objective of Section 4 is to derive the translational holonomy of various closed loops around a Kerr rotating mass. The final result is that the only significant translational holonomy arises from an equatorial spacelike loop, and the translation is timelike and proportional to the angular momentum of the mass. The computations of translational holonomy use only the “exterior” Kerr solution: if the massive object is a black hole, the computations need not use the region near the event horizon; if the massive object is an extended body without a singularity, the computations do not use any properties of a hypothetical but currently undiscovered interior Kerr solution.

Section 4.2 computes the translational holonomy for equatorial loops. Section 4.3 shows that translational holonomy components for other loops are of sufficiently low order that they do not affect the continuum limit calculations that derive torsion from translational holonomy.

Therefore the presence of angular momentum on microscopic scales (i.e. microscopic compared to the scale of validity of the continuum approximation, gives rise to torsion in the continuum limit. More specifically, in coordinates (*t, x, y, z*) angular momentum in the (*x, y*) plane at rest relative to these coordinates generates translational holonomy that corresponds to the torsion component $T_{xy}{}^{t}$.

The subsequent calculations use the Kerr solution in Boyer-Lindquist coordinates [Misner, Thorne, and Wheeler 1973].

(9) $$ds^2 = -\frac{\Delta}{\rho^2}\,[dt - a\,\sin^2(\theta)\,d\varphi]^2 + \frac{\sin^2(\theta)}{\rho^2}\,[(r^2 + a^2)\,d\varphi - a\,dt]^2 + \frac{\rho^2}{a}\,dr^2 + \rho^2\,d\theta^2$$

where $0 \leq \varphi \leq 2\pi$ and $0 \leq \theta \leq \pi$,

$$a := S/m$$

$$\Delta := r^2 + a^2 - 2\,m\,r + q^2$$

$$\rho^2 := r^2 + a^2\,\cos^2(\theta)$$

This form of the metric uses four orthogonal 1-forms, which can be used to form an orthonormal basis for the tangent vectors at each point of Ξ.

4.2. Translational holonomy of equatorial spacelike loops

We shall calculate the translational holonomy of an equatorial loop of constant coordinate radius r. Define an equatorial loop

(10) $$C(s) = (0,\ r,\ \pi/2,\ 2\pi s), \qquad \text{for } 0 \leq s \leq 1$$

Develop this curve into a flat Minkowski space Ξ' with coordinates $(t',\ x',\ y',\ z')$. Start the development at the point p' with coordinates $(t'_0,\ x'_0,\ y'_0,\ z'_0)$.

The map $L\colon\ T_p\Xi \to T_{p'}\Xi'$ introduced in Section 3.1 is defined by the matrix of partial derivatives

(11) $$\frac{\partial(t', x', y', z')}{\partial(t, r, \theta, \varphi)} = \begin{array}{c|cccc} & t & r & \theta & \varphi \\ \hline t' & \frac{c2}{\nu} & 0 & 0 & \frac{c1}{\nu} \\ x' & 0 & 1 & 0 & 0 \\ y' & -\frac{c1}{\nu} & 0 & 0 & -\frac{c2}{\nu} \\ z' & 0 & 0 & 1 & 0 \end{array}$$

where

(12) $$c1 := \frac{a}{r^2}\left(-\,r - m + \frac{q^2}{r}\right)$$

(13) $$c2 := -\frac{\Delta^{1/2}}{r}$$

(14) $$\nu := (c2^2 - c1^2)^{1/2}$$

Note that $|c2| > |c1|$, hence ν is real. Now we present the main theorem of this work. We express the result of the theorem in terms of these two expressions:

(15) $$K1 \equiv \frac{2\pi a}{r\,\nu}\left[-\,c2\,\Delta^{1/2} + \frac{c1}{a}\,(r^2 + a^2)\right]$$

$$= \frac{2\pi a}{r\,\nu}\left[2\,m\,\frac{q^2}{r} - \left(1 + \frac{a^2}{r^2}\right)\left(m - \frac{q^2}{r}\right)\right]$$

(16)
$$K2 \equiv \frac{a}{r\nu^2}\left[c1\ \Delta^{½} - \frac{c2}{a}\ (r^2 + a^2) \right]$$

$$= \frac{\Delta^{½}}{r\nu^2}\left[r + \left[\frac{a^2}{r^2}\ (- m + \frac{q^2}{r^2}) \right] \right]$$

Theorem 1. The development of the loop C into the Minkowski manifold Ξ' yields the curve

$$C(s) = (t'(s),\ x'(s),\ y'(s),\ z'(s))$$

which is given by

$$t'(s) = K1\ s + t'_0$$

$$x'(s) = K2\ \cos(2\pi\nu s) + x'_0$$

$$y'(s) = K2\ \sin(2\pi\nu s) + y'_0$$

$$z'(s) = z'_0$$

While t' grows linearly with the parameter s, x' and y' sweep out a circular arc which does not close (unless ν happens to equal 1). In the case where $q^2,\ a << r$, the slope $dt'/ds = K1$ is proportional to the angular momentum per unit mass $a = S/m$.

Proof. The proof consists of finding an exact solution of the differential equation (1′) for development of the equatorial loop into Minkowski space. The computation uses connection coefficients that have spacetime indices for the first index, and orthonormal frame indices for the second and third indices. See Appendix C for the connection coefficients and intermediate results in the computations. End of proof.

The translational holonomy (or "failure-to-close vector") of the curve C is $C'(1) - C'(0)$.

- The translational holonomy in the timelike direction is $K1$.
- The translational holonomy in the plane of rotation is $K2\ (\sin^2(2\pi\nu) + (\cos(2\pi\nu)-1)^2)^{1/2}$. This vector is a secant connecting the starting and ending points of the equatorial loop; hence it is approximately tangent to the loop at the starting point of the loop.
- The translational holonomy in the axial direction is zero.

As a check, consider the case of a nonrotating massive object: $a=0$, $K1=0$, and

$$K2 = \frac{r}{(1 - 2\ m/r + q^2/r^2)^{½}}$$

We borrow terminology from metallurgy that describes these structures.

- A line defect whose Burghers vector (translational holonomy) is orthogonal to the plane of the loop traversed is called a screw dislocation. As you wind around the center of the defect, you climb a screw whose axis is orthogonal to the plane of rotation.
- A line defect whose Burghers vector lies in the plane of the loop traversed is called an edge dislocation.

Using the Kerr solution of general relativity, the equatorial loop around a rotating mass generates a screw dislocation with a timelike Burghers vector of magnitude $K1$, plus an edge dislocation of magnitude $K2$. Later we shall see that, in the continuum limit of many small rotating masses, the screw dislocation density is finite and the edge dislocation density vanishes.

In the limit $r \to \infty$:

(17) $$\lim_{r\to\infty} \frac{\Delta}{r^2} = 1$$

(18) $$\lim_{r\to\infty} \nu^2 = 1$$

(19) $$\lim_{r\to\infty} r\,K1 = -2\pi a m$$

(20) $$\lim_{r\to\infty} \frac{K2}{r^2} = 0$$

In units where $c = K = h = 1$, the basic units of time, length and mass are: 1.35121 x 10^{-43} s, 4.085083 x 10^{-35} m, 5.45622 x 10^{-8} kg. The approximation q^2, $a \ll r$ is valid in most situations except for spin ½ fields which are localized to within one Compton wavelength. If such a highly localized field is surrounded by a loop whose radius equals one Compton wavelength, then $a = r/2$.

Table 1. Comparison of m, q^2, and Size for Rotating Systems

	m	q^2	a	r
Electron	2 x 10^{-23}	10^{-3}	3 x 10^{22}	6 x 10^{22}
Neutron	2 x 10^{-20}	0	2 x 10^{19}	3 x 10^{19}
Sun	2 x 10^{37}	0	7 x 10^{36}	2 x 10^{43}
Earth/Sun	2 x 10^{37}	0	1 x 10^{36}	4 x 10^{45}
Galaxy	2 x 10^{49}	0	1 x 10^{51}	1 x 10^{55}

4.3. Translational holonomy of spacelike polar loops

Consider a loop which passes through both the north and south polar axes of a rotating object. In particular, consider the curve

$$C(s) = \begin{cases} (0,\ r,\ 2\pi s,\ 0), & \text{for } 0 \le s \le 0.5 \\ (0,\ r,\ 2\pi(1-s),\ \pi), & \text{for } 0.5 \le s \le 1 \end{cases}$$

Develop this curve into the Minkowski space Ξ'.

Theorem 2. The translational holonomy of the polar loop C is zero.

Proof: See Appendix D.

Theorems 1 and 2 give us the holonomy in essentially all the cases where a closed spacelike curve surrounds a Kerr rotating mass. The holonomy around any spacelike planar loop of constant coordinate radius is, to linear approximation, $\cos(\theta)$ times the holonomy of the equatorial loop, where theta is the angle between the axis of the rotating object and the spacelike normal to the plane of the loop.

4.4. Correction of a numerical factor in the original derivation

The original derivation contains a discrepancy of a factor of two in the spin-torsion equation. This is due to introducing in an inconsistent way a factor of two into the definition of torsion in terms of connection coefficients. This relationship should be

(21) $$T_{\mu\nu}{}^{\lambda}(\xi) = \Gamma_{\mu\nu}{}^{\lambda}(\xi) - \Gamma_{\nu\mu}{}^{\lambda}(\xi)$$

whereas the original derivation inserted a factor of ½ on the right side of this equation.

5. Transition from discrete to continuous distributions of rotating matter

5.1. Discrete distributions of matter that approximate continuous matter

The most convincing way to relate the holonomy around a rotating mass to torsion in continuous distributions of rotating matter is to construct a distribution of many small masses, and take the limit as the distribution becomes finer while the mass density and angular momentum density remain constant. This method of deriving a continuum model from a discrete distribution is widely used in continuum mechanics.

In the continuum limit of a discrete distribution of Kerr solutions, the distribution has more and smaller Kerr solutions. In this limiting process, m/r^3 and q/r^3 remain constants characteristic of the continuum limit, denoted m_C and q_C. So we can replace m by $m_C\ r^3$ and q by $q_C\ r^3$.

When considering a uniform arrays of Kerr solutions over an unbounded volume, it is necessary to set the charge $q = 0$. A finite electric charge density in an infinite volume creates infinite energy densities, and this causes many of the limits to become infinite. Electric charge is not essential to our argument, so we set it to zero. (If we considered patches of spacetime with alternating charges, we expect we could include in the limiting process charges with $q^2 << r$ without creating infinite energy densities).

In the limit $r\to\infty$, the key quantities for computing torsion are $K1/(\pi\ r^2)$ and $K2/(\pi\ r^2)$ because torsion is translational holonomy per unit area, $K1$ and $K2$ scale the translational holonomy, and the area of the equatorial loop as seen by an observer at infinity is $\pi\ r^2$. The results (valid only when $q=0$) are:

(22) $$\lim\ v^2 = 1$$

(23) $$\lim\ \frac{K1}{r^2} = 2\ \pi a\ m_C$$

(24) $$\lim\ \frac{K2}{r^2} = 0$$

The computations of continuum limits use only the exterior Kerr solution and do not use any properties of the Kerr solution inside the event horizon.

5.2. Translational holonomy around a discrete rotating mass in the limit of a small mass

Theorem 1 provides expressions for the translational holonomy of equatorial loops around a rotating mass. The timelike component is $K1$, and the integral surrogate for torsion is

(25) $$|T_{r\varphi}{}^t| \sim \frac{K1}{\pi\ r^2} = \frac{m\ a}{r^3}\ \frac{2}{v}\left[\ 2 + \frac{q^2}{r\ m} + \left(1 + \frac{a^2}{r^2}\right)\left(1 - \frac{q^2}{r\ m}\right)\ \right]$$

Assuming $q^2,\ m,\ a << r$, then

(26) $$|T_{r\varphi}{}^t| \sim \frac{6\ m\ a}{\pi\ r^2} = 8\ \pi\ \frac{m\ a}{4/3\ \pi\ r^3} \sim 8\ \pi\ K\ \ \text{(spin density)}$$

A similar calculation for the "in-plane" surrogate torsion yields

(27) $$T_{r\varphi}{}^t = (8/3)\ \pi\ r\ \frac{m\ a}{(4/3)\ \pi\ r^3}$$

which vanishes in the limit as $r \to 0$ (with the limit taken so that the "mass density" remains constant).

5.3. Comparison of results with Einstein–Cartan theory

The field equations of Einstein–Cartan theory are derivable from Lagrangian variation of the scalar curvature, as in general relativity. The defining difference is that the torsion is determined by Lagrangian variation in Einstein–Cartan theory, whereas it is set to zero *ab initio* in general relativity. The field equations of Einstein–Cartan theory are

(28) $$G_i^{\mu} = 8\,\pi\,K\,P_i^{\mu}$$

(29) $$S_{ij}^{\mu} = 8\,\pi\,K\,Spin_{ij}^{\mu}$$

where G and P are the Einstein tensor and momentum tensor respectively, $Spin$ is the intrinsic angular momentum ("spin") tensor, and S is the modified torsion tensor.

We shall compare the relation between angular momentum and translational holonomy for discrete masses with the relation between intrinsic angular momentum density and torsion in Einstein Cartan theory. Using Theorem 1, Theorem 2, and the calculations in Section 5.2, the integral surrogate for torsion gives the relation

(30) $$S_{ij}^{\mu} = 8\,\pi\,K\,Spin_{ij}^{\mu}$$

This differs from the corresponding field equations of Einstein–Cartan theory in that the torsion trace does not appear in the relation derived from the discrete holonomy calculation.

The absence of the torsion trace terms in the heuristic result is due to the simplicity of the rotating mass model which is used. The "in-plane" torsion in the discrete calculation corresponds to the torsion trace in the continuum limit. We saw in Section 4.2 that, while the in-plane holonomy was present in the discrete calculation, it was of such low order that it vanished in the passage to the continuum limit. (Furthermore, it did not depend upon the angular momentum $m\ a$, but only on the mass m.) This is not surprising, in view of the fact that the torsion trace also vanishes for all Dirac fields. The torsion trace apparently describes more complex rotational moments than the simple first-order moments of the Kerr solution or of Dirac fields.

The torsion trace terms are needed in the spin-torsion field equation in order to derive conservation of angular momentum from the contracted first Bianchi identity. (Arguing backward from Einstein–Cartan theory, variation of the scalar curvature generates the torsion trace terms needed to derive angular momentum conservation.)

Let us summarize our results up to this point. Beginning with definitional relations between holonomy and curvature, we have derived the main features of Einstein–Cartan theory from general relativity. To accomplish this, we used some conventional methods for relating models of discrete masses to models of continuous distributions of matter. The discrete mass model captured all the first order angular momentum terms which appear for simple rotating masses and for Dirac fields. No other assumptions were used.

6. Relationship between spin and torsion

In order to associate classical spin with affine torsion, we must distinguish different types of spin. The need for this distinction is illustrated by electrodynamics fields, which are commonly said to possess spin, but which do not generate affine torsion.

The distinction between types of spin is based on differing geometric meanings of tensor indices.

All tensor indices in continuum mechanics represent either conserved currents or spacetime flux boxes through which current flows are measured using exterior derivatives (gradients, divergences, curls). Connection forms and curvatures also fit this pattern, though the spacetime boxes are used to measure holonomy, not flux. Strictly geometrical concepts – such as defining strain as the Lie derivative of a metric tensor, or as the difference of two metric tensors – are similar to flux boxes in that they involve only configurations in spacetime.

When we formulate mechanics on fiber bundles, the distinction between currents and flux boxes translates into a distinction between base space tensor indices (flux boxes) and fiber tensor indices (conserved currents). This is a natural distinction; for example, no one confuses base space and fiber indices in Yang-Mills gauge theories because the transformation groups, and usually the dimensionality of the representation spaces, differ for the two types of indices.

- "Base space indices" or "spacetime indices" (denoted here by lower case Greek letters λ, μ, ν, …), represent spacetime configurations, mostly flux boxes through which current flow is measured.

Derivatives with respect to these indices occur only in exterior derivatives or divergences (which are Hodge duals of exterior derivatives), or in Lie derivatives. When these indices are free indices in a derivative, they are not covariant differentiated with any connection. However, it is often convenient in computations to differentiate such free indices with respect to the Levi-Civita connection because this yields the same result for exterior derivatives and divergences as writing out the Hodge duals and using ordinary coordinate derivatives.

- "Fiber indices" (denoted here by lower case Roman letters a, b, c …, or for spinors A, B, C…), represent conserved currents, and no derivatives are taken with respect to these indices. When these indices are free indices in a derivative, they are covariant-differentiated with the full connection including affine torsion. In classical physics, these indices always represent momentum, or in antisymmetric pairs, angular momentum.

Here are some examples of the distinction between base space and fiber indices.

- Momentum $P_a{}^{\mu}$: fiber index a represents linear momentum; spacetime index μ represents a hypersurface through which flux is measured.
- Spin density $J_{ab}{}^{\mu}$: an antisymmetric pair of fiber indices a, b represent intrinsic angular momentum, and spacetime index μ represents a flux hypersurface.
- The Maxwell potential A_{μ} and the Maxwell field $F_{\mu\nu}$ (= exterior derivative of A_{μ}) have only spacetime indices.
- Dirac spinor $\psi^{A\mu}$: spinor fiber index A contributes spin; spacetime index μ denotes a spacetime hypersurface.
- Yang-Mills potentials $i\ A^{P}{}_{Q\mu}$ are connection coefficients; Yang–Mills fields $i\ F^{P}{}_{Q\mu\nu}$ are curvatures that represents infinitesimal unitary rotations. Spacetime indices μ, ν denote a spacetime loop around which unitary holonomy is measured. Indices P and Q represent directions in the fiber space for unitary symmetry. If we view Maxwell theory as a Yang-Mills theory with group U(1), then the Maxwell potential becomes $i\ A^{P}{}_{\mu}$, and the Maxwell field becomes $i\ F^{P}{}_{Q\mu\nu}$, where P and Q represent the (unique) direction in the fiber space.

Physicists usually count any tensor index whose symmetry group is a rotation group as contributing to spin. We distinguish three basic types of spin.

(i) *Spacetime spin* is carried by fiber indices (denoted a, b, c, A … in our notation). It usually represents a conserved current (more precisely, the spin part of the total angular momentum conserved current), and never represents a spacetime hypersurface.

(ii) In unitary gauge theories, gauge spin is carried by fiber indices of the gauge group (denoted P, Q… in our notation).

(iii) *Representation spin* consists of any index that arises from representation of a rotation group that does not satisfy the requirements of the first two types of spin. This kind of spin is spin only in the sense of group representation theory. For example, the electromagnetic 4-potential is a connection coefficient on a U(1) bundle, and its index is a spacetime index that indicates the direction of differentiation. Therefore the electromagnetic 4-potential has no spacetime spin. However, it is commonly regarded as a spin 1 field because it has a 4-vector index. We call this "representation spin," because the orthogonal group representation is the only motivation for calling this a spin 1 field, whereas it has no spin at all in terms of spacetime physics.

7. Torsion as a driver of cosmic inflation

Poplawski has shown that closed cosmological models based on Einstein-Cartan theory have a finite minimum radius when they contract, and then rapidly re-expand. Poplawsi has analyzed models in which matter is modeled as a classical Weyssenhoff spin fluid [Poplawski 2010], and more realistic models in which spinning matter is modeled as spinor fields [Poplawski 2012]. The torsion energy density dominates behavior at small radii, and generates extreme rates of expansion similar to that in inflation theory. Cosmic inflation explains the spatial flatness, homogeneity and isotropy of the universe, and solves the horizon problem.

Conventional hypotheses about the mechanism causing cosmic inflation require the existence of a speculative scalar field and introduce additional parameters. Cosmological models that include torsion are based on a classical theory that can be derived from general relativity with no additional assumptions or parameters, and they generate cosmological models with minimum radius and rapid re-expansion.

Researchers on inflation theory expect that inflation is caused by a scalar quantum field, the "inflaton" field. The fact that closed Friedman cosmologies based on Einstein–Cartan theory exhibit inflationary expansion suggests that quantized torsion is the inflaton field. Although torsion is a 3-tensor field antisymmetric in two indices, spin ½ fields generate torsion that is fully antisymmetric, so such torsion has the character of a pseudo-vector field. Torsion appears in covariant derivatives and curvature similar to the way connection coefficients do. Fully contracted quadratic torsion terms might appear as a scalar field. Arguments that suggest that the inflaton field is a scalar field should be reviewed to in light of the hypothesis that torsion is the inflaton field.

In spring 2014, the BICEP2 project announced that it has detected polarized gravitational waves that originated during cosmic inflation shortly after the Big Bang [BICEP2 Collaboration 2014]. As of fall 2014, peer review has revealed that the swirling patterns of comic background radiation detected by BICEP2 could conceivably be caused by larger amounts of interstellar dust than the BICEP2 analysis had included in its analysis. It is hoped that this issue may be resolved by joint analysis of the BICEP2 data with that of the Planck project.

8. Conclusion

Einstein–Cartan theory (EC) currently satisfies a substantial array of criteria for adoption:

a) Torsion and the relationship between torsion and spin in EC can be derived from general relativity with no added assumptions or parameters, in the case of a distribution of Kerr rotating masses.
b) EC extends general relativity to describe exchange of orbital and intrinsic angular momentum, which fixes the most outstanding problem in general relativity as the master theory of classical physics.
c) EC is the minimal extension of general relativity that can fix this problem, because it arises either as the mathematical completion of general relativity, or by merely relaxing the ad-hoc assumption that torsion is zero, and because Einstein–Cartan theory is identical with general relativity where spin density is zero.
d) Cosmological models based in Einstein–Cartan theory provide an explanation for inflation theory based on classical geometry, without relying on speculative quantum fields. This result suggests that quantized torsion is the inflaton field; and that recent observations by the BICEP2 project of gravitational waves generated by inflation may be the first empirical observations of the effects of torsion.
e) EC generates predictions that can in principle validate or falsify the theory, but they cannot be tested by observation due to limitations of current technology.

9. Acknowledgements

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Much of the material in this article appears in [Petti 1986].

Appendix A: Mathematical notation

We will use the following notation:

Ξ — a smooth (pseudo-) Riemannian manifold of dimension n with local coordinates ξ^μ, metric $g_{\mu\nu}$, and metric connection 1-form[1,2] $\omega : TP(\Xi, G) \to L(G)$ whose connection coefficients are denoted by $\Gamma_{\lambda\nu}{}^{\mu}(s)$ and covariant differentiation is denoted by D. Denote a point as $\xi \in \Xi$.

$T(\Xi)$ — the tangent bundle of the manifold Ξ.

$GL(n)$ — linear frame group of the linear frame bundle of Ξ

Y — a linear space R^n with linear coordinates y^i on which $GL(n)$ acts on the left.

$P(\Xi, GL(n))$ — the linear frame bundle over Ξ with structure group $GL(n)$. The bundle projection is denoted $\pi : P \to \Xi$.

$B(\Xi, GL(n), Y)$ — is the bundle associated with $P(\Xi, GL(n))$ with fiber Y. The bundle projection is denoted $\pi : P \to \Xi$.

$A(n)$ — affine group that is bundle-isomorphic to the affine frame bundle of Ξ

X — an affine space A^n with affine coordinates x^i on which $A(n)$ acts on the left.

$P(\Xi, A(n))$ — a principal bundle over Ξ with structure group $A(n)$. We do not use the affine frame bundle of Ξ, because the affine frame bundle has a fixed solder form, and we shall vary the solder form.

$B(\Xi, A(n), X)$ — the bundle associated with $P(\Xi, A(n))$ with fiber X.

Ξ' — is a flat smooth (pseudo-) Riemannian manifold of the same dimension as Ξ with local coordinates ξ'^μ, metric $g'_{\mu\nu}$, and metric connection 1-form ω whose connection coefficients are denoted by $\Gamma'_{\lambda\nu}{}^{\mu}(s)$ and whose covariant differentiation is denoted by D'. Denote a point in Ξ' by ξ'.

$C: [0,1] \to \Xi$ — a smooth curve in Ξ, with $C(0) = \xi_0$ and tangent vector field u.

$C': [0,1] \to \Xi'$ — a smooth curve in Ξ', with $C'(0) = \xi'_0$.

u' — tangent vector field along the curve C'.

$L: T_\xi\Xi \to T_{\xi'}\Xi'$ — a linear isometry between tangent spaces at $\xi \in \Xi$ and $\xi' \in \Xi'$.

---

[1] A connection on a principal bundle $P(\Xi,G)$ is an n-dimensional distribution **HD** of vectors on P such that (a) none of the vectors in **HD** are verticle (that is, none of the vectors in **HD** are annihilated by the bundle projection π); and (b) translation of **HD** by right operation of any $g \in G$ leaves **HD** invariant. The second condition is called the equivariance condition for the connection.

[2] The connection form $\boldsymbol{\omega}$: A connection **HD** on a principal bundle $P(\Xi,G)$ defines a 1-form $\boldsymbol{\omega}$ on P with values in $L(G)$ that is defined by two conditions: (a) for any tangent vector V to P that is vertical (that is, $\pi(V)=0$), $\boldsymbol{\omega}(V)$ = the vector in $L(G)$ whose right action on P defines the vector V. In a sense, this condition means $\boldsymbol{\omega}$ is the identity operator on vertical vectors. (b) For any $g \in G$, the right of g on P leaves $\boldsymbol{\omega}$ invariant; more precisely, $\boldsymbol{\omega} \bullet dRg = ad(g^{-1}) \bullet \boldsymbol{\omega}$. The second condition is called the equivariance condition for the connection form.

Appendix B: Connections, curvature and torsion on fiber bundles

We shall assume that the reader is familiar with the basic results of the theory of connections on fiber bundles. [Bishop and Crittenden 1964] and [Kobayashi and Nomizu 1963 and 1969]. The purpose of this section is to focus upon the relation between holonomy and curvature in the case of linear frame bundles. If the reader is not familiar with the theory of connections on fiber bundles, the rest of the paper should be coherent without this section.

B-1 Relationship between curvature and torsion

The most general and elegant definition of holonomy is given in terms of the theory of connections on fiber bundles. Let P be a principal bundle over Ξ with structure group G, endowed with a connection. Define a smooth curve C in Ξ starting and ending at the point ξ. Starting at a point p_0 in the fiber over ξ, form the horizontal lift of the curve C into P. The lifted curve will end at a point p_1 in the fiber over ξ. Then there is a unique element $g \in G$ such that

$$p_0\, g = p_1 \,. \tag{31}$$

By the equivariance condition in the definition of the connection, the element g is independent of the choice of p_0, and depends only on the connection on P and the curve C in Ξ.

The group element g is the holonomy of the loop C. If we have a linear representation of the structure group, then the holonomy g is transformation by which any basis of the representation space is changed upon parallel translation around C.

The most general and elegant definition of curvature is also given in terms of the theory of connections on fiber bundles. There are two equivalent ways to construct the curvature:

(i) Choose a point ξ in Ξ and two vectors V and W at ξ. Construct a one-parameter family of closed loops $C_t(s)$ through ξ which are tangent at ξ to the plane of $V \wedge W$, and which converge to ξ as $t \to 0$. Let g_t be the holonomy of the loop C_t. Then the curvature R is defined as

$$-R(V \wedge W) = \lim_{t\to 0} \frac{g_t}{\mathrm{area}(t)} \tag{32}$$

where $\mathrm{area}(t)$ = area of the loop C_t in any smooth coordinate system whose coordinate vectors at ξ include V and W. The curvature depends only on the 2-form $V \wedge W$, and not on the choice of the vectors V and W, or the curves C_t. The limit exists and is finite when appropriate smoothness conditions are imposed upon the connection.

This construction yields a simple intuitive interpretation of curvature: curvature is holonomy per unit area, in the limit where the area encircled by the loop goes to zero.

(ii) Let ω = the connection 1-form on the bundle P.

$$\omega\colon\ TP \to L(G) \tag{33}$$

The curvature is the horizontal component of the 2-form $d\omega$ (exterior derivative of ω):

$$\Omega(\,\cdot\,,\,\cdot\,) = d\omega(\mathrm{Hor}(\,\cdot\,),\,\mathrm{Hor}(\,\cdot\,)) \tag{34}$$

where Hor is the horizontal projection map for tangents to the bundle P.

The horizontal 2-form Ω on P uniquely determines a Lie-algebra-valued 2-tensor field on Ξ. If the bundle is a linear frame bundle, then the Lie algebra components of the curvature can be identified with (1, 1) rotation tensor fields on Ξ. This gives a (1,3) tensor field R on Ξ, which is the curvature tensor usually used in Riemannian geometry.

The second definition of curvature is the customary one in differential geometry. The first one offers richer intuitive insights for the purposes of this paper.

We can derive the first definition of curvature from the second using the relation

$$d\omega(X,\ Y) = \tfrac{1}{2}\ (X\ \omega(Y) - Y\ \omega(X) - \omega([X,\ Y]))$$

which holds for all 1-forms ω and all smooth vector fields X and Y. If ω is the connection form and X and Y are the horizontal lifts of coordinate vector fields in Ξ, then we get

$$d\omega(X,\ Y) = -\tfrac{1}{2}\ \omega([X,\ Y]) \qquad (35)$$

For the X and Y specified, [X, Y] is vertical, and ω is a vertical projection operator (identifying the Lie algebra of the structure group with the vertical fiber). $[X,\ Y]$ is defined by traversing an integral curve of X followed by those of Y, $-$X, and $-$Y, subtracting the coordinates of the initial location from those of the end point given by the integration process, and taking the limit, per unit area enclosed, as the lengths of integral curves traversed approach zero. When projected into Ξ, this construction for [X, Y] yields a family of closed loops at the point p in M, like the loops used in the first definition of curvature. The construction for [X, Y] amounts to parallel translation around these closed loops, and $\omega([X,\ Y])$ is the holonomy per unit area in the limit of small areas of the loops traversed. The conventional normalization for R is that

$$R(V,\ W) = 2\ d\omega(X,\ Y) \qquad (36)$$

where V and W are projections in Ξ of the vectors X and Y tangent to P. Hence R is minus the holonomy per unit area.

B-2 Relation between development, curvature, and torsion for linear connections

Linear connections are characterized by two conditions:

(i) The structure group G is a subgroup of the linear automorphisms of R^n onto itself (commonly represented by $GL(n, R)$ = group of n-by-n real matrices).

(ii) There is a solder 1-form : $TP \rightarrow R^n$ which is horizontal and equivariant. This form identifies a point p in P with a basis of tangents to Ξ at $\pi(p)$.

The principal bundle is identifiable as a subbundle of the bundle of linear bases of the tangents of Ξ.

There are two ways to define the torsion of a linear connection:

(i) At a point $\xi \in \Xi$, form a family of loops $C_t(s)$ which converge to p and are tangent to $V \wedge W$, as in Section 3.2 above. Develop the curves C_t into a flat (pseudo-) Euclidean n manifold. The developed curves generally do not close. Torsion can be defined as

$$-\ T(V \wedge W) = \lim_{t \to 0} \frac{(C'_t(1) - C'_t(0))}{\mathrm{area}(t)} \qquad (37)$$

Thus, torsion is the amount by which the developed curve of a loop in Ξ fails to close, per unit area enclosed by the loop, in the limit as the enclosed area approaches zero. Torsion is infinitesimal translational holonomy per unit area. We can think of torsion as translational curvature. Indeed, in the theory of affine connections, where the structure group is a subgroup of the inhomogeneous automorphisms of R^n, torsion appears as the curvature components associated with the translations in the Lie algebra of the structure group.

(ii) Torsion can be defined as the horizontal part of the exterior derivative of the fundamental 1-form θ:

$$\mathrm{Tor}(\ \cdot\ ,\ \cdot\ ) = d\theta(\mathrm{Hor}(\ \cdot\ ),\ \mathrm{Hor}(\ \cdot\ )) \qquad (38)$$

> The horizontal 2-form Tor on B determines a unique (1, 2) tensor field T on Ξ, by transforming the R^n–valued index of Tor into tangent vectors of Ξ. *T* is the torsion tensor normally used in Riemann-Cartan geometry.

The second definition of torsion can be used to derive the first definition in terms of translational holonomy, much the same as in the case of curvature. This construction requires use of the inhomogeneous linear group as the structure group.

Appendix C: Proof of theorem 1 (translational holonomy of an equatorial loop)

C-1 Holonomic coordinate bases and orthonormal frames

A basis of the tangent space is called "holonomic" if and only if the Lie brackets of the basis vectorfields are zero; equivalently, if and only if there exist local coordinate systems that yield the basis vectors.

Throughout the entire article, we use the following symbols.

$$a := s/m$$

$$\Delta := r^2 + a^2 - 2\,m\,r + q^2$$

$$\rho^2 := r^2 + a^2 \cos^2(\theta)$$

$$A := -2\,m\,r + q^2$$

$$c := \cos(\theta),\quad s := \sin(\theta)$$

Two bases will be used for the tangent vectors and covectors on the Kerr manifold. The first is the coordinate basis: $(\partial/\partial t,\ \partial/\partial r,\ \partial/\partial\theta,\ \partial/\partial\varphi)$; with dual basis $(dt,\ dr,\ d\theta,\ d\varphi)$.

The second basis is an orthonormal frame field.

$$e_0 = \frac{1}{\rho\,\Delta^2}\left[(r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi}\right]$$

$$e_1 = \frac{\Delta^{½}}{\rho}\ \frac{\partial}{\partial r}$$

$$e_2 = \frac{1}{\rho}\ \frac{\partial}{\partial\theta}$$

$$e_3 = \frac{1}{\rho\sin(\theta)}\left[a\sin^2(\theta)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi}\right]$$

The dual basis of the orthonormal frame field is:

$$e^0 = \frac{\Delta^{½}}{\rho}\,[\,dt - a\sin^2(\theta)\,d\varphi\,]$$

$$e^1 = \frac{\rho}{\Delta^{½}}\,dr$$

$$e^2 = \rho\,d\theta$$

$$e^3 = \frac{\sin(\theta)}{\rho}\,[\,(r^2 + a^2)\,d\varphi - a\,dt\,]$$

C-2 Lie brackets of frame fields

A basis of the tangent space is called "anholonomic" if and only if the Lie brackets of the basis vector fields are non-zero. Orthonormal bases for tangents equivalently, if and only if there exists local coordinate systems that yield the basis vectors. Orthonormal frame fields normally have

The Lie brackets of the frame fields are specified frame brackets coefficients fb as in the following formula.

$$[e_i,\ e_j] = fb_{ij}{}^k\ e_k \qquad\qquad i,j,k = 0,1,2,3$$

Note that $fb_{ij}{}^k = -\ fb_{ji}{}^k$. The non-vanishing frame bracket coefficients are given below.

Figure 1: Frame bracket coefficients for the orthonormal frame fields

$$fb_{01}{}^k = \begin{pmatrix} \dfrac{a^2\ (r-m)\ c^2 + m\ r^2 - r\ (q^2+a^2)}{\Delta^{1/2}\ \rho^3} \\ 0 \\ 0 \\ \dfrac{2\ a\ r\ s}{\rho^3} \end{pmatrix} \qquad fb_{02}{}^k = \begin{pmatrix} -\dfrac{a^2\ c\ s}{\rho^3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$fb_{12}{}^k = \begin{pmatrix} 0 \\ -\dfrac{a^2\ c\ s}{\rho^3} \\ -\dfrac{\Delta^{1/2}\ r}{\rho^3} \\ 0 \end{pmatrix} \qquad fb_{13}{}^k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{\Delta^{1/2}\ r}{\rho^3} \end{pmatrix}$$

$$fb_{23}{}^k = \begin{pmatrix} \dfrac{2\ a\ \Delta^{1/2}\ c}{\rho^3} \\ 0 \\ 0 \\ -\dfrac{(r^2+a^2)\ c}{\rho^3\ s} \end{pmatrix}$$

C-3 Connection coefficients

The most convenient way to write the connection coefficients is to use the coordinate basis for the direction of covariant differentiation (first index), and the frame field basis for the rotation of frames (second and third indices).

$$\text{Christoffel symbol} = \Gamma_{kb}{}^a$$

where k is a coordinate index and a and b are frame field indices.

Below are the connection coefficients using the first spacetime coordinate basis (above) for the directions of covariant differentiation, and the orthonormal frame basis for fiber directions (the second and third indices).

Figure 2: Christoffel symbols for Kerr solution in mixed coordinate/frame field basis

$$(39) \qquad \Gamma_{tb}{}^{a} = \begin{array}{|c|c|c|c|} \hline 0 & -\frac{1}{\rho^2}\left(m + \frac{r\ a}{\rho^2}\right) & 0 & 0 \\ \hline -\frac{1}{\rho^2}\left(m + \frac{r\ a}{\rho^2}\right) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{a\ c\ A}{\rho^4} \\ \hline 0 & 0 & \frac{a\ c\ A}{\rho^4} & 0 \\ \hline \end{array}$$

— b →

$$(40) \qquad \Gamma_{rb}{}^{a} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & \frac{a}{\rho^2}\ \frac{s\ r}{\Delta^{1/2}} \\ \hline 0 & 0 & \frac{a^2}{\rho^2}\ \frac{s\ c}{\Delta^{1/2}} & 0 \\ \hline 0 & \frac{a^2\ s\ c}{\rho^2\ \Delta^{1/2}} & 0 & 0 \\ \hline \frac{a}{\rho^2}\ \frac{s\ r}{\Delta^{1/2}} & 0 & 0 & 0 \\ \hline \end{array}$$

$$(41) \qquad \Gamma_{\theta b}{}^{a} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & \frac{a\ c\ \Delta^{1/2}}{\rho^2} \\ \hline 0 & 0 & -\frac{r\ \Delta^{1/2}}{\rho^2} & 0 \\ \hline 0 & \frac{r\ \Delta^{1/2}}{\rho^2} & 0 & 0 \\ \hline \frac{a\ c\ \Delta^{1/2}}{\rho^2} & 0 & 0 & 0 \\ \hline \end{array}$$

$$(42)\quad \Gamma_{\varphi b}{}^{a} = \begin{bmatrix} 0 & \dfrac{-a\,(a^2(r-m)\,c^2 + r\,(r^2+mr+q^2))\,s^2}{\rho^4} & -\dfrac{a\,s\,c\,\Delta^{1/2}}{\rho^2} & 0 \\ \dfrac{-a\,(a^2(r-m)\,c^2 + r\,(r^2+mr+q^2))\,s^2}{\rho^4} & 0 & 0 & -\dfrac{r\,s\,\Delta^{1/2}}{\rho^2} \\ -\dfrac{a\,s\,c\,\Delta^{1/2}}{\rho^2} & 0 & 0 & \dfrac{c}{\rho^4}\left[a^2s^2\Delta-(r^2+a^2)^2\right] \\ 0 & \dfrac{r\,s\,\Delta^{1/2}}{\rho^2} & -\dfrac{c}{\rho^4}\left[a^2s^2\Delta-(r^2+a^2)^2\right] & 0 \end{bmatrix}$$

C-4 Development of equatorial loop into Minkowski space

The starting point and initial tangent of the equatorial loop used in Theorem 1 are given below.

$$(43)\quad \underset{\text{(Kerr coords)}}{C^{\mu}(0)} = \begin{vmatrix} 0 \\ r \\ 0 \\ 0 \end{vmatrix} \qquad \underset{\text{(Kerr coords)}}{d\,C^{\mu}(0)/d\sigma} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 2\,\pi \end{vmatrix} \qquad \underset{\text{(frames)}}{d\,C^{a}(0)/d\sigma} = \begin{vmatrix} \dfrac{-2\,\pi\,a\,\Delta^{1/2}}{r} \\ 0 \\ 0 \\ \dfrac{2\,\pi\,(r^2+a^2)}{r} \end{vmatrix}$$

$$(44)\quad \underset{\text{(Kerr cords)}}{\mathrm{Accel}^{\mu}(\sigma)} = \begin{vmatrix} \dfrac{-2\,\pi\,a\,\Delta^{1/2}}{r} \\ 0 \\ 0 \\ \dfrac{2\pi a\,(r^2+a^2)}{r} \end{vmatrix} \qquad \underset{\text{(frames)}}{\mathrm{Accel}^{a}(\sigma)} = \begin{vmatrix} 0 \\ \dfrac{-4\pi^2\,\Delta^{1/2}\,(r^4 - a^2\,m\,r + a^2\,q^2)}{r^4} \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ -4\,\pi^2\,k2\,\nu^2 \\ 0 \\ 0 \end{vmatrix}$$

We want to use the mapping $A^{\mu}{}_{\nu}(\sigma)$ that parallel translates vectors at the start of the equatorial loop to any point on the loop with parameter value σ. The indices on $A^{\mu}{}_{\nu}$ are orthonormal frame indices.

Lemma 1 states that $A(\sigma)$ satisfies the differential system

$$\frac{dA^{\mu}{}_{\nu}}{ds}(s) + u^{\lambda}(s)\;\Gamma_{\lambda\sigma}{}^{\mu}(s)\;A^{\sigma}{}_{\nu}(s) = 0$$

In Kerr coordinates, on the specified equatorial loop, this equation is:

$$(45)\qquad \frac{dA^{\mu}{}_{\nu}}{ds}(s) + 2\,\pi\;\Gamma_{\varphi\sigma}{}^{\mu}(s)\;A^{\sigma}{}_{\nu}(s) = 0$$

The matrix $A(\sigma)$ below integrates parallel translation for the chosen equatorial loop satisfies this equation with initial condition $A(0) = \text{identity}$.

Notation: $Z := -r + q^2/r - m$ $\qquad cc := \cos(2\pi\nu\,\theta),\ ss := \sin(2\pi\nu\theta)$

$$(46)\quad A^{\mu}{}_{\nu}(\sigma) = \begin{array}{|c|c|c|c|}\hline \frac{a\ Z^2\ (1 - cc)}{\nu^2\ r^4} + 1 & -\frac{a\ Z\ ss}{\nu\ r^2} & 0 & -\frac{a\ \Delta^{1/2}\ Z\ (1-cc)}{\nu^2\ r^3} \\ \hline -\frac{a\ Z\ ss}{\nu\ r^2} & \frac{(a^2\ Z^2 - \Delta\ r^2)(1-cc)}{\nu^2\ r^4}+1 & 0 & \frac{\Delta^{1/2}\ ss}{\nu\ r} \\ \hline 0 & 0 & 1 & 0 \\ \hline \frac{a\ \Delta^{1/2}\ (1 - cc)}{\nu^2\ r^3} & -\frac{\Delta^{1/2}\ ss}{\nu\ r} & 0 & 1 - \frac{\Delta\,(1 - cc)}{\nu^2\ r^2} \\ \hline \end{array}$$

Pull vector $\mathrm{Accel}(\sigma)$ at $C(\sigma)$ back to the starting point $C(0)$ by multiplying by $A^{-1}(\sigma)$.

$$(47)\quad A^{-1}(\sigma)\ \mathrm{Accel}(\sigma) = \begin{array}{|c|}\hline \frac{4\ \pi^2\ a\ k2\ \nu\ (r^2 + m\ r - q^2)\ \sin(2\ \pi\ \nu\ \sigma)}{r^3} \\ \hline -\frac{4\ \pi^2\ k2\ (r^4(r^2-2mr+q^2) - a^2(m\ r-q^2)(2r^2 + m\ r - q^2))\ \cos(2\ \pi\ \nu\ \sigma)}{r^6} \\ \hline 0 \\ \hline -\frac{4\ \pi^2\ k2\ \nu\ \Delta^{1/2}\ \sin(2\ \pi\ \nu\ \sigma)}{r} \\ \hline \end{array}$$

The next step is to choose a fixed isometry $L^{\mu}{}_{\nu}$ between the tangent vectors at initial points $C(0)$ in spacetime Ξ and $C'(0)$ in the development space Ξ'. We have chosen the starting point and initial direction of the developed curve so that the solutions in Minkowski coordinates reflect the rotational symmetry of the problem.

$$(48)\quad L^{\mu}{}_{\nu} = \begin{array}{|c|c|c|c|}\hline -\frac{\Delta^{1/2}}{\nu\ r} & 0 & 0 & \frac{a\ Z}{\nu\ r^2} \\ \hline 0 & 1 & 0 & 0 \\ \hline -\frac{a\ Z}{\nu\ r^2} & 0 & 0 & \frac{\Delta^{1/2}}{\nu\ r} \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array}$$

Since we use Minkowski coordinates $(t',\ x',\ y',\ z')$ on the flat space Ξ', parallel translation in local coordinates is trivial. The acceleration of curve $C(\sigma)$ is mapped from $C(\sigma)$ to $C(0)$ to $C'(0)$ to $C'(\sigma)$ via that mapping $A'(s)\ L\ [A(s)]^{-1}$, is:

The acceleration of the developed curve in Minkowski space is:

$$(49)\quad \mathrm{Accel}'(\sigma) = \left|\begin{array}{c} 0 \\ 4\ \pi^2\ k2\ \nu^2\ \cos(2\ \pi\ \nu\ \sigma) \\ 4\ \pi^2\ k2\ \nu^2\ \sin(2\ \pi\ \nu\ \sigma) \\ 0 \end{array}\right|$$

The initial condition `d C'(0)/dσ` is determined by the orthogonal map L, and the choice of the initial point `C'(0)` of the developed curve is arbitrary. We choose both of these to express the rotational symmetry.

The initial conditions for development of the equatorial loop into Minkowski space are:

$$(50) \qquad C'(0) = \begin{bmatrix} -k1 \\ 0 \\ 2\ \pi\ k2\ \nu \\ 0 \end{bmatrix} \qquad d\ C'(0)/d\sigma = \begin{bmatrix} -k1 \\ 0 \\ 2\ \pi\ k2\ \nu \\ 0 \end{bmatrix}$$

The development of the equatorial loop into Minkowski space is:

$$(51) \qquad C'(\sigma) = \begin{vmatrix} -k1\ \sigma \\ k2\ \cos(2\ \pi\ \nu\ \sigma) \\ k2\ \sin(2\ \pi\ \nu\ \sigma) \\ 0 \end{vmatrix}$$

C-5 Translational holonomy of an equatorial loop developed into Minkowski space

The translational holonomy (the "failure-to-close vector") of development of the equatorial loop into Minkowski space is `C'(1) - C'(0)`.

$$(52) \qquad C'(1) - C'(0) = \begin{vmatrix} -k1 \\ k2\ (\cos(2\ \pi\ \nu) - 1) \\ k2\ \sin(2\ \pi\ \nu) \\ 0 \end{vmatrix}$$

This completes the proof of Theorem 1.

After the initial publication of this derivation [Petti 1986], the manual calculations in this proof were replicated in Macsyma 2.4.1a. At that time, the basic algebra, matrix operations and ODE solvers used in these computations had been used and debugged for three decades. The tensor analysis capabilities (which were substantially extended to include frame fields and torsion near the end of this period) are used in these computations only to compute the frame fields and connection coefficients of the Kerr metric.

Appendix D: Proof of theorem 2 (translational holonomy of polar loops)

For development of a curve in the θ direction extending from θ_0 to θ_1, the development matrix `A` has the form

$$(53) \qquad A^a{}_b = \begin{vmatrix} \cosh(F) & 0 & 0 & \sinh(F) \\ 0 & \cosh(G) & -\sin(G) & 0 \\ 0 & \sin(G) & \cos(G) & 0 \\ \sinh(F) & 0 & 0 & \cosh(F) \end{vmatrix}$$

where

$$(54)\qquad F = \left| -\frac{\Delta}{r^2 + a^2} \tanh^{-1}\left[\frac{a \sin(\theta)}{(r^2+a^2)^{½}}\right]\right|_{\theta_0}^{\theta_1}$$

$$(55)\qquad G = \left| -\frac{\Delta}{r^2 + a^2} \tanh^{-1}\left[\frac{r \tan(\theta)}{(r^2+a^2)^{½}}\right]\right|_{\theta_0}^{\theta_1}$$

The tensor indices a and b refer to frame field directions. When θ_0 and θ_1 are multiples of π, both F and G vanish, so the matrix A is the identity. Therefore the development of a closed loop in the θ direction are also be closed. Therefore the translational holonomy is zero.

Appendix E: Comments on the derivation and discussion

The derivation of Einstein–Cartan theory has two main limitations: some convergence processes in the derivation are not completely rigorous (section E–1); there is no direct empirical evidence for Einstein–Cartan theory (section E–2). Further comments and discussion on the derivation can be found in section E-3.

E-1 Lack of empirical evidence for Einstein–Cartan theory

Normally a physical theory is accepted if it is the simplest candidate that explains prior observations and makes new predictions that are verified empirically and that currently-accepted theories fail to explain.

Absence of direct experimental evidence for Einstein–Cartan theory limits the degree of certainty assigned to Einstein–Cartan theory. Absence of direct experimental evidence does not challenge the validity of the derivation.

The most promising places to seek experimental evidence for Einstein–Cartan theory are places having high spin density, such as the early universe and the interior of neutron stars. Direct empirical evidence supporting Einstein–Cartan theory probably is not obtainable with current technology because torsion vanishes where spin density is zero, and because torsion forces are weak like gravitational forces.

The BICEP2 project has observed gravitational waves originating from inflation [BICEP2 Collaboration 2014]. Einstein–Cartan theory provides a mechanism for cosmic inflation based on classical geometry [Poplawski 2010; 2012]. If quantized torsion passes all tests for the inflation field that drives cosmic inflation, then these observations can be considered the first empirical evidence for Einstein–Cartan theory.

Even in the absence of direct empirical evidence, a mathematical derivation from an accepted theory without added assumptions is strong evidence, especially since Einstein–Cartan theory is the minimal extension of general relativity that can model exchange of orbital and intrinsic angular momentum.

E-2 Limited rigor in some steps of the derivation

The first of the three steps in the derivation (listed in section 2) – computation of translational holonomy around closed paths for one Kerr rotating mass – is rigorous mathematics. (See section 4 and Appendix C). The relation between translational holonomy and torsion is rigorous. The limit $r \rightarrow \infty$ in section 4.2 is justified; once a length scale is chosen for a loop around which holonomy is computed, each Kerr rotating mass can be made arbitrarily small (and numerous) compared to that length scale. The observer is at infinity compared to the scale of the mass, which is the normal assumption made to interpret rotational moments of a rotating body as angular momentum.

The second step – computing continuum limits of a distribution of Kerr rotating masses – is not completely rigorous, because the computation of continuum limits of a distribution of rotating masses does not include a rigorous convergence argument. (See section 5.) However, this step uses the same method as computation of mass density of a dust cosmological model derived from a distribution of exterior Schwarzschild solutions. Angular momentum density is just the product of the parameter a (= angular momentum per unit mass as seen by an observer at infinity) and the mass density.

The third step is just a matter of observation of the equations derived in the second step.

One difference between continuum limits in Schwarzschild and Kerr dust models is that exact interior Schwarzschild solutions are known, but no exact interior Kerr solution. The lack of known exact interior Kerr solutions is not a fundamental problem for the derivation, for two reasons.

- Mathematically rigorous solutions of field theories are routinely defined by convergent sequences of approximate solutions; a solution need not be representable as exact symbolic solutions (as discussed further in section E-3-e).
- If general relativity did not have interior Kerr solutions (exact symbolic solutions or solutions defined as limits of convergent approximations), then general relativity would be incapable of describing static, rotating cylindrically symmetric bodies that do not contain a singularity, and that have zero moments at infinity except for total mass, charge and angular momentum.

The proper role of rigor in mathematics and physics was discussed by over a dozen mathematicians and physicists (most of them Field Medalists) in the *Bulletin of the American Mathematical Society* [Jaffe 1993; Atiyah 1994; Thurston 1994]. The consensus view was (a) rigor has an important role in assuring the reliability of results; (b) rigor in practice has degrees and is not a binary property; (c) the intellectual processes of innovation and of rigorous proof often differ; (d) an excessive focus on rigor in the innovation process would stifle innovation.

E-3 Other comments about the derivation

Below are additional comments in the derivation followed by this author's responses.

a) The derivation integrates through singularities.

   If general relativity has an interior Kerr solution (as argued above), then no singularities occur in the derivation. If Kerr singularities are present, then the derivation computes line integrals whose paths surround singularities, divided by areas as seen by an observer at infinity. The computations are virtually identical to those used to compute mass densities of dust models derived from a distribution of Schwarzschild solutions. See section E-2 for further discussion.

b) Spin is a quantum phenomenon; it has no place in classical physics

   Any classical model that has orbital angular momentum on a smaller scale than the scale of the model must include the density of subscale orbital angular momentum. The classical theory of spin fluids routinely models subscale orbital angular momentum as intrinsic angular momentum, or classical spin.

   The Noether theorem associates rotational symmetry with conservation of orbital a.m. + intrinsic angular momentum. If we adopt the name "nips" for the classical density of subscale orbital a.m., then orbital a.m. + nips is conserved. This "name game" is unnecessary.

c) Spin–orbit coupling is solely an electromagnetic phenomenon.

   Below are two classical examples of non–electromagnetic exchange of orbital angular momenta on different scales.

   i) Earth's moon now always faces the same side toward earth, because viscous damping caused by tidal forces has damped out other modes of lunar rotation. The lost rotational a.m. of the moon must be converted into orbital or rotational a.m. of the earth. We can rule out that the rotational a.m. of the earth could be the only receiver of moon's lost rotational a.m. by the following thought experiment: Let the earth be rotationally symmetric and let the viscosity of terrestrial rocks approach zero; in this case the viscous mechanism acting on the moon cannot alter the rotational a.m. of the earth. In this case, there must be exchange of rotational a.m. of the moon and orbital a.m. of the earth.

ii) The essence of fluid turbulence is that orbital a.m. on a large scale is transferred to orbital a.m. on smaller scales, down to scales below which it cannot be tracked by the classical theory of fluids [Monin 1971].

In both examples, the exchange of angular momentum is mediated by properties of continuum matter, not by gravitation. However, a complete classical gravitational theory must accommodate the exchange between orbital a.m. and subscale orbital a.m. in its conservation law for angular momentum.

I invite interested researchers to develop formal proofs that, in a dust model of rotating galaxies, exchange can occur between rotational angular momentum of galaxies and orbital angular momentum of the distribution of galaxies.

d) Equating spin density with torsion doesn't work; for example, the Maxwell field has spin 1, but it does not generate affine torsion in Einstein–Cartan theory.

We must distinguish base space tensor indices and fiber tensor indices. All tensor indices that represent conserved currents are covariant differentiated, and spacetime boxes through which fluxes are measured are not covariant differentiated. This is similar to Yang-Mills theory, where physicists are accustomed to distinguishing base space indices and current indices. See Section 6 for a more explicit discussion.

e) A distribution of Kerr solutions is not an exact solution in general relativity.

Mathematical analysis routinely defines solutions of differential equations as convergent sequences of approximate solutions, without having exact symbolic solutions. Theorems on existence and uniqueness of solutions are based on convergent sequences of approximate solutions where the function belongs to a Sobolev space (whose norm includes L_2 norms of derivatives of functions). An engineering-level introduction to Sobolev norms without advanced measure theory can be found in [Strang 1973].

f) Deriving torsion from general relativity does not establish Einstein–Cartan theory

Einstein–Cartan theory resolves the failure of general relativity to model exchange of orbital and intrinsic angular momentum in the least invasive way. No theory – including Newtonian mechanics, special relativity, general relativity, and quantum field theory – can counter the possibility that we might eventually find a better extension. We must apply “Occam's Razor” that simplicity and fit with contemporary evidence should temper skepticism.

g) Deriving torsion from general relativity in this way precludes extension to larger gauge group.

The work herein constructively derives torsion from general relativity; it does not preclude extending the symmetry group to include conformal or projective symmetries.

h) It is not possible to derive a non-symmetric connection from symmetric connection by a limiting process.

This statement is similar to the assertion that a manifold with Riemannian curvature cannot be derived as the continuum limit of a collection of Euclidean pieces because Riemannian geometry is more general than Euclidean geometry. An elementary example of such as construction is to build a cone from a flat sheet in the shape of a circular sector with center angle $2\pi - \theta$ for small $\theta > 0$. The resulting cone has a Dirac delta function of Riemannian curvature of magnitude θ at its apex and is flat elsewhere.

i) Einstein–Cartan theory is no longer important

A complete theory of classical physics must include exchange of orbital angular momentum and intrinsic (sub-scale orbital) angular momentum. Einstein–Cartan theory is currently the simplest extension of classical general relativity that solves this problem, and that can be mathematically derived from general relativity with no additional assumptions.

Einstein–Cartan theory appears able to drive cosmic inflation, which explains observations of spatial flatness, homogeneity and isotropy. The theory uses only classical differential geometry.

Any successful quantum theory of gravitation that includes spin must include affine torsion or a precursor of torsion, because its classical limit must include affine torsion.

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