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• Q1)

a) $f(x,y) = 5y^{4/5}$ $f_y = 4y^{-\frac{1}{5}} = \frac{4}{y^{1/5}}$

$f(x,y) = 5y^{4/5}$ is cont. everywhere. Thus the IVP has least one solution in any neighbourhood of (0,0).

We may choose

$R = \{(x,y) \mid |x| < 5, |y| < 5\}$

$f_y = \frac{4}{y^{\frac{1}{5}}}$ is not defined at $y=0$.

Thus, the given IVP has at least one solution, but EU Thm does not say anything

$\frac{dy}{dx} = 5y^{\frac{4}{5}} \Rightarrow \frac{dy}{5y^{\frac{4}{5}}} = dx \Rightarrow y^{\frac{1}{5}} = x + C$ plug in (0,0) $\Rightarrow 0 = C$. So

$y = x^5$, $y = 0$ are both solutions.

b) $f(x,y) = 5y^{4/5}$ $f_y = 4y^{-\frac{1}{5}} = \frac{4}{y^{\frac{1}{5}}}$

$f(x,y) = 5y^{4/5}$ is cont. everywhere.

$f_y = \frac{4}{y^{\frac{1}{5}}}$ is not cont. at $y=0$.

Both $f(x,y)$ and f_y are both cont. in

$R = \{(x,y) \mid |x| < 1$ and $|y-1| < \frac{1}{2}\}$

$R = \{(x,y) \mid -1 < x < 1$ and $\frac{1}{2} < y < \frac{3}{2}\}$

There is a unique solution around R, EU Thm appliable

$\frac{dy}{dx} = 5y^{\frac{4}{5}} \Rightarrow \int \frac{dy}{5y^{\frac{4}{5}}} = \int dx \Rightarrow \frac{1}{5} 5y^{1/5} = x + C$

$y^{\frac{1}{5}} = x + c \quad y(0) = 1 \Rightarrow c = 1$

$y^{1/5} = x + 1$

$y(x) = (x+1)^5$ is the unique solution.

Q2)

a) when $|x| \leq 1 \Rightarrow g(x) = 1$

since $\frac{dy}{dx} + y = 1, \; y(0) = 0$

$\mu(x) = e^{\int dx} = e^x$ multiply both sides with e^x

$e^x \frac{dy}{dx} + e^x y = e^x \Rightarrow \frac{d}{dx}(e^x y) = e^x$

$\Rightarrow e^x y = e^x + c, \; c \in \mathbb{R}$

$y = 1 + ce^{-x}$ plug in $(0,0)$

$0 = 1 + c \Rightarrow c = -1$

$y = 1 - e^{-x}$ when $|x| \leq 1$

when $|x| > 1$, $g(x) = 0$

$\frac{dy}{dx} + y = 0 \quad \mu(x) = e^{\int dx} = e^x$, multiply both sides.

$e^x \frac{dy}{dx} + e^x y = 0 \Rightarrow \frac{d}{dx}(e^x y) = 0 \Rightarrow e^x y = c, c \in \mathbb{R}$

$y = e^{-x} c$

Since solution is continious

$$\lim_{x \to 1^+} y(x) = \lim_{x \to 1^-} y(x)$$ &

since when $|x| \leq 1$ $y = 1 - e^{-x}$

$|x| > 1$ $y = ce^{-x}$

$$\lim_{x \to 1^+} y(x) = \frac{c}{e}$$

$$\lim_{x \to 1^-} y(x) = 1 - \frac{1}{e}$$

$$\frac{c}{e} = 1 - \frac{1}{e}$$

$$c = e - 1$$

Hence

when $|x| \leq 1$ $y = 1 - e^{-x}$

$|x| > 1$ $y = (e-1)e^{-x}$

Q2)

b) $\frac{dy}{dx} = \frac{1+\frac{3y}{x}}{1-\frac{y}{x}}$ is homogenous of degree 0 since it depends on $\frac{y}{x}$ only.

$z = \frac{y}{x}$, $y = xz$ & $\frac{dy}{dx} = z + x\frac{dz}{dx}$

$z + x\frac{dz}{dx} = \frac{1+3z}{1-z}$

$x\frac{dz}{dx} = \frac{1+3z}{1-z} - z = \frac{z^2+2z+1}{1-z}$

$\frac{1-z}{z^2+2z+1} dz = \frac{dx}{x}$

$-\int \frac{z-1}{z^2+2z+1} = \int \frac{dx}{x}$

$\int \left(\frac{2z+2}{2(z^2+2z+1)} - \frac{2}{(z^2+2z+1)}\right) dz = \int \frac{dx}{x}$

Solve

$\int \left(\frac{z+1}{z^2+2z+1}\right) dz \Rightarrow u = z^2+2z+1 \quad \frac{du}{dz} = 2z+2$

$dz = \frac{1}{2z+2} du \Rightarrow \int \frac{1}{2u} du = \frac{1}{2}\ln|u|$

$= \frac{\ln(z^2+2z+1)}{2}$

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Solve

$$2\int \frac{1\,dz}{z^2+2z+1} = 2\int \frac{1\,dz}{(z+1)^2}$$

$u = z+1 \quad \frac{du}{dx} = 1 \Rightarrow 2\int \frac{1}{u^2}du = \frac{-2}{z+1}$

Solution becomes

$$= -\frac{\ln(z^2+2z+1)}{2} - \frac{2}{z+1} - \ln|x| + C = 0$$

Plug in $z = \frac{y}{x}$

$$\Rightarrow -\frac{\ln(\frac{y}{x}+1)^2}{2} - \frac{2}{\frac{y}{x}+1} - \ln|x| + C = 0$$

$$\Rightarrow -\ln(\frac{y}{x}+1) - \frac{2x}{y+x} - \ln|x| + C = 0$$

Q3)

a) $M = 3xy - y^2$ $N = x(x-y) = x^2 - xy$

$\frac{\partial M}{\partial y} = 3x - 2y \neq \frac{\partial N}{\partial x} = 2x - y$

$\Rightarrow$ The d.e is not exact.

b)

$I(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} =$

$= e^{\int \frac{dx}{x^2 - xy}(3x - 2y - (2x - y))} = e^{\int \frac{x-y}{x^2 - xy}dx}$

$= e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$

$x(3xy - y^2)dx + x^2(x-y)\,dy = 0$

$\underbrace{(3x^2y - y^2x)}_{M_1}dx + \underbrace{(x^3 - x^2y)}_{N_1}dy = 0$

$\frac{\partial M_1}{\partial y} = 3x^2 - 2yx = \frac{\partial N_1}{\partial x} = 3x^2 - 2xy \Rightarrow$

it is exact

$\phi(x,y) = \int 3x^2y - y^2x\,dx = \frac{3x^3}{3}y - \frac{x^2}{2}y^2 + g(y)$

$= x^3y - \frac{x^2y^2}{2} + g(y)$

$\frac{\partial \phi}{\partial y} = x^3 - x^2y + g'(y) = N_1 = x^3 - x^2y$

$g'(y) = 0$ $g(y) = C$

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Thus, the general solution is

$$x^3y - \frac{x^2y^2}{2} + c = 0$$