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Q1)

a) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -24 & 8 \\ 0 & -11-\lambda & 4 \\ 0 & -24 & 9-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -11-\lambda & 4 \\ -24 & 9-\lambda \end{vmatrix} + 24\begin{vmatrix} 0 & 4 \\ 0 & 9-\lambda \end{vmatrix}$

$-8\begin{vmatrix} 0 & -11-\lambda \\ 0 & -24 \end{vmatrix} = (1-\lambda)[(9-\lambda)(-11-\lambda)+96]$

$=(1-\lambda)[-99-9\lambda+11\lambda+\lambda^2+96] = (1-\lambda)[\lambda^2+2\lambda-3]$

$=(1-\lambda)(\lambda+3)(\lambda-1)=0$ $\quad\lambda=-3$ is eigen value with multiplicity 1.

$\lambda=1$, is eigen vale with multiplicity 2.

b)

For $\lambda=1$: Solve $(A-I)X=0$

$A-I = \begin{bmatrix} 0 & -24 & 8 \\ 0 & -12 & 4 \\ 0 & -24 & 8 \end{bmatrix} \xrightarrow{-2R_2} \begin{bmatrix} 0 & -24 & 8 \\ 0 & 24 & -8 \\ 0 & -24 & 8 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 0 & -24 & 8 \\ 0 & 24 & -8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2+R_1}$

$\begin{bmatrix} 0 & -24 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_1} \begin{array}{ccc} s & & t \\ \end{array} \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_3 = t \quad x_1 = s \quad x_2 = \frac{t}{3}$

$S_1 = \{(s, \frac{t}{3}, t) \mid s,t \in \mathbb{R}\} = \{ s\underbrace{(1,0,0)}_{v_1} + t\underbrace{(0,\frac{1}{3},1)}_{v_2} \mid s,t \in \mathbb{R}\}$

$=$ span$\{v_1, v_2\}$ is the eigenspace spanned by v_1 & v_2, where v_1 & v_2 are linearly independent where $v_1 \neq k v_2$ for $k \in \mathbb{R}$.

For $\lambda = -3$ Solve $(A+3I)X = 0$

$$A+3I = \begin{bmatrix} 4 & -24 & 8 \\ 0 & -8 & 4 \\ 0 & -24 & 12 \end{bmatrix} \xrightarrow{-3R_2} \begin{bmatrix} 4 & -24 & 8 \\ 0 & 24 & -12 \\ 0 & -24 & 12 \end{bmatrix} \xrightarrow{R_3+R_2}$$

$$\begin{bmatrix} 4 & -24 & 8 \\ 0 & 24 & -12 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{12}R_2]{\frac{1}{4}R_1} \begin{bmatrix} 1 & -6 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_3 &= s \\ x_2 &= \frac{s}{2} \\ x_1 &= s \end{aligned}$$

$S_2 = \{(s, \frac{1}{2}s, s) \mid s \in \mathbb{R}\} = \{s\underbrace{(1, \tfrac{1}{2}, 1)}_{v_3} \mid s \in \mathbb{R}\}$

$= \text{span}\{v_1\}$ is the eigenspace spanned by v_3.

c) $B = B_{S_1} \cup B_{S_2} = \{v_1, v_2, v_3\}$ contains 3 lin independent eigenvectors of a 3×3 matrix A. Hence A is diag'ble with

$$P = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/3 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$$

d) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1/3 & 1/2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 3 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -3 & 1 \end{array}\right] \xrightarrow{-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 6 & -2 \end{array}\right]$$

$$\xrightarrow[R_2-\frac{3}{2}R_3]{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & 2 \\ 0 & 1 & 0 & 0 & -6 & 3 \\ 0 & 0 & 1 & 0 & 6 & -2 \end{array}\right]$$

$\underbrace{\hphantom{xxxxxx}}_{P^{-1}}$

Proof

$PP^{-1} = I = P^{-1}P$

$$PP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/3 & 1/2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 & 2 \\ 0 & -6 & 3 \\ 0 & 6 & -2 \end{bmatrix} = \begin{bmatrix} 1+0+0 & -6+6 & 2-2 \\ 0 & -2+2 & 1-1 \\ 0 & -6+6 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}P = \begin{bmatrix} 1 & -6 & 2 \\ 0 & -6 & 3 \\ 0 & 6 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/3 & 1/2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2+2 & 1-3+2 \\ 0 & -2+3 & 0-3+3 \\ 0 & 2-2 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e) $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/3 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = diag(1, 1, -3)$.

$$A^{2020} = PD^{2020}P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/3 & 1/2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3^{2020} \end{bmatrix} \begin{bmatrix} 1 & -6 & 2 \\ 0 & -6 & 3 \\ 0 & 6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3^{2020} \\ 0 & 1/3 & -\frac{3^{2020}}{2} \\ 0 & 1 & -3^{2020} \end{bmatrix} \begin{bmatrix} 1 & -6 & 2 \\ 0 & -6 & 3 \\ 0 & 6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -6+6\cdot 3^{2020} & 2-2\cdot 3^{2020} \\ 0 & -2+6\cdot 3^{2020} & 1-2\cdot 3^{2020} \\ 0 & -6+6\cdot 3^{2020} & 3-2\cdot 3^{2020} \end{bmatrix}$$

f) $$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -24 & 8 \\ 0 & -11-\lambda & 4 \\ 0 & -24 & 9-\lambda \end{vmatrix} = (1-\lambda)(\lambda+3)(\lambda-1) = 0$$

$p(\lambda) = -\lambda^3 - \lambda^2 + 5\lambda - 3$ By Cayley-Hamilton

$-A^3 - A^2 + 5A - 3I = 0 \Rightarrow -A^2 - A + 5I - 3A^{-1} = 0$

$A^{-1} = \left(\frac{1}{-3}(A^2+A-5I)\right)$

$$= -\frac{1}{3}\left(\begin{bmatrix}1&48&-16\\0&25&-8\\0&48&-15\end{bmatrix} + \begin{bmatrix}1&-24&8\\0&-11&4\\0&-24&9\end{bmatrix} - \begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}\right)$$

$$= -\frac{1}{3}\begin{bmatrix}-3&24&-8\\0&9&-4\\0&24&-11\end{bmatrix} = \begin{bmatrix}1&-8&8/3\\0&-3&4/3\\0&-8&11/3\end{bmatrix} = A^{-1}$$

Q2) Matrix A should satisfy $p(x)=0$, also note

a) if A is invertible, then $A^{-1} = \frac{1}{|A|}\cdot adj(A)$

A is inv $\Leftrightarrow |A| \neq 0$

$P_c(\lambda) = |A-\lambda I| = \lambda^3 - 7\lambda^2 + 5\lambda + 9 = 0$ $\quad P(A) = |A - AI| = 0$

$p(A) = 0 \Rightarrow A^3 - 7A^2 + 5A - 9I = 0$

$A(A^2 - 7A + 5I) = 9I$

$A^{-1} = \left(\frac{A^2 - 7A + 5I}{9}\right) \Rightarrow 9A^{-1} = (A^2 - 7A + 5I)$

$9\,\frac{adj(A)}{|A|} = A^2 - 7A + 5I$

$adj(A) = \frac{|A|(A^2 - 7A + 5I)}{9}$

$P(0) = |A - 0.I| = |A| = 9$

$adj(A) = A^2 - 7A + 5I$

b) If the $n \times n$ matrix A has the characteristic polynomial

$$p(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_2\lambda^2 + c_1\lambda + c_0$$

then

$$p(A) = (-1)^n A^n + c_{n-1}A^{n-1} + \dots + c_1 A + c_0 I = 0$$

A is non singular $\Rightarrow A^{-1}$ exists

multiply both sides by A^{-1}

$$(-1)^n A^{n-1} + c_{n-1}A^{n-2} + \dots + c_1 I + c_0 A^{-1} = 0 \cdot A^{-1}$$

$$-c_0 A^{-1} = (-1)^n A^{n-1} + c_{n-1}A^{n-2} + \dots + c_1 I$$

Note:

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} \text{ if } A \text{ is inversible}$$

$$-c_0 \frac{\operatorname{adj}(A)}{|A|} = (-1)^n A^{n-1} + c_{n-1}A^{n-2} + \dots + c_1 I$$

$$\operatorname{adj}(A) = \frac{|A|}{-c_0}\left((-1)^n A^{n-1} + c_{n-1}A^{n-1} + \dots + c_1 I\right)$$

To find $|A|$

$$|A - \lambda I| = p(\lambda) = (-1)^n \lambda^n + \dots + c_1\lambda + c_0$$

$\lambda = 0 \Rightarrow |A| = c_0$ A is invertible $\Leftrightarrow c_0 \neq 0$

Hence

$$\operatorname{adj}(A) = -\left((-1)^n A^{n-1} + c_{n-1}A^{n-1} + \dots + c_1 I\right)$$

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3) According to definition of Eigenvalue,

If A is an $n \times n$ matrix, a real or complex number λ is called eigenvalue of A if $Av = \lambda v$ where v is NON ZERO vector,

Hence there exists a non zero vector v such that

$$Av = \lambda v \quad \text{multiply with } A$$

$$A^2 v = \lambda (Av)$$

$$A^2 v = \lambda^2 v$$

A is indempotent $A^2 = A$

$$Av = \lambda^2 v$$

$$\lambda v = \lambda^2 v$$

$$\lambda(1-\lambda) v = 0 \quad v \neq 0, \text{ divide by } v$$

$$\lambda(1-\lambda) = 0$$

$$\lambda = 0 \text{ or } 1.$$