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CS
Math 225 Proj. 4

Q1) Note

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X}$$

Find eigenvalues of matrix $A \Rightarrow |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (-\lambda)(\lambda^2-1) - 1(-\lambda-1) + 1(1+\lambda) = 0$

$\Rightarrow -\lambda^3 + \lambda + \lambda + 1 + \lambda + 1 = 0$

$\Rightarrow (\lambda+1)(-\lambda^2 - \lambda + 2) = 0$

$\Rightarrow (\lambda+1)^2(\lambda-1) = 0 \Rightarrow \lambda = -1, -1, 2$ so the eigenvalues are $-1, -1, 2$

For $\lambda = -1$ solve $(A+I)X = 0$

$$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad x + y + z = 0$$

$$\text{Then } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

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Then $v_1 = (0,1,-1)$, $v_2 = (1,-1,0)$ & v_1 & v_2 are independent.

- $S_1 = \{t(0,1,-1) + s(1,-1,0) \mid s,t \in \mathbb{R}\} = \text{span}\{v_1, v_2\}$

Then $x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$, $y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-t}$, since $\lambda = -1$

For $\lambda = 2$ Solve $(A - 2I)x = 0$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

- $\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$
$z = t$
$-3y + 3z = 0 \quad y = z = t$
$x + t - 2t = 0 \quad x = t$

Than $S_2 = \{t(1,1,1) \mid t \in \mathbb{R}\} = \text{span}\{v_1\}$, since $\lambda = 2$

Then $z = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$

check if x, y, z are independent using wronksian

- $W(x,y,z) = \begin{vmatrix} 0 & e^{-t} & e^{2t} \\ e^{-t} & -e^{-t} & e^{2t} \\ -e^{-t} & 0 & e^{2t} \end{vmatrix}$

$= 0 - e^{-t}(e^{t} + e^{t}) + e^{2t}(-e^{-2t})$

$= -e^{-t} \cdot 2e^{t} - e^{2t} \cdot e^{-2t} = -3 \neq 0$, so the solutions are linearly indep.

so

- $w(t) = c_1 x(t) + c_2 y(t) + c_3 z(t)$
$= c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$

Hence general solutions are :

$x(t) = c_2 e^{-t} + c_3 e^{2t}$
$y(t) = c_1 e^{-t} - c_2 e^{-t} + c_3 e^{2t}$
$z(t) = -c_1 e^{-t} + c_3 e^{2t}$

use initial values $x(0)=0$, $y(0)=1$, $z(0)=-1$
to find c_1, c_2, c_3

$x(0) = c_2 + c_3 = 0$
$y(0) = c_1 - c_2 + c_3 = 1$ $\Rightarrow$ $2c_3 + c_1 = 1$
$z(0) = -c_1 + c_3 = -1$

$2c_3 - c_2 = 0$
$c_2 = 2c_3$
$3c_3 = 0$ $c_3 = 0$ $c_2 = 0$
$c_1 = 1$

So the general solution is

$w(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$

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Q2)

Solve the assoc. hom d.e. $y^{(4)} - 4y^{(3)} + 15y'' - 22y' + 10y = 0$

$L(r) = r^4 - 4r^3 + 15r^2 - 22r + 10r = 0$

$= (r-1)(r^3 - 3r^2 + 12r - 10) = 0$

$= (r-1)^2 (r^2 - 2r + 10) = 0$

$1, 1$

Solve $r^2 - 2r + 10$ $\quad \Delta = 4 - 4.10 = -36$

$x_{1/2} = \dfrac{2 \pm 6i}{2} = 1 \pm 3i$

$\Rightarrow y_c = (c_1 + c_2 x)e^x + e^x(c_3 \cos 3x + c_4 \sin 3x)$

2) $F(x) = 2\cos^2 x + e^x$ is a solution of some hom. d.e. Find it.

$F(x) = \cos(2x) + 1 + e^x$

$\pm 2i \quad 0 \quad 1$

$g(s) = s(s-1)(s-(2i))(s+2i) = 0$

$g(s) = s(s-1)(s^2+4) = 0$

$g(D)y = D(D-1)(D^2+4)y = 0$

$g(D)F(x) = 0 \Rightarrow g(D)L(D) = 0$

$\Rightarrow D(D-1)(D-2i)(D+2i)(D-1)^2(D^2-2r+10) = 0$

roots; $0, 1, 2i, -2i, 1+3i, 1-3i$

$y(x) = y_c + y_p$

$$= \underbrace{(c_1 + c_2 x e^x) + e^x(c_3 \cos 3x + c_4 \sin 3x)}_{y_p}$$

$$+ \underbrace{c_5 + c_6 x^2 e^x + c_7 \cos 2x + c_8 \sin 2x}_{y_c}$$

Q3)

$$x^2y'' + xy' + y = \ln x, \; x > 0$$

Try to convert this to non-homogenous equation with constant coefficients. In other words convert this to

$\varnothing(D)y = (a_nD^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)y = F(x)$

where $a_0 \dots a_n$ are constants.

For this use substitution $x = e^k \Rightarrow \ln x = k$

$$\frac{dk}{dx} = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{dy}{dk}\frac{dk}{dx} = \frac{dy}{dk\,x}$$

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dk}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dk}\right) =$$

$$= -\frac{1}{x^2}\frac{dy}{dk} + \frac{1}{x}\frac{d^2y}{dk^2}\frac{dk}{dx}$$

$$\Rightarrow -\frac{1}{x^2}\frac{dy}{dk} + \frac{1}{x^2}\frac{d^2y^2}{dk^2}$$

$$\boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y^2}{dk^2} - \frac{dy}{dk}}$$

Plug in to $x^2y'' + xy' + y = \ln x$

$$\frac{d^2y^2}{dk^2} - \frac{dy}{dk} + \frac{dy}{dk} + y = k$$

$$\Rightarrow \frac{d^2y^2}{dk^2} + y = k$$

Now, it is converted to the desired format

Solve

$$\frac{d^2y}{dk^2} + y = k$$

① $L(r) = r^2 + 1 = 0 \quad r = \pm i$

$y_c = c_1 \sin k + c_2 \cos k$

② $F(k) = k$ put $y(k) = k$ and observe that is a particular solution since

$\frac{d^2y}{dk^2} = 0 \quad y(k) = k$

$\frac{d^2y}{dk^2} + y = k$ ✓

$y_p = k$

③ $y = \underbrace{c_1 \sin k + c_2 \cos k}_{y_c} + \underbrace{k}_{y_p}$ Plug in $k = \ln x$

$y = c_1 \sin(\ln x) + c_2 \cos(\ln x) + \ln x$

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