CPSC 121: Models of Computation

Unit 9a

Mathematical Induction – Part 1

Based on slides by Patrice Belleville and Steve Wolfman

Pre-Class Learning Goals

- By the start of class, you should be able to
  - Convert sequences to and from explicit formulas that describe the sequence.
  - Convert sums to and from summation/Σ notation.
  - Convert products to and from product/Π notation.
  - Manipulate formulas in summation/product notation by adjusting their bounds, merging or splitting summations/products, and factoring out values.
  - Given a theorem to prove *and the insight into how to break the problem down in terms of smaller problems*, write out the skeleton of an inductive: the base case(s),the induction hypothesis, and the inductive step

Quiz 9 Feedback

- Generally:
- Issues:
- Essay Question:
  - As usual, we will revisit the open-ended question shortly.

In-Class Learning Goals

- By the end of this unit, you should be able to:
  - Formally prove properties of the non-negative integers (or a subset like integers that have appropriate self-referential structure) —including both equalities and inequalities—using either weak or strong induction as needed.
  - Critique formal inductive proofs to determine whether they are valid and where the error(s) lie if they are invalid.

? Addressing the Course Big Questions ?

- CPSC 121: the BIG questions:
  - How can we convince ourselves that an algorithm does what it's supposed to do?
  - How do we determine whether or not one algorithm is better than another one?
- Mathematical induction is a very useful tool when proving the correctness or efficiency of an algorithm.
- We will see several examples of this.

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Outline

- **Introduction and Discussion**
  - **Example: single-elimination tournaments**.
  - Example: max swaps for sorting n items
- A Pattern for Induction
- Induction on Numbers

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Example: Single-Elimination Tournament

- Problem: single-elimination tournament
  - Teams play one another in pairs
  - The winner of each pair advances to the next round
  - The tournament ends when only one team remains.

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How do we start?

- Let’s try some examples with small numbers

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Example (cont`)

- What is the maximum number of teams in a ***0***-round single-elimination tournament ?

  A. 0 teams

  B. 1 team

  C. 2 teams

  D. 3 teams

  E. None of the above.

Example (cont`)

- What is the maximum number of teams in a ***1***-round single-elimination tournament ?

  A. 0 teams

  B. 1 team

  C. 2 teams

  D. 3 teams

  E. None of the above.

Example (cont`)

- What is the maximum number of teams in a ***2***-round single-elimination tournament ?

  A. 0 teams

  B. 1 team

  C. 2 teams

  D. 3 teams

  E. None of the above.

Example (cont`)

- What is the maximum number of teams in a ***n***-round single-elimination tournament ?

  A. n

  B. 2n

  C. n^2

  D. 2^n

  E. None of the above.

How can we prove it?

- How can we prove it for every n ?
  - We will use a technique called mathematical induction.
- We show **some basic cases** first (for 0,1,2)
- Then we show that if the statement is true for case n-1 then it is true for case n (**inductive step**)

- Basic Cases how many we need?):
  - n= 0
  - n = 1
  - …

Inductive Step: Case for n-1 → Case for n

- If at most $\mathbf{2^{n-1}}$ teams can participate in a tournament with **n-1** rounds, then at most $\mathbf{2^{n}}$ teams can participate in a tournament with **n** rounds?
- If we want to *prove* this statement, which of the following techniques might we use?

  A. Antecedent assumption

  B. Witness proof

  C. WLOG

  D. Proof by cases

  E. None of the above.

Working out the proof:

- Proof :
  - Consider an unspecified tournament with **n-1** rounds. Assume that

    $\mathbf{2^{n-1}}$ teams can participate
  - How many teams we need to have a tournament with **n** rounds?
  - We can think of a tournament with **n** rounds as follows:
    - Two tournaments with n-1 rounds proceed in parallel.
    - The two winners then

      play the n-th round
  - Since each tournament with n-1 rounds has 2^{n-1} teams, then the tournament with n rounds has $2^* 2^{n-1} = 2^n$ teams.

Completing the proof:

- **Inductive Step**: if at most 2^{n-1} teams can play in an $(n-1)$-round tournament, then at most 2^n teams can play in an n-round tournament.
- **Proof:**
  - Assume at most 2^{n-1} teams can play in an $(n-1)$-round tournament.
  - An n-round tournament is two $(n-1)$-round tournaments where the winners play each other (since there must be a single champion).
  - By assumption, each of these may have at most 2^{n-1} teams. So, the overall tournament has at most $2^*2^{n-1} = 2^n$ teams. QED!

Are We Done?

Here's the logical structure of our original theorem:
$\forall n \in \mathbf{N},\ \mathbf{Max}(n-1, 2^{n-1}) \rightarrow \mathbf{Max}(n, 2^n)$.

Does that prove $\forall n \in \mathbf{N},\ \mathbf{Max}(n, 2^n)$?

a. Yes.
b. No.
c. I don't know.

What More Do We Need?

We need to adjust it to
$\forall n \in \mathbf{N},\ (n > 0) \land \mathbf{Max}(n-1, 2^{n-1}) \rightarrow \mathbf{Max}(n, 2^n)$.

Why doesn't this work for 0?

If n is 0, n-1 is not a valid case for this problem. Our formula breaks down if n is 0.

What do we do about the base case of our data definition?

We need to provide a separate proof.

Completing (?) the Proof (again)

Base Case : At most one team can play in a 0-round tournament.

Proof:

Every tournament must have one unique winner. A zero-round tournament has no games; so, it can only include one team: the winner. QED!

Now Are We Done?

Here's the logical structure of our theorems:

(1) $\mathbf{Max}(0, 1)$.

(2) $\forall n \in \mathbf{Z}^0,\ (n > 0) \land \mathbf{Max}(n-1, 2^{n-1}) \rightarrow \mathbf{Max}(n, 2^n)$.

Do these prove $\forall n \in \mathbf{Z}^0,\ \mathbf{Max}(n, 2^n)$?

a. Yes.
b. No.
c. I don't know.

One Extra Step We'll Do

Really, we are done.

But just to be thorough, we'll add:

Termination: *n* is a non-negative integer, and each application of the inductive step reduces it by 1. Therefore, it must reach our base case (0) in a finite number of steps.

Step-by-Step?

Here's the logical structure of our theorems:

Max$(0,2^0)$.

$\forall n \in \mathbb{Z}^0$, $(n > 0) \wedge$ **Max**$(n-1, 2^{n-1}) \rightarrow$ **Max**$(n, 2^n)$.

Do these prove **Max**$(1,2^1)$?

a. Yes.
b. No.
c. I don't know.

Step-by-Step?

Here's the logical structure of our theorems:

Max$(0,2^0)$.

$\forall n \in \mathbb{Z}^0$, $(n > 0) \wedge$ **Max**$(n-1, 2^{n-1}) \rightarrow$ **Max**$(n, 2^n)$.

Plus, we know **Max**$(1,2^1)$.
Do all of these prove **Max**$(2,2^2)$?

a. Yes.
b. No.
c. I don't know.

Step-by-Step?

Here's the logical structure of our theorems:

Max$(0,2^0)$.

$\forall n \in \mathbb{Z}^0$, $(n > 0) \wedge$ **Max**$(n-1, 2^{n-1}) \rightarrow$ **Max**$(n, 2^n)$.

Plus, we know **Max**$(1,2^1)$ and **Max**$(2,2^2)$.
Do all of these prove **Max**$(3,2^3)$?

a. Yes.
b. No.
c. I don't know.

Step-by-Step?

Here's the logical structure of our theorems:

$\mathtt{Max(0,2^0)}$.

$\forall n \in \mathbb{Z}^0,\ (n > 0) \land \mathtt{Max}(n-1, 2^{n-1}) \rightarrow \mathtt{Max}(n, 2^n)$.

From this, can we prove $\mathtt{Max}(n, 2^n)$ for any particular integer n?

a. Yes.
b. No.
c. I don't know.

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Tournament Proof Summary

Theorem: At most 2^n teams play in an n-round tournament.

Proof: We proceed by induction.

Base Case: A zero-round tournament has no games and so can only include one (that is, 2^0) team: the winner. So, at most 2^0 teams play in a 0-round tournament. ✓

Inductive Step: WLOG, let n be an arbitrary positive integer (n > 0)

Assume that at most 2^{n-1} teams play in an (n-1)-round tournament (**Induction Hypothesis).** We'll show it is true for n.

Proof:

An n-round tournament is two (n-1)-round tournaments where the winners play each other. By the IH, each of these has at most 2^{n-1} teams. So, the overall tournament has at most $2 * 2^{n-1} = 2^n$ teams. ✓

[By the principle of MI the theorem holds.]

QED

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Example 2: Sorting n items

- How many swaps do we need to sort n items?
  - Suppose we place items from left to right.
    - The items already placed are ordered.
    - We swap each new item with its neighbour until it is at the right place.
  - The i-th item may be swapped with all previous i-1 items.
  - So the total number of swaps is

$$\sum_1^n (i-1) = \sum_0^{n-1} j = \frac{n(n-1)}{2}$$

  - Hence we need to prove that $\sum_0^n i = \frac{n(n+1)}{2}$

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Example 2: Sorting n items

- Which facts do we need to prove?

  A. $\sum_0^0 i = 0$

  B. For every $n \geq 0$ if $\sum_0^{n-1} i = \frac{(n-1)n}{2}$, then $\sum_0^n i = \frac{n(n+1)}{2}$

  C. For every $n > 0$ if $\sum_0^{n-1} i = \frac{(n-1)n}{2}$, then $\sum_0^n i = \frac{n(n+1)}{2}$

  D. Both (a) and (c)

  E. None of the above.

Example 1: Sorting n items

- Proof:
  - ***Base case***: n = 0
    - Clearly : $\sum_0^0 i = 0 = \frac{n(n+1)}{2}$
  - ***Induction step***:
    - Pick an unspecified n > 0. Assume that (***inductive hypothesis):*** $\sum_0^{n-1} i = \frac{(n-1)n}{2}$
    - Then
      - $\sum_0^n i = (\sum_0^{n-1} i) + n$
      - $= \frac{(n-1)n}{2} + n$ (by the inductive hypothesis)
      - $= \frac{2n + (n-1)n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$
  - Hence by the principle of M.I., the theorem holds. QED

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An Induction Proof Pattern

Type of Problem: Prove some property of a structure that is naturally defined in terms of itself.

Part 1: Insight: how does the problem "break down" in terms of smaller pieces? Induction doesn't help you with this part. It is **not** a technique to figure out patterns, only to prove them.

Part 2: Proof.

Base case(s) : Establish that the property is true for your base case(s).

Inductive Step: Establish that if a problem of size n is made out of pieces of smaller size(s), prove that if the property is true for the smaller piece(s) , it is also true for the piece of size n.

[*Termination:* Also Establish that you could create a finite proof out of these steps for any value of interest)].

A Pattern For Induction

P(n) is _____________.

Theorem: P(n) is true for all n ≥ _______.

Proof: We prove it (or proceed) by induction on n.

Base Case(s) (P(···) is true for _______):

Prove each base case via your other techniques.

Inductive Step:

For an arbitrary n > _______,

Assume P(···) is true for __________ (inductive hypothesis)

We'll prove that P(n) is true.

WLOG, let n be greater than ___________.

Assume P(···) is true for _________________.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by the assumption (IH).

Build back up to show that P(n) is true.

This completes the induction proof. QED

A Pattern For Induction

P(n) is true for _______

Which base cases? Almost certainly the smallest n. Otherwise, you don't know yet. Do the *rest* of the proof now. Come back to the base case(s) when you know which one(s) you need!

For an arbitrary n > _______,

What must n be larger than? The largest of your base cases. (Why? So you don't assume the theorem true for something that's too small, like a *negative one* round tournament.) But, you don't know all your base cases yet. So…do the *rest* of the proof now. Come back to this bound once you know your base case(s).

A Pattern For Induction

assume P(·) is true for __________.

Which values are we going to assume P(·) is true for? Whichever we need. How do we know the ones we need? We don't, yet. So… do the *rest* of the proof now. Come back to the assumption when you know which one(s) you need!

Break P(n) down in terms of the smaller case(s)

How do we break the problem down in terms of smaller cases? *THIS* is the real core of every induction problem. Figure this out, and you're ready to fill in the rest of the blanks!

Examples: Breaking down into a problem one smaller

- You want to prove P(n) for all n ≥ k.
  - We prove that P(n) is true for n = k.
  - We prove that P(n) is true if P(n-1) is true.
- This is the simple most common style of induction, in which we define the problem of size n in terms of the same problem problem of size n-1
- It's called "weak (or regular) induction".
- Later we'll see that some problems cannot be defined in terms of the n-1 instance of them. We may need more than one smaller problems to define the problem of size n.
- In these cases we use a slightly different type of induction called "strong induction".

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ _______.

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s): Prove P(·) is true for _______**:**
Prove each base case via your other techniques.

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s): Prove P(·) is true for n=**1:**
Prove each base case via your other techniques. We only need n=1 because n=2 works based on n=1, and all subsequent cases also eventually break down into the n=1 case.

Inductive Step: For n > _______, *assume* P(·) is true for ____________, then we prove that P(n) is true:

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **1**)**:**
Prove each base case via your other techniques.

Inductive Step: For n > **1**: assume P(**n-1**) is true and we'll prove P(n) is true:

WLOG, let n be greater than ____________.

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **1**):

Prove each base case via your other techniques.

Inductive Step For n > **1**, assume P(**n-1**) is true and we'll prove P(n) is true:

WLOG, let n be greater than **1**.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Example: Sum of Odd Numbers

Problem: What is the sum of the first **n** odd numbers?

First, find the pattern. Then, prove it's correct.

The first **1** odd number?

The first **2** odd numbers?

The first **3** odd numbers?

The first **n** odd numbers?

Historical note: Francesco Maurolico made the first recorded use of induction in 1575 to prove this theorem!

Sum of Odd Numbers:Insight

Problem: Prove that the sum of the first **n** odd numbers is n^2.

How can we break the sum of the first, second, …, n^th odd number up in terms of a simpler sum of odd numbers?

Sum of Odd Numbers:Insight

Problem: Prove that the sum of the first **n** odd numbers is n^2.

The sum of the first **n** odd numbers is the sum of the first **n-1** odd numbers plus the n^{th} odd number.

(See our recursive formulation of Σ from the last example!)

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case(s) : Theorem is true for **?:**

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case : Theorem is true for **n=1:** The sum of the first 1 odd natural numbers is 1, which equals 1^2. ✓

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case : Theorem is true for **n=1:** The sum of the first 1 odd natural numbers is 1, which equals 1^2. ✓

Inductive Step For k > **?**: assume P(**?**) is true and we'll prove P(k) is true:

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case : Theorem is true for **n=1:** The sum of the first 1 odd natural numbers is 1, which equals 1^2. ✓

I**Inductive Step:** For k > **1**: assume the sum of first k-1 odds is $(k-1)^2$ and we'll prove that the sum of first k odds is k^2 .

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case : Theorem is true for **n=1:** The sum of the first 1 odd natural numbers is 1, which equals 1^2. ✓

Inductive Step For k > **1**: assume the sum of first k-1 odds is $(k-1)^2$ and we'll prove that the sum of first k odds is k^2 .

WLOG, let k be greater than **1**.

Break P(k) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(k) is true.

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case : Theorem is true for **n=1:** The sum of the first 1 odd natural numbers is 1, which equals 1^2. ✓

Inductive Step For k > **1**: assume the sum of first k-1 odds is $(k-1)^2$ and we'll prove that the sum of first k odds is k^2 .

WLOG, let k be greater than **1**. Then

$$\sum_{i=1}^{k}(2i-1) = ???$$

Sum of Odd Numbers

Theorem: For all positive integers n, the sum of the first n odd natural numbers is n^2.

Proof: We proceed by induction on n.

Base Case : Theorem is true for **n=1:** The sum of the first 1 odd natural numbers is 1, which equals 1^2. ✓

Inductive Step For k > **1**: assume the sum of first k-1 odds is $(k-1)^2$ and we'll prove that the sum of first k odds is k^2 .

WLOG, let k be greater than **1**. Then

$$\sum_{i=1}^{k}(2i-1) = [\sum_{i=1}^{k-1}(2i-1)] + (2k-1)$$

$= (k-1)^2 + (2k-1)$ (by the assumption (IH))

$= k^2 - 2k + 1 + 2k - 1 = k^2$

QED

Worked Example : Sum of 2 powers

Theorem: The sum of the first n powers of 2 is $2^{n+1} - 1$, for all non-negative integers n.

Proof: We proceed by induction on n.

Base Case : Theorem is true for _________**:** __________________

____

Inductive Step For _____: assume ___________________

we'll prove ___________________________________:

WLOG, let ____ be greater than ____________. Then

QED

Worked Example : What is Wrong ?

Theorem: All horses are the same colour.

Proof: We proceed by induction on n, the size of the group of horses.

Base Case : Theorem is true for $n = 1$. All horses in any group of one horse are obviously the same colour. ✓

Inductive Step: For any $k \geq 2$, assume that all horses in any group of size k-1 are the same colour, we'll show that for groups of k horses.

- Consider an arbitrary group of k horses with $k \geq 2$.
- Remove any one horse from it. What remains is a group of k-1 horses, which are all the same colour by the IH. Only the set-aside horse may be a different colour.
- Now, return the horse to the group and remove a different horse. Again, the remaining horses are all the same colour, but from the previous step we already know that this time the set-aside horse is also the same colour. Therefore, all horses in any group of size k are the same colour.

QED