CPSC 121 Sample Final Examination
December 2005

Name: SAMPLE SOL'N Student ID: ________________
Signature: ________________

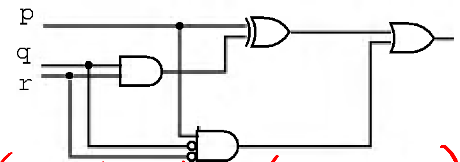
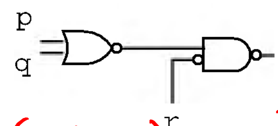
- You have 150 minutes to write the 6 questions on this examination. A total of 100 marks are available.
- **Justify all of your answers.**
- You are allowed to bring in one hand-written, double-sided 8.5 x 11 sheet of notes, and nothing else.
- Keep your answers short. If you run out of space for a question, you have written too much.
- The number in square brackets to the left of the question number indicates the number of marks allocated for that question. Use these to help you determine how much time you should spend on each question.
- Use the back of the pages for your rough work.
- **Good luck!**

Question	Marks
1	
2	
3	
4	
5	
6	
Total	

UNIVERSITY REGULATIONS:

- Each candidate should be prepared to produce, upon request, his/her library card.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
- CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
  2. Speaking or communicating with other candidates.
  3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

[16] 1. Consider the following two columns containing logical expressions, set expressions, and circuits. Each element of the first column is equivalent to an element of the second column. For each element of the first column, write down the element of the second column that it is equivalent to, and give a short proof to justify your choice.

A. $(P \oplus Q) \cup R$	1. $p \oplus (q \wedge r)$
B. $p \vee q \vee r$	2. $r \vee (\neg q \longrightarrow \neg p)$
 C. $(p \oplus (q \wedge r)) \vee (p \wedge \neg q \wedge \neg r)$	 3. $\neg(\neg(p \vee q) \wedge \neg r))$
D. $(p \longrightarrow q) \vee \neg p \vee r$	4. $(P \cup Q \cup R) - (P \cap Q \cap \overline{R})$

$A \equiv 4.$

$$(P \oplus Q) \cup R \equiv \{x \in \mathcal{U} \mid (x \in P \oplus x \in Q) \vee x \in R\}$$

$$\equiv \{x \in \mathcal{U} \mid ((x \in P \vee x \in Q) \wedge \neg(x \in P \wedge x \in Q)) \vee x \in R\}$$

$$\equiv \{x \in \mathcal{U} \mid (x \in P \vee x \in Q \vee x \in R) \wedge (\neg(x \in P \wedge x \in Q) \vee x \in R)\}$$

$$\equiv \{x \in \mathcal{U} \mid (x \in P \vee x \in Q \vee x \in R) \wedge (x \notin P \vee x \notin Q \vee x \in R)\}$$

$$\equiv \{x \in \mathcal{U} \mid (x \in P \vee x \in Q \vee x \in R) \wedge \neg(x \in P \wedge x \in Q \wedge x \notin R)\}$$

$$\equiv (P \cup Q \cup R) - (P \cap Q \cap \bar{R}).$$

$B \equiv 3.$

$$\neg(\neg(p \vee q) \wedge \neg r) \equiv \neg(\neg p \wedge \neg q \wedge \neg r) \equiv p \vee q \vee r.$$

$C \equiv 1.$

$(p \oplus (q \wedge r)) \vee (p \wedge \sim q \wedge \sim r) \equiv$

$\equiv (p \wedge \sim(q \wedge r)) \vee (\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$

$\equiv (p \wedge (\sim(q \wedge r) \vee (\sim q \wedge \sim r))) \vee (\sim p \wedge q \wedge r)$

$\equiv (p \wedge (\sim q \vee \sim r \vee (\sim q \wedge \sim r))) \vee (\sim p \wedge q \wedge r)$

$\equiv (p \wedge (\sim q \vee \sim r)) \vee (\sim p \wedge (q \wedge r)) \equiv (p \wedge \sim(q \wedge r)) \vee (\sim p \wedge (q \wedge r))$

$\equiv p \oplus (q \wedge r)$

$D \equiv 2.$ $(p \to q) \vee \sim p \vee r \equiv \sim p \vee q \vee \sim p \vee r \equiv \sim p \vee q \vee r$

$\equiv r \vee \sim\sim q \vee \sim p \equiv r \vee (\sim q \to \sim p)$

[20] 2. Proof techniques

[10] a. Let U be the set $\{x \in \mathbf{Z}^+ \mid 1 \leq x \leq 100\}$. Prove the following statement:

$$\forall S \in \mathcal{P}(U) \quad (S \neq U \longrightarrow \exists T \in \mathcal{P}(U) \quad |T| = |S| + 1)$$

using one of the proof techniques discussed in class. Recall that $|X|$ denotes the cardinality of a set X, and that $\mathcal{P}(X)$ is the power set of X. Indicate which proof technique you are applying. Part of the marks for this question will be given for the structure of the proof; the remainder will be given for the actual contents.

By contradiction:

Assume: $\sim \forall S \in P(U) \ (S \neq U \to \exists T \in P(U) \ |T| = |S|+1)$

$\exists S \in P(U) \sim (S = U \vee \exists T \ |T| = |S|+1)$

(1) $\exists S \in P(U) (S \neq U \wedge \sim \exists T \ |T| = |S|+1)$.

Since $S \in P(U)$, $S \subseteq U$.

(2) $\forall x, x \in S \to x \in U$. (by defn of $\subseteq$)

(3) $\exists x, x \in S \oplus x \in U$ (since $S \neq U$)

$\exists x, x \in U \wedge x \notin S$ (by (2) & (3))

Let $T = S \cup \{x\}$, then: $T \subseteq U \wedge |T| = |S|+1$.

This contradicts (1). QED.

[10] b. Prove that for any positive integer n

$$\sum_{i=0}^{n} \frac{1}{i!} \leq 3 - \frac{1}{n}$$

Recall that $0! = \cancel{0}\,1$, and for $n > 0$, $n! = 1 * 2 * 3 * \cdots * n$. If you want, you can use the fact that for every $n \geq 1$

$$\frac{1}{(n+1)!} \leq \frac{1}{n(n+1)}$$

without proving it.

BC: $n = 1$, $\displaystyle\sum_{i=0}^{1} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} = 1 + 1 = 2 = 3 - 1 = 3 - \frac{1}{1}$

$$\sum_{i=0}^{1} \frac{1}{i!} \leq 3 - \frac{1}{1} \quad \checkmark$$

IS: Assume $\displaystyle\sum_{i=0}^{n-1} \frac{1}{i!} \leq 3 - \frac{1}{n-1}$

$$\sum_{i=0}^{n} \frac{1}{i!} = \sum_{i=0}^{n-1} \frac{1}{i!} + \frac{1}{n!} \leq 3 - \frac{1}{n-1} + \frac{1}{n!} \leq 3 - \frac{1}{n-1} + \frac{1}{n(n+1)}$$

$$= \frac{3(n-1)(n+1)n - n(n+1) + n - 1}{(n+1)n(n-1)} = \frac{3n^3 - 3n - n^2 - n + n - 1}{(n+1)n(n-1)}$$

$$= \frac{3n^3 - n^2 - 3n - 1}{(n+1)n(n-1)}$$

$$\boxed{\text{NOTE: } 3 - \frac{1}{n} = \frac{3n-1}{n} = \frac{(3n-1)(n+1)(n-1)}{(n+1)n(n-1)} = \frac{3n^3 - n^2 - 3n + 1}{(n+1)n(n-1)}}$$

$$\leq \frac{3n^3 - n^2 - 3n + 1}{(n+1)n(n-1)}$$

$$= 3 - \frac{1}{n} \qquad \text{QED}$$

[24] 3. Figure 1 shows a sequential logic circuit (i.e. one with state held in D flip-flops). The circuit has inputs `up` and `down` and outputs q_0 and q_1. We will refer to the outputs combined together as `q` and interpret `q` as a two's complement, binary value.

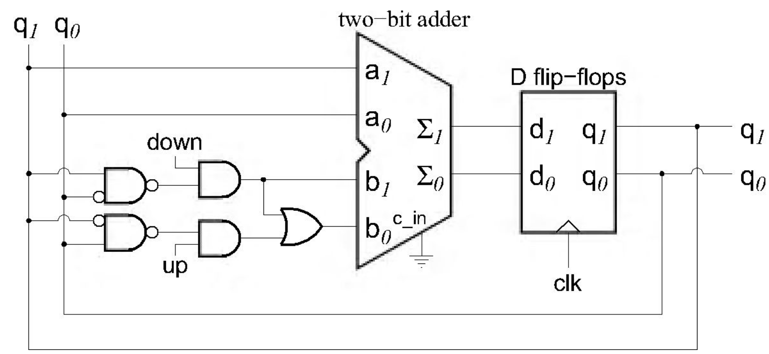


Figure 1: Sequential circuit for question 3

[6] a. Complete the table below as follows:

- For the column labeled `q`, write the value of `q` interpreted as a two's complement binary integer.
- Let us call b_1 and b_0 the two bits of the lower input to the adder in the circuit. For the columns labeled b_1 and b_0, write the values of these signals as a boolean function of the inputs `up` and `down`, assuming that the values of q_0 and q_1 are as written on that row. You can abbreviate `up` as `u` and `down` as `d`.

We have filled in the first row as an example:

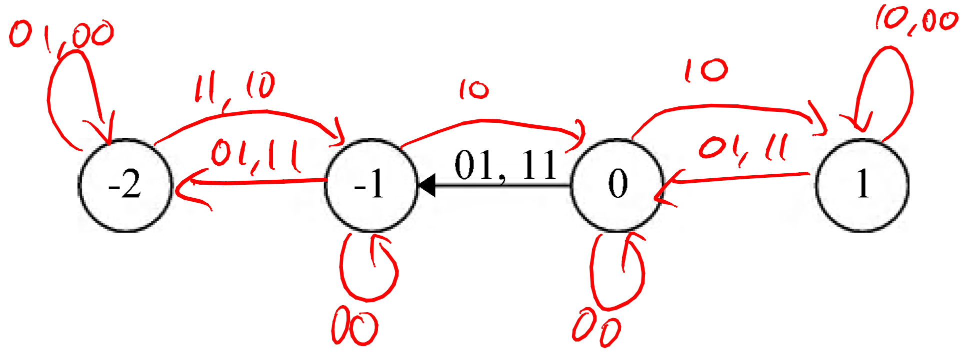
q_1	q_0	q	b_1	b_0
F	F	0	d	u ∨ d
F	T	1	d	d
T	F	-2	F	u
T	T	-1	d	u∨d

[6] b. For each possible pairs of values of up and down, this circuit can be seen as computing a function of the input q. Fill in the following table, indicating the function that is being computed.

up	down	f(q)
F	F	f(q) = q
F	T	max(q-1, -2)
T	F	min(q+1, 1)
T	T	{ q-1 if q > -2; -1 otherwise }

q	FF	FT	TF	TT
0	0	-1	1	-1
1	1	0	1	0
-2	-2	-2	-1	-1
-1	-1	-2	0	-2

[9] c. Draw the state transition diagram (finite state machine) that corresponds to this circuit. Each state has been labeled with the corresponding value of q (interpreted as a two's complement signed integer). You need to add the arrows between states. Each arrow should be labeled with a pair of letters that causes this state transition to take place. For instance, we have labeled the one arrow 01, 11 to indicate that if the current state is 0, and at the next clock tick we have up = 0 and down = 1, or up = 1 and down = 1, then the next state will be -1.

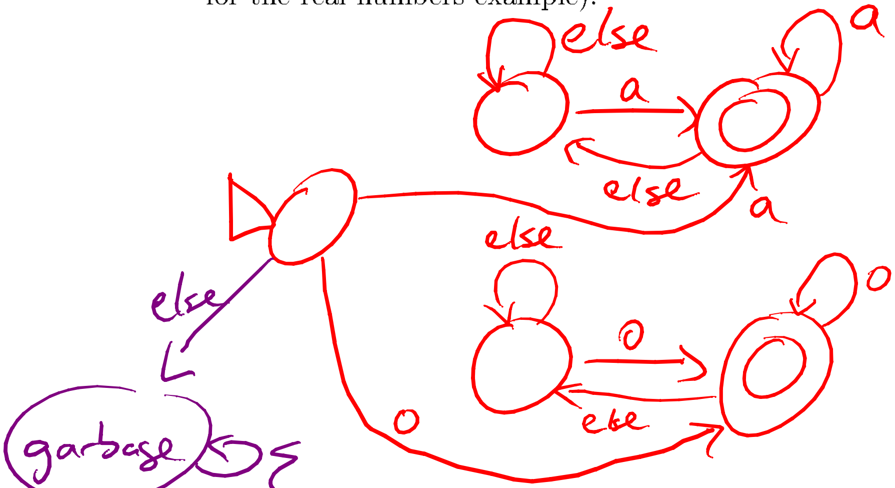


[3] d. Give a **brief** explanation of what this circuit does. Your answer should be no longer than five simple sentences (shorter answers are possible).

"Moves" q up or down one step at each clock tick (without "wrapping around") according to "up" and "down". When up & down are both on, down takes precedence except for input -2, which "bounces" up.

[8] 4. Draw a finite state machine that accepts the set of all strings of letters whose first and last characters are both `a`, **or** whose first and last characters are both `o`. For instance, your machine should be in an accepting state after seeing the strings "`avada kedavra`" and "`oculus reparo`", but not after seeing the strings "`sonorus`" or "`imperio`".

You may write the label `else` on an edge to indicate that the transition will be taken if the next character is any character other than one already used as a label on another edge that starts from the same state (i.e. the same way we did in class for the real numbers example).



NOTE: the garbage state need not be written explicitly.

[16] 5. For each of the following relations, explain whether or not it is (1) reflexive, (2) symmetric and (3) transitive. Then tell us if it is an equivalence relation or not.

[7] a. Let S be the set of residents of Sipiwesk (Manitoba). We define a relation B over S by $(s_1, s_2) \in B$ if and only if residents s_1 and s_2 live on the same block.To keep the problem simple, assume that each resident of Sipiwesk lives in exactly one residence. A "block" is a set of houses on a given street (between intersections with other streets).

(1) Yes, everyone lives on her own block

(2) Yes, order is irrelevant in the relation (I live on your block iff you live on mine)

(3) Yes. If x lives on y's block and y lives on z's block then they all live on the same block; so x lives on z's block.

This <u>is</u> an equivalence rel'n b/c of (1), (2), & (3).

[7] b. The relation $\mathcal{R}$ on $\mathbf{Z}^+$ defined by $(a, b) \in \mathcal{R}$ if and only if ab is a perfect cube, i.e. $ab = x^3$ for some positive integer x.

① No. $(2,2) \notin R$ since $\sqrt[3]{4} \notin \mathbb{Z}^+$.

② Yes, because multiplication is commutative, $ab = ba$.

③ No. $(2,4) \in R \wedge (4,16) \in R$, but $(2,16) \notin R$

R is <u>not</u> an equivalence relation b/c of ① & ③.

[16] 6. Prove or disprove each of the following statements about two functions $f : B \to C$ and $g : A \to B$:

a. If $\underbrace{f \text{ is one-to-one and } g \text{ is one-to-one}}_{\text{Assume}}$, then $\underbrace{f \circ g \text{ is one-to-one}}_{\text{Prove}}$

TRUE

① $\forall x \in B, \forall y \in B, f(x) = f(y) \to x = y$. by assumption

② $\forall x \in A, \forall y \in A, g(x) = g(y) \to x = y$. by assumption

Assume $f \circ g(x) = f \circ g(y)$

$f(g(x)) = f(g(y))$

$g(x) = g(y)$ by ①

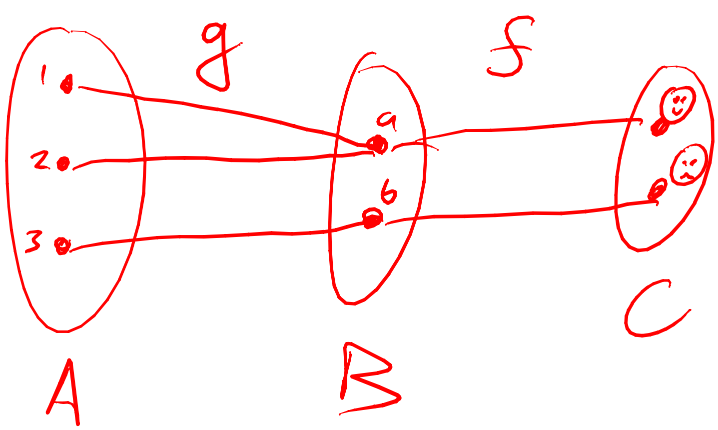
$\to x = y$ by ②

$\therefore \forall x \in A, \forall y \in A, f \circ g(x) = f \circ g(y) \to x = y$.

QED, $f \circ g$ is 1-1.

b. If f is one-to-one and g is onto, then $f \circ g$ is one-to-one.

FALSE



$f \circ g(1) =$

$f \circ g(2) =$