Tutorial Week 8

1. Assume n is an integer. Prove if n^2 is odd, then n is odd.

   Try to prove the contrapositive: we know that the truth value of a conditional statement is the same as its contrapositive statement. So, proving the contrapositive to be true is the same as proving the original statement to be true.
   The contrapositive of the original statement: if n is not odd, then n^2 is not odd.
   n is *not* odd, which means n is an even number: $n = 2k$. $n^2 = (2k)^2 = 2(2k^2)$
   $\therefore n^2$ is even, the contrapositive statement is true, thus the original statement is true.

2. Assume n is an integer. Prove that if $3n + 2$ is odd, then n is odd

   Let us prove the contrapositive: n is not odd, then $3n + 2$ is not odd.
   n is not odd, so n is even: $n = 2k$.

   $$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

   $\therefore$ $3n + 2$ is even. QED.

3. Prove that $\sqrt{2}$ is irrational

   Let us assume $\sqrt{2}$ is rational, $\therefore \sqrt{2} = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$.
   Remember, for this to work, p and q must be co-prime, i.e. share no common factors

   $$\begin{aligned} \Rightarrow \frac{p}{q} &= \sqrt{2} \\ (\frac{p}{q})^2 &= 2 \\ (\frac{p^2}{q^2}) &= 2 \\ p^2 &= 2q^2 \\ \Rightarrow \text{that } p \text{ is even let } p &= 2c \text{ for some } c \in \mathbb{Z} \\ \text{so } (2c)^2 &= 2q^2 \\ 4c^2 &= 2q^2 \\ 2c^2 &= q^2 \end{aligned}$$

   $\therefore$ q is even, $\Rightarrow$ both q and p are divisible by 2, which contradicts p and q have no common factors.

4. Prove the following Big-Oh statement:

   $\log_{10} n \in O(\log_2 n)$

Begin by writing out the definition of Big-O
$f \in O(g) : \exists n_0 \in \mathbb{N}, \exists c \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \to f(n) \leq c \cdot g(n)$

One approach is to choose c, and show that that works, the other is to show that it works for 'some' c, then through math prove this. Either way, we must show the logs equivalent.
In our proof, rewrite the definition of Big-O by plug in the $f(n)$ and $g(n)$. Thus, $\log_{10} n \in O(\log_2 n)$ can be written as:

$$\exists n_0 \in \mathbb{N} \exists c \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \to \log_{10} n \leq c \cdot \log_2 n$$

We choose $n_0 = 1, c = 1$. Without loss of generality, choose an unspecified natural number n. Assume $n \geq n_0 = 1$, we need to prove that $\log_{10} n \leq \log_2 n$. Since $n \geq 1$, it follows $\log_2 n \geq 0$.

$$\begin{aligned}
\log_{10} n &= \frac{\log_2 n}{\log_2 10} \\
&= \frac{1}{\log_2 10} \cdot \log_2 n \\
&\leq 1 \cdot \log_2 n \quad \dagger \\
&= \log_2 n
\end{aligned}$$

† we can do this, because all we care about is an upper bound, it does not have to be an exact runtime

Thus, $\log_{10} n \leq \log_2 n$ when $n_0 = 1, c = 1$.