CPSC 121: Models of Computation

Unit 6

Rewriting Predicate Logic Statements

Based on slides by Patrice Belleville and Steve Wolfman

Pre-Class Learning Goals

- By the start of class, you should be able to:
  - Determine the negation of any quantified statement.
  - Given a quantified statement and an equivalence rule, apply the rule to create an equivalent statement (particularly the De Morgan's and contrapositive rules).
  - Prove and disprove quantified statements using the "challenge" method (Epp, 4th edition, page 119).
  - Apply universal instantiation, universal modus ponens, and universal modus tollens to predicate logic statements that correspond to the rules' premises to infer statements implied by the premises.

Quiz 6 Feedback

- Overall:
- Issues:
- Open-ended question:

In-Class Learning Goals

- By the end of this unit, you should be able to:
  - Explore alternate forms of predicate logic statements using the logical equivalences you have already learned plus negation of quantifiers (a generalized form of the De Morgan's Law).
  - Prove arguments with quantifiers.

? Related to CPSC 121 Big Questions ?

- How can we convince ourselves that an algorithm does what it's supposed to do?
  - We continue discussing how to prove various types of predicate logic statements that arise when we discuss algorithm correctness.

Outline

- **Thinking of quantifiers differently.**
- Rules and Transformations
- The challenge method.

Relation between ∀, ∃, ∧, ∨

- Suppose D contains values x1, x2, ..., xn
- What does ∀x ∈ D, P(x) really mean?
  - It's the same as
    P(x1) ^ P(x2) ^ ... ^ P(xn).
  - Similarly,
    ∃x ∈ D, P(x) ≡ P(x1) v P(x2) v ... v P(xn)
- Thinking of quantifiers this way explains
  - Negation
  - Universal instantiation
  - Universal Modus Ponens, Tollens

Negation

- ~∀x ∈ D, P(x) ≡ ~(P(x1) ^ P(x2) ^ ... ^ P(xn))
  ≡ ~P(x1) v ~P(x2) v ... v ~P(xn)
  ≡ ∃x ∈ D, ~P(x)
- ~∃x ∈ D, P(x) ≡ ~(P(x1) v P(x2) v ... v P(xn))
  ≡ ~P(x1) ^ ~P(x2) ^ ... ^ ~P(xn)
  ≡ ∀x ∈ D, ~P(x)

Negation

- What can we do with the negation in:

  ~ ∃c ∈ R+ ∃n0∈ N ∀n ∈ N, n ≥ n0 → f(n) ≤ cg(n) ?

  A. It cannot be moved inward.
  B. It can only move across one quantifier because the generalized De Morgan's law can only handle one quantifier.
  C. It can only be moved across all three quantifiers because a negation can't appear between quantifiers.
  D. It could be moved across one, two or all three quantifiers.
  E. None of the above.

Negation

Which of the following are equivalent to:

$\sim\exists n_0 \in Z^0, \forall n \in Z^0, n > n_0 \to F(a_1, a_2, n).$

A. $\forall n_0 \in Z^0, \sim\forall n \in Z^0, n > n_0 \to F(a_1, a_2, n).$

B. $\forall n_0 \in Z^0, \exists n \in Z^0, \sim(n > n_0) \to F(a_1, a_2, n).$

C. $\forall n_0 \in Z^0, \exists n \in Z^0, \sim(n > n_0 \to F(a_1, a_2, n)).$

D. $\exists n_0 \in Z^0, \forall n \in Z^0, \sim(n > n_0 \to F(a_1, a_2, n)).$

E. $\forall n_0 \in Z^0, \exists n \in Z^0, n > n_0 \land \sim F(a_1, a_2, n).$

Exercise

- Let A be the set of amoebae, and Parent(x, y) be true if amoeba x is amoeba y's parent.
- Use logical equivalences to show that these two translations of "an amoeba has only one parent" are logically equivalent:

$\forall x \in A, \exists y \in A, Parent(y, x) \land (\forall z \in A, Parent(z, x) \to y = z)$

$\forall x \in A, \exists y \in A, Parent(y, x) \land (\sim\exists z \in A, Parent(z, x) \land y \neq z)$

Outline

- Thinking of quantifiers differently.
- **Rules and Transformations**
- The challenge method.

Universal Instantiation

- If a is an element of D then:

$$\frac{\forall x \in D,\ P(x)}{P(a)}$$

- Proving it is a valid inference:
  - Suppose $\forall x \in D$, P(x) is true.
  - Hence $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$ holds.
  - If $a = x_i$ is an element of D, then by specialization we have $P(x_i)$.

Is Existential Instantiation a Valid Rule?

- Consider an existential instantiation rule:

$$\frac{\exists x \in D,\ P(x) \qquad a \in D}{P(a)}$$

A. This argument is valid: P(a) is true.

B. This argument is invalid: P(a) is false.

C. This argument is invalid: P(a) might be false.

D. This argument is invalid for another reason.

Existential Generalization

- If a is an element of D then:

$$\frac{P(a)}{\exists x \in D,\ P(x)}$$

- Proving it is a valid inference:
  - Suppose P(a) is true and $a = x_i$
  - Hence $P(x_1) \vee \ldots \vee P(x_i) \vee \ldots \vee P(x_n)$ holds.
  - Therefore $\exists x \in D$, P(x) is true.

Universal Generalization

- If y is a ***non-specific*** (***arbitrary***) element of D then:

$$\frac{P(y) \quad \text{for a non-specific } y}{\forall x \in D,\ P(x)}$$

- Proving it is a valid inference:
  - Suppose P(y) is true a non-specific $y \in D$
  - Since y can be anyone of the elements of D, $P(x_1) \wedge \ldots \wedge P(x_n)$ holds.
  - Therefore $\forall x \in D$, P(x) is true.

Universal Modus Ponens/Tollens

- If a is an element of D then:

  $\forall x \in D,\ P(x) \rightarrow Q(x)$
  $P(a)$
  $\overline{Q(a)}$

- Proof:
  1. $\forall x \in D,\ P(x) \rightarrow Q(x)$ — premise
  2. $P(a)$ — premise
  3. $P(a) \rightarrow Q(a)$ — 1, universal instantiation
  4. $Q(a)$ — 3, modus ponens

- The proof for universal modus tollens is similar.

Is Existential Modus Ponens Valid?

- Is this rule valid?

  $\exists x \in D,\ P(x) \rightarrow Q(x)$
  $P(a)$
  $\overline{Q(a)}$

  A. This argument is valid, and Q(a) is true.
  B. The argument is valid, but the 1st premise can not be true; so Q(a) might be false.
  C. This argument is invalid because Q(a) is false.
  D. This argument is invalid because the premises can be true and Q(a) can be false.
  E. The argument is invalid for another reason.

Quantifier Rules (the only new rules we need)

<u>Universal Instantiation</u>

For any $a \in D$:

$\forall x \in D,\ P(x)$
$\overline{P(a)}$

<u>Existential Instantiation</u>

For an unspecified new (witness) $w \in D$:

$\exists\ x \in D,\ P(x)$
$\overline{P(w)}$

<u>Universal Generalization</u>

For any **arbitrary** $x \in D$:

$P(x)$
$\overline{\forall x \in D,\ P(x)}$

<u>Existential Generalization</u>

For any $a \in D$:

$P(a)$
$\overline{\exists x \in D,\ P(x)}$

Logical Equivalences

- Applying logical equivalences to predicate logic:
  - Suppose we have
    - $\forall x \in D,\ P(x) \rightarrow Q(x)$
  - and we know that
    - $P(x) \rightarrow Q(x) \equiv \sim P(x) \vee Q(x)$
  - Can we infer
    - $\forall x \in D,\ \sim P(x) \vee Q(x)$?
  - Can we infer
    - $\sim \forall x \in D,\ P(x) \vee Q(x)$?
- Is any of these valid?

Logical Equivalences

- Which propositional logic equivalences apply to predicate logic?
  - A. De Morgan's
  - B. $\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x)$
  - C. Commutative, Associative, and the “definition of conditional”
  - D. All propositional logic equivalences apply to predicate logic, but… we have to be sure to carefully “line up” the parts of the logical equivalence with the parts of the logical statement.
  - E. None of the above.

Applying Rules of Inference

- Suppose a and b are elements of D and we know

  $P(a)$

  $\forall x \in D, P(x) \rightarrow \exists y \in D, Q(x,y)$
  - Can we infer
    - A. $Q(a, b)$?
    - B. $\exists y \in D, Q(a, y)$?
- What if we have

  $P(a)$

  $\forall x \in D, P(x) \rightarrow \forall y \in D, Q(x,y)$
  - Can we infer
    - A. $Q(a, b)$?
    - B. $\forall y \in D, Q(a, y)$?

Rules of Inference

- Which rules of inference apply to predicate logic?
  - A. Modus ponens and modus tollens only.
  - B. All rules apply, but only if they follow universal quantifiers, not existential quantifiers.
  - C. All rules apply, but only if they follow existential quantifiers, not universal quantifiers.
  - D. All rules apply, but… we have to be sure to match the parts of the rule with correct logical statements.
  - E. None of the above.

Outline

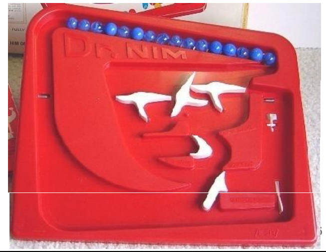
- Thinking of quantifiers differently.
- Rules and Transformations
- **The challenge method.**

The Challenge Method

- A predicate logic statement is like a game with two players.
  - you (trying to prove the statement true)
  - your adversary (trying to prove it false).
- The two of you pick values for the quantified variables working from the left to right (i.e. inwards).
  - You pick the values of existentially quantified variables.
  - Your adversary picks the values of universally quantified variables

The Challenge Method

- The Challenge method (continued):
  - If there is a strategy that allows you to always win, then the statement is true.
  - If there is a strategy for your adversary that allows him/her to always win, then the statement is false.
- What does it mean to a winning strategy



The Challenge Method

- Example 1: $\exists x \in Z, \forall n \in Z^+, 2^x < n$
  - How would we say this in English?
  - How would we prove this theorem?
- Example 2: $\forall n \in N, \exists x \in N, n < 2^x$
  - How would we say this in English?
  - How would we prove this theorem?

The Challenge Method

- Example 3: $\exists x \in N, \forall n \in N, n < 2^x$
  - How would we say this in English?
  - Is this statement true?
- How do we prove a statement is false?

Reading for Quiz 7

- Online quiz #7 is due ________________________
- Readings for the quiz:
  - Epp, 4th edition: 4.1, 4.6, Theorem 4.4.1
  - Epp, 3rd edition: 3.1, 3.6, Theorem 3.4.1.
  - Rosen, 6th edition: 1.6, 1.7. 3.4 (theorem 2 only).
  - Rosen 7th edition: 1.7, 1.8, 4.1 (theorem 2 only).