CPSC 121: Models of Computation

Unit 12: Functions

Based on slides by Patrice Belleville and Steve Wolfman

PART 1
REVIEW OF TEXT READING

These pages correspond to text reading and are not covered in the lectures.

What is a Function?

Mostly, a function is what you learned it was all through K-12 mathematics, with strange vocabulary to make it more interesting…

A function **f:A → B** maps values from its domain **A** to its co-domain **B**.

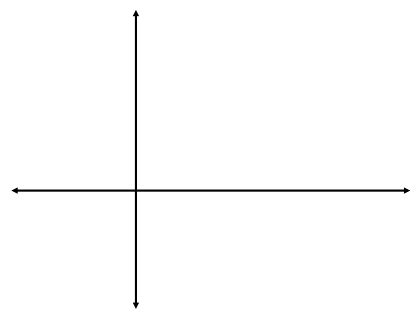
	Domain	Co-domain
$f(x) = x^3$		
$f(x) = x \bmod 4$		
$f(x) = \lfloor x \rfloor$		

Plotting Functions

$f(x) = x^3$

$f(x) = x \bmod 4$

$f(x) = \lfloor x \rfloor$



Not every function is easy to plot!

What is a Function?

Not every function has to do with numbers…

A function $f:A \rightarrow B$ maps values from its domain A to its co-domain B.

	Domain	Co-domain
$f(x) = {\sim}x$		
$f(x,y) = x \vee y$		
$f(x)$ = x's phone #		

What is a Function?

A function $f:A \rightarrow B$ maps values from its domain A to its co-domain B.

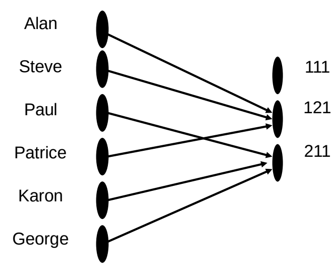
$$f(control,\ data1,\ data2) = ({\sim}control \wedge data1) \vee (control \wedge data2)$$

Domain?

Co-domain?

What is a Function?

A function $f:A \rightarrow B$ maps values from its domain A to its co-domain B.



Domain?

Co-domain?

Other examples?

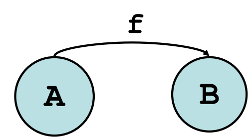
What is a Function?

A function $f:A \rightarrow B$ maps values from its domain A to its co-domain B.

f can't map one element of its domain to more than one element of its co-domain:

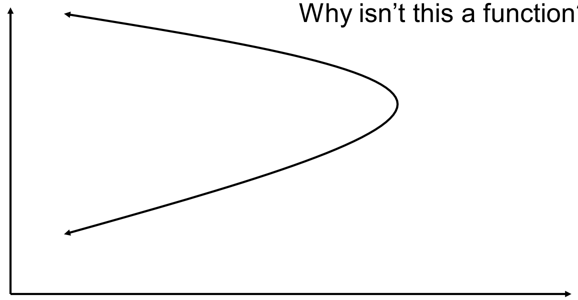
$$\forall x \in A,\ \forall y_1, y_2 \in B,\ [(f(x) = y_1) \wedge (f(x) = y_2)] \rightarrow (y_1 = y_2).$$

Why insist on this?



Not a Function

Why isn't this a function?



(The Laffer Curve: a non-functional tax policy.)

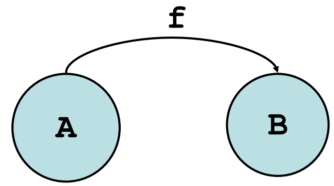
Function Terminology

A function $\mathbf{f:A \rightarrow B}$ maps values from its domain $\mathbf{A}$ to its co-domain $\mathbf{B}$.

For $\mathbf{f}$ to be a function, it must map every element in its domain:

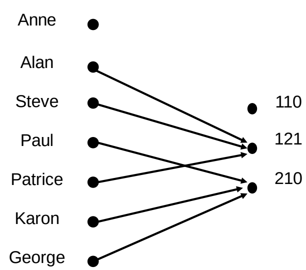
$\forall \mathbf{x} \in \mathbf{A},\ \exists \mathbf{y} \in \mathbf{B},\ \mathbf{f(x) = y}.$

Why insist on this?



Warning: some mathematicians would say that makes $\mathbf{f}$ "total".

Not a Function

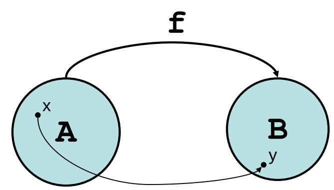


Function Terminology

A function $\mathbf{f:A \rightarrow B}$ maps values from its domain $\mathbf{A}$ to its co-domain $\mathbf{B}$.

$\mathbf{f(x)}$ is called the image of $\mathbf{x}$ (under $\mathbf{f}$).

$\mathbf{x}$ is called the pre-image of $\mathbf{f(x)}$ (under $\mathbf{f}$).

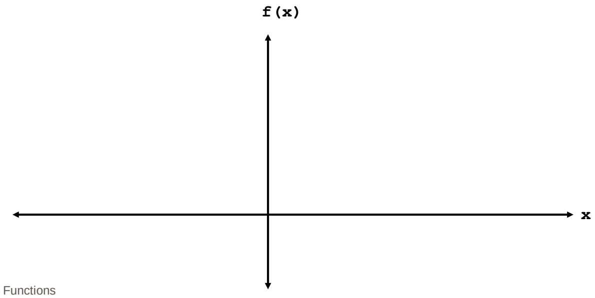


Trying out Terminology

$f(x) = x^2$

What is the image of 16?

What is the range of f?



PART 2
IN CLASS PAGES

Pre-Class Learning Goals

- By the start of class, you should be able to:
  - Define the terms domain, co-domain, range, image, and pre-image
  - Use appropriate function syntax to relate these terms (e.g., $f : A \rightarrow B$ indicates that f is a function mapping domain A to co-domain B).
  - Determine whether $f : A \rightarrow B$ is a function given a definition for f as an equation or arrow diagram.

Quiz 10

- In General:
- Specific issues:

In-Class Learning Goals

- By the end of this unit, you should be able to:
  - Define the terms injective (one-to-one), surjective (onto), bijective (one-to-one correspondence), and inverse.
  - Determine whether a given function is injective, surjective, and/or bijective.
  - Determine whether the inverse of a given function is a function.

Outline

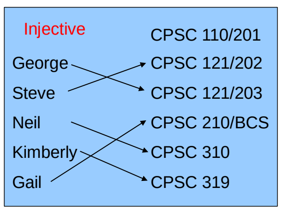
- Injective Functions
- Surjective Functions
- Bijective Functions
- Inverse Operations.

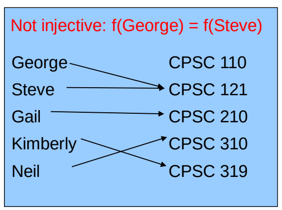
Injective Functions

Some special types of functions:

- A function $f : A \rightarrow B$ is injective (one-to-one) if

  $\forall x \in A, \forall y \in A, x \neq y \rightarrow f(x) \neq f(y)$.
- In the arrow diagram: at most one arrow points to each element of B.

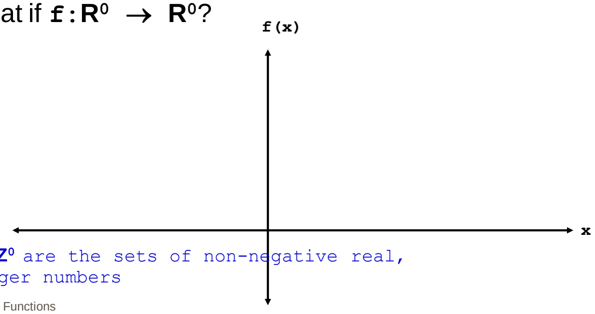




Trying out Terminology

$\mathbf{f}:\mathbf{R} \rightarrow \mathbf{R}^0$
$\mathbf{f(x)} = \mathbf{x}^2$
Injective?

What if $\mathbf{f}:\mathbf{R}^0 \rightarrow \mathbf{R}^0$?



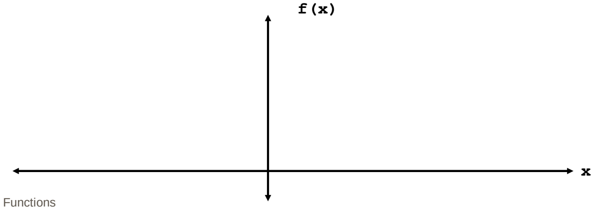
R^0, $\mathbf{Z}^0$ are the sets of non-negative real, integer numbers

Trying out Terminology

`f(x) = |x|` (the absolute value of **x**)

Injective?

a. Yes, if $\mathbf{f:R \rightarrow R^0}$
b. Yes, if $\mathbf{f:R^0 \rightarrow R}$
c. Yes, for some other domain/co-domain
d. No, not for any domain/co-domain
e. None of these is correct



Trying out Terminology

f:{s|s is a 121 student} → {A+, A, …, D, F}

f(s) = s's mark in 121

If we know that there are 300 students in 121 is **f** injective?

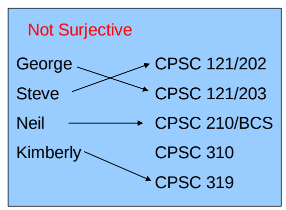
a. Yes
b. No
c. Not enough information

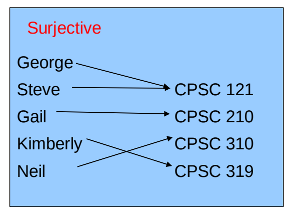
Outline

- Injective Functions
- Surjective Functions
- Bijective Functions
- Inverse Operations.

Surjective Functions

- A function $f : A \rightarrow B$ is surjective (onto) if $\forall y \in B, \exists x \in A, f(x) = y$.
  Can we define it in terms of range and co-domain?
- In the arrow diagram: at least one arrow points to each element of B.



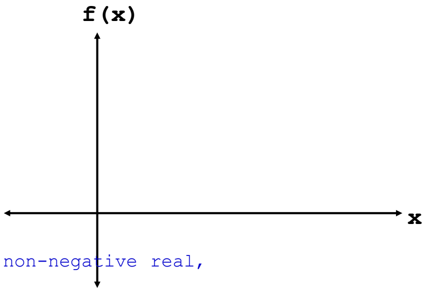


Trying out Terminology

f:**R** → **R**^{0}
f(x) = x^2

Surjective?

What if **f**:**R** → **R**?
What if **f**:**Z** → **Z**^{0}?



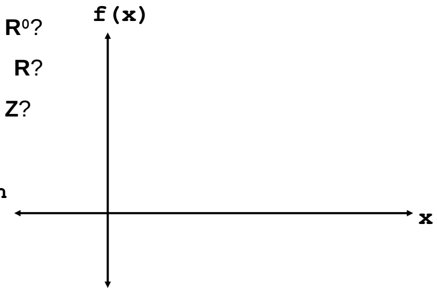
R^0, **Z**^{0} are the sets of non-negative real, integer numbers

Trying out Terminology

f(x) = $\lfloor$**x**$\rfloor$

Is **f** surjective?

a. **Yes, for f:R** → **R**^{0}?
b. **Yes, for f:R**^{0} → **R**?
c. **Yes, for f:R** → **Z**?
d. **No, not for any domain/co-domain**
e. **None of these is correct**



Trying out Terminology

f:{s|s is a 121 student} → **{A+, A, …, D, F}**
f(s) = s's mark in 121

- If we know that there are 300 students in 121, is **f** surjective?

a. Yes
b. No
c. Not enough information

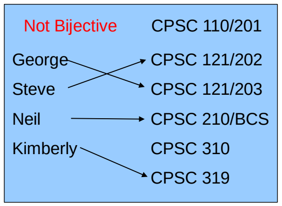
- Could we *ever* know that f was surjective just by knowing **f**'s domain and co-domain?

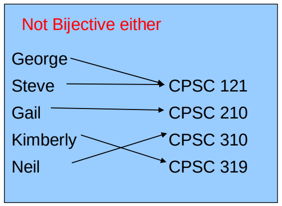
Outline

- Injective Functions
- Surjective Functions
- Bijective Functions
- Inverse Operations.

Bijective Functions

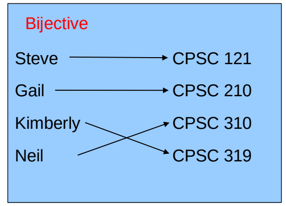
- A function $f : A \to B$ is bijective (also one-to-one correspondence) if it is both one-to-one and onto (both injective and surjective).
- In the arrow diagram: exactly one arrow points to each element of B.





Bijective Functions

- This is bijective

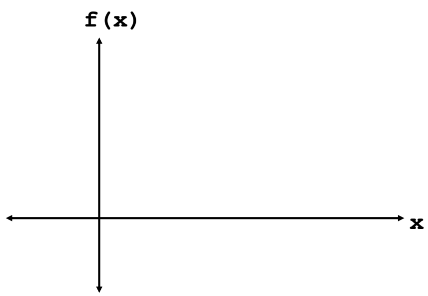


Trying out Terminology

$f(x) = x^2$

$f: ? \to ?$

Bijective for what domain/co-domain?



Outline

- Injective Functions
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- Inverse Operations.

Inverse of a Function

- The inverse of a function f: A → B, denoted f^{-1}, is
  $f^{-1}: B \to A$.
  $f^{-1}(y) = x \leftrightarrow f(x) = y$.

- In other words:
  - If we think of a function as a list of pairs.
    E.g. $f(x) = x^2$: { (1, 1), (2, 4), (3, 9), (4, 16), ... }
  - Then f^{-1} is obtained by swapping the elements of each pair:
    f^{-1} = { (1, 1), (4, 2), (9, 3), (16, 4), ... }

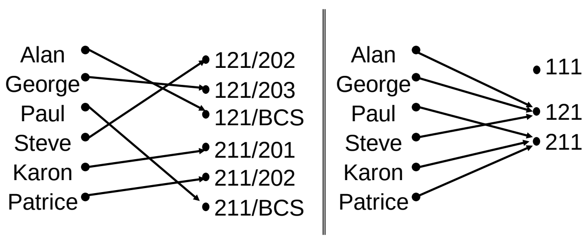
Inverse of a Function

- Is f^{-1} a function?
  A. Yes, always.
  B. No, never.
  C. Yes, but only if f is injective.
  D. Yes, but only if f is surjective.
  E. Yes, but only if f is bijective.

- Can we prove it?

Trying out Terminology

What's the inverse of each of these **f**s?

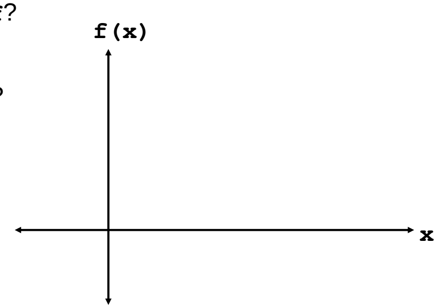


Trying out Terminology

$f(x) = x^2$

What's the inverse of **f**?

What should the domain/co-domain be?



Appendix 3: An Inverse Proof

- **Theorem**: If $\mathbf{f} : \mathbf{A} \rightarrow \mathbf{B}$ is bijective, then $\mathbf{f}^{-1} : \mathbf{B} \rightarrow \mathbf{A}$ is a function.
- **Proof:** We proceed by antecedent assumption.
  - Assume $\mathbf{f} : \mathbf{A} \rightarrow \mathbf{B}$ is bijective.
  - Consider an arbitrary element $\mathbf{y}$ of $\mathbf{B}$.
    Because $\mathbf{f}$ is surjective, there is some $\mathbf{x}$ in $\mathbf{A}$ such that $\mathbf{f(x) = y}$.
    Because $\mathbf{f}$ is injective, that is the only such $\mathbf{x}$.
  - $\mathbf{f}^{-1}\mathbf{(y) = x}$ by definition; so, $\mathbf{f}^{-1}$ maps every element of $\mathbf{B}$ to exactly one element of $\mathbf{A}$.

QED