CPSC 121: Models of Computation

Unit 9b: Mathematical Induction - part 2

Based on slides by Patrice Belleville and Steve Wolfman

Outline

- **Strong Mathematical Induction.**
- Pattern and Examples
- More examples using induction.
- Further exercises.

Strong Mathematical Induction

- The induction we have seen so far handles problems which can be broken down to sub-problems of size 1 less that the original problem size.
- How do we handle more general problems which can be defined in terms of one or more smaller similar problems with various but smaller sizes?
- We need to make our induction technique more general.

Strong Mathematical Induction

- When we want to prove

  $\forall n \in Z^+, Q(n)$

  We use a slightly different induction step.
  - Instead of proving that
    - $\forall n \in Z^+, Q(n-1) \rightarrow Q(n)$
  - We prove that
    - $\forall n \in Z^+, (Q(1) \wedge Q(2) \wedge \ldots \wedge Q(n-1)) \rightarrow Q(n)$
- That is, we now assume that the theorem is true for all the numbers smaller than n and prove it for n
- We can also show that this type of induction is a valid proof technique.

Outline

- Strong Mathematical Induction.
- **Pattern and Examples**
- More examples using induction.
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Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 24 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ _______.

Breaking down into all smaller problems

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Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for _______**):**
Prove each base case via your other techniques.

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 24 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Cases: Prove P(·) is true for **22 , 23 and 24 (and possibly more base cases that are not reachable from 22 using the inductive step**)
Prove each base case via your other techniques. For n=23, we may just need n=22 and so on.

Inductive Step: For n > _______, if P(·) is true for ____________, then P(n) is true.

Examples: Breaking down into all smaller problems

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Prove each base case via your other techniques.

Inductive Step: For n > **24**: if P(·) is true for **every integer from 24 up to n-1**, then P(n) is true:

WLOG, let n be greater than ____________.

Assume P(·) is true for __________________.

Examples: Breaking down into all smaller problems

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Inductive Step: For n > **24**: if P(·) is true for **every integer from 24 up to n-1**, then P(n) is true:

WLOG, let n be greater than **24**.

Assume **for all integers i where 24 ≤ i < n, P(i) is true**. We'll prove P(n)

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) *is true.*

This completes our induction proof. QED

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P(⌊n/2⌋) and P(⌈n/2⌉) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

But, your insight may come in *any* form.
Maybe you need problems half as large or one-third.
Maybe you need problems that are 7 smaller.
Maybe you need the problems that are 1, 2, and 3 smaller.

Regardless, the pattern is the same!

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Theorem: P(n) is true for all n ≥ _______.

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Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

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Prove each base case via your other techniques.

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Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for n = **7, 8, 9, 10, 11, 12, 13):**
Prove each base case via your other techniques. (We need all the way up to 13 because only at 14/2 do we reach a base case. From 15 on, we always eventually hit a base case.)

Inductive Step (for n > _______, if P(·) is true for ____________, then P(n) is true):

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Prove each base case via your other techniques.

Inductive Step (for n > **13**: if P(·) is true for **⌊n/2⌋ and ⌈n/2⌉**, then P(n) is true):

WLOG, let n be greater than ____________.

Assume P(·) is true for __________________.

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P(⌊n/2⌋) and P(⌈n/2⌉) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

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Inductive Step (for n > **13**: if P(·) is true for **⌊n/2⌋ and ⌈n/2⌉**, then P(n) is true):

WLOG, let n be greater than **13**.

Assume P(·) is true for **⌊n/2⌋ and ⌈n/2⌉**.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Example 1

- Every positive integer n greater than 1 can be written as a product of primes.
- What base case(s) should we use?
  A. n = 1
  B. n = 2
  C. n = 2, 3 or 5.
  D. n is prime.
  E. None of the above.

Example 1

- Every positive integer n greater than 1 can be written as a product of primes.
- What is the inductive step?
  A. For every integer k >2, if k-1 is a product of primes, then k is a product of primes
  B. For every integer k ≥ 2, if k-1 is a product of primes, then k is a product of primes
  C. For every integer n >2, if every integer k, 2 ≤ k ≤ n-1, is a product of primes, then n is a product of primes.
  D. For every integer n ≥ 2, if every integer k, 2 < k ≤ n-1, is a product of primes, then n is a product of primes .
  E. None of the above.

Example 1

- Proof: we prove the result by induction on n.
  - **Base case:** n = 2
    - Since 2 is prime, the statement is true.
  - **Induction step:**
    - Let n be any integer greater than 2. Suppose that every number from 2 to n-1 is a product of primes. We'll show that n is a product of primes
    - Case 1: ______________________

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  - **Base case:** n = 2 is prime.
    - Since 2 is prime, the statement is true.
  - **Induction step:**
    - Let n be any integer greater than 2. Suppose that every number from 2 to n-1 is a product of primes. We'll show that n is a product of primes
    - Case 1: n is prime. Then the statement is true.

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  - **Base case:** n = 2 is prime.
    - Since 2 is prime, the statement is true.
  - **Induction step:**
    - Let n be any integer greater than 2. Suppose that every number from 2 to n-1 is a product of primes. We'll show that n is a product of primes
    - Case 1: n is prime. Then the statement is true
    - Case 2: n is composite. Then

      n =

Example 1

- Proof: we prove the result by induction on n.
  - **Base case:** n = 2 is prime.
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  - **Induction step:**
    - Let n be any integer greater than 2. Suppose that every number from 2 to n-1 is a product of primes. We'll show that n is a product of primes
    - Case 1: n is prime. Then the statement is true
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      n = a*b such that 1< a < n and 1< b < n
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    - Let n be any integer greater than 2. Suppose that every number from 2 to n-1 is a product of primes. We'll show that n is a product of primes
    - Case 1: n is prime. Then the statement is true
    - Case 2: n is composite. Then

      n = a*b such that 1< a < n and 1< b < n
    - By the induction hypothesis:

      a = p1*p2*…*pm where pi is prime

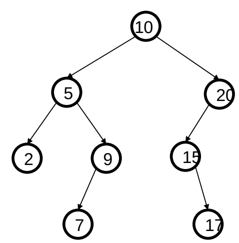
      b = q1*q2*…*qr where qi is prime
    - and

      n = p1*p2*…*pm*q1*q2*…*qr

  QED

Binary Trees

- CPSC 110 review: A binary tree is a data structure that is defined recursively as following
- A binary tree is either
  - Empty, or
  - A node with some data, and two children that are themselves binary trees.



Proving Correctness : Binary trees

- Example 2: Consider the following function:

```
(define (tree-size t)
  (if (null? t)
     0
    (+ 1
      (tree-size (left-child t))
      (tree-size (right-child t)))))
```

- How can we prove that it correctly computes the number of (non-null) nodes of the tree?

Example : Binary trees

- We prove this using mathematical induction on the size of the tree t.
  - **Base case**: t is null
    - In this case t contains exactly 0 nodes.
    - The algorithm returns 0. Therefore it is correct
  - **Induction step**:
    - Let **t** be any binary tree with size greater than 0.
    - Assume the algorithm works for trees that are smaller than t.
    - Because the left sub-tree of t is smaller than t, the 1st recursive calls returns the size of the left sub-tree of t.

Example : Binary trees

- Induction step (continued)
  - Similarly the right sub-tree of t is smaller than t, and so the 2nd recursive call returns the size of the right sub-tree of t.
  - The algorithm then returns 1 + the sum of the values returned by the recursive calls.
  - This is exactly the size of t (1 for the root + the sum of the sizes of the two sub-trees).
- Hence our algorithm computes correctly the size of every tree. QED

Worked Example : What is Wrong ?

Theorem: All integers greater than or equal to 2 are even.

Proof: We proceed by induction on n.

Base Case : Theorem is true for the first case where n = 2. Since 2 = 2*1, 2 is even.

Inductive Step For any k >2, assume that k-2 is even and we'll show that k is even.

- WLOG, let k be any integer > 2 .
- By the inductive hypothesis, k-2 is even.
- Therefore k -2 = 2 m for some integer m
- Then k = 2m -2 = 2(m-1).
- Since m-1 is an integer, k is even

QED

Recall: Practical Induction

How can we figure out an inductive proof?

- Start at the inductive step!
- Look at a "big" problem (of size n).
- Figure out how to break it down into smaller pieces.
- Assume those smaller pieces work. That will end up as your Induction Hypothesis.
- Figure out which problems cannot be broken down (usually small ones!). Those will end up as your basis step(s).

Outline

- Strong Mathematical Induction.
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- **More examples using induction**.
- Further exercises.

Example : Geometric series

- Example 2: geometric series
  - We will prove that for every value of $a \neq 0, 1$:

$$\sum_{i=0}^{t} a^i = \frac{a^{t+1} - 1}{a - 1}$$

  - These summations occur frequently when we need to determine the running time of divide-and-conquer algorithms (in CPSC 320).

Example : Geometric series

- Proof:
  - Base case: t = 0
    - o In this case the summation is $a^0 = 1$, and $\frac{a^1 - 1}{a - 1} = 1$
  - Induction step:
    - o Pick any t > 0. Assume that the statement is true for t-1
    - o Now will show that the statement is true for t
    - o $\sum_{i=1}^{t} a^i = (\sum_{i=0}^{t-1} ai\ ) + a^t\ =$

$$= \frac{a^t - 1}{a-1} + a^t = \frac{a^t - 1 + a^t(a-1)}{a-1} = \frac{a^{t+1} - 1}{a-1}$$

  - QED

Example: Using Inequalities

- Example 3: Prove that
  $\forall n \geq 4,\ 2^n < n!$
- Rules for proving inequalities:
  - Start from one side (say the left side)
  - Work step by step towards the other.
  - When dealing with <, you are allowed to make the expression larger, but never smaller.
    - Example: if I am smaller than you, then I am still smaller than you when you stand on a bench.

Inequalities

- Proof: by induction on n.
  - Base case: $n = 4$
    - $2^4 < 4!$ because $16 < 24$
  - Induction step: we want to prove that for any $k > 4$, if $2^{k-1} < (k-1)!$ then $2^k < k!$
    - Induction hypothesis: assume that $2^{k-1} < (k-1)!$
    - Then
      $2^k = 2(2^{k-1}) < 2(k-1)! < k(k-1)! = k!$
      this is where we "approximate"
      the induction hypothesis is used here
  - Hence by the principle of M.I., $\forall n \geq 4,\ 2^n < n!$

Example : Binary Search

- Example: binary search
  - Suppose we have something like a list, but whose i-th element and length can be found in a single step.
    - This structure is called a vector in Racket.
    - It is similar to an ArrayList in Java.
  - We assume that we have such a vector, sorted in increasing order.
    - Examples: ("Ann", "Charles", "Dora", "Gregor", "Wei").
  - We want to find the position of a given element (for instance, "Dora").

Binary Search

- Claim: the following algorithm (formerly known as Binary Search) works:

```
(define (binary-search avector first-pos last-pos x)
  (if (> first-pos last-pos)
    #f
    (if (= first-pos last-pos)
      (if (= x (vector-ref avector first-pos))   first-pos   #f)
      (let ((mid-pos  (quotient  (+  first-pos  last-pos)  2)))
        (if (= x (vector-ref avector mid-pos))
            mid-pos
           (if (< x (vector-ref avector mid-pos))
              (binary-search avector first-pos (- mid-pos 1) x)
              (binary-search avector (+ mid-pos 1) last-pos x)…)
```

Binary Search

- Proof: by induction on the size of the part of the vector that we are searching.
  - Base cases: size ≤ 1
    - If size is 0, then x can not be in that part of the vector, so returning #f is correct.
    - If size is 1, then there is only one possible location for x, and we check this position.
  - Induction step: let **v** be any vector of size ≥ 2
    - Suppose that the algorithm will find x (if it is in the vector) for every vector with fewer than size elements.
    - We'll show that the algorithm will find x in any vector with size elements.

Binary Search

- Proof (continued)
    - If x is at position mid-pos of v, then the algorithm returns mid-pos.
    - Otherwise, x is either smaller than the element at position mid-pos, or larger.
    - If x is smaller then either x is in the first half of **v** or not in v at all
      - Algorithm returns the result of searching the first half of **v** which by the IH is the correct result
    - If x is larger then either x is in the second half of **v** or not in at all
      - Algorithm returns the result of searching the second half of **v** which by the IH is the correct result
  - Hence by the principle of M.I., the algorithm returns the correct value. QED

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Additional Examples

- Prove that for every n ≥ 1, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$
- (very challenging): Prove that binary search makes at most ⌈log2 (size+1)⌉ comparisons if size ≥ 1.

  Give a proof by induction on the size of the vector.