Menu January 28, 2009

Topics:

Predicates and Quantified Statements (cont'd)

Reading:

Today: Epp 2.2–2.4

Reminders:

Assignment 1 due Friday, January 30, 17:00
In-class Quiz 1 Wednesday, February 4
Midterm exam Tuesday, February 24 (evening)
READ *the WebCT Vista course announcements board*

As noted above, Assignment 1 is due Friday, January 30, 17:00. Here is further information from the "Assignments, Labs, Online Quizzes" section of the course web site.

> The hand-in boxes for CPSC 121 are in the basement of the ICCS building and are labeled with the course number, section number and instructor's name. They are in the hallway outside of room 011.
>
> Assignment submissions must be stapled below the CPSC 121 assignment cover page or a clearly legible reproduction of the same information.
>
> Late submissions are not accepted.

Predicates and Quantified Statements (cont'd)

Lecture 9 ended with two logical equivalences. . .

Two Important Logical Equivalences

$$\overline{\forall x, P(x)} \equiv \exists x, \overline{P(x)}$$

$$\overline{\exists x, P(x)} \equiv \forall x, \overline{P(x)}$$

These two logical equivalences are referred to as *Generalized De Morgan's Laws*

NOTE: We've already observed that universal quantification generalizes conjunction and that existential quantification generalizes disjunction. The generalized De Morgan's Laws also follow from this observation

These two logical equivalences determine, respectively, the negation of a universally quantified statement, $\forall x, P(x)$, and the negation of an existentially quantified statement, $\exists x, P(x)$.

Epp considers the universally quantified conditional statement, $\forall x, P(x) \rightarrow Q(x)$, to be an important special case. The negation of a universally quantified conditional statement is derived, as follows:

Another Important Logical Equivalence

$$
\begin{aligned}
\overline{\forall x, P(x) \rightarrow Q(x)} &\equiv \exists x, \overline{P(x) \rightarrow Q(x)} \\
&\equiv \exists x, \overline{\overline{P(x)} \vee Q(x)} \\
&\equiv \exists x, P(x) \wedge \overline{Q(x)}
\end{aligned}
$$

Thus, to prove that a universally quantified conditional statement $\forall x \in D, P(x) \rightarrow Q(x)$ is false, it is sufficent to show that there exists an x in the domain of discourse, D, for which $P(x)$ is true and $Q(x)$ is false.

We now consider statements containing multiple quantifiers. As often is the case, there is a convention adopted for their intepretation.

Statements Containing Multiple Quantifiers

Convention:

"When a statement contains more than one quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur."

– Epp, page 97

Statements Containing Multiple Quantifiers (cont'd)

Interpret $\forall x \in D, \exists y \in E, P(x, y)$ as $\forall x \in D\, (\exists y \in E, P(x, y))$

Interpret $\exists x \in D, \forall y \in E, P(x, y)$ as $\exists x \in D\, (\forall y \in E, P(x, y))$

Note: The "convention" simply is agreed upon short-hand to avoid having to fully parenthesize statements containing multiple quantifiers

The negations of multiply quantified statements are derived, as follows:

Negation of Multiply Quantified Statements

$$
\begin{aligned}
\overline{\forall x \in D, \exists y \in E, P(x, y)} &\equiv \exists x \in D, \overline{\exists y \in E, P(x, y)} \\
&\equiv \exists x \in D, \forall y \in E, \overline{P(x, y)}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\exists x \in D, \forall y \in E, P(x, y)} &\equiv \forall x \in D, \overline{\forall y \in E, P(x, y)} \\
&\equiv \forall x \in D, \exists y \in E, \overline{P(x, y)}
\end{aligned}
$$

Supplementary examples were included as an addendum to Lecture 9. Let's consider a few of those examples now.

Example 1:

Let $F(x)$: x is a fierce creature
$L(x)$: x is a lion
$C(x)$: x drinks coffee

($D =$ set of all creatures)

Translate: “All fierce creatures are not lions”

NOTE: We claim there are two interpretations

Translation 1: $\forall x, (F(x) \rightarrow \overline{L(x)})$ *Interpretation 1:* “Nothing that is fierce is a lion”

Translation 2: $\exists x, (F(x) \wedge \overline{L(x)})$ *Interpretation 2:* “There is some creature that is fierce but not a lion”

Example 1 (cont’d):

Let’s look at Translation 2 more closely:

$$\exists x, (F(x) \wedge \overline{L(x)})$$

$$\equiv \quad \exists x, \overline{F(x) \rightarrow L(x)} \qquad \text{by } \overline{p \rightarrow q} \equiv p \wedge \overline{q}$$

$$\equiv \quad \overline{\forall x, F(x) \rightarrow L(x)} \qquad \text{by generalized De Morgan}$$

Thus, the ambiguity arises from whether the negation applies to $L(x)$ alone (as in Translation 1) or to the entire statement (as in Translation 2)

Example 2:

Negation often is ambiguous in English. Consider the sentence, “Nothing is too good for you.”

Interpretation 1: “You are so fine that there isn’t anything good one could do for you that goes beyond what you deserve.”

Interpretation 2: “You are so bad that even doing nothing for you is more than you deserve.”

ASIDE: These two interpretations have dramatically different meanings!

This form of statement occurs frequently in advertising. Consider a claim of the form “Nothing cleans better than <insert name of commercial cleaning product>.” One might well respond, “In that case I’ll use nothing (and save my money).”

Example 3:

Let $T(x, y)$: x has tasted y

($D =$ set of all creatures)

Question: Is $\forall x, T(x, y)$ a proposition?

Answer: No.

The variable x in $\forall x, T(x, y)$ is *bound* by the universal quantifier. But, the variable y is *unbound* (i.e., *free*). Unless all variables in a quantified statement are bound, the statement is **not** a proposition

To be bound, a variable must:

1. have its domain of discourse, D, specified
2. be governed by a quantifier ($\forall$ or $\exists$)

Finally, an example to make the point that the order of the quantifiers matters.

Example 4:

Let $P(x, y) : x \ + \ y = 0$

($D = \mathbf{Z}$, the set of integers)

Consider: $\forall x \exists y P(x, y)$

Recall: Interpret this as $\forall x(\exists y P(x, y))$

Question: Is this proposition true?

Answer: Yes

Proof: Let x be an arbitrary integer. Choose $y = -x$. Then $x + y = x + (-x) = 0$ so that $P(x, y)$ is true

Since x was arbitrary, we have shown $\forall x(\exists y P(x, y))$, as required

NOTE: The y we choose depends on x. But, this is perfectly OK in this example

Example 4 (cont'd):

Consider: $\exists x \forall y P(x, y)$

Recall: Interpret this as $\exists x(\forall y P(x, y))$

Question: Is this proposition true?

Answer: No

Proof: There is no fixed integer, x, independent of y, such that for an arbitrary integer y, $x + y = 0$

Let's consider rules of inference involving quantified statements...

Rules of Inference for Quantified Statements

Rule of Inference	Name
$\dfrac{\forall x, P(x)}{\therefore P(c)}$	Universal Instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x, P(x)}$	Universal Generalization
$\dfrac{\exists x, P(x)}{\therefore P(c) \text{ for some new element } c}$	Existential Instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x, P(x)}$	Existential Generalization

Rules of Inference for Quantified Statements (cont'd)

Rule of Inference	Name
$\forall x, P(x) \rightarrow Q(x)$ $\underline{P(a)}$ $\therefore Q(a)$	Universal Modus Ponens
$\forall x, P(x) \rightarrow Q(x)$ $\underline{\sim Q(a)}$ $\therefore \sim P(a)$	Universal Modus Tollens
$\forall x, P(x) \rightarrow Q(x)$ $\underline{\forall x, Q(x) \rightarrow R(x)}$ $\therefore \forall x, P(x) \rightarrow R(x)$	Universal Transitivity

Question: Are there any issues here (or do we accept these inference rules as "obvious")?

Answer: Yes, there are issues.

"Existential Instantiation" is controversial. Proofs that invoke this rule are non-constructive. Constructivist mathematics rejects this rule of inference. Constructivists assert that it is necessary to find (or "construct") a mathematical object to prove that it exists. When one assumes that an object does not exist and derives a contradiction from that assumption, one still has not found the object and therefore not proved its existence, according to constructivists. To prove $\exists x \in D, P(x)$ constructively one must present a particular $a \in D$ together with a proof of $P(a)$.