LINEAR ALGEBRA

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Contents

1 8/21/23 - Mon

1.1 Field and its Properties

Definition (Field)**.** $\mathbb{R}\mathbb{C}\mathbb{Q}$ *k: is a field if k has operations and satisfies*

1. *k contains* 0&1
2. *a+0=1,a* $a \cdot 1 = a$
3. *a+b=b+a, (a+b)* $\cdot$ *c=a* $\cdot c + b \cdot c$
4. $a \neq 0, a$ *has multiplicative inverse i.e.*$a \in K$ $a \cdot a^{-1} = 1, a^{-1} \in K$
5. $\forall a \in k$ *has an additive inverse -a*
6. *associativity for + and* $\cdot$

examples that $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields
$\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ is a field

$$i^{-1} = -i | (a + bi)^{-1} = \frac{a - bi}{a^{\alpha} + b^{\alpha}}$$

$\mathbb{Z}$ is not a field. Because not all element has a multiplicative inverse

1.2 vector space

Definition (Vector spaces)**.** *let K be a field.* $K^n = \{(a_1, a_2, a_3 \dots a_n) | a_i \in K\}$ *where as a is a vector*
$\underbrace{(1, 0, \dots, 0)}_{e_1}\underbrace{(0, 1, \dots, 0)}_{e_2} \dots \underbrace{(0, 0, \dots, 1)}_{e_n}$
also $\vec{0} \in K$

addition:$(a_1, a_2 \dots a_n) + (b_1, b_2, b_3 \dots b_n) = (a_1 + b_1, a_2 + b_2 \dots a_n + b_n)$
scalar multiplication: $c \in Kc \cdot (a_1, a_2, a_3 \dots a_n) = ca_1, ca_2, ca_3 \dots ca_n$ They satisfy the following requirement

1. $\vec{a} + \vec{a} = \vec{a}$
2. $\vec{b} + \vec{a} = \vec{a} + \vec{b}$
3. $c \cdot (\vec{a} + \vec{b}) = c \cdot \vec{a} + c \cdot \vec{b}$
4. $c_1 c_2 \cdot \vec{a} = c_1 \cdot (c_2 \cdot \vec{a})$
5. $(c_1 + c_2) \cdot \vec{a} = c_1 \vec{a} + c_2 \vec{a}$
6. $1 \cdot \vec{a} = \vec{a}$
7. $\vec{a} + -\vec{a} = 0$
8.

9.

with all the prereq, K^n is a vector space over K

Definition (general vector space)**.** *a set V with origion 0* $\in V$ *together, closed addition and scalar multiplication*
i.e.$\vec{V} + \vec{W} \in V, c \cdot v \in V$ *also* $c \in K, v, w \in V$ *is called vector space over K if all the above holds*
any element $v \in V$ *is called vectors of V*
δ

:
e.g.

1. $\mathbb{R}C(\mathbb{R})-$ { continuous function on R is a v space over $\mathbb{R}$ }

2. $f + g \in C(\mathbb{R})$

3. $a \in \mathbb{R}$ a $\cdot$ f, a function $f \in C(\mathbb{R})$ is a vector

more general X is a set k(X)={ x $\rightarrow$ k } is a v space over K $\forall f, g \in k(x)$
$(f + g)(x) = f(x) + g(x)$
$(c \cdot f)(x) = c \cdot f(x)$

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2.1 fields

last class recall that V is a vector space over $\underbrace{K}_{\text{field, k= } \mathbb{R} \text{ or } \mathbb{C}}$ note in this class, $\mathbb{R}$, $\mathbb{C}$ is our field
$\underbrace{V}_{\text{vector}} \cdot \underbrace{W}_{\text{vector}} \in V$
also $V + W \in V$
$0 \in V$

2.2 subspaces

Definition (subspaces)**.** *V is a vector space over k we say the subset* $W \subseteq V$ *is a subspace if it is closed under*

- *addition*

- *multiplication*

$$v + w \in W$$

$$v \cdot w \in W$$

$\forall v, w \in W, a \in K$
note this definition also implies that $0 \in W$ *e.g.* $V = k^n$ $W = \{(a_1, a_2, a_3 \ldots a_n) \in K^n | \Sigma_{i=1}^n a_i = 0\}$ $w \subseteq v$ *subspace*

2.3 Linear Combination

we have vectors $v_1, v_2, v_3 \dots v_n \in V$ and scalars $a_1, a_2, a_3 \dots a_n \in K$ and we call $a_1 + v_1, a_2 + v_2, a_3 + v_3$ **linear combination** of $v_1, v_2 \dots$ e.g. we have $e_1(1,0) \wedge e_2(0,1) \in k^2$ example we have $(3,2) = \underbrace{3}_{\text{scalar}} \underbrace{e_1}_{\text{vector}} + 2e_2$

> **Proposition 2.1.** given $v_1, v_2 \dots v_n$ W=set of all possible linear combination of $v_1 \dots v_n$ then, W is a subspace of V.

Proof. given $a_1v_1 \dots a_nv_n$ and $b_1 \dots b_n$ $=(a_1+b_1)v_1 + (a_2+b_2)v_2 + \dots$ $(a_n+b_n)v_n$ is also an linear combination
property 2. $c(a_1v_1 + a_nv_n) = c(a_1)v_1 + c(a_n)v_n$

□

2.4 Dot Product

$\vec{a}(a_1, a_2 \dots a_n)\vec{b}(b_1, b_2 \dots b_n)$
$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 \dots a_nb_n$

Remark (properties of dot product)**.**

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\underbrace{(c \cdot \vec{a})}_{\text{scalar multiples}} \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

Definition (orthogonal)**.** *we say 2 vectors $\vec{a}, \vec{b} \in k^n$ are* ***perpendicular*** *or* ***orthogonal*** *if their dot product is 0 in k^n*
in notation $e_i \cdot e_j = 0$
hence we write $\vec{a} \perp \vec{b}$

recall $W = \{(a_1, a_2 \dots a_n) | a_1 + a_n = 0 \subseteq k^n \equiv \{\vec{a} | \vec{a} \cdot (1,1,1)\}\}$ more generally $\vec{b}(b_1, b_2 \dots b_n)$
$W\{\vec{a} \in k^n | \vec{a} \cdot \vec{b} = 0\} \subseteq k^n$
give n 2 sub spaces w_1 and w_2 we have 2 operations

1. $$w_1 \cap w_2$$
2. $$w_1 + w_2$$

Notice that both of those operations preserves sub spaces.
$(w_1 + w_2) + (w_1' + w_2') = (w_1 + w_1') + (w_2 + w_2')$

2.5 linear independence

Definition (Linear independence)**.** *V is a vector space over K, we say that* $v_1, v_2 \dots v_n \in V$ *are* ***linearly dependent*** *over K if there exists* $a_1, a_2 \dots a_n$ *such that not all of them are zero and* $a_1v_1 + a_2v_2 + a_3v_3 + \dots a_nv_n = 0$
otherwise we call it linearly independent

Remark. *we have* k^2 *where as we have (1,0)(0.1) V(2,5) such that* $= 2e_1 + 3e_2$ *hence we know that*

$$v - 2e_1 - 3e_2 = 0$$

Thus $e_1 \wedge e_2$ *are not linearly independent*

Remark. *notice that* e^t *and* e^{2t} *functions are linearly independent*

Proof. suppose that there are linearly dependent then we have a,b such that

$$ae^t + be^{2t} = 0$$

factor out a e^t we have

$$a + be^t = 0$$

taking derivative of both sides we have

$$be^t = 0$$

but $e^t \neq 0$ hence b=0 and a=0 which we have arrived at an contradiction ↯ ☐

Definition (alternative definition of vector space)**.** *V is a vector space if*

1. $v_1, v_2 \dots v_n$ *are linearly independent*
2. $v_1, v_2 \dots v_n$ ***generates*** *V*
   (a) *i.e. any vector* $v \in V$ *is a linear combination of* $v_1, v_2 \dots v_n$
   (a) *e.g.* $e_1, e_2, e_3 \dots e_n$ *are linearly independent and clearly*
      i. $e_1, e_2, e_3 \dots e_n$ *generates V*

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Last class recall V is a vector space over K $v_1, v_2 \dots v_n \in V$

Definition (Linear Combination)**.** $a_1v_1 + a_2v_2 + \dots + a_nv_n$ $W = \{\sum a_iv_i | a_i \in K\} \subseteq V$
$v_1, v_2 \dots v_n$ *are linearly dependent if* $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ *for* $\forall \neq 0a_i$
otherwise we call $v_1, v_2 \dots v_n$ *are linearly independent*

Definition (basis).

$$v_1, v_2 \dots v_n$$

is a ***basis*** *if and only if:*

1. $v_1, v_2 \dots v_n$ *are linearly independent*

2. $v_1, v_2 \dots v_n$ ***generates*** V

Theorem 3.1. Assume that $v_1, v_2 \dots v_n$ are linearly independent $\in V$ then $a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n$ then $a_i = b_i \forall i\ a_ib_i \in K$

Proof.
$(\mathrm{a}_1v_1 + a_2v_2 + \cdots + a_nv_n) - (b_1v_1 + b_2v_2 + \cdots + b_nv_n)$
$= (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$
$= 0$

$\square$

Note: Linearly independent of $v_1, v_2 \dots v_n \Rightarrow a_i - b_i = 0$
$\Leftrightarrow a_i = b_i$

uniqueness of a_i . $if\,\mathrm{v}_1, v_2 \dots v_n$ is a basis of V, then $\forall v \in V$
$V = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for unique $a_i \in K$

Definition (coordinates). *if* $v_1, v_2 \dots v_n$ *is a basis, if* V*=*$a_1v_1 + a_2v_2 + \cdots + a_nv_n$
we call $a_1, a_2 \dots a_n$ *the coordinates of* v *with reference of this basis*
e.g.$\{\underbrace{(1,1)}_{v_1}, \underbrace{(1,-1)}_{v_2}\}$ *is a basis of* K^2

Proof. 1. linear independence
suppose $\exists a, b$ s.t. $a \cdot (1,1) + b \cdot (1,-1) = 0, K = \mathbb{R}$ or $\mathbb{C}$
$(a + b)(a - b) = 0$
$(a + b = 0)$
$(a - b = 0)$
$(a = 0, b = 0)$
Contradiction ↯
2.v_1, v_2 generates K
Given $(a, b) \in K^2$
$(a, b) = \frac{a+b}{2}(1,1) + \frac{a-b}{2}(1,-1)$ $\square$

3.1 finitely generated vspace

Definition (Finitely generated). *We say* V *is* ***finitely generated*** *over* K *if there exists* $v_1, v_2 \dots v_n \in V$ *which generates to* V *and its finite*

Theorem 3.2. Suppose that $v_1, v_2 \dots v_n$ generates V. Let $\{v_1 \dots v_r\}$ be the maximal subset of linearly independent of vectors in $\{v_1, v_2 \dots v_n\}$ then $v_1 \dots v_r$ form a basis

Proof. By assumption, we know that $v_1, v_2 \dots v_r$ are linearly independent
$\forall k, k > r$
$v_1, v_2 \dots v_r, v_k$are linearly dependent
$i.e. a_1 + v_1 + \cdots + a_r v_r + b v_k = 0$ for some a,b $\neq 0$ in fact b $\neq 0$

$$v_k = -\frac{a_i}{b_i} v_i - \cdots - \frac{a_r}{b} v_r$$

which implies $v_1, v_2 \dots v_n$ generates V
Hence $v_1, v_2 \dots v_r$ is a basis

□

3.2 dimension of v space

Theorem 3.3 (linearly dependent for n>m)**.** Let v be a vector space over K and let $\{v_1 \dots v_m\}$ be a basis of V, let $\{w_1 \dots w_n\}$ be vectors in V, assume $n > m$ then $w_1 \dots w_n$ are linearly dependent

proof by contradiction. Assumes that w_1, w_n are linearly dependent $(\star)$
For simplicity, let m=2 $n > 2$ and assume that $w_i \neq 0 \forall i$
First of all $w_1$can be written as

$$w = a_1 v_1 + a_2 v_2$$

Since a_1, a_2 cannot both be 0 WLOG we may assume that $a_1 \neq 0$ then

$$v_1 = \frac{1}{a_1} w_1 - \frac{a_2}{a_1} v_2$$

Because v_1, v_2 generates V by the definition of v space $\rightarrow$ w_1, v_2 generates V if we do this repeatly
Thus

$$w_2 = b_1 w_1 + b_2 w_2$$

where as $b_2 \neq 0$

$$v_2 = \frac{1}{b_2} w_2 - \frac{b_1}{b_2} w_1$$

This means that

$$w_1, w_2 \text{generates V which contradicts} \star$$

↯

□

Corollary (. 1.2 cardinality of the basis) Any 2 basis of V have the same cardinality

4 8/28/23 - Mon

Last class
Let V be a V space over K
recite the definition of basis lol

Theorem 4.1 (A)**.** and let $v_1, v_2 \dots v_m$ be a basis of V and let $w_1, w_2 \dots w_n$ be any vectors in V and if $n > m$ then $w_1, w_2 \dots w_n$ are linearly independent

Proof. Last we have proven m=2
case m=3 $\{v_1, v_2, v_3\}$ is a basis for $n > 3$
$w_1 = a_1v_1 + a_2v_2 + a_3v_3 \Rightarrow v_1 = \frac{1}{a_1}w_1 - \frac{a_2}{a_1}v_2 - \frac{a_3}{a_1}v_3$
WLOG assume that $a_1 \neq 0 \Rightarrow w_1, v_2, v_3$ generates V

$$w_2 = b_1w_1 + b_2v_2 + b_2v_3$$

WLOG assume that $b_2 \neq 0$

$$V_2 = \frac{1}{b_2}w_2 - \frac{b_1}{b_2}w_1 - \frac{b_3}{b_2}v_3$$

Thus $w_1w_2v_3$ **generates** V

$$w_3 = c_1w_1 + c_2w_2 + c_3v_3$$

which gives us

$$v_3 = \star w_1 + \star w_2 + \star v_3$$

which means that $w_1w_2w_3$ **Generates** v_1
and $w_4 = w_1 + w_2 + w_3 \rightarrow \lightning$

□

This allows us to arrive at an immediate corllary

Corollary i. any 2 basis of V have the same cardinally

Proof. $\#\mathcal{B} = \{v_1, v_2 \dots v_n\}$ be a basis and let
$\#\mathcal{B}' = \{w_1, w_2 \dots w_n\}$
by the above theorem, we can immediately conclude that

$$\#\mathcal{B} = \#\mathcal{B}'$$

□

4.1 dimensions & maximal set

Definition (Maximal set)**.** $v_1, v_2 \dots v_n$ *are linearly independents* $\in V$
we say that $v_1, v_2 \dots v_n$ *form a* ***maximal set*** *of linearly independent vectors of V.*
i.e. $\forall w \in V w_1 v_1, v_2 \dots v_n$ *are linearly dependent*

Theorem 4.2 (B)**.** Any maximal set of linearly independent vectors of V is a basis

Proof. let $v_1, v_2 \dots v_n$ be a maximal set of linearly dependent vectors of V be a basis then, for all $w \in v$ are linearly dependent

$w_1 v_1, v_2 \dots v_n$ are linearly dependent
$bw + a_1v_1 + a_2v_2 + \cdots + a_nv_n \to w = \star v_1 + \cdots + \star v_n$
are linearly independent hence generates V

□

Theorem 4.3 (C)**.** let V be a vspace over K and let $\dim V = n$
let $v_1, v_2 \dots v_n$ be any set of linearly dependent vectors $\in V$ then $v_1, v_2 \dots v_n$ is a basis

Proof. By theorem A, we know that $\{v_1, v_2 \dots v_n\}$ is a maximal set of linearly dependent vectors
Then by theorem B $\{v_1, v_2 \dots v_n\}$ is a basis □

Note: #maximal set $= \dim V$

Corollary K. let W be a subspace of V, if $\dim w = \dim V$ then $V = W$
i.e. Any proper subspace of w has $\dim W < \dim V$

Proof. suppose that $\dim W = \dim V = n$
then $\exists w_1, w_2 \dots w_n$ such that it is a basis of W
$w_1, w_2 \dots w_n$ is also a basis of v so $W = V$

□

Corollary L. suppose that $\dim V = n$ let $v_1 \dots c_r, r < n$ be linearly independent , then we can find vectors $v_{r+1} \dots v_n$ such that $v_1 \dots v_r, v_{r+1} \dots v_n$ forms a basis of V

Proof. $\{v_1 \dots v_r\}$ is NOT a maximal set of linearly independent vectors then $\exists v_{r+1}$ such that v_1, v_r, v_{r+1} are linearly independent if $r + 1 = n$ then we are done
otherwise we can find vectors $v_{r+1} \dots v_n$ such that $v_1, v_2 \dots v_n$ are linearly independent □

Theorem 4.4 (D)**.** let V be a vspace over K such that $\dim V = n$ and W is a proper subspace of V
Then W has a basis and $\dim w < n$

Proof. if W=0, then we are done
Otherwise suppose that $W \neq 0$
There exists a $w_1 \in W \neq 0$
tbc..... □

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6 9/1/23 - Fri

recall: Let V be a finite dimension vector space over K
$\dim V = \#\mathcal{B}, \mathcal{B} = \{v_1, v_2 \dots v_n\}$ is a basis

Theorem 6.1. Any max set of linearly independent vectors is a basis

Theorem 6.2. if $\dim V = n$ and $v_1, v_2 \dots v_n \in V$ are linearly independent then $v_1, v_2 \dots v_n$ is a basis

Corollary i. $\dim v = n, v_1, v_2 \dots v_r$and $v < n$ are linearly independent , then $\exists v_{r+1} \dots v_r$ such that $v_1 \dots v_r, v_{r+1}, \dots, v_n$ is a basis

Theorem 6.3. $\dim V = n$ and let W be any proper subspace of V, then W has a basis and $\dim W < n$

Proof. suppose W has no max set of linearly independent vectors then $\exists$ vectors $v_1, v_2, v_3 \dots$ such that

$$\{v_1\} \subset \{v_1, v_2\} \subset \{v_1, v_2, v_3\}$$

are linearly independent but this contradicts $\dim V = n$ ↯Thus W has a max set $\{w_1, w_2, \dots w_r\}$ of linearly independent vectors $r \leq n$ which is a basis since $w \subsetneq v$ $w_1, w_2, \dots w_r$ does not generate V
Hence $\{w_1, w_2, \dots w_r\}$ is not a basis of V
in particular $r < n$ ☐

6.1 sums & direct sums

Let V be a vector space over K, Let W,U be subspaces of V
recall $w + u = \{w + u | w \in W, u \in U\}$

Definition (direct sum)**.** *let W,U be subspaces of V, we say V is a direct sum of W and U if*

1. *V=W+U*
2. *$\forall v \in V$ can be written as a sum of w=w+u in a **unique** way*

we denote this $V = W \oplus U$

Theorem 6.4. let let W,U be subspaces of V, if $V = W + U$ and $W \cap U = 0$ then $V = W \oplus U$

Proof. $V = u_1 + w_1 = u_2 + w_2 \to w_1 - w_2 = u_1 - u_2 \land w \cap u = 0 \to w_1 = w_2, u_1 = u_2$
This is a uniqueness proof

☐

Theorem 6.5. let V be a vector space, for any subspace $W \subseteq V$ there exists a **Compliment U** of W such that $V = W \oplus U$

Proof. By previous theorem $\exists$ a basis $\{w_1, w_2, \dots w_r\}$of W which can be extended to a basis $\{w_1, w_2, \dots w_r, w_{r+1} \dots w_n\}$of V such that U=span$\{w_{r+1} \dots w_n\}$Then, $V = W \oplus U$

☐

Note: The author omitted a step that needed to prove that $U \cap W = 0$because the instructor's handwriting is unreadable ☹

Theorem 6.6 (Dimensions of Direct sum v spaces)**.** If $V = W \oplus U$ then $\dim V = \dim U + \dim W$

Proof. Choose a basis$\{u_1, u_2 \dots u_s\}$ of U and a basis $\{w_1, w_2, \dots w_t\}$ of W. Then $\{u_1, u_2 \dots, u_s, w_1, w_2 \dots, w_t\}$ forms a basis for V □

Remark. *Given subspaces* $w_1, w_2, w_k \subseteq V$
$w_1, w_2 + \dots w_k = \{w_1 + w_2 + \dots + w_k | w_i \in w, 1 \leq i \leq k\}$ *is a subspace of* V

Definition. *We say that* V *is a direct sum of* $w_1, \dots w_k$*. If* $\forall v \in V$ *The summation* $V = w_1 + \dots + w_k$ *is unique*
We write $V = w_1 \, |w\rangle\langle w|_2 \, |w\rangle\langle w|_3 \, |\dots\rangle\langle\dots| \, |w\rangle\langle w|_{ki} \, |w_1 \in w_i$

e.g.

$$\mathbb{R}^3 = \underbrace{l_x}_{\mathbb{R}_{e_1}} |$$

$l_{y \mathbb{R}_{e_2}}$ |
$l_{z \mathbb{R}_{e_3}}$

Theorem 6.7. $w_1 \dots w_k$ be subspaces of V if $V = w_1 + \dots w_k$ and $w_i \cap (\sum_{j \neq i} w_j)$ then $V = w_1 \, |\dots\rangle\langle\dots| \, |w\rangle\langle w|_k$

Proof. k=3
$V = w_1 + w_2 + w_3 = w_1' + w_2' + w_3'$
$\rightarrow w_1 - w_1' = w_2 - w_2' = w_3 - w_3'$ □

Lemma *. $w_1 \cap (w_2 + w_3) = 0$
then $v = w_1 = w_2 + w_3$

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recall: direct sum $W, U, \subseteq V$ $V = W \, |U\rangle\langle U|$ if
$\exists$ a unique $w \in W u \in U$ s.t.
V=w+u and $w \cap u = 0$
Given 2 vectors w, u
Let $w \times u$ be a direct product
$w \times u = \{(w, u) | w \in W, u \in U\}$
$W \times U$ can be endowed w/ a vector space structures
Additives $(w, u) + (w' + u') = (w + w', u + u')$
scalar multiplication a(w,u)=(aw,au)
$W \times U$ is a vector scpace over K
$ex : \dim W \times U$ $\{w_1, w_2 \dots w_n\}$ be a basis of W

$\{u_1, u_2 \dots u_m\}$ be a basis of U

$$\{(w_1, 0) \dots (w_n, 0)(0, u_1 \dots (0, u_m))\}$$

is a basis of $W \times U$
<u>in fact</u> W can be identified w/ $\{(w, 0)|w \in W\} \subseteq W \times U$
U can be identified w/ $\{(0, u)|u \in U\} \subseteq W \times U$
under such identification $W \times U = W\,|U\rangle\!\langle U|$
$W \subseteq W \times U$
$W \to (W, 0)$

Remark. *Given $v_1, v_2 \dots v_n$ we can define their produce $V_1 \times V_2 \times V_3 \dots \times V_n$ to be a vector space*

7.1 Matricies

we call matricies

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1_n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

This is a $m \times n$ matrix over a field K$(\mathbb{R}, \mathbb{C}, \mathbb{Q} \dots)$
Where a row vectors are
$a_1 = (a_{11}, a_{12} \dots a_{1m})$
$\dots$
$a_m = a_{m1}, a_{m2}, \dots, a_{mn}$
Where column vectors are
$a^1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{bmatrix}$
$\dots$
$a^n = \begin{bmatrix} a_{1_n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$

Definition (square matrix)**.** *if m=n then A is a square matrix*

Definition (zero matrix)**.** $A_{ij} = 0 \forall i, j$ *Then a is a zero matrix*

Definition (diagonal matrix)**.** *the square matrix A is called diagonal if*

$$A = \begin{bmatrix} x & & & \\ & \dots & & \\ & & \dots & \\ & & & x \end{bmatrix}$$

Definition (Upper triangular matrix)**.** *The square matrix a A is upper triangular iff*

$$A = \begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix}$$

Definition (Lower triangular matrix)**.** *The square matrix A is lower triangular iff*

$$A = \begin{bmatrix} x & & & \\ x & x & & \\ x & x & x & \\ x & x & x & x \end{bmatrix}$$

A $m \times n$ matrix is transposed when

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \sim A^t = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

we denote this $a^t \equiv^t a \equiv A^T \dots$
Let $\mathcal{M}_{m\times n}(K) = \{m \times n$ matricies over K$\}$
Addition of scalar muplication, on $\mathcal{M}_{m\times n}(K)$
$A + B(A = a_{ij}, B = b_{ij})$
$= a_{ij} + b_{ij}\ \forall c \in K, c \cdot A = (ca_{ij})$
Zero matrix $\mathcal{O} \in \mathcal{M}_{m\times n}(K)$
A $m \times n$ matrix A is called symmatri iff $A = A^t$

7.2 sys. of Linear Eqns

Given $a_{11}x_1 + \dots + a_{1n}x_n = b_1$
$\dots$
$a_{m1}x_1 + \dots + a_{mn}x_n = b_1$

if $\forall b_i = 0$ we call this system homogeneous
which can be written as

$$(\star)x_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \dots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

To find a solution of such eqn is equiv to express $\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$ as a linear combination of

$$A' = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix} \dots \begin{bmatrix} a_{1m} \\ a_{2m} \\ \dots \\ a_{mn} \end{bmatrix}$$

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Theorem 8.1 (Linear independence of solns)**.** In a linear system $\star$, assume that m=n, and $A^1, A^2, \dots A^n$ are linearly independent then, $\star$ has a unique solution.

Proof. Let $A^1, A^2, \dots A^n \in \mathbb{K}^n$ and they are linearly independent . Thus they form a basis $\rightarrow B = c_1A^1 + \dots c_nA^n$ for unique numbers
i.e. $(c_1, c_2, \dots, c_n)$ is a unique solution. □

8.1 Matrix multiplications

Let

$$A = a_1, a_2 \dots a_n \in \mathbb{K}^n$$

$$B = b_1, b_2 \dots b_n \in \mathbb{K}^n$$

recall their dot product $A \cdot B = a_1b_1 + a_2b_2 + \dots + a_nb_n$
They have some nice properties

1. $A \cdot B = B \cdot A$
2. $c \in \mathbb{K}, (cA)B = c(AB) = A(cB)$

Definition (Matrix multiplication)**.** *Given 2 matrices*
$A = a_{ij} m \times n$ *matrix*
$B = b_{ij} n \times k$ *matrix*
We define a matrix multiplication AB *as*

$$AB = \begin{bmatrix} A_1B^1 & A_1B^2 & \dots & A_1B^k \\ A_2B^1 & A_2B^2 & \dots & A_2B^k \\ \dots & \dots & \dots & \dots \\ A_mB^1 & A_mB^2 & \dots & A_mB^k \end{bmatrix}$$

e.g. $a = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix} b = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$
$ab = \begin{bmatrix} 15 & 15 \\ 4 & 12 \end{bmatrix}$

In general let $A : m \times n, B : n \times k AB - m \times k$
Let A $= a_{ij}$ be a $m \times n$ matrix

let $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$ which is a column vector $n \times 1$ matrix

Their product AB is a $\begin{bmatrix} A^1 B \\ A^2 B \\ \dots \\ A^n B \end{bmatrix}$ col vector

A system of linear equation $\star$ can be written as

$$Ax = B$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Where as $c : (c_1, c_2, \dots, c_m)$ is a row vector
cA=$(cA^1, cA^2, \dots, cA^n)$ has a $\# = n$
can be alternatively written as the prduct of

$$(c_1, c_2, \dots, c_m) \cdot \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots \end{bmatrix} \equiv \begin{bmatrix} A^1 & A^2 & \dots & A^m \end{bmatrix}$$

Theorem 8.2. A $m \times n$ matrix B $n \times k$ matrix A(B+C)=AB+AC
Notation A matrix $\rightsquigarrow A_{ij} = ij$ entries of A
if $A = a_{ij}$ then $(AB)_{ij} = A_i B_j$

Proof. $(A(B + C))_{ij}$
$= A_i(B + C)^j$
$= A_i(B^j + C^j)$
$= A_i B^j + A_i C^j$
$= (AB)_{ij} + (AC)_{ij} = (AB + AC)_{ij}$ $\square$

Theorem 8.3 (commutativity of scalar multiplication)**.** let $c \in \mathbb{K}$
$(cA)B = A(cB)$
Assume $A.B = m \times n$ matrix and let $C = n \times k$ matrix then $(A + B)C = AC + BC$

Theorem 8.4 (commutativity of matrix multiplication)**.** let ABC be mutually manipulable matrices then $(AB)C = A(BC)$

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Recall $A = (a_{ij})$ be a $m \times n$ matrix $B = b_{ij}$ be a $n \times k$ matrix
$AB = (c_{ij})$ of a matrix

$(AB)_{ij} = A_i B^j = (a_1, a_2, \dots, a_n) \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{nj} \end{bmatrix}$

> **Theorem 9.1.** A: $m \times n$ matrix B:$n \times k$ matrix C:$k \times l$ matrix
> $(AB)C = A(BC)$ (assoc.)

Proof. A=a_{ij} B=$b_{ij} C = c_{ij}$
$AB_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}$ $((AB)C)_{ij} = \sum_{t-1}^{k} (AB)_{it} C_{tj} = \sum_{t=1}^{k} (\sum_{s=1}^{n} a_{is} b_{st}) c_{tj} = \sum_{t=1}^{k} \sum_{s=1}^{n} a_{is} b_{st} c_{tj}$
similarly
$(A(BC))_{ij} = \sum_{s=1}^{n} a_{is} (BC)_{sj} = \sum_{s=1}^{n} a_{is} (\sum_{t=1}^{k} b_{st} c_{kj})$

=$\sum_{s=1}^{n} \sum_{t=1}^{k} a_{is} b_{st} c_{tj}$ The summation can be switched ☐

let $A = ija$ be a $m \times n$ matrix
$ijA^t = a_{ji}$ then $A^t = n \times m$ matrix
$B = n \times k$ matrix
$B^t = k \times n$ matrix

> **Theorem 9.2.** $(AB)^t = B^t A^t$

Proof. $ij(AB)^t = (AB)_{ji} = \sum_{s=1}^{n} a_{js} b_{si}$
$ij(B^t A^t) = \sum_{s=1}^{n} (B^t)_{is} (A^t)_{sj} = \sum_{s=1}^{n} b_{si} a_{js}$

☐

9.1 Linear maps

Definition (Linear maps)**.** *Let v,w, be vector spaces over K, a map $F : V \to W$ is called linear if*

1. $F(V + U) = F(U) + F(V) \forall v, u \in V$
2. $F(av) = aF(v), \forall a \in K, v \in V \equiv F(au + bv = aF(u) + bF(v))$

Remark. $F(0) = 0$

e.g. let $P : K^3 \to K^2$
1.$(x, y, z) \mapsto (x, y)$
2. $\mathbb{C}^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$
$f \mapsto \frac{df}{dx}$
3. $A = (a, b, c) \in K^3$
$F_A : K^3 \to K$ given by $F_A(x, y, z) = ax + by + cz = A \cdot (x, y, z)$

hence F_A is linear
let $A = ija$ be a $m \times n$ matrix
we define a map $F_A : \underbrace{K^n \to K^m \text{which is linear}}_{x \mapsto AX = \begin{bmatrix} A_1 \cdot x \\ a_2 \cdot x \\ \dots \\ A_m \cdot x \end{bmatrix}}$

Let V be a vector space over K
Identity map $V \to V$ 0 map $V \to V, V \to 0$ they are all linear
Given a basis $\mathcal{B} : \{v_1, v_2 \dots v_n\}$ of V
$F_{\mathcal{B}} : V \to K^n$
$v \mapsto (x_1, x_2 \dots x_n)$ where $v = x_1 v_1 + \cdots + x_n v_n$
and we know $F_{\mathcal{B}}$ is linear
$v = \sum x_i v_i, w = \sum y_i v_i, v + w \sum (x_1 + y_i) v_i$
hence $F_{\mathcal{B}}(w) = F_{\mathcal{B}}(V)$
see pic Given V, W such that they are vector spaces over K
$L(V, W) = \{\text{Linear maps from V to W}\}$
Then $L(V, W)$ is a vector space over K
So we have fcns F,G, $(F + G)(v) = F(v) + G(v)$
$(AF)(v) = aF(v)$
$0 \in L(v, w)$
$0 : v \to w$
$v \mapsto 0$

Theorem 9.3. v,w, as arb. and let $\mathcal{B} = \{v_1, v_2 \dots v_n\}$ be a basis of V, and let $\{w_1, w_2 \dots w_n\}$ be a arb set of vectors in W.
There exists a unique linear map $F : V \to W$ such that $F(v_1) = w_1, f(v_2) = w_2 \dots f(v_n) = w_n$

Proof. F(v)=$a_1 w_1 + \cdots + a_n w_n$
where $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ Then one way check F is linear

□

see pic

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Recall: A linear map $F : V \to W$, $V, W/K$
$F(au + bv) = aF(u) + bF(v) a, b \in K, u, v \in V$
$L(v, w) = \{\text{Linear maps from} V \to W\}$
vector spaces over K

10.1 Kernel and image of linear maps

Let F $V \to W$ Linear

Definition (kernel)**.** $\ker F = \{v \in V | f(v) = 0\} \subseteq V$

Lemma . $\ker F$ is a subspace of V

Proof. Given u,v $\in \ker F$, $\forall a, b, \in K$ $F(au+bv) = aF(u)+bF(w) = 0 \to au+bv \in \ker F$ $\square$

Lemma . $F : V \to W$ is injective if and only if $\ker F = 0$

Proof. $\to$ suppose F is injective, then the only element that maps to $0 is 0$

$\leftarrow$
$\forall u, v \in V$ suppose $F(u) = F(v)$ then by injectivity of $F(u - v) = 0$
since $\ker F = 0, u - v = 0 \to u = v$ $\square$

e.g. $A = (2, 1, -1) \in K^3$
$F_A : K^3 \to K$
$(x, y, z) \mapsto (2x + y - z)$
$\ker F_A = \{(x, y, z) \in K^3 | 2x + y - z = 0\}$
similiarly $A = ija : m \times n$ matrix
$F_A : K^m \to K^n$
X:col vector
$X \mapsto AX$
$\ker F_A = \{X \in K^n | A \cdot X = 0\}$
$= \{\text{solution of } \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_n = 0 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{bmatrix}\}$ call it $\star$ $\ker F_A = 0 \iff \star$ has trivial solution $\iff$
F_A is injective $\iff$ col vectors are linearly independent

Theorem 10.1. $F : V \to W$ linear s.t. $\ker F = 0$
if $v_1, v_2 \ldots v_n \in V$ are linearly independent then $F(v_1), f(v_2) \ldots, f(v_n)$ are also linearly independent

Proof. by contradiction suppose we have $a_1 f(v_1) + \cdots + a_n f(v_n) = 0$
by linearility we have $f(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = 0$ then $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$
hence $a_1 = \cdots = a_n = 0$ $\square$

10.2 image

$F : V \to W$ linear
$ImF = F(v) = \{f(v) | v \in V\}$

Lemma I. m F is a subspace of W

Proof. $\forall F(v), F(u) \in imF, \forall a, b \in K$
$aF(u) + bF(v) = F(au + bv) \in imF$

$\square$

Given $v_1, v_2 \dots v_n \in V$
$F : K^n \to V$
$(a_1, a_2 \dots a_n) \mapsto a_1v_1 + a_2v_2 + \dots + a_nv_n$
imF= {linear combinations of $v_1, v_2 \dots v_n$}
$= span\{v_1, v_2 \dots v_n\}$
$F_A : k^n \to k^m$

$$X = \begin{bmatrix} x_1 \\ x_1 \\ \dots \\ x_n \end{bmatrix} \mapsto A \cdot X \text{ note that } A \cdot X = \star\star \text{ or a lin comb of the col vectors}$$

im$F_A = span$\{column of vectors\}
Given $A = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$
$K^3 \to k$
$(x, y, z) \mapsto 2x + y - z$
$\dim \ker F_A = 2$
Im $F_A = k$ dim Im $F_A = 1$

Theorem 10.2. $F : v \to w$ linear
$\dim \ker F$+dim Im F=dim V

Proof. Choose a basis $\{w_1, w_2 \dots w_n\}$ of Im F also a basis $v_1, v_2 \dots v_n$ of $\ker F$ Choose There exists $u_1, u_2 \dots u_m \in V$
s.t. $F(u_1) = w_1 \dots f(um) = w_m$
we claim that $\{u_1, u_2, \dots u_m, v_1 \dots v_n\}$ is a basis of V

1. it generates V
   $\forall v \in V, f(v) \in$im F
   $F(v) = a_1w_1 + \dots + a_mw_n$
   $= a_1f(u_1) + \dots + a_mF(u_m)$
   $= F(a_1u_1 + \dots a_mu_m) \to v - \sum a_iu_i \in \ker F \to v - \sum a_iu_i = \sum b_jv_j$
2. $\{v_1, v_2 \dots v_n, u_1 \dots, u_m\}$ is linearly independent
   suppose $\sum a_iv_i + \sum b_ju_j = 0$

we apply F hence $F(b_iu_j) = 0$
$= \sum b_jF(u_j)$
$= \sum b_jw_j = 0$
by linearly independent of$\{w_j\}, b_j = 0 \forall j \to \sum a_iv_i = 0, a_i = 0 \forall i$

$\square$

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$A = a_{ij} : m \times n$ matrix $F_A : K^n \to k^m$
$X \mapsto AX$
$\ker F_A = \{$solns of AX=0$\}$
Im F=span$\{$sol vectors $\} \to n = \dim\{$sols of AX=0$\} + \dim span\{$col of A$\}$
Let $n = m$ AX=0 has only trivial sol $\iff$ cols of A is a basis of K^n

Theorem 11.1. $F : V \to W$: Linear Map
assume $\dim V = \dim W$ if $\ker F = 0 or$ ImF=W then F is a bijection.

Proof. Let $\underbrace{\ker F = 0}_{\text{F is injective}}$ by thm $\iff$ $\dim ImF = \underbrace{\dim V = \dim W}_{\text{By assumptions}}$
$\iff ImF = W$
i.e. F is injective $\iff$ F is surjective

□

11.1 Composition of linear maps

Theorem 11.2. Given 2 linear maps :F$U \to V$ G:$V \to W$ Their composition $G \circ F : U \to W$ is linear

Proof. $\forall u_1, u_2 \in U, a_1 a_2 \in K$
$G \circ F(a_1u_1 + a_2u_2)$
$=G(F(a_1u_1 + a_2u_2))$
$=G(a_1F(u_1) + a_2F(u_2))$
$=a_1G(F(u_1)) + a_2G(F(u_2))$
$=a_1(G \circ F)(u_1) + a_2(G \circ F)(u_2)$
hence $G \circ F$ is linear

□

Theorem 11.3. $F : V \to W$ linear and bijective then its inver se $G : W \to V$ is also linear

Proof. $\forall w_1 w_2 \in W a_1, a_2 \in K$
$G(a_1w_1a_2w_2)$ want to proof $= a_1G(w_1) + a_2G(w_2)$
we apply $F, F(G(a_1w_1 + a_2w_2)) \equiv a_1w_1 + a_2w_2$
where as $a_1G(w_1)+a_2G(w_2)$ apply F$F(a_1G(w_1)+a_2G(w_2)) = a_1F(G(w_1))+a_2F(G(w_2)) = a_1w_1 + a_2w_2$
since F is bijective we are done

□

e.g. $K^2 \xrightarrow{F} K^2$
$(x, y) \mapsto (2x - y, x + y)$

is that a bijection?

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x-y \\ x+y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are linearly independent $\to F$ is bijective

2. Only trivial solution = $\ker F = 0$ hence F is bijective

Definition (isomorphism)**.** *A linear map F is a called isomorphism or inveritble if F is also a bijection*
ie F: invertible $\to F^{-1}$ is linear

$A \in \mathcal{M}_{m\times n}(k) \rightsquigarrow F_A \in \mathcal{L}(k^n, k^m)$
$\mathcal{M}_{m\times n}(k) \to \mathcal{L}(k^n, k^m)$
linear

Theorem 11.4. $\mathcal{M}_{m\times n}(k) \to L(k^n, k^m)$
is injective i.e. $F_A = F_B \to A = B \forall A, B, \in \mathcal{M}_{m\times n}(k)$

Proof. since F is linear it is suffcient to show that $F_A = 0 \to A = 0$

$$F_A(X) = AX = \begin{bmatrix} A_1 \cdot X \\ A_2 \cdot X \\ \dots \\ A_n X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \forall X \in K^n$$

$x \in k^n$ for each $i = 1 \dots m \to A_i X = 0, \forall X \in k^n \to A_i = 0$

□

Lemma . If $a_1 x_1 + \dots a_n x_n = 0$
$\forall x_i \in K$
then $a_i = \dots = a_n = 0$

Theorem 11.5. $F : \mathcal{M}_{m\times n}(k) \to L(k^n, k^m)$
$A \mapsto F_A$ is surjective
ie for any linear map $\mathcal{Q} : k^n \to k^m$ Q=F_A for some A.

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$F : V \to W, G : W \to U$ be linear maps
Composition $G \circ F : V \to U$

1. $G \circ F$ is linear

2. $G \circ (a_1F_1 + a_2F_2) = a_1G \circ F_1 + a_2G \circ F_2$

$(b_1G_1 + b_2G_2) \circ F = b_1G_1 \circ F + b_2G_2 \circ F$
$\mathcal{M}_{m\times n}(k) \xrightarrow{\varphi} \mathcal{L}(k^n, k^m)$
$A \mapsto F_A$
e.g. $A = \begin{bmatrix} 2 & 1 \\ -1 & 5 \\ 1 & 0 \end{bmatrix} \rightsquigarrow F_A : k^2 \to k^3$
$F_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$
$F_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$
In general
Given $A = (a_{ij}) : m \times n$ matrix
$F_A(e_i) =$ith column of A
e.g. $e_i = \begin{bmatrix} 0 \\ \dots \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Theorem 12.1. $\varphi : \mathcal{M}_{m\times n}(k) \to \mathcal{L}(k^n, k^m)$ is injective i.e. $F_A = F_B \forall A, B \in \mathcal{M}_{m\times n}(k)$

Theorem 12.2. φ is onto i.e. $\forall$ linear maps $F : k^n \to k^m$ there exists a $m \times n$ matrix A such that $F = F_A$

Lemma . Given a linear map $F : k^n \to k$
$F = \underbrace{A \cdot X}_{\text{dot product}}$ where $A = F(e_1), F(e_2) \dots, F(e_n) \in k^n$

e.g. $F : k^n \to k$
$F(x_1 \dots x_n) = a_1v_1 + a_2v_2 + \cdots + a_nv_n$
$F(e_1) = a_1, F(e_2) = a_2 \dots$

Proof. We can write $X = x_1e_1 + \cdots + x_ne_n$
$F(x) = x_1F(e_1) + \cdots + x_nF(e_n)$
$=F(2_1), \dots, F(e_n) \cdot (x_1 \dots x_n)$

$\square$

proof of theorem

Proof. Let $F : k^n \to k^m$ be a linear map
Let $P : k^m \to k$ be the ith projection
$p_i \cdot \begin{bmatrix} y_1 \\ \dots \\ y_n \end{bmatrix} = y_i$
Then $p_i \circ F : k^n \to k$ is linear, by the lemma $p_i \circ F(X) = A_i \cdot X_i$ for some $A_i \in k^n$
In fact $F(x) = \begin{bmatrix} A_1 \cdot X \\ A_2 \cdot X \\ A_3 \cdot X \end{bmatrix} = AX, A + \begin{bmatrix} A_1 \\ A_2 \\ \dots \\ A_m \end{bmatrix}$ □

$a_1, \dots, a_n \in k^n$
$A = \begin{bmatrix} A_1 \\ A_2 \\ A_m \end{bmatrix} : m \times n$ matrix
$A_1 \begin{bmatrix} a_z 11 & a_{12} & \dots & a_{1n} \end{bmatrix} \in k^n$
$A_2 \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \in k^n \end{bmatrix}$
$\dots$

Theorem 12.3 (A+B)**.** let$\Phi : \mathcal{M}_{m \times n}(k) \to \mathcal{L}(k^n.k^m)$
is an isomorphism of vector spaces over k
for any $F \in \mathcal{L}(k^n.k^m)$
there exists a unique $m \times n$ matrix A
such that $F = F_A$ we call A to be associated matrix of F

e.g. $F : k^3 \to k^2$
$(x, y, z) \mapsto (x + y, z)$
find the associated matrix of F

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : 2 \times 3$$

$F(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}) F(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
In general $F : k^n \to k^m$
$F = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}$ e.g. $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$
a rotation by θ counter-clockly
what is the matrix A?

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

e.g. $R(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Lemma . Let $Am \times n$ matrix and $Bn \times l$ matrix
$F_A : k^n \to k^m F_B : k^l \to k^n$
then $F_A \circ F_B = F_{AB}$

Proof. for every $x \in k^n F_{AB} = (AB)X = A(BX) = (F_A \circ F_B)(x)$ □

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warning: Midterm 9/27 chap I-IV
recall $\mathcal{M}_{m\times n}(k) \xrightarrow{\sim} L(k^n, k^m)$
hence $A \mapsto^{\varphi} F_A$
$F_{AB} = F_A \circ F_B$
let n=m, F_A is invertible iff A is invertible

Proof. $\to$
$F_A : K^n \to K^n$ invertible
there exists $G : k^n \to k^n$ such that $F_A \circ G = Id, G \circ F_A = Id$ $G + F_B$ fora unique matrix B
then $F_A \circ F_B = F_{AB} = Id = F_I$
$F_B \circ F_A = F_{BA} = Id = F_I$

□

Theorem 13.1. $A : n \times n$ matrix and let A^i be the ith col of A, then A is invertible iff $A^1 \dots A^n$ are linearly independent

Proof. consider the associated linear map

$$F_A : K^n \to K^n X \mapsto AX$$

$$F_A(e_i) = A^i$$

As explained previously A is invertible iff F_A is invertible. F_A is invertible then $A^1 \dots A^n$ are linearly independent
suppose we have $c_1 A^1 + \cdots + c_n A^1 = 0$
then we know that $c_1 F_A(e_1) + \cdots + c_n F_A(e_n) = 0 \iff F_A(c_1 e_1 + \cdots + c_n e_n) = 0 \iff c_1 e_1 + \cdots +_n e_n = 0 \iff c_1 = \cdots = c_n = 0$
$\leftarrow$ suppose $A^1 \dots A^n$ are linearly independent then they for a basis of k^n. There exists a linear map $G : k^n \to k^n$ s.t.

$$G(A^1) = e_1 \dots G(A^n) = e_n$$

clearly $F_A \circ G = I$ $G \circ F_A = I$

$\square$

e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible since A^1, A^2, A^3 are linearly independent

e.g.$F(x,y,z) = (x-2y, y-z, 2z), F_A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

This matrix is invertible because $\dim \mathbb{R}^3 = \dim \mathbb{R}^3 \to$ demension is the same also $\ker F = \mathcal{O} \to$ this is injective hence this is bijective and A is invertible

13.1 Bases, matrices and linear maps

V is a vector space over K and let $\mathcal{B}$ be a basis $\{v_1, v_2 \dots v_n\}$

$k^n \xrightarrow{\varphi} V$ is an isomorphism iff $(a_1, a_2 \dots a_n) \mapsto a_1v_1 + a_2v_2 + \dots + a_nv_n$ that $\{v_1, v_2 \dots v_n\}$ is a basis

Proof. φ is injective $\iff$ $v_1, v_2 \dots v_n$ are linearly independent

φ is surjective $\iff$ $v_1, v_2 \dots v_n$ generates v. $\square$

Given a linear map $F : V \to W$

Let $\mathcal{B}$ be a basis of V, $\mathcal{B}'$ be a basis of W

Let $\dim V = n$ $\dim W = m$

$$V \xrightarrow{F} W$$

$k^n \xrightarrow{F^{\mathcal{B}}_{\mathcal{B}'}} k^m$ Let $M^{\mathcal{B}}_{\mathcal{B}'}$ be a matrix associated to $F^{\mathcal{B}}_{\mathcal{B}'}$

Definition. *$M^{\mathcal{B}}_{\mathcal{B}'}$ (F)is the matrix associated to F with respect to $\mathcal{B}, \mathcal{B}'$*

Exercise . $V \subseteq k^3, V\{(x,y,z)|x+y+z=0\}$ $F : k^3 \to V$

$F(x,y,z) = (x-y, y-z, z-x)$ we have standard basis $e_1 = (1,0,0), e_2(0,1,0), e_3(0,0,1)$

$v_1 = (1,-1,0), v_2 = (0,-1,1)$ clearly forms a basis of V. $F(1,0,0) = (1,0,-1) = v_1 - v_2$

$F(0,1,0) = (-1,1,0) = -v_1$

$F(0,0,1) = (0,-1,1) = v_2$

we claim that $M^{\mathcal{B}}_{\mathcal{B}'}(F) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

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V: a vector space over K

$\mathcal{B} = \{v_1, v_2 \dots v_n\}$

$v \in V, v = x_1 + v_2 + \dots + x_nv_n$

$X_{\mathcal{B}} = (x_1, \ldots, x_n$ coordinates of v w,r, to $\mathcal{B}$
e.g. $k^2\{v_1(1,1), v_2(1,-1)\}$a basis $=\mathcal{B}$ $v(1,0) = \frac{1}{2}v_2 + \frac{1}{2}v_2$ $X_{\mathcal{B}} = (\frac{1}{2}, \frac{1}{2})$
Given a linear map$F : V \to W$
$\mathcal{B} = \{v_1, v_2 \ldots v_n\}$ a basis of V
$\mathcal{B}' = \{w_1, w_2 \ldots w_n\}$ a basis of W
then $\exists \mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}$ $m \times n$ marix associated to F and $\mathcal{B}, \mathcal{B}'$

$$F : \underbrace{V}_{k^n} \quad \underbrace{\to}_{\xrightarrow{\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F)}} \quad \underbrace{W}_{k^m}$$

Theorem 14.1. $\forall v \in V$

$$X_{\mathcal{B}'}(F(v)) = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) X_{\mathcal{B}}(v)$$

Proof. By linearliry it suffies to check for $v_1, v_2 \ldots v_n$ we write

$$F(v_1) = a_{11}w_1 + \ldots a_{m1}w_m$$

$$\ldots$$

$$F(v_n) = a_{m1}w_1 + \ldots a_{mn}w_m$$

$$X_{\mathcal{B}'}F(v_1) = (a_{11}, a_{21} \ldots a_{m1})$$

$$X_{\mathcal{B}'}F(v_i) = (a_{1i}, a_{2i} \ldots a_{mi}), 1 \leq i \leq n$$

Hence

$$X_{\mathcal{B}}(v_i) = (0, 0, \underbrace{1}_{\text{i-th}}, 0, \ldots, 0)$$

In fact $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

Thus

$$X_{\mathcal{B}}F(v) = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) X_{\mathcal{B}}(v)$$

□

e.g. In k^2 we have an identity map $k^2 \to k^2$ $\mathcal{B} = \{e_1(1,0), e_2(0,1)\}$
$\mathcal{B}' = \{(1,1)(0,1)\}$ determine $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(Id)$
$(1,0) = (1,1) - (0,1)$
$(0,1) = (0,1)$
then we know that $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}id = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ transition matrix from $\mathcal{B} \to \mathcal{B}'$
In general let $\mathcal{B}.\mathcal{B}'$ to be 2 bases of V $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(Id)$ is called transition matrix from $\mathcal{B} \to \mathcal{B}'$ e.g.
$F : k^3 \to k^3$
$(x, y, z) \mapsto (z, x)$

$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ $\mathcal{B}' = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$ determine $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'} F$
$F(1,1,0) = (0,1) = 0(1,-1) + (0,1)$
$F(0,1,1) = (1,0) = 1(1,-1) - 1(0,-1)$
$F(0,0,1) = (1,0) = 1(1,-1) - 1(0,-1)$
Hence matrix $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

14.1 Properties

$F : G : V(\mathcal{B}) \to W(\mathcal{B}')$
$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(aF + bG) = a\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) + b\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(G)$

Proof. $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(aF + bG) = \forall v \in V X_{\mathcal{B}}(v)$ then $X_{\mathcal{B}'}(aF + bG)(v) = X_{\mathcal{B}'}(aF(v)) + vG(v)$
$= aX_{\mathcal{B}'}F(v)_b X_{\mathcal{B}'}G(v)$
see pic □

Theorem 14.2. Given linear maps

$$V \xrightarrow{F} W \xrightarrow{G} U$$

and let $\mathcal{B}$ be a basis of V
$\mathcal{B}'$ be a basis of W
$\mathcal{B}''$ be a basis of U
$\dim V = n, \dim \dim W = m \dim U = l$

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) \sim m \times n, \mathcal{M}^{\mathcal{B}'}_{\mathcal{B}''}(G) \sim l \times m, \mathcal{M}^{\mathcal{B}''}_{\mathcal{B}'''}(G \circ F) \sim l \times n$$

then
$$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(G)\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(G \circ F)$$

Proof. $\forall v \in V$ $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(G \circ F)X_{\mathcal{B}}(v)$
then for $G \circ F$
$=X_{\mathcal{B}''}(G \circ F)(v)$
$= X_{\mathcal{B}''}G(F(v)) = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}GX_{\mathcal{B}'}F(v)$ □

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Last class $F : V \to W$ be a linear map
$\mathcal{B} = \{v_1, v_2 \dots v_n\}$ a basis for V
$\mathcal{B}' = \{w_1, w_2 \dots w_n\}$ a basis for W
Then $F(v_1) = a_1 w_1 + a_{21} w_{21} + \cdots + a_{m1} w_m$

Theorem 15.1. $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

Proof. Let $W = Id$ $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(Id)$=transformation matrix from $\mathcal{B} \to \mathcal{B}'$ $V = W$ then
$\mathcal{B}' = \mathcal{B} = \{v_1, v_2 \dots v_n\}$
$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F)$

$$F(v_1) = a_{11}v_1 + \cdots + a_{n1}v_n$$

$$\vdots$$

$$F(v_n) = a_{1n}v_1 + \cdots + a_{nn}v_n$$

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

Which is the matrix associated F and $\mathcal{B}$ □

e.g. $F : k^2 \to k^2$
$F(x, y) = (x + y)(y - x)$
$\mathcal{B} = \{\binom{1}{1}\binom{1}{-1}\}$ compute $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}F$
since $F(\binom{1}{1} = \binom{2}{0} = \binom{1}{1} + \binom{1}{-1} F\binom{1}{-1} = \binom{0}{-2} = -\binom{1}{1} + \binom{1}{-1}$ Hence matrix $= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
recall let

$$V \xrightarrow{F} V' \xrightarrow{G} V''$$

Let $\mathcal{B} \to V, \mathcal{B}' \to V'\mathcal{B}'' \to V''$ Hence $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}} G \circ F = \mathcal{M}^{\mathcal{B}'}_{\mathcal{B}''}(G)\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}\ (F)$

Corollary . Let $V \xrightarrow{Id} V \xrightarrow{Id} V \mathcal{B} \to \mathcal{B}' \to \mathcal{B}$
Then $\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id)\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(Id) = I_n$
$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(id)\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id) = I_n$ ie $\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id) = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(Id)^{-1}$
Let $\mathcal{B}\mathcal{B}'$ be 2 basis

Theorem 15.2 (change of basis)**.** $\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(F) = N^{-1}\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F)N$, where N $=\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id)$

Proof. $\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id)\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(F)\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(Id) = \mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(F)\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(id) = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F)$ Thus $\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(F) = N^{-1}\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F)N$ □

In the prev example we want to verify
$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F) = N^{-1}\mathcal{M}^{\mathcal{B}_0}_{\mathcal{B}_0 0}(F)N$
$N = \mathcal{M}^{\mathcal{B}}_{\mathcal{B}_{st}}(Id) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, N^{-1} = \mathcal{M}^{\mathcal{B}_{st}}_{\mathcal{B}}(Id) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ We verify

$$\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} =^{?} \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$
$$= Id \cdot \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$
$$= \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

Definition (diagoniability of a matrix)**.** *A linear map $F : V \to V$ is called diagonizible if there exists a basis $\mathcal{B} = \{v_1, v_2 \dots v_n\}$ $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F)$ is a diagonal matrix*
i.e. $F(v_1) = c_1 v_1$
$F(v_2) = c_2 v_2$
$F(v_n) = c_n v_n$
for some $c_1 \dots c_n \in K$

e.g.
$F : k^2 \to k^2$ $F(x, y) = (y, x)$ in this case $F \sim \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ can we find a basis such that F can be diagonalized. $\binom{1}{1}\binom{1}{-1}$ $F\binom{1}{1} = \binom{1}{1}$, $F\binom{1}{-1} = -\binom{1}{-1}$
$\{\binom{1}{1}\binom{1}{-1}\}$ diagonizes F
Let $\mathcal{B}$ be stand basis and let $\mathcal{B}' = \{\binom{1}{1}\binom{1}{-1}\}$
then we can see that

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^{-1} \underbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}_{\text{standard basis}} \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$N = \mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id) = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$
e.g. $F : k^2 \to k^2$
we claim tht $F(x, y) = (0, x)$ is not diagable

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Theorem 16.1. $F : V \to V$ and let $\mathcal{B}, \mathcal{B}'$ be basis. we know that $\mathcal{M}^{\mathcal{B}'}_{\mathcal{B}'}(F)$ $N^{-1}\mathcal{M}^{\mathcal{B}}_{\mathcal{B}}(F)N$ where as $N = \mathcal{M}^{\mathcal{B}'}_{\mathcal{B}}(Id)$

e.g. we have $V = k^n, \mathcal{B} = \{\begin{bmatrix}\end{bmatrix}\} 10 \dots 0, \begin{bmatrix}0\\ 1\\ \dots\\ 0, \dots\end{bmatrix}$ $\mathcal{B}' = \{v_1 = \begin{bmatrix}g_{11}\\ \vdots\\ g_{n1}\end{bmatrix}\}, \dots, v_n = \begin{bmatrix}g_{1n}\\ \vdots\\ g_{nn}\end{bmatrix}$
$F : k^n \to k^n$
$F = F_A \iff F(X) = AX$

where as $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{mn} \end{bmatrix}$ $F(v_1) = b_{11}v_1 + b_{21}v_2 + \cdots + v_{n1}v_n$

$\vdots$

$F(v_n) = b_{1n}v_1 + b_{2n}v_2 + \cdots + v_{nn}v_n$ $B = b_{ij}$

where $\mathcal{M}_{\mathcal{B}}^{\mathcal{B}}(F) = A$ $\mathcal{M}_{\mathcal{B}'}^{\mathcal{B}'}(F) = B$ $N = \mathcal{M}_{\mathcal{B}}^{\mathcal{B}'}(Id) = \begin{bmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ g_{21} & g_{22} & \dots & g_{2n} \\ \dots & \dots & \dots & \dots \\ g_{n1} & g_{n2} & \dots & g_{nn} \end{bmatrix}$ $B = N^{-1}AN$

16.1 Scalar product and orthogonality

Let V be as vector space over K such as $\mathbb{R}, \mathbb{C}, \mathbb{Q}$

Definition. *A scalar product of a vector space V is a binary operation $<, >: V \times V \to K$ such that*

1. $< av_1 + bv_2, w >= a < v, w > +b < v_2, w >$

$v, w{=}w, v$

the scalar product$<, >$ is k degenerate if and only if TFAE: $< v, w >= 0$ for any $w \in V \to v = \mathcal{O}$ $\forall v \neq \mathcal{O} \in V \exists w \in V s.t. < v, w >\neq \mathcal{O}$

e.g. 1. $V = k^n x = (x_1, \dots, x_n), y = (y_1 \dots, y_n)$
$X \cdot Y- < X < Y >{=}x_1Y_1 + x_2y_2 + x_ny_n$ $\forall x = (X_1, X_2, X_n \neq 0)$
$x \cdot e_i = s_i \neq 0 if X_i \neq -0$ then $<>$ is nondegenerative 2. $v = k^2 < x, y >= x_1y_2 x = (0, 1) < x, y >= 0 \forall Y$ degenerative!
3. Let $\mathcal{S}$ to be any finite set
$K(\mathcal{S}) = \{$functions on $\mathcal{S}\}$
$< f, g >= \sum_{s \in \mathcal{S}} f(s)g(s)$ is a scalar product $< f, 1 >= \sum_{t \in \mathcal{S}} f(k)1_s(t)$
$f(s) \neq 0 \to <, >$ is not degenerative

Definition. *let V be a vector spaec with a scalar product we say $v, w \in V$ are perpenicular /orthogonal if $< v, w >= \mathcal{O}$ denoted by $v \perp w$*

$S \subset V, S^{\perp} = \{v \in V | v \perp w \forall w \in S\}$ e.g. $S = v_1, v_2 \dots v_n$ $S^{\perp} = \{v \in V | v \perp v_1, \dots, v \perp v_m\}$
claim $S^{\perp}$ is a subspace in V
Fact: Let U be the span of S then

$$S^{\perp} = U^{\perp}$$

we use bilineraity of scalar products
let

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + \cdots + a_{2n}x = 0$$

$$\vdots$$

$$a_{n1}x_1 + \cdots + a_{mn}x_n = 0$$

In fact solution to this homogeneus lienar eqn $\{x \in k^n | x \cdot A_1 = x \cdot A_2 \ldots X \cdot A - n = 0\} = \{A_1 \ldots A_n\}$

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recall $FV \to V$ and let $\mathcal{B}, \mathcal{B}'$ be 2 basis of V
$\mathcal{M}_{\mathcal{B}'}^{\mathcal{B}'}(F) = N^{-1}\mathcal{M}_{\mathcal{B}}^{\mathcal{B}}(F)N$ where $N = \mathcal{M}_{\mathcal{B}}^{\mathcal{B}'} ID$

recall $v \in V$ $X_{\mathcal{B}'}F(v_1) = \mathcal{M}_{\mathcal{B}'}^{\mathcal{B}}F(X)X_{\mathcal{B}}(V)$
Take F=Id
$\to X_{\mathcal{B}'}(V) = \mathcal{M}_{\mathcal{B}'}^{\mathcal{B}} IDX_{\mathcal{B}}(V)$
e.g. $V = k^2 \mathcal{B} = \{\binom{1}{0}\binom{0}{1}\}, \mathcal{B}' = \{\binom{1}{1}\binom{1}{-1}\}$ $V = \{\begin{bmatrix} x \\ y \end{bmatrix}\}$
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} X_{\mathcal{B}}(v)$

$$a_{11}x_1 + \cdots + a_{1n}X_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + \cdots + a_{nn}X_n = 0$$
$$\updownarrow$$
$$Ax = 0, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$\updownarrow$$
$$X \cdot A_1 = X \cdot A_2 = \cdots = X \cdot A_3 = 0$$

where as
$$X \in \{A_1 \ldots A_n\} = span\{A_1 \ldots A_n\}^{\perp}$$

IF columns of A are linearly independent then this sys of equns has only trival soluns $\iff$ $span\{A_1 \ldots A_n\}^{\perp} = \mathcal{O}$ Further If $A_1 \ldots A_n$ are linearly independent then span $\{A_1 \ldots A_n\} = k^n = V$
$V^{\perp} = \{\mathcal{O}\}$ scalar product is non-degenerative
For now on we will define $K = \mathbb{R}$

Definition (orthogonality). *Let $\mathcal{B} = \{v_1, v_2 \ldots v_n\}$ be as basis of V we say that $\mathcal{B}$ is an orthogonal basis iff $< v_i, v_j >= 0$ for $i \neq j$*

Definition (positive definite). *A Scalar product on $V/\mathbb{R}$ is called positive definite iff $< v, v >> 0 \forall v \neq 0 \in V$*

e.g. $X = (x_1 \ldots x_n) \neq 0$ $x \cdot c = x_1^2 + \ldots x_n^2 > 0$ their dot ptoduct on $\mathbb{R}$ is positive definite.

Definition (norm). *The norm of $V/\mathbb{R}$ $\|V\| = \sqrt{< v, v >}$*

In previous example $\|x\| = \sqrt{x_1^2 + x_2^2 + s_x^2}$ let $c \in \mathbb{R}$ we have $\|cv\| = \|c\| \cdot \|v\|$ $\|v - w\| =$ distance from w to v unit vector

Theorem 17.1 (pythagorams thm). If $v \perp w$ we have $\|v - w\|^2 = \|v + w\|^2 = \|v^2\| + \|w^2\|$

Proof. $\|v + w\|^2 =< v + w, v + w >$
$=< v, v > +2 < v, w > + < w, w >$
since $v \perp w$ $< v, w >= 0$

hence $=< v, v > + < w, w >$
$= \|v\|^2 + \|w\|^2$ $\square$

17.1 Projection

We can project v onto W denote proj$\binom{v}{w} = cw$

$$< v - proj(v) < v >>= 0$$

$$= v - < w, w >= 0$$

hence $C = \frac{<v,w>}{<w,w>}$

Definition. $\frac{<v,w>}{<w,w>}$ *is the projection compment of V along w and proj* $\binom{v}{w} = \frac{<v,w>}{<w,w>}$ *is the projection of v along w*

e.g. $v = (2, 1)w = (1, 1), \frac{<v,w>}{<w,w>} = \frac{3}{2}$

Theorem 17.2. v/$\mathbb{R}$ is non-degenerative scalar product if $\{v_1, v_2 \dots v_n\}$ are orthogonal along w for $w \in V$
$w_1 = a_1 v - 1 + \dots + a_n v_n$
$a_i = \frac{<w_i,v_i>}{<v_i,v_i>}$ where as $a_i v_i = proj \frac{w}{a_i}$

Proof. $< w, v_i >$
$< a_1 v_1 + \dots + a_n v_n, v_i >$
$=< a_i v_i, v_i >$
$= a_i < v_i, v_i >$
$a_i = \frac{w v_i}{v_i v_i}$ $\square$

Theorem 17.3 (Schwartz Inequality)**.** $\forall v, w \in V$ we have $| < v, w > | \leq \|v\| \cdot \|w\|$

Proof. by cases: case $1w = 0$ trival
case 2 w is the unit vector $< w, w >= 0$ then proj$\binom{v}{w} =< v, w > w$
Then we apply the pythagoram thm $\|v\| \leq \left\|proj\binom{v}{w}\right\| = \|< v, w > w\| =< v, w >$
case $3w \neq 0$ then we know that $\frac{w}{\|w\|}$ is the unit vector by case 2
$| < \frac{w}{\|w\|}, v > | \leq \|v\|$
$= \frac{|<w,v>|}{w} \leq \|v\| \iff | < v, w > | \leq \|v\| \|w\|$

$\square$

Definition. $\frac{<v,w>}{<w,w>}$ *is the projection compment of V along w and proj* $\binom{v}{w} = \frac{<v,w>}{<w,w>}$ *is the projection of v along w*

e.g. $v = (2, 1)w = (1, 1), \frac{<v,w>}{<w,w>} = \frac{3}{2}$

> **Theorem 17.4.** v/$\mathbb{R}$ is non-degenerative scalar product if $\{v_1, v_2 \dots v_n\}$ are orthogonal along w for $w \in V$
> $w_1 = a_1 v - 1 + \cdots + a_n v_n$
> $a_i = \frac{<w_i, v_i>}{<v_i, v_i>}$ where as $a_i v_i = proj\frac{w}{a_i}$

Proof. $< w, v_i >$
$< a_1 v_1 + \cdots + a_n v_n, v_i >$
$=< a_i v_i, v_i >$
$= a_i < v_i, v_i >$
$a_i = \frac{w v_i}{v_i v_i}$ □

> **Theorem 17.5** (Schwartz Inequality)**.** $\forall v, w \in V$ we have $| < v, w > | \leq \|v\| \cdot \|w\|$

Proof. by cases:
case $1 w = 0$ trival
case 2 w is the unit vector $< w, w >= 0$ then proj$\binom{v}{w}$ $=< v, w > w$
Then we apply the pythagoram thm $\|v\| \leq \left\|proj\binom{v}{w}\right\| = \|< v, w > w\| =< v, w >$
case $3 w \neq 0$ then we know that $\frac{w}{\|w\|}$ is the unit vector by case 2
$| < \frac{w}{\|w\|}, v > | \leq \|v\|$
$= \frac{|<w,v>|}{w} \leq \|v\|$

$$\iff | < v, w > | \leq \|v\| \|w\| \tag{1}$$

□

we know that the projection matrix

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Recall: last class, $<, >$is positive definite scalar product on V
proj$\binom{v}{w} = \frac{<v,w>}{w,w} \cdot w$

$v_1, v_2 v \in V$ are linearly independent , $v_1 \perp v_2$

$$\text{proj}\Big(\Big(_{w=} \underbrace{\frac{< \overset{v}{v_1} v_2 >}{< v_1, v_1 >}}_{proj\binom{v}{v_1}}\Big) + \underbrace{\frac{< v_1, v_2 >}{< v_2 v_2 >}}_{proj\binom{v}{v_2}}\Big)$$

we claim that $u_i = v - \frac{<v,v_i>}{<v_1,v_1>} \cdot v_i - \frac{<v_i v_2>}{<v_2,v_2>} \cdot v_2$ is perpendicular to v_1 and v_2

Proof. $< u, v_i >=< v, v_1 > - < \frac{<v_1,v_1>}{<v_1,v_1>} \cdot v_1, v_1 > -0 = 0$ similarly $< u, v_2 >= 0$ □

Corollary . v_1, v_2, u is an orthogonal basis of span $v_1.v_2, v$ whereas$u = v - \frac{<v,v_1>}{<v_i,v_i>} v_1 - \frac{<v,v_2>}{<v_2,v_2>} v_2$ This is called the gram smidt procedure

e.g. in $\mathbb{R}^3$ $v_1 = (1,1,0), v_2 = (0,1,1), v_3 = (1,0,1)$
Goal: construct an orthogonal basis w_1, w_2, w_3 out of v_1, v_2, v_3

$$u_1 = v_1 = (1,1,0)u_2 = v_2 - \frac{< v_2, v_1 >}{< v_1 v_1 >} v_1 = (0,1,1) - \frac{1}{2} < 1,1,0 > = (-\frac{1}{2}, \frac{1}{2}, 1)$$

(2)

$$w_3 = v_3 - \frac{< v_3, u_1 >}{< u_1, u_1 >} u_1 - \frac{< v_3, u_2 >}{< u_2, u_2 >} u_2 = (1,0,1) - \frac{1}{2}(1,1,0) - \frac{\frac{1}{2}}{\frac{3}{2}} = (\frac{1}{2}, \frac{1}{2}, 1)$$

$(1,0,1) - (\frac{1}{2}, \frac{1}{2}, 0) - < -\frac{1}{6}, \frac{1}{6}, \frac{1}{3} >$
$= (1 - \frac{1}{2} + \frac{1}{6}, -\frac{1}{2} - \frac{1}{6}, 1 - \frac{1}{3})$
$\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}$
$\to (1,1,0), (-\frac{1}{2}, \frac{1}{2}, 1)(\frac{2}{3}, \frac{-2}{3}, \frac{2}{3})$)are the orthogonal basis

Lemma . suppose that we have v_1, v_2, v_k are mutually orthogonal vectors in V, then v_1, v_2, v_k are linearly independent

Proof. if we have $a_1 v_1 + a_2 v_2 + \dots a_k v_k = 0$ then we know that $< a_1 v_1 + 12 v_2 + \dots a_k v_k = a_i < v_i, v_i >= a_i = 0$ ☐

Theorem 18.1. $V/\mathbb{R}$ be a vector space with a positive definite scalar product with $W \subseteq V$ subspace and let $\{w_1, w_2, \dots w_m\}$ be an orthogonal basis of W. Then there exists $w_{m+1} \dots w_n$ such that $\{w_1, w_2, \dots w_m, w_{m+1}, \dots, w_n\}$ is an orthogonal basis

Proof. By a theorem we have proven earlier there exists vectors $v_{m+1}, \dots v_n$ such that $\{w_1 \dots w_m, v_{m+1}, v_n\}$ is a basis of V
By gram schmidt, we have
$w_{m+1} = v_{m+1} - \frac{< v_{m+1}, w_1 >}{< w_1, w_1 >} w_1 - \dots - \frac{< v_{m_1}, w_m >}{< w_m, w_m >} w_n$
we have $w_{n+1} \perp w_1, w_2, \dots w_m$

we use induction to find another vector ☐

Corollary F. or vector space $v/\mathbb{R}$ with positive-definite scalar products there always exists an orthogonal basis.

Definition (orthonormal basis). *An orthogonal basis $\{v_1, v_2 \dots v_n\}$ is called orthonormal if $\|v_i\| = 1 \forall i$*

for example $\{v_1, v_2 \dots v_n\}$ be a basis of V

then v_1, v_2, v_3 are linearly independent
$u_1 = v_1$
$u_2 = v_2 - \frac{< v_2, v_1 >}{< v_1, v_1 >} v_1$
$u_3 = v_3$

hence step 1 $\begin{bmatrix} 1 & \star & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ step 2: $w_1 = u_1$
$w_2 = u_2$
$w_3 = u_3 - \star u_2 - \star u_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{bmatrix}$

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Gram schmidt orthogonalitites where $\langle , \rangle$ are positive definite
say that We have $v_1, v_2 \dots v_n$ are orthogonal are linearly independent
Then we can find vectors that

$$v_{n+1} - proj_{v_i}(v_{n+1}) - proj_{v_2}(v_{n+1}) - \cdots - proj_{v_n}(u_{n+1})$$

see pic

Theorem 19.1 (orthogonality implies linearly independent). Any mutually orthogonal nonzero vectors $v_1, v_2 \dots v_k \in V$ can be extended to an orthogonal basis $\{v_1, v_2 \dots v_n\}$

let $(v, \langle , \rangle)$ be positive definite
let $\mathcal{B} = \{v_1, v_2 \dots v_n\}$ be an orthogonal basis
$\forall v \in V v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$
$\forall w \in v = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n$
$\langle v, w \rangle \; x_1 y_1 \langle v_1, v_1 \rangle + x_2 y_2 \langle v_2, v_2 \rangle + \cdots + x_n y_n \langle v_n, v_n \rangle$
$= x_1 y_1 + \dots x_n y_n$
if $\mathcal{B}$ is orthonormal

Theorem 19.2. Let V be a vector space over $\mathbb{R}$ with positive definite scalar product $\langle , \rangle$ of $\dim v = n$ let $w \in v$ be a subspace of dimension r, then $\dim w^{\perp} = n - r$

Remark. *() (w $\langle , \rangle$)*
if

$$\langle , \rangle$$

is positive definite $\rightarrow$ non degenerative then $\langle , \rangle|_w$ is also positive definite (in particular non-degenerative)

Proof. give (w, $\langle , \rangle$) is also positive definite, there exists an orthogonal basis $v_1, v_2, \dots, v_r$ Thses vectors can be exitended to an orthogonal basis of V $v_1, v_2, \dots, v_r, v_{r+1}, \dots v_n$ we claim that$w^{\perp}$ =span$\{v_{r+1}, \dots, v_n\}$
proof of claim:

It is clear that $v_{r+1}, \dots v_n \in W^{\perp}$
$\forall u \in W^{\perp}$
we write that $u = a_1 u_1 + \dots + a_r u_r + + a_{r+1} u_{r+1} + \dots + a_n v_n$
since $1 \leq i \leq r \langle u, v_i \rangle = a_i \langle v_i, v_i \rangle = \mathcal{O} \to a_i = \mathcal{O}, 1 \leq i \leq r$
i.e.$u = a_{r+1} u_{r+1} + \dots a_n v_n \in$ span $\{v_{r+1}, v_n\}$

□

19.1 application in Complex numbers: Hertian product

example $z = a + bi \in \mathbb{C}$

we have $z \cdot z = a^2 - b^2 + 2abi$
$z \cdot \bar{z} = (a + bi)(a - bi) = (a^2 b^2 > 0)$ if z $\neq 0$

Definition (Hertian product). *let $V = \mathbb{C}^n = (y_1, y_2 \dots, y_n)$ then we know that $\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = x_1 \bar{y_1} + \dots x_n \bar{y_n} = \|x_1\|^2 + \|x_2\|^2 + \dots$ if $x_i = y_i \forall i$*

example we have $\mathbb{C}^2$
$\langle (x_1, y_2), (y_1, y_2) \rangle = \frac{x_1 \bar{y_1} + x_2 \bar{y_2}}{\langle (y_1, y_2)(x_1, x_2) \rangle}$
$\langle (x_1, x_2) + (x_1', y_2'), (y_1, y_2) \rangle$

$$
\begin{aligned}
&= \langle (x_1 + x_1', y_2 + y_2'), (y_1, y_2) \rangle \\
&= (x_1 + x_1')\bar{y_1} + (x_2 + y_2')\bar{y_2} \\
&= \langle (x_1, x_2), (y_1, y_2) \rangle + \langle (x_1',') \rangle
\end{aligned}
$$

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Last class reminder we know that the hernitian product on $V : vspace/C$ $\langle , \rangle V \times V \to \mathbb{C}$

- $\langle u, v \rangle = \overline{\langle u, v \rangle}$
- $\langle au, v \rangle = a \langle u, v \rangle$
- $\langle a_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$:

Definition. *The herition product $\langle , \rangle$ on V is positive definite if $\langle u, v \rangle = 0 \forall v \neq 0$*

$\forall vu, \in V$ if $\langle u, v \rangle = 0$ then we sat that $u \perp v$
$\|v\| = \sqrt{\langle v, v \rangle}$
phthagorans theorem if $v \perp w$ then $\|v \mp w\|^2] = \|v\|^2 + \|w\|^2$
the schwarz inequality $|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$
This also suggest that we have the gram schmidt product for positive definite hermian product
e.g. we have $\mathbb{C}^2$
$\langle (x, y), (x', y') \rangle = x\bar{x'} + y\bar{y'}$ is positive definite
$v_1 = (1, 1 + i) v_2 = (1, i)$ they are not orthogonal

another example we have $w_1 = v_1 = (1, 1+i)$
$w_2 - v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1$
$= (1, i) - \frac{1+i(1-i)}{1+(1+1)}(1, 1+i)$
$= (1, i) - \frac{2+i}{3}(1, 1+i)$

$= (1, 1) - (\frac{2+i}{3}, \frac{1+3i}{3})$
$(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$

Theorem 20.1. Let $V/\mathbb{C}$ be a vector space with hermition product $\langle, \rangle$ given non zero mutually othogonal vectors $v_1, v_2, \dots v_k$ then v_1, v_2, v_k can be etendedinto an orthogonal basis $\{v_1, v_2 \dots v_n\}$

Corollary 1. . V has an orthongonal basis
2. $W \subseteq V W^{\perp} = \{v \in V | v \perp W\}$

20.1 application to sysm. equation

we have $a_{11}x_1 + \cdots + a_{1n}x_n = 0$
$\vdots$
$a_{m_1}x_1 + \cdots + a_{mn}x_1 = 0$
we claim that $\{\text{soln of}(\star)\} + \dim$ span of column $= n$ $F_A : k^n \to k^m$

Definition (Ranks)**.** *column rank (A)=dim of span of column vectors*
row rank of (A)=dim of span of row vectors

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

we have col rank $A = 2$ row rank $A = 2$

Theorem 20.2 (column · rank (A))**.** we have col rank = row rank

Proof. Null space of (A)=$\{(x_1, x_2, x_3) | x_1 A^1 + \cdots + x_n A^n = 0\}$
solution of space $= \{(x_1, x_2, \dots x_n) | X \cdot A_1 = X \cdot A_2 = \cdots = X \cdot A_m = 0\}$
Null (A)=$\{A_1, A_2 \dots A_m\}^{\perp} = span\{A_1 \dots A_m\}^{\perp}$ when $k = \mathbb{R}$ recall $\mathbb{R}^n = W_{row} \oplus W_{row}^{\perp}$
$\to \mathbb{R}^n = W_{row} \oplus Null(A) \to n = rowrank(A) + null(A)$ rewrite $\star$
we have $n = colrank(A) + null(A) \to$ row rank (A)=col rank(A)

□

rank for general k the theorem holds

Theorem 20.3. v/K $\langle, \rangle$ scalar product be non degenerative then $\dim w + \dim w^{\perp} = \dim V$

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reminder $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & \dots & a_{mn} \end{bmatrix}$ and colrank $A =$ dimspan of sol vectors $\subseteq k^m$
and rowrank $A =$ dimspan of wooo vectors $\subseteq k^n$

Theorem 20.4. Col rank $A = Rowrank(A)$

rank $A = col$ rank $A = row$ rank A

last time we proven that theorem for $k = \mathbb{R}$ we have a theorem that generalize this to any field.

Theorem 20.5. let V/K $\langle , \rangle$ be a scalar product w$\subseteq$ C subspace them we know that $\dim w + \dim w^{\perp} = \dim v$

e.g. $k = \mathbb{C}$ dotprouct, given that $v = \mathbb{C}^2$, $W = \mathbb{C}(1, i)$
$W^{\perp} = \mathbb{C}(1, i) = W.V \neq W + W^{\perp}$
we still see that $w + \dim W^{\perp} = 2(1, i) \cdot (1, i) = 0$

20.2 standard hermition product

$$W^{\perp} = \mathbb{C}(1, -i), V = W \oplus W^{\perp}$$

20.3 Bilinearmaps

recall: Given a $m \times n$ matrix A, we can associate a linear map $F_A : k^n \to k^m$
$X \mapsto AX$
Recall: V/K vector space with $\langle , \rangle$ scalar product $\langle , \rangle : V \times V \to K$

Definition (bilinear maps)**.** *Let U, V, W be Vector spaces on k, A map g :$U \times V \to W$ is bi linear map if and only if $\forall u \in u, g(u, \cdot) : V \to W$ is linear and $\forall v \in V, g(\cdot, v) : U \to W$ is linear.*

$w = k, g : U \times V \to K$

$$\mapsto$$

$U = V, g : V \times V \to K$

e.g. a $m \times n$ matrix

$$\mapsto$$

$g_A = k^m \times k^n \to K$

and let (X,Y)$\mapsto XAY = (x_1, x_2, \dots x_n) \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m_1} & a_{m_2} \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_n \end{bmatrix}$ e.g.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$g_A : k^2 \to k^3 \to k$
$((x_1 x_2), \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) \mapsto (x_1, y_2) \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = X \cdot F_A(Y) = (x_1 x_2 2x_1 + y_2) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$
$= x_1 y_1 + x_2 y_2 + 2x_1 y_3 + x_2 y_3$

In general:

$$g_A(X, Y) = \sum a_{ij} x_i y_j \forall 1 \leq i \leq m, 1 \leq j \leq n$$

In k^n we shave standard basis $e_1, e_2, \dots e_n$ $e_1 = (1, \dots, 0) e_2 = (0, 1, \dots, 0) \dots$ in k^n we have f_1, f_2, f_n where $f_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$ $f_2 = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}$ $\dots$ what is $g_A(e_i, f_j) = a_{ij}$

$$g_A(X, Y) = g_A(\sum x_i e_i, \sum y_j f_j)$$

bi linearility $\sum_{ij} g_A(e_i f_j) x_i y_j = \sum ij x_i y_j$

Theorem 20.6. Given any bilinear map $g : k^m \to k^n \to k$ there exists a unique matrix $m \times n$ A such that $g = g_A$

Proof. set $A = (g(e_i, f_j)) \forall 1 \leq i \leq, 1 \leq j \leq n : m \times n$ matrix
we claim that $g = g_a$
since $g(e_i, f_j) = g_A(e_i f_j) \forall i, j$
by bi linearility $g = g_A$ uniqueness: suppose $g_a = g_b$
$g_A(e_i, f_j) = a_{ij}$
$g_b(e_i, f_j) = b_{ij}$
$g_A = g_B \Rightarrow a_{ij} = b_{ij} i.j \iff A = B_i$ □

by the above theorem we immediately get
all $m \times n$ matrices are bijectie with bilinear maps :$k^m \times k^n \to k$ and linear maps $k^n \to k^m$
wereas $F \mapsto G(X, Y) = X \cdot F(Y)$
$A = m11 - 31$
$g_a : k^2 \times k^2 \to k$
$g_A((x_1 x_2), \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = x_1 y_2 + 2x_1 y_2 - 3x_2 y_2 + x_2 y_2$

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Given any $m \times n$ matrix $A = a_{ij}$

we have $g_A(X,Y) = X^t AY = (x_1, x_2 \dots x_n) \begin{bmatrix} a_{11} & \dots & a1n \\ \dots & \dots & \dots \\ a_{m_1} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$

If A is Symmetric Then, bi linear map$g_A k^n \times k^n \to k$
then g_A is a linear pruduct $\iff$ symmetric bilinear map
$x^t AY$
$g_A(X,Y) = x^t AY$
recall $(AB)^t = B^t A^t$
e.g. $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$
$g_A = (\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix})$
$= x_1y_1 + 2x_1y_2 + 2y_2y_1 - x_2y_2$

21.1 General Orthogonal basis

Given a scalar product on $V/K\langle , \rangle : V \times V \to K$

Question . can we find an orthogonal basis with respect to $\langle , \rangle$

we know that if the scalar product is positive definite on VR we have proven the existance by gram-schmidt
e.g. $V = k^2$
$\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rangle = x_1y_1 - x_2y_2$ since this scalar product came from an symmetric metric $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

since $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$is an orthogonal basis

when $k = \mathbb{R}$ we claim its not positive definite because $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle < 0$

2. $v = k^2, \langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rangle = x_1y_1$

$$\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$

is orthogonal.

Definition (scalar product)**.** *A scalar product $\langle , \rangle$ on V is called null if $\langle v, v \rangle = 0 \forall v \in V$ and V with this $\langle , \rangle$ is called null space.*

Remark. *if $\langle,\rangle$ is null then $\langle v,w\rangle = 0 \forall v,w \in V$*

Proof.

$$\langle v-u, v-u\rangle = 0 = \langle v,v\rangle + \langle w,w\rangle - 2\langle v,w\rangle$$

$$\langle v,w\rangle = \frac{\langle v,v\rangle + \langle u,u\rangle}{2} = 0$$

□

> **Theorem 21.1.** For any scalar product $\langle,\rangle$ on V $V \neq 0$ there always exists an orthogonal basis

Proof. we will use induction on $\dim V = n$
when $\dim = 1$This is trival
suppose than $\dim V \geq 2$ if V is a null space then any basis is orthogonal
Otherwise there exists non zero vector $v_1 \in V s.t.\ \langle v_1, v_1\rangle \neq 0$

$$V \xrightarrow{P} V_1 = k \cdot v_1$$

$$v \mapsto \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

P is a linear map.$\ker p = \{v | \langle v_1, v_1\rangle = 0\} := v_1^{\perp}$
Hence P is surjective
$\rightarrow \dim v_1^{\perp} = n-1$
we claim that

$$V = V_1 \oplus V_1^{\perp}$$

$$v = \frac{\langle v_1, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \in V_1 + (v - \frac{\langle v_1, v_1\rangle}{\langle v_1, v_1\rangle} v_1) \in V_1^{\perp}$$

By induction $V_1^{\perp}$ has an orthogonal basis $\{v_1, v_2 \dots v_n\}$, Then $\{v_1, v_2 \dots v_n\}$ is an orthogonal basis

□

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recall V : space over K
$V* = L(V,K) = \{\varphi V \rightarrow K\}$ Given a basis $\mathcal{B} = \{v_1, v_2 \dots v_n\} of V$
A dual basis called $\mathcal{B} = \{\varphi_1, \varphi_2, dots \varphi_n\}$ a basis of V*
Given $W \subseteq V$ a subspace

Definition (peep). $W^{\perp} * \subseteq V* = \{\varphi \in V * | \varphi(w) = 0\}$

Theorem 22.1. $\dim W + \dim W^{\perp}* = \dim V$

for example $V = k^3 v_1 = (1,1,0), v_2 = (0,1,1), v_3 = (0,0,1)$ and let $\{e_1, e_2, e_3\}$ be standard basis then we have $\{ \varphi_1, \varphi_2, \varphi_3\} be a dual basis of V *$
$and let$W= span $\{$ v$_1, v_2\} \subseteq V$ be 2 dim and we have $W^{\perp}* = \{\varphi \in V * | \varphi(v_1) = 0 \wedge \varphi(v_2) = 0\} \subseteq V*$
$\varphi = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3$
$\varphi(v_1) = \varphi(e_1) + \varphi(e_2) = a_1 + a_2 = 0$
$\varphi(v_2) = a_2 + a_3 = 0$

Let $\{\psi_1, \psi_2, \psi_n\}$ be dual basis of $\{v_1, v_2, v_3\}$ can you find a transition matrix M that takes $\{ \varphi_1, \varphi_2, \varphi_3\} \to \{\psi_1, \psi_2, \psi_3\}$?

Let $\langle , \rangle$ be non degenerative scalar product on V recall that $W^{\perp} = \{v \in V | \langle v, W\rangle = 0\}$

Theorem 22.2. $\dim W + \dim W^{\perp} = \dim V$

Proof.

$$V \to V*$$

$v \mapsto L_v(u) = \langle v, u\rangle$ $W^{\perp} \to W^{\perp}*$ we claim $L : W^{\perp} \to W^{\perp}* \; \forall v \in W^{\perp}, L_v \in W^{\perp}*$
since $L_v(w) = \langle v, w\rangle = 0$ if suffices to check that$L(w^{\perp}) = W^{\perp}$ for any $\varphi \in W^{\perp}* \subseteq V*$
$\varphi = L_v$ for some $v \in V$
$\varphi \in W^{\perp}* \iff \varphi(w) = 0 \iff L_v(W) = 0 \iff \langle v, W\rangle = 0 \iff v \in W^{\perp}$

□

Corollary A. =matrix whereas $a_{ij} \in K$ row rank = col rank = rank A

Proof. null space (A)= $\ker(F_A), F_A = k^n \to k^m \iff \{X | AX = A_m X = 0\}$
$\iff$ span $\{A_1, \dots, Am\} \iff W^{\perp}_{row}$ hence $\dim \ker(F_A) = n-$col rank
$\dim W^{\perp} = n-$ row rank □

22.1 Quadratic Form

e.g. $Q : k^2 \to k$
$(x, y) \mapsto (x^2 - 4xy + y^2)$
$x^2 \pm y^2$

Definition (Quadratic form)**.** *V a v space over K a function QV → k is a quadratic form if there exists a scalar product ⟨, ⟩ on V such that* $Q(v) = \langle v, v\rangle$

e.g. $k^2 \; \langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (x_1, x_2) \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rangle$
$Q(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = (x_1, x_2) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1y_2 + cx_2^2$ if $a = 1, c = 2, b = 0$
we get $Q(x_1, x_2) = x_1^2 - x_2^2$ on K^n all quadratic forms can be described as

$$Q()$$

$=x^tAX$ where A is a symmetric matrix $X = \begin{bmatrix} x_1 \\ \cdots n \end{bmatrix} = a_{ij}, a_{ij} = a_{ji}$

$= \sum_i = 1^n a_{ii}x_i^2 + \sum_{i<j} 2a_{ij}x_iy_j$

Proposition 22.3. Any scalar product $\langle , \rangle$ arise from a quadratic form Q on V s.t. $Q(v) = \langle v, v\rangle$

Proof. Let Q(v)=$\langle v, v\rangle$

Then $\langle v, w\rangle = \frac{1}{2}(\langle v + w, w + v\rangle - \langle v, v\rangle - \langle w, w\rangle) = \frac{1}{2}Q(v + w) - Q(v) - Q(w")$

□

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Recall $Q : V \to K$ is a quadratic form if $Q(v) = \langle v, v\rangle$ for some scalar product $\langle , \rangle \in V$

If V $=K^n$

{ scalar products on K^n }∼ { symbolic matrix } hence $Q \in$ { quadratic forms on k^n }

$\langle v, w\rangle = \frac{1}{2}(Q)(v + w) = Q(v) - Q(w)$

Let A be a symmetric matrix

$Q = (x_1 \dots x_n) = X^tAX$

$= \sum_{i=1}^n a_{ii}x_ix^2 + 2\sum a_{ij}x_iy_j$

e.g. $Q(x, y) = x^2 - 3xy + 2y^2$

we have $A = \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 2 \end{bmatrix}$

Let V be a vector space over K $\langle , \rangle$ is a scalar product

$\{v_1, v_2 \dots v_n\}$ be a basis of V

we know that $(\langle v_i, v_j\rangle)$ is a symmetric matrix

In particular we proved that there exists basis orthogonal product $\{v_1, v_2 \dots v_n\}$ such that $(\langle v_i, v_j\rangle)$ is diagonal

e.g.

For now we assume $k = \mathbb{R}$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\langle X, Y\rangle = x_1y_2 + x_2y_1$ on k^2

question so is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an orthogonal basis is orthogonal basis. so $\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\rangle = -2$

$\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\rangle = 2$

2. Let 2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we already know that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are orthogonal

$\langle e_1, e_1\rangle = 1 > 0$, $\langle e_2, e_2\rangle = 1 > 0$

3. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

let $\langle x, y\rangle = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$

indeed $v_1 = \begin{bmatrix}1\\0\end{bmatrix}, v_2 = \begin{bmatrix}1, 1\end{bmatrix}$ is an orthogonal basis.

$\langle v_1, v_1\rangle = -1$, $\langle v_2, v_2\rangle = 0$

Another basis for this would be $w_1 = \begin{bmatrix}0\end{bmatrix}, w_2 = \begin{bmatrix}1, 1\end{bmatrix}$

$\langle w_1, w_2\rangle = 0$, $\langle w_1, w_2\rangle = -1$, $\langle w_2, w_2\rangle = 0$

Theorem 23.1 (Sylvester's theorem)**.** Let V be a vector space over $\mathbb{R}$ with scalar product. Let $v_1, v_2, \ldots, v_n$ be orthogonal basis and then

$$n^+ = \#\{v_1|\langle v_i, v_i\rangle > 0\}$$

$$n^0 = \#\{v_i|\langle v_i, v_i = 0\rangle\}$$

$$n^- = \#\{v_i|\langle v_i, v_i\rangle < 0\}$$

we call then positive, nullity index and negativity index they do not dependent on the choice of orthogonal basis.

e.g. $Q = (x, y, z) = 2x^2 - y^2$

lets determine n^+, n^0n^-

In this case the associated matrix is

$$A = \begin{bmatrix}2 & & \\ & -1 & \\ & & 0\end{bmatrix}$$

we see that $\langle e_1, e_1\rangle = 2$

$\langle e_2, e_2\rangle = -1, \langle e_3, e_3\rangle = 0$

hence $n^+ = n^- = n^0 = 1$

note that anytime we have daag matrix then standard matrices are orthogonal basis

Observe we see that $n^+ + n^0 + n^- = n$ so we only need to prove 2 of them.

Proposition 23.2. 1. Let $V_0 = \{v \in V|\langle v, w\rangle = 0 \forall w \in W\}$

let { $v_1, v_2 \ldots v_n$ } be an orthogonal basis then $n^0 = \dim V_0$

Proof. Suppose that $\{v_1, v_2 \ldots v_n\} = \{v_1 \ldots v_s, v_{s+1}, \ldots, v_n\}$ is ordered so that $\langle v_i, v_i\rangle \neq 0 1 \leq i \leq s$

$\langle v, v_i\rangle = 0 s < i \leq n$then we see that $v_{s+1}, \ldots v_n \in V_0$

we claim that V_0 =span $\{v_{s+1} \ldots, v_n\} \forall v \in V_0$ we we write $v = a_1v_1 + \ldots a_sv_s + a_{s+1}v_{s+1} + a_nv_n$

$$0 = \langle v_i, v_i\rangle = a_i\langle v_i, v_i\rangle \to a_i = 0$$

$\square$

Theorem 23.3 (hairy ball)**.** theorem hence we alternatively i see is $that hence \mathcal{M}_{\mathcal{B}''}^{\mathcal{B}'}, which gives us tath$

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We proof that $n+$ is inde. Of orthogonal basis. Suppose we are given n two orthogonal basis.
$\{v_1, v_2 \dots v_n\}$
$w_1, w_2 \dots w_n\}$
Those basis can be ordered such that

$$\langle v_i, v_i \rangle > 0 1 \leq i \leq r$$

$$\langle v_i, v_i ``\rangle < 0 r + 1 \leq i \leq s$$

$$\langle v_i, v_i = 0 \rangle s + 1 \leq i \leq n$$

$$\langle w_i, w_i \rangle > 0 1 \leq i \leq r'$$

$$\langle w_i, w_i \rangle < 0 r' + 1 \leq t \leq s'$$

$$\langle w_i, w_i \rangle = 0 s' + 1 \leq i \leq n$$

we only need to prove that r=r'
we clarm that $v_1, v_2, v_r, w_{r+1}, w_m$ are linearly independent .
suppose there is a linearly dependent
$a_1 v_1 + \cdots + a_r v_r + b_{r+1} w_{r+1} + \cdots + b_n w_n = 0$
$\rightarrow a_1 v_1 + \cdots + a_r v_r = (b_{r'+1} w_{r'+1} + \dots{}_b {}_n w_n)$
the rest see pic
if $\exists a_i > 0, 1 \leq i \leq r$
then $\leq > 0$
but rhs ≤ 0 this is a contradiction.
Thus $a_1 = a_2 = a_r = \cdots = 0 \rightarrow b_{r+1} = \cdots = b_n = 0$
Therefore $r + (n - r') leqn \rightarrow r \leq r'$
similaryly $f' \leq r \rightarrow r = r'$

24.1 application of Sylvesters thm

Let $Q(x_1 \dots x_n = x^t AX)$ Let A be a symmetric marix and $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
then the scalar product: $\langle X, Y \rangle = x^t AY$
There exists an orthogonal basis

$v_1, v_2 \dots v_n \in \mathbb{R}^n$ such that $\langle v_i, v_j \rangle = 0, i \neq j$
$\langle v_i, v_i \rangle = d_i$

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \dots & \\ & & & d_n \end{bmatrix} P$$

$\langle v_i, v_j \rangle = v_i^t A v_j = e_{{}^t P^t A P e_j}$=ij entry of $P^t AP$
$Pe_j = v_j$

$$\begin{bmatrix} p_{11} & \dots & p_1 n \end{bmatrix} \dots \dots \dots p_{n_1} \dots p_{mn} \cdot e_i = v_i$$

claim: A is symmetric $n \times n$ real matrix then exists a matrix P such that $P^t AP = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \dots & \\ & & & d_n \end{bmatrix}$ $p =$(Orthogonal basis)
we can find orthonormal basis i.e. $\langle v_i, v_i \rangle = 1, -1, 0$
Given orthogonal basis $\{v_i\}$

$$\tilde{v_i} = \begin{cases} v_i & \langle v_i, v_i \rangle = 0 \\ \frac{v_i}{\sqrt{\langle v_i, v_i \rangle}} & \langle v_i, v_i > 0 \rangle \\ \frac{v_i}{\sqrt{-\langle v_i, v_i \rangle}} & \langle \rangle \end{cases} \text{ Thus } \exists P s.t. P^t AP = a matrix with 1, 0, -1 in the diagonal$$

24.2 determint

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$\det(A) = ad - bc$
Then $\det k^2 \times k^2 \to K$
we claim that det is bilinear
proof of distributivity and homogenity

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