LINEAR ALGEBRA

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Contents

1 8/21/23 - Mon

1.1 Field and its Properties

Definition (Field)**.** $\mathbb{R}\mathbb{C}\mathbb{Q}$ *k: is a field if k has operations and satisfies*

1. *k contains* 0&1
2. *a+0=1,a* $a \cdot 1 = a$
3. *a+b=b+a, (a+b)* $\cdot$ *c=a* $\cdot c + b \cdot c$
4. $a \neq 0, a$ *has multiplicative inverse i.e.*$a \in K$ $a \cdot a^{-1} = 1, a^{-1} \in K$
5. $\forall a \in k$ *has an additive inverse -a*
6. *associativity for + and* $\cdot$

examples that $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields
$\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ is a field

$$i^{-1} = -i|(a+bi)^{-1} = \frac{a - bi}{a^\alpha + b^\alpha}$$

$\mathbb{Z}$ is not a field. Because not all element has a multiplicative inverse

1.2 vector space

Definition (Vector spaces)**.** *let K be a field.* $K^n = \{(a_1, a_2, a_3 \dots a_n) | a_i \in K\}$ *where as a is a vector*
$\underbrace{(1, 0, \dots, 0)}_{e_1}\underbrace{(0, 1, \dots, 0)}_{e_2} \dots \underbrace{(0, 0, \dots, 1)}_{e_n}$
also $\vec{0} \in K$

addition:$(a_1, a_2 \dots a_n) + (b_1, b_2, b_3 \dots b_n) = (a_1 + b_1, a_2 + b_2 \dots a_n + b_n)$
scalar multiplication: $c \in Kc \cdot (a_1, a_2, a_3 \dots a_n) = ca_1, ca_2, ca_3 \dots ca_n$ They satisfy the following requirement

1. $\vec{a} + \vec{a} = \vec{a}$
2. $\vec{b} + \vec{a} = \vec{a} + \vec{b}$
3. $c \cdot (\vec{a} + \vec{b}) = c \cdot \vec{a} + c \cdot \vec{b}$
4. $c_1 c_2 \cdot \vec{a} = c_1 \cdot (c_2 \cdot \vec{a})$
5. $(c_1 + c_2) \cdot \vec{a} = c_1 \vec{a} + c_2 \vec{a}$
6. $1 \cdot \vec{a} = \vec{a}$
7. $\vec{a} + -\vec{a} = 0$
8.

9.

with all the prereq, K^n is a vector space over K

Definition (general vector space). *a set V with origion 0 $\in V$ together, closed addition and scalar multiplication*
i.e.$\vec{V} + \vec{W} \in V, c \cdot v \in V$ *also* $c \in K, v, w \in V$ *is called vector space over K if all the above holds*
any element $v \in V$ *is called vectors of V*

e.g.

1. $\mathbb{R}C(\mathbb{R})-$ { continuous function on R is a v space over $\mathbb{R}$ }
2. $f + g \in C(\mathbb{R})$
3. $a \in \mathbb{R}$ a $\cdot$ f, a function $f \in C(\mathbb{R})$ is a vector

more general X is a set k(X)={ x $\to$ k } is a v space over K $\forall f, g \in k(x)$
$(f + g)(x) = f(x) + g(x)$
$(c \cdot f)(x) = c \cdot f(x)$

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2.1 fields

last class recall that V is a vector space over $\underbrace{K}_{\text{field, k= } \mathbb{R} \text{ or } \mathbb{C}}$ note in this class, $\mathbb{R}$, $\mathbb{C}$ is our field
$\underbrace{V}_{\text{vector}} \cdot \underbrace{W}_{\text{vector}} \in V$
also $V + W \in V$
$0 \in V$

2.2 subspaces

Definition (subspaces). *V is a vector space over k we say the subset* $W \subseteq V$ *is a subspace if it is closed under*

- *addition*
- *multiplication*

$$v + w \in W$$

$$v \cdot w \in W$$

$\forall v, w \in W, a \in K$
note this definition also implies that $0 \in W$ *e.g.* $V = k^n$ $W = \{(a_1, a_2, a_3 \dots a_n) \in K^n | \Sigma_{i=1}^{n} a_i = 0\}$ $w \subseteq v$ *subspace*

2.3 Linear Combination

we have vectors $v_1, v_2, v_3 \dots v_n \in V$ and scalars $a_1, a_2, a_3 \dots a_n \in K$ and we call $a_1 + v_1, a_2 + v_2, a_3 + v_3$ **linear combination** of $v_1, v_2 \dots$ e.g. we have $e_1(1,0) \wedge e_2(0,1) \in k^2$ example we have $(3,2) = \underbrace{3}_{\text{scalar}} \underbrace{e_1}_{\text{vector}} + 2e_2$

Proposition 2.1. given $v_1, v_2 \dots v_n$ W=set of all possible linear combination of $v_1 \dots v_n$ then, W is a subspace of V.

Proof. given $a_1v_1 \dots a_nv_n$ and $b_1 \dots b_n$ $=(a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \dots$ $(a_n + b_n)v_n$ is also an linear combination

property 2. $c(a_1v_1 + a_nv_n) = c(a_1)v_1 + c(a_n)v_n$

□

2.4 Dot Product

$\vec{a}(a_1, a_2 \dots a_n)\vec{b}(b_1, b_2 \dots b_n)$
$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 \dots a_nb_n$

Remark (properties of dot product)**.**

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\underbrace{(c \cdot \vec{a})}_{scalar\ multiples} \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

Definition (orthogonal)**.** *we say 2 vectors $\vec{a}, \vec{b} \in k^n$ are **perpendicular** or **orthogonal** if their dot product is 0 in k^n*
in notation $e_i \cdot e_j = 0$
hence we write $\vec{a} \perp \vec{b}$

recall $W = \{(a_1, a_2 \dots a_n) | a_1 + a_n = 0 \subseteq k^n \equiv \{\vec{a} | \vec{a} \cdot (1,1,1)\}\}$ more generally $\vec{b}(b_1, b_2 \dots b_n)$
$W\{\vec{a} \in k^n | \vec{a} \cdot \vec{b} = 0\} \subseteq k^n$

give n 2 sub spaces w_1 and w_2 we have 2 operations

1.
$$w_1 \cap w_2$$
2.
$$w_1 + w_2$$

Notice that both of those operations preserves sub spaces.
$(w_1 + w_2) + (w_1' + w_2') = (w_1 + w_1') + (w_2 + w_2')$

2.5 linear independence

Definition (Linear independence)**.** *V is a vector space over K, we say that* $v_1, v_2 \dots v_n \in V$ *are* ***linearly dependent*** *over K if there exists* $a_1, a_2 \dots a_n$ *such that not all of them are zero and* $a_1v_1 + a_2v_2 + a_3v_3 + \dots a_nv_n = 0$
otherwise we call it linearly independent

Remark. *we have* k^2 *where as we have (1,0)(0.1) V(2,5) such that* $= 2e_1 + 3e_2$ *hence we know that*

$$v - 2e_1 - 3e_2 = 0$$

Thus $e_1 \wedge e_2$ *are not linearly independent*

Remark. *notice that* e^t *and* e^{2t} *functions are linearly independent*

Proof. suppose that there are linearly dependent then we have a,b such that

$$ae^t + be^{2t} = 0$$

factor out a e^t we have

$$a + be^t = 0$$

taking derivative of both sides we have

$$be^t = 0$$

but $e^t \neq 0$ hence b=0 and a=0 which we have arrived at an contradiction ↯ □

Definition (alternative definition of vector space)**.** *V is a vector space if*

1. $v_1, v_2 \dots v_n$ *are linearly independent*
2. $v_1, v_2 \dots v_n$ ***generates*** *V*
   (a) *i.e. any vector* $v \in V$ *is a linear combination of* $v_1, v_2 \dots v_n$
   (a) *e.g.* $e_1, e_2, e_3 \dots e_n$ *are linearly independent and clearly*
      i. $e_1, e_2, e_3 \dots e_n$ *generates V*

3 8/25/23 - Fri

Last class recall V is a vector space over K $v_1, v_2 \dots v_n \in V$

Definition (Linear Combination)**.** $a_1v_1 + a_2v_2 + \dots + a_nv_n$ $W = \{\sum a_iv_i | a_i \in K\} \subseteq V$
$v_1, v_2 \dots v_n$ *are linearly dependent if* $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ *for* $\forall \neq 0a_i$
otherwise we call $v_1, v_2 \dots v_n$ *are linearly independent*

Definition (basis).

$$v_1, v_2 \dots v_n$$

is a ***basis*** *if and only if:*

1. $v_1, v_2 \dots v_n$ *are linearly independent*
2. $v_1, v_2 \dots v_n$ ***generates*** V

Theorem 3.1. Assume that $v_1, v_2 \dots v_n$ are linearly independent $\in V$ then $a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n$ then $a_i = b_i \forall i\ a_ib_i \in K$

Proof.
$(\mathrm{a}_1v_1 + a_2v_2 + \cdots + a_nv_n) - (b_1v_1 + b_2v_2 + \cdots + b_nv_n)$
$= (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$
$= 0$

□

Note: Linearly independent of $v_1, v_2 \dots v_n \Rightarrow a_i - b_i = 0$
$\Leftrightarrow a_i = b_i$

uniqueness of a_i . $if \mathrm{v}_1, v_2 \dots v_n$ is a basis of V, then $\forall v \in V$
$V = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for unique $a_i \in K$

Definition (coordinates). *if* $v_1, v_2 \dots v_n$ *is a basis, if* V=$a_1v_1 + a_2v_2 + \cdots + a_nv_n$
we call $a_1, a_2 \dots a_n$ *the coordinates of* v *with reference of this basis*
$e.g. \{\underbrace{(1,1)}_{v_1}, \underbrace{(1,-1)}_{v_2}\}$ *is a basis of* K^2

Proof. 1. linear independence
suppose $\exists a, b$ s.t. $a \cdot (1,1) + b \cdot (1,-1) = 0, K = \mathbb{R}$ or $\mathbb{C}$
$(a+b)(a-b) = 0$
$(a+b=0)$
$(a-b=0)$
$(a=0, b=0)$
Contradiction ↯
2.v_1, v_2 generates K
Given $(a,b) \in K^2$
$(a,b) = \frac{a+b}{2}(1,1) + \frac{a-b}{2}(1,-1)$ □

3.1 finitely generated vspace

Definition (Finitely generated). *We say* V *is* ***finitely generated*** *over* K *if there exists* $v_1, v_2 \dots v_n \in V$ *which generates to* V *and its finite*

Theorem 3.2. Suppose that $v_1, v_2 \dots v_n$ generates V. Let $\{v_1 \dots v_r\}$ be the maximal subset of linearly independent of vectors in $\{v_1, v_2 \dots v_n\}$ then $v_1 \dots v_r$ form a basis

Proof. By assumption, we know that $v_1, v_2 \dots v_r$ are linearly independent
$\forall k, k > r$
$v_1, v_2 \dots v_r, v_k$are linearly dependent
$i.e. a_1 + v_1 + \cdots + a_r v_r + b v_k = 0$ for some a,b $\neq 0$ in fact b $\neq 0$

$$v_k = -\frac{a_i}{b_i} v_i - \cdots - \frac{a_r}{b} v_r$$

which implies $v_1, v_2 \dots v_n$ generates V
Hence $v_1, v_2 \dots v_r$ is a basis

$\square$

3.2 dimension of v space

Theorem 3.3 (linearly dependent for n>m)**.** Let v be a vector space over K and let $\{v_1 \dots v_m\}$ be a basis of V, let $\{w_1 \dots w_n\}$ be vectors in V, assume $n > m$ then $w_1 \dots w_n$ are linearly dependent

proof by contradiction. Assumes that w_1, w_n are linearly dependent ($\star$)
For simplicity, let m=2 $n > 2$ and assume that $w_i \neq 0 \forall i$
First of all $w_1$can be written as

$$w = a_1 v_1 + a_2 v_2$$

Since a_1, a_2 cannot both be 0 WLOG we may assume that $a_1 \neq 0$ then

$$v_1 = \frac{1}{a_1} w_1 - \frac{a_2}{a_1} v_2$$

Because v_1, v_2 generates V by the definition of v space $\rightarrow$ w_1, v_2 generates V if we do this repeatly
Thus

$$w_2 = b_1 w_1 + b_2 w_2$$

where as $b_2 \neq 0$

$$v_2 = \frac{1}{b_2} w_2 - \frac{b_1}{b_2} w_1$$

This means that

$$w_1, w_2 \text{generates V which contradicts} \star$$

ϟ

$\square$

Corollary (. 1.2 cardinality of the basis) Any 2 basis of V have the same cardinality

4 8/28/23 - Mon

Last class
Let V be a V space over K
recite the definition of basis lol

Theorem 4.1 (A)**.** and let $v_1, v_2 \dots v_m$ be a basis of V and let $w_1, w_2 \dots w_n$ be any vectors in V and if $n > m$ then $w_1, w_2 \dots w_n$ are linearly independent

Proof. Last we have proven m=2
case m=3 $\{v_1, v_2, v_3\}$ is a basis for $n > 3$
$w_1 = a_1v_1 + a_2v_2 + a_3v_3 \Rightarrow v_1 = \frac{1}{a_1}w_1 - \frac{a_2}{a_1}v_2 - \frac{a_3}{a_1}v_3$
WLOG assume that $a_1 \neq 0 \Rightarrow w_1, v_2, v_3$ generates V

$$w_2 = b_1w_1 + b_2v_2 + b_2v_3$$

WLOG assume that $b_2 \neq 0$

$$V_2 = \frac{1}{b_2}w_2 - \frac{b_1}{b_2}w_1 - \frac{b_3}{b_2}v_3$$

Thus $w_1w_2v_3$ **generates** V

$$w_3 = c_1w_1 + c_2w_2 + c_3v_3$$

which gives us

$$v_3 = \star w_1 + \star w_2 + \star v_3$$

which means that $w_1w_2w_3$ **Generates** v_1
and $w_4 = w_1 + w_2 + w_3 \to ϟ$

□

This allows us to arrive at an immediate corllary

Corollary i. any 2 basis of V have the same cardinally

Proof. $\#\mathcal{B} = \{v_1, v_2 \dots v_n\}$ be a basis and let
$\#\mathcal{B}' = \{w_1, w_2 \dots w_n\}$
by the above theorem, we can immediately conclude that

$$\#\mathcal{B} = \#\mathcal{B}'$$

□

4.1 dimensions & maximal set

Definition (Maximal set)**.** $v_1, v_2 \dots v_n$ *are linearly independents* $\in V$
we say that $v_1, v_2 \dots v_n$ *form a* ***maximal set*** *of linearly independent vectors of V.*
i.e. $\forall w \in V\, w_1v_1, v_2 \dots v_n$ *are linearly dependent*

Theorem 4.2 (B)**.** Any maximal set of linearly independent vectors of V is a basis

Proof. let $v_1, v_2 \dots v_n$ be a maximal set of linearly dependent vectors of V be a basis
then, for all $w \in v$ are linearly dependent
$w_1 v_1, v_2 \dots v_n$ are linearly dependent
$bw + a_1 v_1 + a_2 v_2 + \dots + a_n v_n \to w = \star v_1 + \dots + \star v_n$
are linearly independent hence generates V

□

Theorem 4.3 (C)**.** let V be a vspace over K and let $\dim V = n$
let $v_1, v_2 \dots v_n$ be any set of linearly dependent vectors $\in V$ then $v_1, v_2 \dots v_n$ is a basis

Proof. By theorem A, we know that $\{v_1, v_2 \dots v_n\}$ is a maximal set of linearly dependent vectors
Then by theorem B $\{v_1, v_2 \dots v_n\}$ is a basis □

Note: #maximal set $= \dim V$

Corollary K. let W be a subspace of V, if $\dim w = \dim V$ then $V = W$
i.e. Any proper subspace of w has $\dim W < \dim V$

Proof. suppose that $\dim W = \dim V = n$
then $\exists w_1, w_2 \dots w_n$ such that it is a basis of W
$w_1, w_2 \dots w_n$ is also a basis of v so $W = V$

□

Corollary L. suppose that $\dim V = n$ let $v_1 \dots c_r, r < n$ be linearly independent , then we can find vectors $v_{r+1} \dots v_n$ such that $v_1 \dots v_r, v_{r+1} \dots v_n$ forms a basis of V

Proof. $\{v_1 \dots v_r\}$ is NOT a maximal set of linearly independent vectors then $\exists v_{r+1}$ such that v_1, v_r, v_{r+1} are linearly independent if $r + 1 = n$ then we are done
otherwise we can find vectors $v_{r+1} \dots v_n$ such that $v_1, v_2 \dots v_n$ are linearly independent □

Theorem 4.4 (D)**.** let V be a vspace over K such that $\dim V = n$ and W is a proper subspace of V
Then W has a basis and $\dim w < n$

Proof. if W=0, then we are done
Otherwise suppose that $W \neq 0$
There exists a $w_1 \in W \neq 0$
tbc.....

□

5 8/30/23 - Cancelled

6 9/1/23 - Fri

recall: Let V be a finite dimension vector space over K
$\dim V = \#\mathcal{B}, \mathcal{B} = \{v_1, v_2 \dots v_n\}$ is a basis

Theorem 6.1. Any max set of linearly independent vectors is a basis

Theorem 6.2. if $\dim V = n$ and $v_1, v_2 \dots v_n \in V$ are linearly independent then $v_1, v_2 \dots v_n$ is a basis

Corollary i. $\dim v = n, v_1, v_2 \dots v_r$and $v < n$ are linearly independent , then $\exists v_{r+1} \dots v_r$ such that $v_1 \dots v_r, v_{r+1}, \dots, v_n$ is a basis

Theorem 6.3. $\dim V = n$ and let W be any proper subspace of V, then W has a basis and $\dim W < n$

Proof. suppose W has no max set of linearly independent vectors then $\exists$ vectors $v_1, v_2, v_3 \dots$ such that

$$\{v_1\} \subset \{v_1, v_2\} \subset \{v_1, v_2, v_3\}$$

are linearly independent but this contradicts $\dim V = n$ ↯Thus W has a max set $\{w_1, w_2, \dots w_r\}$ of linearly independent vectors $r \leq n$ which is a basis since $w \subsetneq v$ $w_1, w_2, \dots w_r$ does not generate V
Hence $\{w_1, w_2, \dots w_r\}$ is not a basis of V
in particular $r < n$ □

6.1 sums & direct sums

Let V be a vector space over K, Let W,U be subspaces of V
recall $w + u = \{w + u | w \in W, u \in U\}$

Definition (direct sum)**.** *let W,U be subspaces of V, we say V is a direct sum of W and U if*

1. *V=W+U*
2. *$\forall v \in V$ can be written as a sum of w=w+u in a* ***unique*** *way*

we denote this $V = W \oplus U$

Theorem 6.4. let let W,U be subspaces of V, if $V = W + U$ and $W \cap U = 0$ then $V = W \oplus U$

Proof. $V = u_1 + w_1 = u_2 + w_2 \to w_1 - w_2 = u_1 - u_2 \wedge w \cap u = 0 \to w_1 = w_2, u_1 = u_2$
This is a uniqueness proof

□

Theorem 6.5. let V be a vector space, for any subspace $W \subseteq V$ there exists a **Compliment U** of W such that $V = W \oplus U$

Proof. By previous theorem $\exists$ a basis $\{w_1, w_2, \dots w_r\}$of W which can be extended to a basis $\{w_1, w_2, \dots w_r, w_{r+1} \dots w_n\}$of V such that U=span$\{w_{r+1} \dots w_n\}$Then, $V = W \oplus U$

□

Note: The author omitted a step that needed to prove that $U \cap W = 0$because the instructor's handwriting is unreadable ☹

Theorem 6.6 (Dimensions of Direct sum v spaces)**.** If $V = W \oplus U$ then $\dim V = \dim U + \dim W$

Proof. Choose a basis$\{u_1, u_2 \dots u_s\}$ of U and a basis $\{w_1, w_2, \dots w_t\}$ of W. Then $\{u_1, u_2 \dots, u_s, w_1, w_2 \dots, w_t\}$ forms a basis for V □

Remark. *Given subspaces* $w_1, w_2, w_k \subseteq V$
$w_1, w_2 + \dots w_k = \{w_1 + w_2 + \dots + w_k | w_i \in w, 1 \leq i \leq k\}$ *is a subspace of* V

Definition. *We say that* V *is a direct sum of* $w_1, \dots w_k$. *If* $\forall v \in V$ *The summation* $V = w_1 + \dots + w_k$ *is unique*
We write $V = w_1 \left|w\right\rangle\!\left\langle w\right|_2 \left|w\right\rangle\!\left\langle w\right|_3 \left|\dots\right\rangle\!\left\langle\dots\right| \left|w\right\rangle\!\left\langle w\right|_{ki} |w_1 \in w_i$

e.g.

$$\mathbb{R}^3 = \underbrace{l_x}_{\mathbb{R}_{e_1}} |$$

$l_{y\mathbb{R}_{e_2}}$ |
$l_{z\mathbb{R}_{e_3}}$

Theorem 6.7. $w_1 \dots w_k$ be subspaces of V if $V = w_1 + \dots w_k$ and $w_i \cap (\sum_{j \neq i} w_j)$ then $V = w_1 \left|\dots\right\rangle\!\left\langle\dots\right| \left|w\right\rangle\!\left\langle w\right|_k$

Proof. k=3
$V = w_1 + w_2 + w_3 = w_1' + w_2' + w_3'$
$\to w_1 - w_1' = w_2 - w_2' = w_3 - w_3'$ □

Lemma *. $w_1 \cap (w_2 + w_3) = 0$
then $v = w_1 = w_2 + w_3$

7 9/6/23 - Wed

recall: direct sum $W, U, \subseteq V$ $V = W\, |U\rangle\langle U|$ if
$\exists$ a unique $w \in W u \in U$ s.t.
V=w+u and $w \cap u = 0$
Given 2 vectors w, u
Let $w \times u$ be a direct product
$w \times u = \{(w, u) | w \in W, u \in U\}$
$W \times U$ can be endowed w/ a vector space structures
Additives $(w, u) + (w' + u') = (w + w', u + u')$
scalar multiplication a(w,u)=(aw,au)
$W \times U$ is a vector scpace over K
$ex : \dim W \times U$ $\{w_1, w_2 \dots w_n\}$ be a basis of W
$\{u_1, u_2 \dots u_m\}$ be a basis of U

$$\{(w_1, 0) \dots (w_n, 0)(0, u_1 \dots (0, u_m))\}$$

is a basis of $W \times U$
<u>in fact</u> W can be identified w/ $\{(w, 0) | w \in W\} \subseteq W \times U$
U can be identified w/ $\{(0, u) | u \in U\} \subseteq W \times U$
under such identification $W \times U = W\, |U\rangle\langle U|$
$W \subseteq W \times U$
$W \to (W, 0)$

Remark. *Given $v_1, v_2 \dots v_n$ we can define their produce $V_1 \times V_2 \times V_3 \dots \times V_n$ to be a vector space*

7.1 Matricies

we call matricies

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1_n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

This is a $m \times n$ matrix over a field K$(\mathbb{R}, \mathbb{C}, \mathbb{Q} \dots)$
Where a row vectors are
$a_1 = (a_{11}, a_{12} \dots a_{1m})$
$\dots$
$a_m = a_{m1}, a_{m2}, \dots, a_{mn}$
Where column vectors are
$a^1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{bmatrix}$
$\dots$

$$a^n = \begin{bmatrix} a_{1_n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$$

Definition (square matrix)**.** *if m=n then A is a square matrix*

Definition (zero matrix)**.** $A_{ij} = 0 \forall i, j$ *Then a is a zero matrix*

Definition (diagonal matrix)**.** *the square matrix A is called diagonal if*

$$A = \begin{bmatrix} x & & & \\ & \dots & & \\ & & \dots & \\ & & & x \end{bmatrix}$$

Definition (Upper triangular matrix)**.** *The square matrix a A is upper triangular iff*

$$A = \begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix}$$

Definition (Lower triangular matrix)**.** *The square matrix A is lower triangular iff*

$$A = \begin{bmatrix} x & & & \\ x & x & & \\ x & x & x & \\ x & x & x & x \end{bmatrix}$$

A $m \times n$ matrix is transposed when

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \sim A^t = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

we denote this $a^t \equiv^t a \equiv A^T \dots$
Let $\mathcal{M}_{m \times n}(K) = \{m \times n$ matricies over K$\}$
Addition of scalar muplication, on $\mathcal{M}_{m \times n}(K)$
$A + B(A = a_{ij}, B = b_{ij})$
$= a_{ij} + b_{ij}\ \forall c \in K, c \cdot A = (ca_{ij})$
Zero matrix $\mathcal{O} \in \mathcal{M}_{m \times n}(K)$
A $m \times n$ matrix A is called symmatri iff $A = A^t$

7.2 sys. of Linear Eqns

Given $a_{11}x_1 + \dots + a_{1n}x_n = b_1$
$\dots$

$a_{m1}x_1 + \cdots + a_{mn}x_n = b_1$
if $\forall b_i = 0$ we call this system homogeneous
which can be written as

$$(\star)x_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \cdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$$

To find a solution of such eqn is equiv to express $\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$ as a linear combination of

$A' = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix} \cdots \begin{bmatrix} a_{1m} \\ a_{2m} \\ \cdots \\ a_{mn} \end{bmatrix}$

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Theorem 8.1 (Linear independence of solns)**.** In a linear system $\star$, assume that m=n, and $A^1, A^2, \dots A^n$ are linearly independent then, $\star$ has a unique solution.

Proof. Let $A^1, A^2, \dots A^n \in \mathbb{K}^n$ and they are linearly independent . Thus they form a basis $\rightarrow B = c_1A^1 + \dots c_nA^n$ for unique numbers
i.e. $(c_1, c_2, \dots, c_n)$ is a unique solution. $\square$

8.1 Matrix multiplications

Let

$$A = a_1, a_2 \dots a_n \in \mathbb{K}^n$$

$$B = b_1, b_2 \dots b_n \in \mathbb{K}^n$$

recall their dot product $A \cdot B = a_1b_1 + a_2b_2 + \cdots + a_nb_n$
They have some nice properties

1. $A \cdot B = B \cdot A$
2. $c \in \mathbb{K}, (cA)B = c(AB) = A(cB)$

Definition (Matrix multiplication)**.** *Given 2 matrices*
$A = a_{ij} m \times n$ *matrix*
$B = b_{ij} n \times k$ *matrix*

We define a matrix multiplication AB as

$$AB = \begin{bmatrix} A_1B^1 & A_1B^2 & \dots & A_1B^k \\ A_2B^1 & A_2B^2 & \dots & A_2B^k \\ \dots & \dots & \dots & \dots \\ A_mB^1 & A_mB^2 & \dots & A_mB^k \end{bmatrix}$$

$e.g.\ a = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix} b = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$
$ab = \begin{bmatrix} 15 & 15 \\ 4 & 12 \end{bmatrix}$

In general let $A : m \times n, B : n \times kAB - m \times k$
Let A =a_{ij} be a $m \times n$ matrix
let $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$ which is a column vector $n \times 1$ matrix

Their product AB is a $\begin{bmatrix} A^1B \\ A^2B \\ \dots \\ A^nB \end{bmatrix}$ col vector
A system of linear equation $\star$ can be written as

$$Ax = B$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Where as $c : (c_1, c_2, \dots, c_m)$ is a row vector
cA=$(cA^1, cA^2, \dots, cA^n)$ has a $\# = n$
can be alternatively written as the prduct of

$$(c_1, c_2, \dots, c_m) \cdot \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots \end{bmatrix} \equiv \begin{bmatrix} A^1 & A^2 & \dots & A^m \end{bmatrix}$$

Theorem 8.2. A $m \times n$ matrix B $n \times k$ matrix A(B+C)=AB+AC
Notation A matrix $\rightsquigarrow A_{ij} = ij$ entries of A
if $A = a_{ij}$ then $(AB)_{ij} = A_iB_j$

Proof. $(A(B+C))_{ij}$
$= A_i(B+C)^j$
$= A_i(B^j + C^j)$
$= A_iB^j + A_iC^j$
$= (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij}$ □

Theorem 8.3 (commutativity of scalar multiplication)**.** let $c \in \mathbb{K}$
$(cA)B = A(cB)$
Assume $A.B = m \times n$ matrix and let $C = n \times k$ matrix then $(A+B)C = AC + BC$

Theorem 8.4 (commutativity of matrix multiplication)**.** let ABC be mutually manipulable matrices then $(AB)C = A(BC)$

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Recall $A = (a_{ij})$ be a $m \times n$ matrix $B = b_{ij}$ be a $n \times k$ matrix
$AB = (c_{ij})$ of a matrix

$(AB)_{ij} = A_iB^j = (a_1, a_2, \ldots, a_n) \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \cdots \\ b_{nj} \end{bmatrix}$

Theorem 9.1. A: $m \times n$ matrix B:$n \times k$ matrix C:$k \times l$ matrix
$(AB)C = A(BC)$ (assoc.)

Proof. A=a_{ij} B=$b_{ij}C = c_{ij}$
$AB_{ij} = \sum_{s=1}^n a_{is}b_{sj}$ $((AB)C)_{ij} = \sum_{t-1}^k (AB)_{it}C_{tj} = \sum_{t=1}^k (\sum_{s=1}^n a_{is}b_{st})c_{tj} = \sum_{t=1}^k \sum_{s=1}^n a_{is}b_{st}c_{tj}$
similarly
$(A(BC))_{ij} = \sum_{s=1}^n a_{is}(BC)_{sj} = \sum_{s=1}^n a_{is}(\sum_{t=1}^k b_{st}c_{kj})$

=$\sum_{s=1}^n \sum_{t=1}^k a_{is}b_{st}c_{tj}$ The summation can be switched □

let $A = ija$ be a $m \times n$ matrix
$ijA^t = a_{ji}$ then $A^t = n \times m$ matrix
$B = n \times k$ matrix
$B^t = k \times n$ matrix

Theorem 9.2. $(AB)^t = B^tA^t$

Proof. $ij(AB)^t = (AB)_{ji} = \sum_{s=1}^n a_{js}b_{si}$
$ij(B^tA^t) = \sum_{s=1}^n (B^t)_{is}(A^t)_{sj} = \sum_{s=1}^n b_{si}a_{js}$
□

9.1 Linear maps

Definition (Linear maps). *Let v,w, be vector spaces over K, a map $F : V \to W$ is called linear if*

1. $F(V + U) = F(U) + F(V)\forall v, u \in V$
2. $F(av) = aF(v), \forall a \in K, v \in V \equiv F(au + bv = aF(u) + bF(v))$

Remark. $F(0) = 0$

e.g. let $P : K^3 \to K^2$
1.$(x, y, z) \mapsto (x, y)$
2. $\mathbb{C}^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$
$f \mapsto \frac{df}{dx}$
3. $A = (a, b, c) \in K^3$
$F_A : K^3 \to K$ given by $F_A(x, y, z) = ax + by + cz = A \cdot (x, y, z)$
hence F_A is linear
let $A = ija$ be a $m \times n$ matrix
we define a map $F_A : \underbrace{K^n \to K^m \text{which is linear}}_{x \mapsto AX = \begin{bmatrix} A_1 \cdot x \\ a_2 \cdot x \\ \dots \\ A_m \cdot x \end{bmatrix}}$
Let V be a vector space over K
Identity map $V \to V$ 0 map $V \to V, V \to 0$ they are all linear
Given a basis $\mathcal{B} : \{v_1, v_2 \dots v_n\}$ of V
$F_{\mathcal{B}} : V \to K^n$
$v \mapsto (x_1, x_2 \dots x_n)$ where $v = x_1v_1 + \dots + x_nv_n$
and we know $F_{\mathcal{B}}$ is linear
$v = \sum x_iv_i, w = \sum y_iv_i, v + w \sum(x_1 + y_i)v_i$
hence $F_{\mathcal{B}}(w) = F_{\mathcal{B}}(V)$
see pic Given V, W such that they are vector spaces over K
$L(V, W) = \{\text{Linear maps from V to W}\}$
Then $L(V, W)$ is a vector space over K
So we have fcns F,G, $(F + G)(v) = F(v) + G(v)$
$(AF)(v) = aF(v)$
$0 \in L(v, w)$
$0 : v \to w$
$v \mapsto 0$

Theorem 9.3. v,w, as arb. and let $\mathcal{B} = \{v_1, v_2 \dots v_n\}$ be a basis of V, and let $\{w_1, w_2 \dots w_n\}$ be a arb set of vectors in W.
There exists a unique linear map $F : V \to W$ such that $F(v_1) = w_1, f(v_2) = w_2 \dots f(v_n) = w_n$

Proof. F(v)=$a_1w_1 + \cdots + a_nw_n$
where $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ Then one way check F is linear □

see pic

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Recall: A linear map $F : V \to W$, $V, W/K$
$F(au + bv) = aF(u) + bF(v)a, b \in K, u, v \in V$
$L(v, w) = \{\text{Linear maps from} V \to W\}$
vector spaces over K

10.1 Kernel and image of linear maps

Let F $V \to W$ Linear

Definition (kernel)**.** $\ker F = \{v \in V | f(v) = 0\} \subseteq V$

Lemma . $\ker F$ is a subspace of V

Proof. Given u,v $\in \ker F$, $\forall a, b, \in K$ $F(au+bv) = aF(u)+bF(w) = 0 \to au+bv \in \ker F$ □

Lemma . $F : V \to W$ is injective if and only if $\ker F = 0$

Proof. $\to$ suppose F is injective, then the only element that maps to $0 is 0$

$\leftarrow$
$\forall u, v \in V$ suppose $F(u) = F(v)$ then by injectivity of $F(u - v) = 0$
since $\ker F = 0, u - v = 0 \to u = v$ □

e.g. $A = (2, 1, -1) \in K^3$
$F_A : K^3 \to K$
$(x, y, z) \mapsto (2x + y - z)$
$\ker F_A = \{(x, y, z) \in K^3 | 2x + y - z = 0\}$
similiarly $A = ija : m \times n$ matrix
$F_A : K^m \to K^n$
X:col vector
$X \mapsto AX$
$\ker F_A = \{X \in K^n | A \cdot X = 0\}$
$= \{\text{solution of } \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_n = 0 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{bmatrix} \} \text{ call it} \star \ker F_A = 0 \iff \star \text{ has trivial solution} \iff$
F_A is injective $\iff$ col vectors are linearly independent

Theorem 10.1. $F : V \to W$ linear s.t. $\ker F = 0$
if $v_1, v_2 \dots v_n \in V$ are linearly independent then $F(v_1), f(v_2) \dots, f(v_n)$ are also linearly independent

Proof. by contradiction suppose we have $a_1 f(v_1) + \cdots + a_n f(v_n) = 0$
by linearility we have $f(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = 0$ then $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$
hence $a_1 = \cdots = a_n = 0$ ☐

10.2 image

$F : V \to W$ linear
$ImF = F(v) = \{f(v)|v \in V\}$

Lemma I. m F is a subspace of W

Proof. $\forall F(v), F(u) \in imF, \forall a, b \in K$
$aF(u) + bF(v) = F(au + bv) \in imF$

☐

Given $v_1, v_2 \dots v_n \in V$
$F : K^n \to V$
$(a_1, a_2 \dots a_n) \mapsto a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$
imF= {linear combinations of $v_1, v_2 \dots v_n$}
$= span\{v_1, v_2 \dots v_n\}$
$F_A : k^n \to k^m$
$X = \begin{bmatrix} x_1 \\ x_1 \\ \dots \\ x_n \end{bmatrix} \mapsto A \cdot X$ note that $A \cdot X = \star\star$ or a lin comb of the col vectors

im$F_A = span$\{column of vectors\}
Given $A = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$
$K^3 \to k$
$(x, y, z) \mapsto 2x + y - z$
$\dim \ker F_A = 2$
Im $F_A = k$ dim Im $F_A = 1$

Theorem 10.2. $F : v \to w$ linear
$\dim \ker F$+dim Im F=dim V

Proof. Choose a basis $\{w_1, w_2 \dots w_n\}$ of Im F also a basis $v_1, v_2 \dots v_n$ of $\ker F$ Choose There exists $u_1, u_2 \dots u_m \in V$
s.t. $F(u_1) = w_1 \dots f(um) = w_m$
we claim that $\{u_1, u_2, \dots u_m, v_1 \dots v_n\}$ is a basis of V

1. it generates V
   $\forall v \in V, f(v) \in$im F
   $F(v) = a_1 w_1 + \cdots + a_m w_n$
   $= a_1 f(u_1) + \cdots + a_m F(u_m)$
   $= F(a_1 u_1 + \ldots a_m u_m) \to v - \sum a_i u_i \in \ker F \to v - \sum a_i u_i = \sum b_j v_j$
2. $\{v_1, v_2 \ldots v_n, u_1 \ldots, u_m\}$ is linearly independent
   suppose $\sum a_i v_i + \sum b_j u_j = 0$

we apply F hence $F(b_i u_j) = 0$
$= \sum b_j F(u_j)$
$= \sum b_j w_j = 0$
by linearly independent of$\{w_j\}, b_j = 0 \forall j \to \sum a_i v_i = 0, a_i = 0 \forall i$

□

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$A = a_{ij} : m \times n$ matrix $F_A : K^n \to k^m$
$X \mapsto AX$
$\ker F_A = \{$solns of AX=0$\}$
Im F=span$\{$sol vectors $\} \to n = \dim\{$sols of AX=0$\} + \dim span\{$col of A$\}$
Let $n = m$ AX=0 has only trivial sol $\iff$ cols of A is a basis of K^n

Theorem 11.1. $F : V \to W$: Linear Map
assume $\dim V = \dim W$ if $\ker F = 0 or$ ImF=W then F is a bijection.

Proof. Let $\underbrace{\ker F = 0}_{\text{F is injective}}$ by thm $\iff$ $\dim ImF = \underbrace{\dim V = \dim W}_{\text{By assumptions}}$
$\iff ImF = W$
i.e. F is injective $\iff$ F is surjective

□

11.1 Composition of linear maps

Theorem 11.2. Given 2 linear maps :F$U \to V$ G:$V \to W$ Their composition $G \circ F$: $U \to W$ is linear

Proof. $\forall u_1, u_2 \in U, a_1 a_2 \in K$
$G \circ F(a_1 u_1 + a_2 u_2)$
=$G(F(a_1 u_1 + a_2 u_2))$
=$G(a_1 F(u_1) + a_2 F(u_2))$
=$a_1 G(F(u_1)) + a_2 G(F(u_2))$
=$a_1 (G \circ F)(u_1) + a_2 (G \circ F)(u_2)$
hence $G \circ F$ is linear

□

> **Theorem 11.3.** $F : V \to W$ linear and bijective then its inver se $G : W \to V$ is also linear

Proof. $\forall w_1 w_2 \in W a_1, a_2 \in K$
$G(a_1 w_1 a_2 w_2)$ want to proof $= a_1 G(w_1) + a_2 G(w_2)$
we apply $F, F(G(a_1 w_1 + a_2 w_2)) \equiv a_1 w_1 + a_2 w_2$
where as $a_1 G(w_1) + a_2 G(w_2)$ apply F$F(a_1 G(w_1) + a_2 G(w_2)) = a_1 F(G(w_1)) + a_2 F(G(w_2)) = a_1 w_1 + a_2 w_2$
since F is bijective we are done

□

e.g. $K^2 \xrightarrow{F} K^2$
$(x, y) \mapsto (2x - y, x + y)$
is that a bijection?

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x - y \\ x + y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are linearly independent $\to F$ is bijective

2. Only trivial solution $= \ker F = 0$ hence F is bijective

Definition (isomorphism)**.** *A linear map F is a called isomorphism or inveritble if F is also a bijection*
ie F: invertible $\to F^{-1}$ *is linear*

$A \in \mathcal{M}_{m\times n}(k) \rightsquigarrow F_A \in \mathcal{L}(k^n, k^m)$
$\mathcal{M}_{m\times n}(k) \to \mathcal{L}(k^n, k^m)$
linear

> **Theorem 11.4.** $\mathcal{M}_{m\times n}(k) \to L(k^n, k^m)$
> is injective i.e. $F_A = F_B \to A = B \forall A, B, \in \mathcal{M}_{m\times n}(k)$

Proof. since F is linear it is suffcient to show that $F_A = 0 \to A = 0$

$$F_A(X) = AX = \begin{bmatrix} A_1 \cdot X \\ A_2 \cdot X \\ \dots \\ A_n X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \forall X \in K^n$$

$x \in k^n$ for each $i = 1 \dots m \to A_i X = 0, \forall X \in k^n \to A_i = 0$

□

Lemma . If $a_1x_1 + \dots a_nx_n = 0$
$\forall x_i \in K$
then $a_i = \dots = a_n = 0$

> **Theorem 11.5.** $F : \mathcal{M}_{m\times n}(k) \to L(k^n, k^m)$
> $A \mapsto F_A$ is surjective
> ie for any linear map $\mathcal{Q} : k^n \to k^m$ Q=F_A for some A.

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$F : V \to W, G : W \to U$ be linear maps
Composition $G \circ F : V \to U$

1. $G \circ F$ is linear

2. $G \circ (a_1F_1 + a_2F_2) = a_1G \circ F_1 + a_2G \circ F_2$

$(b_1G_1 + b_2G_2) \circ F = b_1G_1 \circ F + b_2G_2 \circ F$
$\mathcal{M}_{m\times n}(k) \xrightarrow{\varphi} \mathcal{L}(k^n, k^m)$
$A \mapsto F_A$
e.g. $A = \begin{bmatrix} 2 & 1 \\ -1 & 5 \\ 1 & 0 \end{bmatrix} \rightsquigarrow F_A : k^2 \to k^3$
$F_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$
$F_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$
In general
Given $A = (a_{ij}) : m \times n$ matrix
$F_A(e_i)$ =ith column of A
e.g. $e_i = \begin{bmatrix} 0 \\ \dots \\ 1 \\ 0 \\ 0 \end{bmatrix}$

> **Theorem 12.1.** $\varphi : \mathcal{M}_{m\times n}(k) \to \mathcal{L}(k^n, k^m)$ is injective i.e. $F_A = F_B \forall A, B \in \mathcal{M}_{m\times n}(k)$

> **Theorem 12.2.** φ is onto i.e. $\forall$ linear maps $F : k^n \to k^m$ there exists a $m \times n$ matrix A such that $F = F_A$

Lemma . Given a linear map $F : k^n \to k$
$F = \underbrace{A \cdot X}_{\text{dot product}}$ where $A = F(e_1), F(e_2) \ldots, F(e_n) \in k^n$

e.g. $F : k^n \to k$
$F(x_1 \ldots x_n) = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$
$F(e_1) = a_1, F(e_2) = a_2 \ldots$

Proof. We can write $X = x_1 e_1 + \cdots + x_n e_n$
$F(x) = x_1 F(e_1) + \cdots + x_n F(e_n)$
$=F(2_1), \ldots, F(e_n) \cdot (x_1 \ldots x_n)$

$\square$

proof of theorem

Proof. Let $F : k^n \to k^m$ be a linear map
Let $P : k^m \to k$ be the ith projection
$p_i \cdot \begin{bmatrix} y_1 \\ \ldots \\ y_n \end{bmatrix} = y_i$
Then $p_i \circ F : k^n \to k$ is linear, by the lemma $p_i \circ F(X) = A_i \cdot X_i$ for some $A_i \in k^n$
In fact $F(x) = \begin{bmatrix} A_1 \cdot X \\ A_2 \cdot X \\ A_3 \cdot X \end{bmatrix} = AX, A + \begin{bmatrix} A_1 \\ A_2 \\ \ldots \\ A_m \end{bmatrix}$ $\square$

$a_1, \ldots, a_n \in k^n$
$A = \begin{bmatrix} A_1 \\ A_2 \\ A_m \end{bmatrix} : m \times n$ matrix
$A_1 \begin{bmatrix} a_z 11 & a_{12} & \ldots & a_{1n} \end{bmatrix} \in k^n$
$A_2 \begin{bmatrix} a_{21} & a_{22} & \ldots & a_{2n} \in k^n \end{bmatrix}$
$\ldots$

Theorem 12.3 (A+B)**.** let$\Phi : \mathcal{M}_{m \times n}(k) \to \mathcal{L}(k^n . k^m)$
is an isomorphism of vector spaces over k
for any $F \in \mathcal{L}(k^n . k^m)$
there exists a unique $m \times n$ matrix A
such that $F = F_A$ we call A to be associated matrix of F

e.g. $F : k^3 \to k^2$
$(x, y, z) \mapsto (x + y, z)$
find the associated matrix of F

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : 2 \times 3$$

$F(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}) F(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In general $F : k^n \to k^m$

$F = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}$ e.g. $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

a rotation by θ counter-clockly

what is the matrix A?

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

e.g. $R(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Lemma . Let $Am \times n$ matrix and $Bn \times l$ matrix
$F_A : k^n \to k^m F_B : k^l \to k^n$
then $F_A \circ F_B = F_{AB}$

Proof. for every $x \in k^n F_{AB} = (AB)X = A(BX) = (F_A \circ F_B)(x)$ □

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warning: Midterm 9/27 chap I-IV
recall $\mathcal{M}_{m\times n}(k) \xrightarrow{\sim} L(k^n, k^m)$

hence $A \mapsto^{\varphi} F_A$

$F_{AB} = F_A \circ F_B$

let n=m, F_A is invertible iff A is invertible

Proof. $\to$
$F_A : K^n \to K^n$ invertible
there exists $G : k^n \to k^n$ such that $F_A \circ G = Id, G \circ F_A = Id$ $G + F_B$ fora unique matrix B
then $F_A \circ F_B = F_{AB} = Id = F_I$
$F_B \circ F_A = F_{BA} = Id = F_I$

□

Theorem 13.1. $A : n \times n$ matrix and let A^i be the ith col of A, then A is invertible iff $A^1 \dots A^n$ are linearly independent

Proof. consider the associated linear map

$$F_A : K^n \to K^n X \mapsto AX$$

$$F_A(e_i) = A^i$$

As explained previously A is invertible iff F_A is invertible. F_A is invertible then $A^1 \dots A^n$ are linearly independent
suppose we have $c_1 A^1 + \cdots + c_n A^1 = 0$
then we know that $c_1 F_A(e_1) + \cdots + c_n F_A(e_n) = 0 \iff F_A(c_1 e_1 + \cdots + c_n e_n) = 0 \iff c_1 e_1 + \cdots +_n e_n = 0 \iff c_1 = \cdots = c_n = 0$
$\leftarrow$ suppose $A^1 \dots A^n$ are linearly independent then they for a basis of k^n. There exists a linear map $G : k^n \to k^n$ s.t.

$$G(A^1) = e_1 \dots G(A^n) = e_n$$

clearly $F_A \circ G = I$ $G \circ F_A = I$

$\square$

e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible since A^1, A^2, A^3 are linearly independent

e.g.$F(x, y, z) = (x - 2y, y - z, 2z), F_A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$
This matrix is invertible because $\dim \mathbb{R}^3 = \dim \mathbb{R}^3 \to$ demension is the same also $\ker F = \mathcal{O} \to$ this is injective hence this is bijective and A is invertible

13.1 Bases, matrices and linear maps

V is a vector space over K and let $\mathcal{B}$ be a basis $\{v_1, v_2 \dots v_n\}$
$k^n \xrightarrow{\varphi} V$ is an isomorphism iff $(a_1, a_2 \dots a_n) \mapsto a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ that $\{v_1, v_2 \dots v_n\}$ is a basis

Proof. φ is injective $\iff v_1, v_2 \dots v_n$ are linearly independent
φ is surjective $\iff v_1, v_2 \dots v_n$ generates v. $\square$

Given a linear map $F : V \to W$
Let $\mathcal{B}$ be a basis of V, $\mathcal{B}'$ be a basis of W
Let $\dim V = n$ $\dim W = m$

$$V \xrightarrow{F} W$$

$k^n \xrightarrow{F^{\mathcal{B}}_{\mathcal{B}'}} k^m$ Let $M^{\mathcal{B}}_{\mathcal{B}'}$ be a matrix associated to $F^{\mathcal{B}}_{\mathcal{B}'}$

Definition. $M_{\mathcal{B}'}^{\mathcal{B}}$ *(F)is the matrix associated to F with respect to* $\mathcal{B}, \mathcal{B}'$

Exercise . V $\subseteq k^3, V\{(x,y,z)|x+y+z=0\}$ $F : k^3 \to V$

$F(x,y,z) = (x-y, y-z, z-x)$ we have standard basis $e_1 = (1,0,0), e_2(0,1,0), e_3(0,0,1)$
$v_1 = (1,-1,0), v_2 = (0,-1,1)$ clearly forms a basis of V. $F(1,0,0) = (1,0,-1) = v_1 - v_2$
$F(0,1,0) = (-1,1,0) = -v_1$
$F(0,0,1) = (0,-1,1) = v_2$
we claim that $M_{\mathcal{B}'}^{\mathcal{B}}(F) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$