Assignment 5

Due: Dec 14, 2020

Question 1: (50 points)

Suppose there are 3 securities S_1, S_2, S_3 with return vector $r = (r_1, r_2, r_3)$, and

$$\mu = E[r] = \begin{bmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{bmatrix}; \Sigma = Cov(r) = \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix}$$

Let $x = (x_1, x_2, x_3)$ be your weight of the portfolio, then the expected portfolio return is $\mu_p = \mu^T x$, and the variance of the portfolio return is $\sigma_p^2 = x^T \Sigma x$.

Suppose you want to find the optimal portfolio weight x such that $\frac{1}{2}\sigma_p^2$ is minimized, subject to $\mu_p \geq 0.05$ and $x_1 + x_2 + x_3 \leq 1$.[1]

So the problem becomes

$$\begin{aligned} \min_x f(x) = \frac{1}{2} x^T \Sigma x = \frac{1}{2}(&0.01x_1^2 + 0.002x_1x_2 + 0.001x_1x_3 \\ &+0.002x_2x_1 + 0.011x_2^2 + 0.003x_2x_3 + \cdots) \\ \text{subject to: } \mu^T x = &\ 0.0427x_1 + 0.0015x_2 + 0.0285x_3 \geq 0.05 \\ &x_1 + x_2 + x_3 \leq 1 \end{aligned}$$

- Write this problem into the form in page 8 to find f, ui and ci. Then use function **constrOptim()** to find the optimal portfolio weights x and the optimal value, you can choose starting point as $(2, -2, 0)$. (Hint: you can use x%*%Sigma%*%x in R to calculate $x^T \Sigma x$)
- Write this problem into the form in page 15 to find D, d, A and b. Then use function **solve.QP()** to find the optimal portfolio weights x and the optimal value.

Question 2: (20 points)

- Suppose Treasury Yield Rates in time

$$t = [0.25, 0.5, 1, 2, 3, 5, 7, 10]$$

[1] Usually we need $x_1 + x_2 + x_3 = 1$, but it is hard when we use function **constrOptim()**

years are given by

$$r = [0.09\%, 0.11\%, 0.16\%, 0.20\%, 0.24\%, 0.36\%, 0.53\%, 0.64\%].$$

Use *both linear and spline* interpolation to approximate the corresponding rates r_{out} at time

$$t_{out} = [0.75, 1.5, 4, 6, 8]$$

years. (You can use either write functions yourself or use built-in functions)

Question 3: (30 points)

Recall in Black-Scholes model, stock price follows geometric Brownian motion

$$S(T) = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W(T)}$$

Then the call option price is given by

$$\begin{aligned} c(S_0, K, T, \sigma, r) &= e^{-rT} E^Q[(S(T) - K)_+] \\ &= e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K)_+ f(x)dx \\ &= e^{-rT} \int_{-d_2}^{+\infty} (S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K)\phi(x)dx; d_2 = \frac{\ln\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}$$

where ϕ is the probability density function of standard normal distribution.

- Calculate the call option price using numerical integration with $S_0 = 100, K = 100, T = 1, r = 0.05, \sigma = 0.2$, compare the result with Black-Scholes formula. (You can use Trapezoidal rule or Simpson's rule, or use built-in functions)