MAT 322 Spring 2018 Midterm II Practice Exam

1. Do problem # 7, p.151 of Munkres text (or check again your solution of it on earlier HW).
Prove that the solid torus is a manifold with boundary.

2. Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y, z, w) = (xyz, yzw).$$

For what points $p = (p_1, p_2) \in \mathbb{R}^2$ is $F^{-1}(p)$ a submanifold of $\mathbb{R}^4$. What is the dimension of such submanifolds?

Compute the tangent space of the submanifold containing the point $(1, 1, 1, 1)$ at that point.

3. Let $\Lambda^2(\mathbb{R}^3)$ be the space of alternating 2-tensors on $\mathbb{R}^3$.
What is $dim\, \Lambda^2(\mathbb{R}^3)$?
Find a basis for $\Lambda^2(\mathbb{R}^3)$.
Is there an alternating 2-tensor $\omega \in \Lambda^2(\mathbb{R}^3)$ such that

$$\omega \wedge \omega \neq 0.$$

4. Let e_i be the standard basis of $\mathbb{R}^n$ with dual basis e_i^*. If $v^* = \sum v_i e_i^*$ is a general element in $(\mathbb{R}^n)^*$, let T be the k-fold tensor product

$$T = v^* \otimes \cdots \otimes v^*.$$

For j fixed, compute $T(e_j, e_j, \cdots, e_j)$.

5. Show that $(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)(v_1, \cdots, v_k)$ is the determinant of the $k \times k$ minor obtained by selecting the $i_1, \cdots, i_k$ columns of the matrix formed by writing the vectors v_j as row vectors.

5. State the definition of (smooth) diffeomorphism F between smooth submanifolds of $\mathbb{R}^n$.
Find a diffeomorphism $F : \mathbb{R}^2 \to S$, where S is the paraboloid of revolution given by $z = x^2 + y^2$.
Prove there is no diffeomorphism between $\mathbb{R}^2$ and the 2-sphere $S^2(1)$ of radius 1.