CS 388C: COMBINATORICS AND GRAPH THEORY
Lecture 3

Scribes: Shilpi Goel, Joyce Whang

January 24, 2012

1 The Kővári-Sós-Turán Upper Bound

Proof. ...Continued from Lecture 2 ...

$$\sum_{x \in V_1} \binom{d(x)}{t} \leq (s-1)\binom{n}{t}$$

Note that the left term corresponds to the number of t-stars in the graph, if we replace $\binom{d(x)}{t}$ by zero, when $d(x) < t$.

A detour: The following box provides a review of convex functions and Jensen's inequality, that we use in the rest of the proof.

Convex Function:
A real-valued function $f(x)$ is convex if for any two vaues a and b in the domain of f:

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$$

for any

$$0 \leq \lambda \leq 1.$$

If we plot the graph of a convex function $f(x)$ and draw a line connecting any two points $(a, f(a))$ and $(b, f(b))$ of the curve of the function, then all the points for values between a and b lie below that line. Another way to know whether $f(x)$ is convex is if its second derivative is non-negative.

Jensen's Inequality:
If f is a convex function and

$$0 \leq \lambda_i \leq 1, \quad \sum_{i=0}^{n} \lambda_i = 1,$$

then:

$$f(\sum_{i=1}^{n} \lambda_i X_i) \leq \sum_{i=1}^{n} \lambda_i f(X_i).$$

Back to the inequality in consideration:

$$\sum_{x \in V_1} \binom{d(x)}{t} \leq (s-1)\binom{n}{t}$$

We can transform the above inequality further using Jensen's Inequality. However, to use Jensen's Inequality, we require that

$$\binom{d(x)}{t}$$

be convex. Since $d(x)$ need not be necessarily greater than t for all values of $x \in \mathrm{V}_1$, we need to define a convex function which we can use in place of the binomial coefficient in the above inequality. Hence, we define a function $f(u)$ such that:

$$f(u) = \begin{cases} \binom{u}{t} & \text{if } u \geq t \\ \\ 0 & \text{if } u < t \end{cases}$$

Now, we can replace

$$\binom{d(x)}{t}$$

with $f(d(x))$. The number of t-stars in the graph are:

$$\#\text{ t-stars} = \sum_{x \in V_1} f(d(x))$$

We apply Jensen's Inequality here and assign $\lambda_x = 1/n$ for every x. We note that these coefficients sum to 1.

$$\begin{aligned} \#\text{ t-stars} &= n \sum_{x \in V_1} \frac{1}{n} f(d(x)) \\ &\geq n f(\sum_{x \in V_1} \frac{d(x)}{n}) \end{aligned}$$

Summing the degrees of x over every x gives us the number of 1s in the matrix (or the number of edges in the bipartite graph).

$$n f(\sum_{x \in V_1} \frac{d(x)}{n}) = n f(\frac{z}{n}) = n f(y), \text{ since } z = ny$$

The original inequality becomes:

$$\begin{aligned} n f(y) &\leq (s-1)\binom{n}{t} \\ \Rightarrow n\binom{y}{t} &\leq (s-1)\binom{n}{t} \end{aligned}$$

Note that we can assume $y \geq t$, since otherwise, the statement of the theorem trivially holds. Now, since

$$\binom{n}{t} = \frac{n(n-1)\ldots(n-t+1)}{t!}$$

we can state that

$$\frac{(n-t+1)^t}{t!} \leq \binom{n}{t} \leq \frac{n^t}{t!}$$

So the inequality becomes:

$$
\begin{aligned}
n\binom{y}{t} &\leq (s-1)\binom{n}{t} \\
\Rightarrow n(y-(t-1))^t &\leq (s-1)n^t \\
\Rightarrow n^{\frac{1}{t}}(y-(t-1)) &\leq (s-1)^{\frac{1}{t}}n \\
\Rightarrow y-(t-1)) &\leq (s-1)^{\frac{1}{t}}n^{1-\frac{1}{t}}
\end{aligned}
$$

Since z = ny:

$$z \leq (s-1)^{\frac{1}{t}}n^{2-\frac{1}{t}} + n(t-1)$$

which is our original conjecture, i.e.

$$k_{s,t}(n) \leq c_{s,t}n^{2-\frac{1}{\min\{s,t\}}}$$

□

2 Shannon's Counting Argument

Boolean circuits are directed acyclic graphs. A boolean function of n variables is a mapping from a binary string of length n to 0 or 1. Shannon's counting argument gives an estimate of the number of gates needed to represent a boolean function.

Theorem 1 (An Easy Upper Bound)**.** *Any boolean function of n variables can be implemented with $\leq n2^n + n$ gates, using a fan-in ≤ 2 and operators ($\wedge$, $\vee$, $\neg$).*

Proof. Given a boolean function in DNF, the upper bound of the number of gates needed to implement it can be derived by taking the sum of the upper bounds of:

- The number of conjunctions in the DNF: There are at most (n-1) conjunctions in every term, and an optimal DNF does not need more than 2^n terms. Thus, there are at most (n-1)2^n number of conjunctions (i.e. $\wedge$).
- The number of disjunctions: There are at most 2^n terms, and hence, (2^n -1) disjunctions (i.e. $\vee$) in the DNF.
- The number of possible negations of the inputs: There are n inputs and hence, n $\neg$ gates corresponding to their negations.

The sum of the three gives an upper bound $n2^n + n$. □

Theorem 2 (Shannon,'49)**.** *[A Lower bound] Almost all boolean functions of n variables require $\geq \Omega(\frac{2^n}{n})$ gates to compute with a fan-in ≤ 2 and gates ($\wedge$, $\vee$, $\neg$).*

Proof. First, instead of almost all boolean functions, we will prove the statement for at least one boolean function.

1. *Count the number of all possible boolean functions of n variables.*
   For n variables, there are 2^n possible input values. Each value may be assigned the truth value 0 or 1 by a function. Hence, there are 2^{2^n} possible ways to fill the truth table for n variables. The number of possible boolean functions for n variables are 2^{2^n}.

2. *Count the number of different boolean circuits (fan-in ≤ 2 and operators ($\wedge$, $\vee$, $\neg$)) with n input variables and m gates.*
   We will do a rough overestimation of the number of boolean circuits here but still, this number will be less than all the possible boolean functions of n variables.

We assign labels to the elements in a circuit — we assume that any gate can be given any label and we specify the incoming edges for each gate.

For a given circuit with m gates, there are at most $\leq (n+5)$ possible labels. For a given gate, there are $\leq m^2$ possible pairs of incoming edges (from the outputs of other gates). Hence, the number of different boolean circuits for m gates and n variables are:

$$\leq ((n+5) \times m^2)^m$$
$$= 2^{m \times (2 \log m + \log (n+5))}$$

If we substitute $m = \frac{2^n}{10n}$, then the expression will be less than 2^{2^n}.

$\square$

NOTE:

- There are many circuits for a single expression but just one expression for a circuit. As long as m is small enough, the number of circuits will be less than the number of functions. Thus, there will be functions of n variables which circuits with m gates will not be able to compute.
- The statement of the theorem mentioned the words "almost all" whereas we proved this property for at least one circuit. "Almost all" in the theorem statement means that as n tends to infinity, the probability that a random function requires circuits of size AT LEAST $\Omega(\frac{2^n}{n})$ tends to 1. Thus, the probability of a random function having circuits of size $o(\frac{2^n}{n})$ tends to 0 (or as n tends to infinity, the fraction of n-ary functions with circuit complexity less than $\Omega(\frac{2^n}{n})$ tends to zero).

This proof of the theorem by Shannon can be viewed as a special case of the **Probabilistic method**: we picked a random function and found the probability of having a circuit of a small size that computes that function. If we pick objects from a uniform distribution, then the probability that objects satisfy a certain property is:

$$\Pr(\text{objects satisfy a certain property}) = \frac{\text{Number of objects that satisfy the property}}{\text{Total number of objects}}$$

In this proof by Shannon:

$$\Pr(f \text{ has a circuit of size} \leq m) = \frac{\text{Number of circuits of size} \leq m}{\text{Total number of functions}}$$

As long as this probability is < 1, then there exist functions that *do not* satisfy this property.