As before, let $\mathcal{M} = \{ \mathbb{P} : \mathbb{P} \geq 0,\ \mathbb{P}(\mathbb{R}^n) = 1 \}$ be the set of probability measures on $(\mathbb{R}^n, \mathcal{F})$. We considered a set $\mathcal{P} \subset \mathcal{M}$ with given mean μ_0 and covariance $\Gamma_0 \in S^n_{++}$ as the uncertainty set.

* How about other uncertainty sets?

  – Usually, we only have estimates of the true mean and covariance, rather than their exact values.

    e.g.: Suppose that we have the samples $\xi_1, \xi_2, \cdots, \xi_N$ of the random variable in question. Then, we can estimate the true mean and covariance by

$$\mu_0 = \frac{1}{N} \sum_{i=1}^{N} \xi_i \quad ; \quad \Gamma_0 = \frac{1}{N} \sum_{i=1}^{N} (\xi_i - \mu_0)(\xi_i - \mu_0)^T.$$

  – Often, we have support information of the probability measure.

    e.g.: Suppose that the random vector represents production levels of various activities. Then, $\mathbb{P}(\mathbb{R}^n_+) = 1$

* To model uncertainties in the mean estimate, consider

$$(\mathbb{E}_{\mathbb{P}}[\xi] - \mu_0)^T \Gamma_0^{-1} (\mathbb{E}_{\mathbb{P}}[\xi] - \mu_0) \leq \gamma_1 .$$

Interpretation: The true mean lies in an ellipsoid centered at μ_0 with radius γ_1. Here, we assume $\Gamma_0 \succ 0$.

* To model uncertainties in the covariance estimate, consider

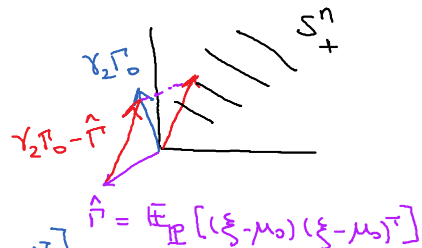
$$\mathbb{E}_{\mathbb{P}}\left[(\xi - \mu_0)(\xi - \mu_0)^T \right] \preceq \gamma_2 \Gamma_0$$

Interpretation: The matrix on the LHS lies in a semidefinite cone determined by Γ_0 and γ_2

Note: $\hat{\Gamma} = \mathbb{E}_{\mathbb{P}}\left[(\xi - \mu_0)(\xi - \mu_0)^T \right]$ is NOT the covariance of ξ

The covariance is

$$\mathbb{E}_{\mathbb{P}}\left[(\xi - \mathbb{E}_{\mathbb{P}}[\xi])(\xi - \mathbb{E}_{\mathbb{P}}[\xi])^T \right].$$



Remark: One may be tempted to consider a "lower" bound on the covariance estimate:

$$\gamma_3 \Gamma_0 \preceq \mathbb{E}_{\mathbb{P}}[(\xi-\mu_0)(\xi-\mu_0)^T]$$

However, there is no known tractable reformulation of this constraint.

* To model support information, we simply use a closed convex set C, which is assumed to contain the support of the underlying probability measure.

Then, we can formulate the following uncertainty set:

$$\mathcal{P} \triangleq \mathcal{P}(C,\mu_0,\Gamma_0,\gamma_1,\gamma_2) = \left\{ \mathbb{P} \geq 0 : \begin{array}{c} \mathbb{P}(C) = 1, \\ (\mathbb{E}_{\mathbb{P}}[\xi]-\mu_0)^T \Gamma_0^{-1} (\mathbb{E}_{\mathbb{P}}[\xi]-\mu_0) \leq \gamma_1, \\ \mathbb{E}_{\mathbb{P}}[(\xi-\mu_0)(\xi-\mu_0)^T] \preceq \gamma_2 \Gamma_0 \end{array} \right\}$$

Now, let $h: \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ be a cost function, where the first argument is the decision vector and the second is the perturbation vector. Let $\mathcal{U} \subseteq \mathbb{R}^p$ be the set of feasible decisions. Consider

$$\text{(DR-M)} \qquad \inf_{x \in \mathcal{U}} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[h(x,\xi)]$$

* How to analyze (DR-M)?

Again, we start with the inner problem:

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[h(x,\xi)] = \begin{array}{ll} \sup & \int_C h(x,\xi)\, d\mathbb{P}(\xi) \\ \text{s.t.} & \int_C d\mathbb{P}(\xi) = 1, \\ & \int_C (\xi-\mu_0)(\xi-\mu_0)^T d\mathbb{P}(\xi) \preceq \gamma_2 \Gamma_0, \\ & \int_C \begin{bmatrix} \Gamma_0 & \xi-\mu_0 \\ (\xi-\mu_0)^T & \gamma_1 \end{bmatrix} d\mathbb{P}(\xi) \succeq 0. \\ & \mathbb{P} \geq 0. \end{array}$$

The second-to-last constraint on the RHS follows from the Schur complement:

$$(\mathbb{E}_{\mathbb{P}}[\xi]-\mu_0)^T \Gamma_0^{-1} (\mathbb{E}_{\mathbb{P}}[\xi]-\mu_0) \leq \gamma_1$$

$$\Longleftrightarrow \begin{bmatrix} \Gamma_0 & \mathbb{E}_{\mathbb{P}}[\xi]-\mu_0 \\ (\mathbb{E}_{\mathbb{P}}[\xi]-\mu_0)^T & \gamma_1 \end{bmatrix} \succeq 0$$

$$\Longleftrightarrow \quad \mathbb{E}_{\mathbb{P}} \begin{bmatrix} \Gamma_0 & \xi - \mu_0 \\ (\xi - \mu_0)^T & \gamma_1 \end{bmatrix} \succeq 0.$$

The RHS can be simplified as

$$\begin{aligned} \sup \quad & \int_C h(x,\xi)\, d\mathbb{P}(\xi) \\ \text{s.t.} \quad & \int_C d\mathbb{P}(\xi) = 1, \quad (r) \\ & \int_C (\xi\xi^T - \xi\mu_0^T - \mu_0\xi^T)\, d\mathbb{P}(\xi) \preceq \gamma_2 \Gamma_0 - \mu_0\mu_0^T, \quad (Q) \\ & \begin{bmatrix} \Gamma_0 & \int_C \xi\, d\mathbb{P}(\xi) - \mu_0 \\ \int_C \xi^T d\mathbb{P}(\xi) - \mu_0^T & \gamma_1 \end{bmatrix} \succeq 0, \quad \left(\begin{bmatrix} P & p \\ p^T & s \end{bmatrix} \right) \\ & \mathbb{P} \geq 0. \end{aligned} \qquad \text{(DR-S)}$$

As before, we write down the Lagrangian of (DR-S):

$$\begin{aligned} \mathcal{L}(\mathbb{P}, r, Q, P, p, s) = & \int_C h(x,\xi)\, d\mathbb{P}(\xi) + r\left(1 - \int_C d\mathbb{P}(\xi)\right) \\ & - \left\langle Q, \int_C (\xi\xi^T - \xi\mu_0^T - \mu_0\xi^T)\, d\mathbb{P}(\xi) - \gamma_2\Gamma_0 + \mu_0\mu_0^T \right\rangle \\ & + \langle P, \Gamma_0 \rangle + 2p^T\left(\int_C \xi\, d\mathbb{P}(\xi) - \mu_0 \right) + s\gamma_1 \end{aligned}$$

with $Q \succeq 0$, $\begin{bmatrix} P & p \\ p^T & s \end{bmatrix} \succeq 0.$

Then, (DR-S) $\Longleftrightarrow$
$$\sup_{\mathbb{P} \geq 0} \ \inf_{\substack{r \in \mathbb{R} \\ Q \succeq 0 \\ \begin{bmatrix} P & p \\ p^T & s \end{bmatrix} \succeq 0}} \mathcal{L}(\mathbb{P}, r, Q, P, p, s)$$

The dual is obtained by interchanging sup and inf. We compute

$$\begin{aligned} \sup_{\mathbb{P} \geq 0} \mathcal{L}(\mathbb{P}, r, Q, P, p, s) = & \int_C \left[h(x,\xi) - r - \xi^T Q \xi + 2\xi^T Q \mu_0 + 2p^T\xi \right] d\mathbb{P}(\xi) \\ & + r + \langle Q, \gamma_2\Gamma_0 - \mu_0\mu_0^T \rangle + \langle P, \Gamma_0 \rangle - 2p^T\mu_0 + s\gamma_1, \end{aligned}$$

which implies the dual of (DR-S) takes the form

$$\begin{aligned} \inf \quad & r + \langle Q, \gamma_2\Gamma_0 - \mu_0\mu_0^T \rangle + \langle P, \Gamma_0 \rangle - 2p^T\mu_0 + s\gamma_1 \\ \text{s.t.} \quad & h(x,\xi) - r - \xi^T Q \xi + 2\xi^T Q \mu_0 + 2p^T\xi \leq 0 \quad \forall \xi \in C \\ & r \in \mathbb{R}, \quad Q \succeq 0, \quad \begin{bmatrix} P & p \\ \cdots \end{bmatrix} \end{aligned}$$

$$\text{s.t.} \quad h(x,\xi) - r - \xi' Q \xi + 2\xi' Q \mu_0 + 2p'\xi \le 0 \quad \forall \xi \in C$$

$$r \in \mathbb{R}, \quad Q \succeq 0, \quad \begin{bmatrix} P & p \\ p^T & s \end{bmatrix} \succeq 0.$$

We can further simplify the above by elimination the variables P, p, s. Indeed, suppose that the above is solvable. Consider the value $s^* \ge 0$ in an optimal solution.

<u>Case 1</u>: $s^* > 0$

By Schur complement, $P \succeq \frac{1}{s} pp^T$. Since $\Gamma_0 \succ 0$, we can take $P^* = \frac{1}{s} pp^T$ in the objective. Then, s^* can be found by

$$\min_{s>0} \frac{1}{s} p^T \Gamma_0 p + s\gamma_1 \Rightarrow s^* = \frac{1}{\sqrt{\gamma_1}} \sqrt{p^T \Gamma_0 p} = \frac{1}{\sqrt{\gamma_1}} \|\Gamma_0^{1/2} p\|_2.$$

Hence, the objective can be expressed as

$$r + \gamma_2 \langle \Gamma_0, Q \rangle + \mu_0^T Q \mu_0 - \mu_0^T (2\underbrace{(p + Q\mu_0)}_{q}) + \sqrt{\gamma_1} \|\Gamma_0^{1/2} (\underbrace{p + Q\mu_0}_{q} - Q\mu_0)\|_2$$

$$= r + \gamma_2 \langle \Gamma_0, Q \rangle + \mu_0^T Q \mu_0 - 2\mu_0^T q + \sqrt{\gamma_1} \|\Gamma_0^{1/2} (q - Q\mu_0)\|_2 \quad \text{—} (\Delta)$$

<u>Case 2</u>: $s^* = 0$

Then, we must have $p^* = 0$. We can then take $P^* = 0$.

Then, the objective can again be written as (Δ).

In summary, we can write the dual as

$$\inf \quad r + \gamma_2 \langle \Gamma_0, Q \rangle + \mu_0^T Q \mu_0 - 2\mu_0^T q + \sqrt{\gamma_1} \|\Gamma_0^{1/2} (q - Q\mu_0)\|_2$$

$$\text{s.t.} \quad h(x,\xi) - r - \xi^T Q \xi + 2\xi^T q \le 0 \quad \forall \xi \in C$$

$$r \in \mathbb{R}, \quad Q \succeq 0, \quad q \in \mathbb{R}^n.$$