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On the energy-angle distribution of Cherenkov radiation in an absorbing medium

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Abstract

The energy-angle distribution of the mean number of Cherenkov photons produced by a relativistic charged particle moving in a non-transparent medium is derived. Calculations show that taking into account absorption in the medium results in broadening of the angular distribution at a fixed photon energy compared to the standard theory for Cherenkov radiation of Frank and Tamm developed for transparent media.

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The standard theory of Cherenkov radiation (CR) developed by Frank and Tamm [1] predicts a fixed value for the emission angle of Cherenkov photons when a radiating relativistic charged particle moves in a transparent medium. The value of the Cherenkov angle is given by the well known formula:

$$\cos\theta_{\mathrm{c}} = \frac{1}{\beta n} \tag{1}$$

where θ_{c} is the angle between the particle and a Cherenkov photon momenta, $\beta = v/c$ is the ratio of the particle velocity v to the speed of light in vacuum c, and n is the refractive index of the medium. Physically the reason for the CR angle fixed value is that in transparent medium interference fully suppresses radiation at other angles.

It is obvious, however, that real media are always non-transparent and absorb light. Therefore, there were multiple attempts to generalize the theory of CR for the case of non-transparent medium. Fermi [2] using the Frank–Tamm method calculated the energy losses of a relativistic charged particle in a medium with a simplified law of dispersion and considered the contribution due to CR. Budini [3] and independently Sternheimer [4] derived the expression for the spectrum of mean energy loss, $\mathrm{d}^2\bar{\Delta}_c/\mathrm{d}\omega\,\mathrm{d}l$, to CR in a non-transparent medium,

$$\frac{\mathrm{d}^2\bar{\Delta}_c}{\mathrm{d}\omega\,\mathrm{d}l} = \frac{\alpha z^2}{\hbar c}\omega\left(1 - \frac{\varepsilon_1}{\beta^2|\varepsilon|^2}\right), \quad \beta^2\varepsilon_1 > 1. \tag{2}$$

Here ω is the CR photon energy, l is the distance along the particle trajectory ($l = vt$, where t is the time), α is the fine structure constant, $\hbar$ is Planck's

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constant, $z = q/|e|$, where q and e are the particle and electron charges, respectively; and $\varepsilon = \varepsilon_1 + i\varepsilon_2$ is the complex dielectric permittivity of the medium. Note that in a transparent medium, $\varepsilon_2 = 0$, $\varepsilon = n^2$.

The energy-angle distribution of energy loss to transverse electro-magnetic excitations at low momenta was considered by Fano in his famous review [5]. In particular, he presented the relation for energy spectrum, $\mathrm{d}^2\bar{\Delta}_\perp/\mathrm{d}\omega\,\mathrm{d}l$ of these excitations:

$$\frac{\mathrm{d}^2\bar{\Delta}_\perp}{\mathrm{d}\omega\,\mathrm{d}l} = \frac{\alpha z^2}{\pi}\frac{\omega}{\hbar c} Im\left\{\left[1 - \frac{1}{\beta^2\varepsilon(\omega)}\right]\ln\left[\frac{1}{1 - \beta^2\varepsilon(\omega)}\right]\right\}. \tag{3}$$

Later Saffouri [6] considered the energy loss of a relativistic charged particle and of a monopole in non-transparent medium with arbitrary magnetic permittivity. A considerable step forward was made in the paper of Kirzhnits [7] who considered the angular distribution of the energy loss of relativistic charged particles to transverse electromagnetic excitation of the medium (photons in the medium).

The aim of the present letter is to derive, using the results of [5,7], the energy-angle distribution of the mean number of CR photons in a non-transparent medium, $\varepsilon_2 > 0$. Instead of using of the Frank–Tamm–Fermi method [1,2] based on calculation of the energy loss as the flux of the Poynting vector over a cylindrical surface surrounding the particle trajectory, we will use the Landau method [8]. The latter is based on the calculation of the energy loss of a relativistic charged particle as the work done by the electric field produced by the particle at its current position. This approach allows us to get the energy-angle distribution of primary quasi-particles generated by the incident particle moving in an absorbing medium. The distribution can be used for simulation of practical Cherenkov detectors especially in the spectral regions near absorption lines.

We will start with the derivation of the energy loss as a function of the transferred energy and emission angle. Consider the mean energy loss $\mathrm{d}\bar{\Delta}/\mathrm{d}l$ produced by a relativistic charged particle moving in a non-transparent medium. According to the Landau method [8] it can be expressed as

$$\frac{\mathrm{d}\bar{\Delta}}{\mathrm{d}l} = -q\mathbf{E}(vt, t)\cdot\frac{\mathbf{v}}{v} \tag{4}$$

where the value of the electric field $\mathbf{E}$ is considered just in the current position of the incident particle. Following the results of [8,9] we calculate the mean energy loss based on the solution of Maxwell's equations for an electro-magnetic field created by a relativistic charged particle moving in a non-magnetic dielectric medium. It is convenient to introduce the vector $\mathbf{A}$ and scalar ϕ potentials:

$$\mathbf{H} = \nabla\times\mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\cdot\phi \tag{5}$$

in the Coulomb gauge:

$$\nabla\cdot\mathbf{A} = 0. \tag{6}$$

The Maxwell's equations for potentials become:

$$\nabla\cdot(\varepsilon\nabla\phi) = -4\pi q\delta^3(\mathbf{r} - \mathbf{v}t), \tag{7}$$

$$-\nabla^2\mathbf{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\mathbf{A}}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\varepsilon\nabla\phi) + \frac{4\pi}{c}q\mathbf{v}\delta^3(\mathbf{r} - \mathbf{v}t) \tag{8}$$

where δ is the Dirac delta function. These equations can be solved introducing the Fourier transforms for potentials:

$$F(\mathbf{r}, t) = \iint\frac{\mathrm{d}\mathbf{k}\,\mathrm{d}\omega}{(2\pi\hbar)^4}F(\mathbf{k}, \omega)\exp\left[\frac{i}{\hbar}(\mathbf{kr} - \omega t)\right] \tag{9}$$

$$\delta^3(\mathbf{r} - \mathbf{v}t) = \int\frac{\mathrm{d}\mathbf{k}}{(2\pi\hbar)^3}\exp\left[\frac{i}{\hbar}\mathbf{k}\cdot(\mathbf{r} - \mathbf{v}t)\right] \tag{10}$$

where $\mathbf{k}$ is the momentum transferred from the incident particle to the medium. Solving (7) and(8) in terms of $\mathbf{A}(\mathbf{k}, \omega)$ and $\phi(\mathbf{k}, \omega)$ and substituting them into the expression for $\mathbf{E}(\mathbf{v}t, t)$

$$\mathbf{E}(\mathbf{v}t, t) = \iint\frac{\mathrm{d}\mathbf{k}\,\mathrm{d}\omega}{(2\pi\hbar)^4}\left\{\frac{i\omega}{\hbar c}\mathbf{A}(\mathbf{k}, \omega) - \frac{i\mathbf{k}}{\hbar}\phi(\mathbf{k}, \omega)\right\} \times \exp\left[\frac{it}{\hbar}(\mathbf{kv} - \omega)\right] \tag{11}$$

we get the mean energy loss in the form [9]:

$$\frac{d\bar{\Delta}}{dl} = -\frac{2iq^2}{(2\pi\hbar)^2 v}\iint d\mathbf{k}\, d\omega \left\{\frac{\omega v^2/c - \omega/c\mathbf{k}\cdot\mathbf{v}/k^2}{c \quad k^2 - \omega^2 c^2 \varepsilon} - \frac{\mathbf{k}\cdot\mathbf{v}}{\varepsilon k^2}\right\}\delta(\omega - \mathbf{k}\cdot\mathbf{v})\exp\left[\frac{it}{\hbar}(\mathbf{kv} - \omega)\right].$$

Here the delta function $\delta(\omega - \mathbf{k}\cdot\mathbf{v})$ reflects a kinematic limitation since we consider the energy transfer ω to be much less than the energy of the incident particle. We represent the integral in the latter expression as:

$$\iint d\mathbf{k}\, d\omega = 2\pi \int_{-\infty}^{\infty} d\omega \int_{\omega/v}^{\infty} k^2\, dk \int_0^1 d\xi$$

where we introduce the angular variable, $\xi = \cos\theta$ (θ is the emission angle, i.e. the angle between $\mathbf{v}$ and $\mathbf{k}$).

To get the energy-angle distribution of the mean energy loss we integrate with respect to k rather than more usual to ξ. Using the delta function constraint, $k = \omega/v\xi$, and the properties of dielectric permittivity ($\varepsilon^*(\omega) = \varepsilon(-\omega)$) we get

$$\frac{d\bar{\Delta}}{dl} = \frac{2}{\pi}\frac{q^2}{\hbar^2 c^2 \beta^2}\int_0^{\infty} \omega\, d\omega \int_0^1 \frac{d\xi}{\xi} \times \left\{ Im\left[\frac{\beta^2(1-\xi^2)}{1-\beta^2\xi^2\varepsilon}\right] - Im\left(\frac{1}{\varepsilon}\right)\right\}. \tag{12}$$

The latter equation, in a slightly different notation, was considered for the first time by Fano [5] and later by Kirzhnits [7]. Consider now the mean energy loss $\bar{\Delta}$ over the particle trajectory with a fixed length L. Using the usual representation:

$$\bar{\Delta} = \int_0^{\infty} d\omega \int_0^1 d\xi \frac{d^2\bar{\Delta}}{d\omega\, d\xi} \tag{13}$$

we get for $d^2\bar{\Delta}/d\omega\, d\xi$, ($q^2 = z^2 e^2$):

$$\frac{d^2\bar{\Delta}}{d\omega\, d\xi} = \frac{2\alpha z^2}{\pi}\frac{L}{\hbar c}\frac{\omega}{\beta^2\xi}\left\{ Im\left[\frac{\beta^2(1-\xi^2)}{1-\beta^2\xi^2\varepsilon}\right] - Im\left(\frac{1}{\varepsilon}\right)\right\} = \frac{d^2\bar{\Delta}_\perp}{d\omega\, d\xi} + \frac{d^2\bar{\Delta}_\parallel}{d\omega\, d\xi}. \tag{14}$$

The first term in the figure brackets is responsible for the energy-angle distribution of the mean energy loss to transverse electro-magnetic excitations of the medium (photons in medium). The second term is responsible for the excitation of plasmons, i.e., longitudinal electro-magnetic excitations.

Now we will consider the transverse energy loss in terms of the energy-angle distribution of the mean number of photons produced by the incident particle. We introduce the mean number $\bar{N}$ of emitted photons in the standard way:

$$\frac{d^2\bar{\Delta}_\perp}{d\omega\, d\xi} = \omega \frac{d^2\bar{N}}{d\omega\, d\xi}.$$

Then from (14) we have for $d^2\bar{N}/d\omega\, d\xi$

$$\frac{d^2\bar{N}}{d\omega\, d\xi} = \frac{2\alpha z^2}{\pi}\frac{L}{\hbar c} Im\left[\frac{1-\xi^2}{\xi(1-\beta^2\xi^2\varepsilon)}\right]. \tag{15}$$

Introducing the new variable $x = \xi^2 = \cos^2\theta$ and expanding the imaginary part we get

$$\frac{d^2\bar{N}}{d\omega\, dx} = \alpha z^2 \frac{L}{\hbar c}\frac{(1-x)\nu x_0}{\pi[(x-x_0)^2 + (\nu x_0)^2]} \tag{16}$$

where $\nu = \varepsilon_2/\varepsilon_1$, and

$$x_0 = \frac{\varepsilon_1}{\beta^2|\varepsilon|^2}. \tag{17}$$

The distribution (16) has a sharp peak at $x = x_0$. Therefore, the most probable emission angle θ_o of Cherenkov photons in a non-transparent medium is defined by

$$\cos\theta_o = \frac{\sqrt{\varepsilon_1}}{\beta|\varepsilon|} \simeq \frac{1-\nu^2/2}{\beta\sqrt{\varepsilon_1}}. \tag{18}$$

The full width at half maximum, FWHM, of the peak is equal to

$$\mathrm{FWHM} = 2\nu x_0 = \frac{2\varepsilon_2}{\beta^2|\varepsilon|^2}. \tag{19}$$

It can be seen from expression (16) that radiation always occurs and experiences angle broadening even at fixed photon energy. However, the main part of the photon flux is emitted in the interval $(1 \pm \nu)x_0$. Therefore, since $\nu \ll 1$ (see below), the radiation can be observed if $x_0 < 1 - \nu x_0$ or

$$\varepsilon_1 > \frac{1+\nu}{1+\nu^2} \tag{20}$$

which is slightly more strict than the condition for a transparent medium $\varepsilon_1 > 1$. If the latter condition is valid, radiation can be observed for a relativistic particle with a velocity satisfying the following condition:

$$\beta^2 > \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2} \tag{21}$$

which for a transparent medium ($\varepsilon_2 = 0$) reduces to the well known expression, $\beta > 1/n$.

The integration of (16) with respect to x results in the energy spectrum of the radiation (we consider the dielectric permittivity as a function of the photon energy only and in the visible-ultraviolet range of interest to be independent of x (no spatial dispersion), $\varepsilon = \varepsilon(\omega)$):

$$\frac{d\bar{N}}{d\omega} = \frac{\alpha z^2}{\pi}\frac{L}{\hbar c}\left\{\left[1 - \frac{\varepsilon_1}{\beta^2|\varepsilon|^2}\right]\arg(1-\beta^2\varepsilon^*) + \frac{\varepsilon_2}{\beta^2|\varepsilon|^2}\ln\left(\frac{1}{|1-\beta^2\varepsilon|}\right)\right\} \tag{22}$$

which coincides with (3) and for $v \ll 1$ approximately corresponds to (2).

In the limit of a transparent medium, when $v \to 0$, we note that

$$\frac{v x_0}{\pi[(x - x_0)^2 + (v x_0)^2]} \to \delta\left(x - \frac{1}{\beta^2 n^2}\right)$$

and get the energy-angle distribution of the mean number of Cherenkov photons:

$$\frac{d^2\bar{N}_c}{d\omega\, dx} = \alpha z^2 \frac{L}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2}\right)\delta\left(x - \frac{1}{\beta^2 n^2}\right). \tag{23}$$

This expression fixes explicitly the emission angle to be $\theta_c = \arccos(1/\beta n)$ in agreement with (1). If $\beta n > 1$, integration of (23) with respect to x results in the well known formula of the Frank–Tamm theory:

$$\frac{d\bar{N}_c}{d\omega} = \alpha z^2 \frac{L}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2}\right), \quad \beta n > 1. \tag{24}$$

The above calculations show that taking into account absorption by a medium ($\varepsilon_2 > 0$) results in a change of the main features of Cherenkov radiation, namely:

1. The most probable emission angle of Cherenkov radiation becomes slightly higher than the Cherenkov angle θ_c. From (18) we have

   $$\cos\theta_o \simeq \frac{1 - v^2/2}{\beta\sqrt{\varepsilon_1}} \lesssim \frac{1}{\beta n} = \cos\theta_c. \tag{25}$$

   Here we use the expression $\varepsilon = (n + i\varkappa)^2$, where $\varkappa > 0$ is the absorption coefficient in terms of the amplitude of electro-magnetic wave. Note that $v \simeq 2\varkappa/n$.
2. The emission angle of Cherenkov radiation at a given photon energy is not exactly fixed at the angle θ_c. The angular distribution of Cherenkov photons experiences broadening (aberration) which is proportional to the ratio of $v = \varepsilon_2/\varepsilon_1$. Qualitatively the main reason of the broadening is violation of full coherence due to absorption of photons emitted from initial parts of the particle trajectory. In terms of quantum mechanics it is easy to see that distribution (16) is of the Breit–Wigner type reflecting the fact that the photon states are not static due to their absorption.
3. The threshold behaviour of Cherenkov radiation becomes more extended. The radiation appears when the particle velocity changes within

   $$\frac{\delta\beta}{\bar{\beta}} \simeq v \tag{26}$$

   where $\bar{\beta} = \sqrt{\varepsilon_1}/|\varepsilon|$ corresponds to $x_0 = 1$.

These corrections however are very small for practical cases since they all are proportional to v. In the optical-ultraviolet range the value of v can be estimated from

$$v \sim \frac{\hbar c}{n\omega\lambda} \tag{27}$$

where λ is the photon absorption length. Since for semi-transparent media $\omega\lambda \gtrsim 2$ eV cm and $\hbar c \sim 2 \times 10^{-5}$eV cm,

$$v \lesssim 10^{-5}$$

and the broadening is in practice very difficult to observe in semi-transparent solids and liquids. Fig. 1 illustrates this showing the angle distributions of Cherenkov photons produced in C_6F_{14} [10] by relativistic particle $v \sim c$ for two values of the photon energy: $\omega = 6.8$ eV (open circles) and $\omega = 7.7$ eV (closed circles and upper x-axis). The curves are normalized on the equal most probable values.

For gases the situation can be more promising. Here the Cherenkov angle is always very small, $\theta \ll 1$ or $x_0 \sim 1$. Then the relative broadening due to the gas absorption $\delta\theta/\theta$ can be estimated from

$$\frac{\delta\theta}{\theta} \sim \frac{v}{\theta^2} \sim \frac{\hbar c}{2\eta\omega\lambda} \tag{28}$$

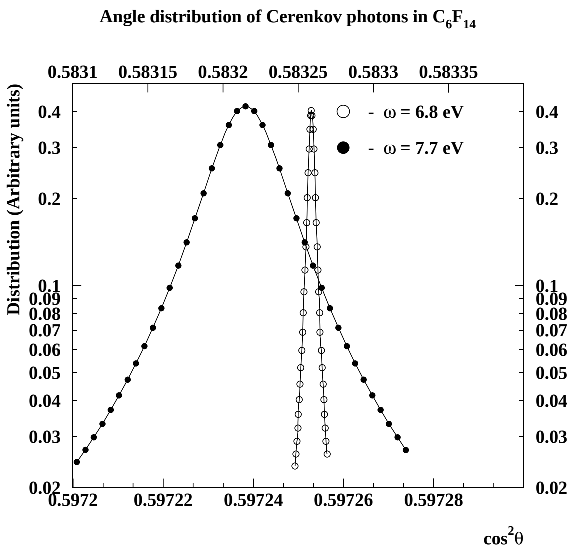

Fig. 1. The angle distributions of Cherenkov photons produced in C_6F_{14} by relativistic particle $v \sim c$ for two values of the photon energy: $\omega = 6.8\,\mathrm{eV}$ (open circles) and $\omega = 7.7\,\mathrm{eV}$ (closed circles and upper x-axis). The curves are normalized on the equal most probable values.

where $\eta = n - 1$ is the refractivity. In most gases of practical use $\eta \sim 10^{-4}$ and therefore the aberration of Cherenkov radiation due to absorption can be observed in gas Ring Imaging Cherenkov (RICH) detectors [11]. Radio frequency Cherenkov photons (meter range) can be also considered as promising for the observation of the broadening due to absorption since v is proportional to the ratio of the wavelength to the absorption length of the photons.

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References

[1] I.M. Frank, I.E. Tamm, Comptes rendus de l'Acad. Sci. U.R.S.S. 14 (1937) 109.
[2] E. Fermi, Phys. Rev. 57 (1940) 485.
[3] P. Budini, Phys. Rev. 89 (1953) 1147.
[4] R.M. Sternheimer, Phys. Rev. 89 (1953) 1148.
[5] U. Fano, Ann. Rev. Nucl. Sci. 13 (1963) 1.
[6] M.H. Saffouri, IL Nuovo Cim. D 3 (1984) 589.
[7] D.A. Kirzhnits, in: Some problems of Nuclear Physics (On the Occasion of the 80th Birthday of I.M. Frank), Nauka, Moscow, 1989, p.144 (in Russian).
[8] L.D. Landau, E.M. Lifshitz, Electrodynamics of Continuos Media, Oxford, New York, 1984 (Chapter 12).
[9] W.W.M. Allison, J.H. Cobb, Ann. Rev. Nucl. Part. Sci. 30 (1980) 253.
[10] A. Di Mauro (CERN), private communication.
[11] J. Seguinot, T. Ypsilantis, Nucl. Instr. and Meth. 142 (1977) 142.
[12] GEANT4 Collaboration, CERN/LHCC 98-44, GEANT4: An Object-Oriented Toolkit for Simulation in HEP; see also the web site: http://geant4.web.cern.ch/geant4/.