Referee's Report for SISC on

TOPOLOGY-PRESERVING DISCRETIZATION FOR THE MAGNETO-FRICTIONAL EQUATIONS ARISING IN THE PARKER CONJECTURE

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February 2, 2025, Manuscript No. M172754

The magneto-frictional model, Equ. (1.3) in the manuscript, is a simplification of the full ideal magneto-hydrodynamic (MHD) model. It replaces the Navier-Stokes equations of fluid flow with the equations for creeping flow. When equipped with perfectly electrically conducting wall boundary conditions plus no-slip boundary conditions for the electric current, the total magnetic helicity (1.1) is conserved by the evolution, and the magnetic field energy will not decay to zero for $t \to \infty$, though the process is dissipative. This peculiar behavior is closely linked to the so-called Parker conjecture.

The authors propose a particular variational formulation of the magneto-frictional model that involves both the magnetic induction field $\boldsymbol{B}$ and the magnetic field $\boldsymbol{H}$ as separate unknowns, as well as the current $\boldsymbol{j}$. They propose (Problem 3.1) a conforming finite-element discretization in the spirit of finite-element exterior calculus (FEEC): $\boldsymbol{B}$ is approximated by discrete 2-forms, while $\boldsymbol{H}$ and $\boldsymbol{j}$ are replaced with discrete 1-forms (all with vanishing traces on the boundary of the domain).

The proposed discretization has remarkable structure-preserving properties. For topologically simple domains (vanishing first Betti number $\beta_1(\Omega)$) helicity is an invariant of the spatially semi-discrete version, which also respects the energy law and the so-called Arnold inequality. This is stated and proved in Section 3.1.

The following Section 3.2 examines domains Ω with $\beta_1(\Omega) > 0$ and $\beta_2(\Omega) = 0$, which denies the existence of a magnetic vector potential for $\boldsymbol{B}$. Instead, the authors consider the Hodge decomposition of $\boldsymbol{B}$, $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A} + \boldsymbol{B}_H$, where $\boldsymbol{B}_H$ is a (discrete) harmonic 2-form. Based on $\boldsymbol{B}_H$ they formulate a generalized helicity, which remains an invariant of the (semi-discrete) evolution.

Section 4 presents carefully selected numerical tests with a focus on the constraint $\operatorname{div} \boldsymbol{B} = 0$, on the preservation of helicity and on capturing the right decay of the magnetic field energy.

Assessment. The manuscript is presented well and all arguments are elaborated fairly clearly. I could not detect any mistakes. It adapts the ideas of Ref.#23 in a straightforward way. It goes beyond Ref. #23 in its treatment of domains with non-trivial topology. Comprehensive numerical tests are reported and show the superiority of the proposed method in terms of preserving the key quantity of helicity. This marks significant progress in the structure-preserving discretization of the magneto-frictional system.

Issues:

- Let me start with a strange observation: The approach of the paper is a specialized and simplified version of the FEM introduced in the seminal Ref. #23, where a structure-preserving "helicity aware" FEEC-style discretization of the full real MHD model is introduced. In fact the discrete variational equations (3.2) are essentially a subset of the variational equations (3.4) of Ref. #23, even the bulk of the notations are borrowed from there! However, Ref. #23 is never mentioned in Section 3, but only cited in the end of Section 2 as a reference for fairly straightforward orthogonal projections. In fact, Ref. #23, apart from helicity and energy laws, also proves the Arnold inequality in Lemma 3, all on topologically trivial domains.

  Is this an embarrassing attempt to hide the fact that the present manuscript is an adaptation of the method of Ref. #23? I hope not. I expect the authors to add detailed references to Ref. #23 whenever they adopt considerations, techniques, and notations from that paper, whose lead author is even among the authors of the present paper. In the Introduction they should also acknowledge that they closely follow Ref. #23.

- This connects to the previous item: The authors repeatedly mention an "Andrews-Farell framework", a technique of "auxiliary variables", and cite Ref.#2 as main source. For instance in the beginning of Section 3 they write that "the framework [2] introduces AVs representing the current ($\boldsymbol{j}_h = \mathbb{Q}_c \, \mathbf{curl} \, \boldsymbol{B}_h$) and magnetic field ($\boldsymbol{H}_h = \mathbb{Q}_c \boldsymbol{B}_h$) respectively", though introducing these auxiliary variables is one of the main ideas in Ref. #23 and obviously inspired by it. Even after perusal of Ref. #2 I cannot see a close connection. Ref. #2 is about the preservation of invariants in evolution problems, but it remains vague about when and how to introduce those "auxiliary variables". I do not think that Ref. #2 gives useful hints for arriving at Problem 3.1. The key to it is Ref. #23! The authors should stop throwing red herrings and give due credit to Ref. #23.

- Title: What is "topology preserving"? The method is "helicity-preserving"! Also drop "Parker conjecture" from the title, because this is hardly addressed in the paper.

- "Contractible" is a concept from *homotopy*, but what matters in the context of the manuscript is co-*homology*. Therefore the authors should express all assumptions they make on the topology of Ω in terms of Betti numbers.

- Since $\boldsymbol{B}$ and $\boldsymbol{H}$ occur, maybe $\boldsymbol{B}$ should be called the "magnetic induction field" to distinguish it from the *magnetic field* $\boldsymbol{H}$.
- In various manipulations, for instance in the proof of Theorem 3.2 and Remark 3.6, integration by parts is instrumental. Generic integration by parts introduces boundary terms, which disappear only thanks to the judicious choice of boundary conditions. This role of the boundary conditions should be pointed out whenever it matters.
- I think (3.5b) can only be derived using (3.2b). Please explain.
- Give a reference for the fact that Gauss collocation Runge-Kutta methods preserve quadratic invariants, of which the helicity is one specimen.
- Sections 4.2.2 and 4.3.2 are closely related to Section 3.1 of Ref. #23.