Numerical Bifurcation Analysis for Heat Transfer by the Edge Plasma of a Tokamak

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Abstract

BA: Abstract.

Contents

Introduction

BA: Introduction.

BA: Motivation.

BA: Previous work and literature review.

BA: Results we might expect to find.

BA: Relevance and impact.

BA: Other topics to note mentioned on the transfer thesis requirements.

BA: Summary.

Part I

Research

Chapter 1

Physical Model and System of Equations

BA: Introduction.

BA: Room for lots of pictures here.

BA: What physics characterise a (magnetised) neutral plasma? Quasi-neutral mix of *separated* electrically-charged phases. (Check out the plasma Wikipedia page.)

BA: Will make the assumptions:

- Only 2 phases (positive and negative) i.e.:
  - The plasma is fully ionised, i.e. neutrals (or at least the effects thereof) are negligible. I've been led to believe this is generally the case in edge plasmas (i.e. outside the divertor). (A bold assumption?)
  - Dust and impurities (or at least the effects thereof) are negligible. (A bold assumption?)
- Only (thermalising) Coulomb collisions are considered- these are generally dominant over the others in a tokamak. (N.B. No fusion.)
- Relativistic effects are negligible. (Is observing that the velocity scale is much smaller than c sufficient justification? Imperial guy at the NEPTUNE workshop appeared to think not.)

1.1 Tokamak BA: (Edge?) Plasma Dynamics

1.1.1 What is a Plasma?

Definition 1.1.1 (Plasma)**.** *"Plasma" refers here to an electrically charged fluid, typically occurring when a fluid is supplied with sufficient energy—from heating or an applied electromagnetic (EM) field—that a significant portion of the atoms* BA: (Not molecules?) *are ionised* BA: (Terminology?)*, causing the the (positive) ion and (negative) electron phase to move independently.*

Plasma is one of the most abundant forms of matter in the universe [CL13], found most frequently in stars [Phi95; Asc06; Pie17] and similarly—as in our case—the star-like environments emulated in a tokamak.

While an applied EM field induces a current as it separates the two charged phases: the ions and electrons, the current simultaneously induces and EM field through Maxwell's equations, creating a complex, coupled, nonlinear system, referred to as magnetohydrodynamics (MHD). (Figure 1.1) BA: [Ref.]

$$\begin{tikzcd} \text{Plasma motion} \arrow[r, bend left, "\textit{Maxwell's equations}"] & \text{EM field} \arrow[l, bend left, "\textit{Ionisation}"] \end{tikzcd}$$

Figure 1.1: Coupling of the plasmsa motion and EM field in MHD.

1.1.2 What Makes a Tokamak Plasma Special?

Certain properties characterise the plasma in a tokamak:

- **Very high heat**: Plasma temperatures within a tokamak are on the order of 10^8K BA: [Ref]; an order of magnitude *higher* than that in the centre of the sun, at around 1.5×10^7K BA: [Ref].

- **Very strong EM fields**: The EM fields used to ionise tokamak plasmas have strengths on the order 1T BA: [Ref], with the world's most powerful magnets being those employed in the world's most powerful tokamaks BA: [Ref].

- **Very low density**: BA: (Why do tokamaks do this? I know there's these theoretical limits on particle density, but I've never seen anyone explain why we have to enforce this?) Tokamak plasmas feature particle densities on the order of 10^{-5}mol^{-1} BA: (Check!) BA: (Is that even the right unit?) BA: [Ref]. The quantity of hydrogen gas used during a JET pulse is often compared with the size BA: (/mass? I'm not sure actually!) of a postage stamp BA: [Ref].

BA: Complicated BCs in a tokamak.

1.2 Kinetic Models: The General Theory

BA: What is a "kinetic" model?

Definition 1.2.1 (Kinetic model)**.** *Here, "kinetic" model refers to a model wherein a fluid is modelled via a particle density function of position and velocity (and time).*

This is typically written as $f(\mathbf{x}, \mathbf{v}; t)$. Obviously, with the high dimensionality, this can be computationally prohibitive, so we'd like to avoid such a model if possible.

BA: Boltzmann equation definition here.

Definition 1.2.2 (Fluid model)**.** *Here, "fluid" model refers to a reduced kinetic model, where a collection of functions of position (and time) only are modelled.*

These typically define certain properties of the kinetic model density function, f, at each position, $\mathbf{x}$, to in some way capture the physics of the full kinetic model. The Navier–Stokes equations, for example, can be found from the assumption that collisions between particles dominate the behaviour of each particle, such that at each position, $\mathbf{x}$, the kinetic model density function, $f|_{\mathbf{x}}$, converges to a high entropy state (often a scaled normal distribution: a Maxwellian) characterised by 3 conserved variables:

- Mass (per unit volume, i.e. density)
- Momentum (per unit volume)
- Energy (per unit volume, i.e. temperature after conservation of mass of momentum)

When collisions between particles do *not* dominate the particle behaviour—as is the case in the very low density, high temperature BA: (Does this increase or decrease the size of the Coulomb collision term?) tokamak plasmas—the assumptions that give rise to these fluid approximations often break down, implying that, without modification, these models fail to capture so-called "kinetic effects" BA: (Such as?).

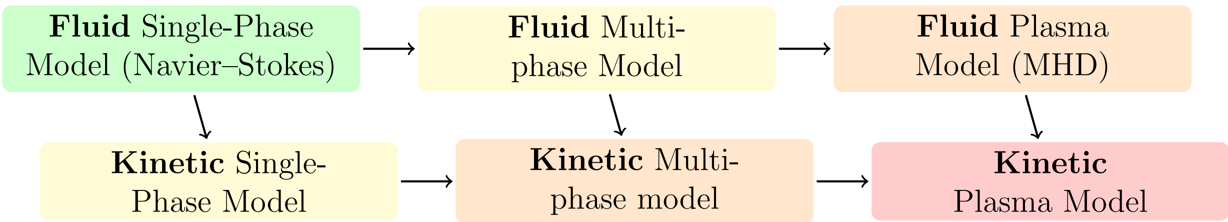

Figure 1.2: BA: Diagram of workflow for creating a kinetic plasma model to account for pressure anisotropy.

1.2.1 Single-Phase Fluids

BA: The resultant kinetic PDE. (Boltzmann equation.)

BA: How we traditionally convert that to a "fluid" model.

BA: Will use this simpler case as a reference study to develop the ideas for the more complicated tokamak plasma case.

BA: The way I've got this section structured with reference to introducing the moments is a bit funny.

Consider first a single-phase fluid. Let a single particle of mass m in this fluid have position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$, and be subject to a force $\mathbf{F}(\mathbf{x}, \mathbf{v}; t)$. We seek a (system of) ODEs governing the behaviour of this particle:

- The position evolves simply according to the ODE

$$d\mathbf{x} = \mathbf{v}dt \tag{1.1}$$

- The velocity evolves according to the ODE, $d\mathbf{v} = \frac{1}{m}\mathbf{F}dt + \frac{1}{m}$**"Collisional forces"**$dt$. BA: (Note here with references about how tough it is to model collisional terms- *got* to make *some* assumptions, *especially* if we want to do simulations!) To model the collisional forces, consider the following 2 assumptions:

  - The molecular chaos hypothesis:

    Definition 1.2.3 (Molecular chaos hypothesis)**.** *The "molecular chaos hypothesis" postulates that the velocities of colliding particles are uncorrelated, and independent of position. BA: [Ref] BA: (Justification that this is valid here? Particle number sufficiently large?)*

  - Collisional forces on the particles are dominated by those from Coulomb collisions: BA: (Is this the right way of phrasing this?)

    Definition 1.2.4 (Coulomb collision)**.** *A "Coulomb collision" is an elastic collision between two charged particles interacting through their own electric field. BA: [Ref] BA: (Justification that this is valid here?)*

  We can then model the collision forces through a Wiener process, with forces proportional to the: BA: (Give justification for this model. That might be tough...)

  - Difference in momentum between that of the particle, $m\mathbf{v}$, and the average particle with which the particle in question is colliding, $m(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau})$ BA: (I think this is right for the stress contribution? Maybe off by a scaling of m or something...), where $\boldsymbol{\tau}(\mathbf{x})$ denotes an internal stress from variation in momentum BA: ($\boldsymbol{\tau}$ should be symmetric, right? But why?) BA: (Should this be $\boldsymbol{\tau}$ or $\boldsymbol{\sigma}$? I've written $\boldsymbol{\tau}$ so as not to clash in notation, but this could change.) BA: (Why?)

    $$\mu m[(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}) - \mathbf{v}] \tag{1.2}$$

  - "Derivative" of some Wiener process BA: (Obviously the "derivative of a Wiener process" is not a rigorously defined concept, so I may see if I can switch around the way this is structured...), representing the random forces from collisions at random angles from the molecular chaos hypothesis

    $$\sigma m \frac{d\mathbf{W}}{dt} \tag{1.3}$$

  BA: Here's where I think I should state that we have the relation $\frac{\mu}{\sigma^2} = \frac{k_B T}{m}$. Is this from conservation of energy? Would like to give a proper derivation of the scale of σ, and am sure I'll have to for the dimensional analysis.

  By convention, mapping $\mathbf{x}, \mathbf{v} \mapsto \mathbf{X}, \mathbf{V}$ to represent the switch from an ODE to an SDE (and accordingly in (1.1)) gives the SDE for the velocity evolution

  $$d\mathbf{V} = \frac{1}{m}\mathbf{F}dt + \mu[(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}) - \mathbf{V}]dt + \sigma d\mathbf{W} \tag{1.4}$$

  This resembles a combination of an Ornstein–Uhlenbeck process and a forcing term.

Applying the Fokker-Planck equation, the particle density function, $f(\mathbf{x}, \mathbf{v}; t)$ evolves according to the PDE

$$\partial_t[f] + \underbrace{\nabla_{\mathbf{x}} \cdot [f\mathbf{v}]}_{\text{Convection}} + \underbrace{\frac{1}{m}\nabla_{\mathbf{v}} \cdot [f\mathbf{F}]}_{\text{Forces}} = \underbrace{-\mu\nabla_{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \cdot [\boldsymbol{\tau}]}_{\text{Viscosity}} + \underbrace{\mu\nabla_{\mathbf{v}} \cdot [f(\mathbf{v} - \mathbf{u})] + \frac{\sigma^2}{2}\Delta_{\mathbf{v}}[f]}_{\text{Collisions}} \tag{1.5}$$

This is a form of the Boltzmann equation, where the approximation to the collisional term is explicitly derived from the above assumptions.

1.2.2 Multiphase Fluids

BA: Similar analysis for a multiphase fluid, in preparation for handling the tokamak plasmas.

Working towards the full plasma kinetic equation, consider a general multiphase fluid, where each phase is subject to a different external force. We index each phase, i, through the upper index $*^{(i)}$.

The collisional forces (1.2) and (1.3) on phase i will be modified to the form: BA: (This will definitely be switched around in the future when I switch around my notation for $\mu^{(i,j)}$, $\sigma^{(i)}$. In particular there should be some symmetry relations on $\mu^{(i,j)}$ from conservation of momentum between phases i, j- I'd like to scale $\mu^{(i,j)}$ such that $\mu^{(i,j)} = \mu^{(j,i)}$.)

$$\mu m[(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}) - \mathbf{v}] \mapsto \sum_j \mu^{(i,j)} m^{(i)} \left[\left(\mathbf{u}^{(j)} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}^{(i,j)}\right) - \mathbf{v}\right] \tag{1.6}$$

$$\sigma m \frac{d\mathbf{W}}{dt} \mapsto \sum_i \sigma^{(i)} m^{(j)} \frac{d\mathbf{W}^{(i)}}{dt} \tag{1.7}$$

BA: We should similarly here have the relation $\frac{\mu^{(i)}}{{\sigma^{(i)}}^2} = \frac{k_B T^{(i)}}{m^{(i)}}$ I believe.

Accordingly, for each phase i the single-phase Boltzmann equation (1.5) takes the multiphase form

$$\begin{aligned}\partial_t \left[f^{(i)}\right] &+ \underbrace{\nabla_{\mathbf{x}} \cdot \left[f^{(i)}\mathbf{v}\right]}_{\text{Convection}} + \underbrace{\frac{1}{m^{(i)}}\nabla_{\mathbf{v}} \cdot \left[f^{(i)}\mathbf{F}^{(i)}\right]}_{\text{Forces}} \\ &= \underbrace{-\sum_j \mu^{(i,j)}\nabla_{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \cdot \left[\boldsymbol{\tau}^{(i,j)}\right]}_{\text{Viscosity}} + \underbrace{\sum_j \mu^{(i,j)}\nabla_{\mathbf{v}} \cdot \left[f\left(\mathbf{v} - \mathbf{u}^{(j)}\right)\right] + \frac{(\sigma^{(i)})^2}{2}\Delta_{\mathbf{v}}\left[f^{(i)}\right]}_{\text{Collisions}}\end{aligned} \tag{1.8}$$

1.2.3 Tokamak Plasmas

BA: Why the fluid/MHD model reductions aren't necessarily valid in tokamak plasmas. (Incorrectly assumed dominant collisional term- get some estimates on the scale of these terms in the edge plasma. Good content under "Mathematical Descriptions" here.)

BA: Many effects not captured my MHD/2 fluid models (check out this diagram off Wikipedia, or again the content under "Mathematical Descriptions" here.):

- Most plasma waves
- Most plasma/kinetic instabilities
- Landau damping/bump-on-tail instability
- Leakage
- Structures (Beams/Double layers)
- Anisotropic pressures

BA: The resultant kinetic PDE. (Boltzmann/Vlasov equations.)
BA: Talk about gyrokinetic model:

- The model's physical basis/mathematics. (Equations provide good insight into the origin of some behavioural effects, e.g. gyro-orbits/drifts.)
- Why we don't use it on the general kinetic equation:
  - High mathematical (more terms in lower dimensions doesn't necessarily mean faster computation)/computational (really don't want to do a 5D simulation) complexity.
  - Errors from neglection of terms. (Non-physical behaviour over long times/resonances and adiabatic invariants can be lost.)
- We *will* however use it for the PIC correction.

1.3 Fluid Models: The *High* Collisionality Limit

It is often the case that on the typical parameter scales of interest, the collisional terms massively dominate the kinetic Boltzmann equation. BA: (Dimensional analysis here? Or at least some note saying that this is natural to expect from the typical *very high* particle densities we encouter in daily life.)

1.3.1 Single-Phase Fluids

When the collisional terms are dominant in the single-phase fluid Boltzmann equation (1.5) we find, up to leading order

$$0 \sim \mu \nabla_{\mathbf{v}} \cdot [f(\mathbf{v} - \mathbf{u})] + \frac{\sigma^2}{2} \Delta_{\mathbf{v}}[f] \tag{1.9}$$

$$f \sim \frac{\rho}{m} \sqrt{\frac{m}{k_B T}}^3 \exp\left(-\frac{m}{2k_B T} \|\mathbf{v} - \mathbf{u}\|^2\right) \tag{1.10}$$

where ρ, $\mathbf{u}$, T—functions of $\mathbf{x}$ (and t)—are moments characterising the conserved quantities of mass, momentum and energy:

$$\rho = \int fm d\mathbf{v} \tag{1.11}$$

$$\rho\mathbf{u} = \int fm\mathbf{v} d\mathbf{v} \tag{1.12}$$

$$\frac{1}{2}\rho\left(\|\mathbf{u}\|^2 + 3\frac{k_B T}{m}\right) = \int f\frac{1}{2}m\|\mathbf{v}\|^2 d\mathbf{v} \tag{1.13}$$

and, by its definition from the energy equation here, BA: (Don't like the way that's phrased...) $k_B \approx 1.381 \times 10^{-23}\mathrm{JK}^{-1}$ is the Boltzmann constant.

1.3.2 Multi-Phase Fluids

1.3.3 Tokamak Plasmas

1.4 Coupled Fluid/Correction Models: A *Low* Collisionality Concept

BA: How we can re-adapt the techniques that traditionally give a fluid model when the collision operator is non-dominant to get an accurate fluid model, to apply modern techniques in fluid simulation?

BA: Expand as a sum of a Maxwellian and some correction!

Fluid Background: A *Fluid* Model

BA: Ideas already well-developed!

BA: Correction contribution not too problematic (hopefully).

Non-Maxwellian Correction: A *Kinetic* Model

BA: Not just kicking the problem down the road- plasma is thermalised/Maxwellian in "most places" for "most physically relevant simulations", so the correction is (compartively) small in "most places".

BA: This works so nicely, because — at least in the continuous case — having the fluid model satisfied should mean that all the relevant moments of the non-Maxwellian correction should remain at 0 for all time, thus meaning we don't have to fiddle around with evaluating the moments of the correction, or particle-on-particle collisions etc. (*very* parallelisable). How well this'll carry over to the discrete simulation, only time will tell.

BA: How do we model this:

- Lattice Boltzmann?
- Some series expansion?

- Particle-in-cell (PIC)?
  One thing worthy of note that I'm concerned about is how a PIC model would handle different tail behaviours for very-high-energy particles. In tokamak plasmas for example, Alfvénic damping of particles with velocities above the Alfvén velocity causes particle energy distributions to have a cut-off that is $\ll \exp\left(-\text{const.}\|\mathbf{v}\|^2\right)$ as $\|\mathbf{v}\| \to \infty$ (see figure 1.3). Salomon seems to believe this has a strong effect on the plasma behaviour, and things Wayne has said would hint in this direction too. Honestly I think the response to this is just to ensure the particles are generated with enough fidelity on the tails to capture these varying behaviours, and otherwise not worry about it, unless the people at Warwick have any better suggestions? I mean honestly, how much can I do about this? Would need to understand the theory a little better first.

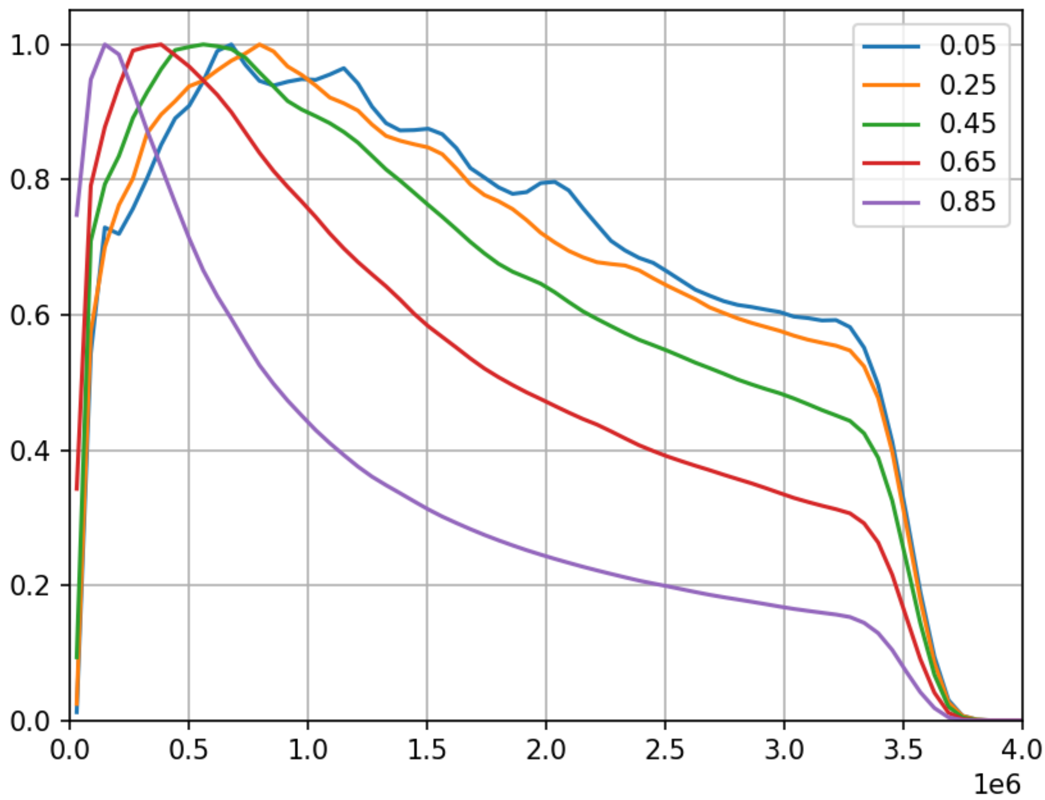

Figure 1.3: Experimental energy distribution function for alpha particles over different flux tubes of varying minor radius within a tokamak, showing the sharp cut-off when the energy reaches a certain threshold corresponding to the Alfvén velocity. BA: (Since this diagram came from Tokamak Energy, I'm very wary that I probably can't use it in my final transfer thesis, but it's useful here for demonstration purposes.)

1.4.1 Single-Phase Fluids

BA: Need to run some simulations this model works, testing for exclusively kinetic effects:

- One region of fluid passing through another- this should *really* highlight the effectiveness of the coupled fluid/correction model!
- Kinetic fluid waves? I'm not sure those are a thing...

1.4.2 Multiphase Fluids

1.4.3 Tokamak Plasmas

BA: Again, need to run some simulations this model works, testing for exclusively kinetic effects:

- Similar can do one region of fluid passing through another- would be interesting to see how this differs from the original single-phase fluid simulations.
- Kinetic plasma waves. There's a *lot* of these!

1.5 Edge Plasma Model

BA: Little about the edge plasma model we're going to consider:

- Looking for symmetric structures on a low aspect ratio?
- Idk there's a lot of ways to consider this- I could very well end up doing different kinds of simulations with all different kinds of BCs in the final product, and seeing how my results match up.

Summary

BA: Summary.

Chapter 2

Numerical Simulation and Preconditioning

BA: Introduction.

2.1 Maxwellian Background: A Fluid Simulation

BA: Important thing of note here is the fact that this is necessarily a *compressible*, and therefore partially *hyperbolic* system, which can cause a lot of difficulties for creating good discretisations and simulations. (C.F. Numerical dissipation.)

2.1.1 Augmented Lagrangian (AL) Preconditioning

BA: Very high Reynolds, so these fluid equations are *primed* for augmented Lagrangian preconditioning.

BA: Problem is, these equations are necessarily compressible- AL preconditioners have never been done for *compressible* fluid simulations before. How to transfer the ideas across is not immediate.

BA: Crucially: Need exact satisfaction of the mass conservation equations- can tackle this by using the vector of momentum, $\rho\mathbf{u}$, instead of velocity, $\mathbf{u}$.

Stationary State Simulations

Transient (State) Simulations

2.1.2 Fast Diagonalisation Method (FDM)

BA: NEPTUNE interested in high-order methods- why?:

- Better approximation properties (provided sufficient regularity- N.B. *Not* necessarily the case with funky BCs in a tokamak, but *should* be fine in my case if I pick a nice model with nice BCs on a nice domain).

- Better numerical approximation properties. (Recall that diagram from the NEPTUNE workshop with the travelling bump).
- Better suited to modern computer architectures. (Ask Pablo for more clarification here.)

Why not?:

- Massively worse computational complexity- very dense matrices, unless we find some way to mitigate this...

2.2 Non-Maxwellian Correction: A Kinetic Simulation

BA: What approach?

2.2.1 Lattice Boltzmann?

2.2.2 Series Expansion?

2.2.3 Particle-in-Cell (PIC)?

BA: Will Saunders at NEPTUNE said there's evidence that this decomposition gives a "low-noise" PIC simulation- quite what this means I don't know — potentially the reduction of high-wavenumber perturbations in the fluid portion? That's the impression I got off James Cook in his NEPTUNE presentation — *however*, the key takeaway is there is *solid evidence* that this is *the way to go*. Just Google "low-noise particle-in-cell" and you find there's a *solid* amount of literature on the idea.

Single-Phase Fluids

Tokamak Plasmas

BA: Would be a great opportunity to work in the ideas of gyrokinetic theory, especially the more mathematical aspects such as the Lie transformations overviewed in Lapillone's thesis (I'm sure there's a more original source for these ideas of course).

BA: A gyro-averaged model would shine here! Plus lots of opportunity for interesting (and actually accurate, in contrast to most physics gyrokinetic model papers) maths and multiscale analysis, especially with the addition of the stochastic Wiener process term. (Not sure I've ever seen a multiscale analysis with a stochastic term in before, I should ask the perturbation methods lecturer!)

BA: Typing out my thoughts here in a emphsemi-formal manner. Will re-phrase once I know what's in the previous sections etc....

Recall the system of SDEs satisfied by the position, $\mathbf{X}$, and velocity, $\mathbf{V}$, of the charged particle as defined in BA: (”earlier section”): BA: (The following is after non-dimensionalisation of course...)

$$d\mathbf{X} = \mathbf{V}dt \tag{2.1}$$

$$d\mathbf{V} = \frac{1}{\varepsilon^2}([\mathbf{E} + \mathbf{V} \wedge \mathbf{B}] + \mu[(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}) - \mathbf{V}])dt + \frac{\sigma}{\varepsilon}d\mathbf{W} \tag{2.2}$$

$\mathbf{V}$ here varies on the timescale of $\mathcal{O}(\varepsilon)$, at the cyclotron frequency, such that if we were to attempt to directly simulate the motion of such a charged particle, we would require a timestepping scheme with timestep frequency on the order of the cyclotron frequency, which would be potentially computationally prohibitive as $\varepsilon \to 0$ in the highly-magnetised limit. BA: (Compare with the value of ε in a typical tokamak and make some computational estimates.)

We can combat this using a multiscale analysis, with the transport on the $\mathcal{O}(1)$ timescale, and the gyrations on the $\mathcal{O}(\varepsilon)$ timescale. This is the assumption that is typically done in deriving the gyrokinetic model and resulting gyrocentre drifts. Defining $T = \frac{1}{\varepsilon}t$, write $\mathbf{X}$, $\mathbf{V}$ as each respectively the sum of a gyro-averaged value dependent only on the transport timescale $\overline{\mathbf{X}}(t;\varepsilon)$, $\overline{\mathbf{V}}(t;\varepsilon)$, and a $\mathcal{O}(1)$ perturbation $\delta\mathbf{X}(t,T;\varepsilon)$, $\delta\mathbf{V}(t,T;\varepsilon)$ oscillating on the cyclotron timescale with amplitudes dependent on t:

$$\left[\overline{\mathbf{X}}_t + \left(\delta\mathbf{X}_t + \frac{1}{\varepsilon}\delta\mathbf{X}_T\right)\right] dt = \left[\overline{\mathbf{V}} + \delta\mathbf{V}\right] dt \tag{2.3}$$

$$\begin{aligned}\left[\overline{\mathbf{V}}_t + \left(\delta\mathbf{V}_t + \frac{1}{\varepsilon}\delta\mathbf{V}_T\right)\right] dt = \frac{1}{\varepsilon^2}(&[\mathbf{E} + \left[\overline{\mathbf{V}} + \delta\mathbf{V}\right] \wedge \mathbf{B}] + \mu[(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}) \\ &- \left[\overline{\mathbf{V}} + \delta\mathbf{V}\right]])dt + \frac{\sigma}{\varepsilon}d\mathbf{W}\end{aligned} \tag{2.4}$$

BA: [Ref] This is modified from the classical gyrokinetic theory by the inclusion of the stochastic collisional terms, giving a multiscale *stochastic* system. Compare with the systems as defined in BA: [pretty much any paper on multiscale methods for SDEs]. We similarly expand the Wiener process $\mathbf{W}$ as the sum of a gyro-averaged $\overline{\mathbf{W}}$ and perturbation $\delta\mathbf{W}$. BA: (Do this by splitting the Fourier series at frequencies $\mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$!

$$\begin{aligned}\left[\overline{\mathbf{V}}_t + \left(\delta\mathbf{V}_t + \frac{1}{\varepsilon}\delta\mathbf{V}_T\right)\right] dt = \frac{1}{\varepsilon^2}(&[\mathbf{E} + \left[\overline{\mathbf{V}} + \delta\mathbf{V}\right] \wedge \mathbf{B}] + \mu[(\mathbf{u} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}) - \\ &\left[\overline{\mathbf{V}} + \delta\mathbf{V}\right]])dt + \frac{\sigma}{\varepsilon}\left[d\overline{\mathbf{W}} + d\delta\mathbf{W}\right]\end{aligned} \tag{2.5}$$

Writing $\mathbf{X}$, $\mathbf{V}$ — and with it $\overline{\mathbf{X}}$, $\overline{\mathbf{V}}$ and $\delta\mathbf{X}$, $\delta\mathbf{V}$ — as power series in powers of ε [1] as

$$*(\cdots;\varepsilon) = \sum_i \varepsilon^i *^{(i)}(\cdots) \left(= *^{(0)}(\cdots) + \varepsilon *^{(1)}(\cdots) + \varepsilon^2 *^{(2)}(\cdots) + \cdots\right) \tag{2.6}$$

we solve each of (2.3) and (2.5) at each order in ε: BA: (A lot of technicalities coming up, maybe good to put in an appendix?)

[1] Note, in the classical case *without* the collisional terms, these need only be powers in ε^2.

(2.3). At order $\mathcal{O}\left(\varepsilon^{(i)}\right)$

$$\overline{\mathbf{X}}_t^{(i)} + \left(\delta \mathbf{X}_t^{(i)} + \delta \mathbf{X}_T^{(i-1)}\right) = \overline{\mathbf{V}}^{(i)} + \delta \mathbf{V}^{(i)} \tag{2.7}$$

Separating into gyro-averaged and fluctuating components:

$$\overline{\mathbf{X}}_t^{(i)} = \overline{\mathbf{V}}^{(i)} \tag{2.8}$$

$$\delta \mathbf{X}_t^{(i)} + \delta \mathbf{X}_T^{(i-1)} = \delta \mathbf{V}^{(i)} \tag{2.9}$$

Resursively from $i = 0$, this gives:

$$\delta \mathbf{X}^{(0)} = \mathbf{0} \tag{2.10}$$

$$\delta \mathbf{V}^{(i)} = \delta \mathbf{X}_t^{(i)} + \delta \mathbf{X}_T^{(i-1)} \tag{2.11}$$

(2.5). ...

Summary

BA: Summary.

Chapter 3

Bifurcation Analysis

BA: Introduction.

BA: Lots to be said about the search for symmetry and ill-posedness here.

BA: Also worthy of note is the fact that the way I handle the PIC side could end up being non-deterministic, which would maybe mess up the deflation algorithm. (Maybe use some kind of deterministic seed- I could run the bifurcation analysis a few times with different seeds, and compare the results?)

Summary

BA: Summary.

Chapter 4

Numerical Implementation

BA: Introduction.

Support for:	Firedrake	Nektar++	NGSolve	deal.ii	Bespoke
Open source	✔		✔		✔
Portable	✔	✔	✔	✔	✘
FEEC elements	✔	✘	✱[5]		✱[1]
Block/AL preconditioning	✔				✱[1]
Multigrid preconditioning	✔				✱[1]
> 3 dimensions	✘			✔	✱[1]
Fluid/PIC interaction	✱[2]	✱[3]			✱[1]
Deflation	✱[4]	✘			✱[1]

Figure 4.1: BA: Applicability of different numerical implementation frameworks. BA: Details about what the ✔'s, ✘'s and ✱'s mean.

BA: What each of the ✱'s means:

✱[1] Obviously all of this is supported in a bespoke implementation, in as far as I would have to implement it, however I would have much more control over how it was done in this case.

✱[2] There's various possibilities here. Patrick mentioned `DMSwarm` in `PETSc`,[1] Pablo mentioned a few other ideas and people to get in contact with about this- I'm confident

[1] Will Saunders has concerns about the way `DMWarm` passed particles between MPI ranks- the particles are distributed from one rank to *all possible receiving ranks* — computationally intense already! — then has no checks for existence and uniqueness for a receiving rank. Apparently solving this would be a parallelisation nightmare too.

that some of the ideas I'd like to try should already be supported in some form, *without* me having to delve into the nitty-gritty of parallelising the PIC.

✱[3] Some of the fluid/PIC integration is implemented in *certain* cases within the ExCALIBUR NEPTUNE project in `NESO` (link), `NESO-Particles` (link) and `NESO-Spack` (link) within `Nektar++`- works on GPUs and everything!

✱[4] Firedrake has DefCon! Obviously, that's going to require some tweaks — as detailed in the Bifurcation Analysis chapter — for the applications I'd like to apply it to, but that's going to be the case for any premade deflation package.

✱[5] I'm not sure if `NGSolve` has support for FEEC elements on *non-simplicial* domains- it's unclear from the documentation.

Summary

BA: Summary.

Part II

Project Overview

Chapter 5

Research Plan

BA: Introduction.

BA: What have I/others done so far, and what is there now to do.

BA: Flowchart!

Summary

BA: Summary.

Summary

BA: Introduction.

BA: Summary.

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