Множества

множества $A=\{0,1,2\}$

подмножества $B=\{0,1\}$

$B\subseteq A$, $A\subseteq A$

$|A|=3$ – мощност

$|\mathbb{N}|=\infty$

Числови множества

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

$\mathbb{I} \to \mathbb{R}\setminus\mathbb{Q}$

$\{0,0,1,2\}=\{0,1,2\}=\{2,0,1\}$

$(0,1,2)$ – наредена тройка

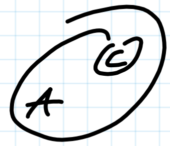
Операции с множества

$\cup$ – или

$\cap$ – и

$\setminus$ – разлика

$\bar{C}=C^c$ –



всичко от A без C

Фамилии (Семейства)

Нека $A_1, A_2, A_3, \dots$

Може да се запише $\{A_i\}_{i=1}^{\infty}$

$B_{m_0}, B_{m_1} \dots B_{m_k}$ $m_i\in M$

$\{B_{m_i}\}_{m_i\in M}$

Квантори: $\forall$, $\exists$

Нека $A\subseteq B \Rightarrow \forall x\in A: x\in B$

$A\nsubseteq B = \exists x\in B. \ x\notin A$

* $A=B \Leftrightarrow A\subseteq B$ и $B\subseteq A$

•1 $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$

$\subseteq$/ Нека $x\in(A\cup B)\cap C$

$\Rightarrow x\in(A\cup B)$ и $x\in C$

$\Rightarrow (x\in A$ или $x\in B)$ и $x\in C$

$(x\in A$ и $x\in C)$ или $(x\in B$ и $x\in C)$

$\Rightarrow x\in(A\cap C)\cup(B\cap C)$

$\Rightarrow (A\cup B)\cap C\subseteq(A\cap C)\cup(B\cap C)$

$\supseteq$ $x\in(A\cap C)\cup(B\cap C)$

$\Rightarrow (x\in A$ и $x\in C)$ или $(x\in B$ и $x\in C)$

$\Rightarrow x\in A$ или $x\in B$ и $x\in C$

$\Rightarrow x\in(A\cup B)\cap C$

$\Rightarrow A\cup B\cap C\supseteq$ —"—

Основни понятия

2) $(A \cap B) \cup C = (A \cup C) \cap (B \cap C)$ - дистрибутивност

3) $A \cup B = B \cup A$; $A \cap B = B \cap A$ - комутативност

4) Де Морган

i $A \setminus (B \cap C) = A \setminus B \cup A \setminus C$

$A \setminus (B \cup C) = A \setminus B \cap A \setminus C$

I Д-во Нека $x \in A \setminus (B \cap C)$

$\Rightarrow x \in A$ и $x \notin B \cap C$

$\Rightarrow x \in A$ и $(x \notin B$ или $x \notin C)$

$\Rightarrow (x \in A$ и $x \notin B)$ или $(x \in A$ и $x \notin C)$

$\Leftrightarrow x \in (A \setminus B) \cup (A \setminus C)$

$\Rightarrow x \in A \setminus (B \cap C) \Leftrightarrow x \in (A \setminus B) \cap (A \setminus C) = A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Декартово произведение

$\mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$

$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) : a_i \in A_i\}$

$f : A \to B$ Ако $x \in A$ $f(x) \in B$

Дефиниционна област (Domain) $\to D_f = \{x \in A \mid f(x) \in B\}$

Област от стойности (Range) $- R_f = \{y \in B \mid \exists x \in A : f(x) = y\}$

Обратима ф-я

$f \; A \to B \quad f^{-1} : B \to A \qquad f(x) = x^3 \Leftrightarrow x^3 = y \Leftrightarrow x = \sqrt[3]{y}$

$f^{-1}(y) = \sqrt[3]{y}$

Инекция $\forall x_1, x_2 \in A \quad x_1 \neq x_2 \quad f_{x_1} \neq f(x_2)$

Сюрекция $\forall y \in Z \ \exists x \in A \quad f(x) = y$

Биекция —

$x \in \overline{A \cup B \cup C} \Rightarrow x \notin A \cup B \cup C \Rightarrow x \notin A$ и $x \notin B$ и $x \in C$

$\Rightarrow x \notin A \cap B \cap C \Rightarrow x \in \overline{A \cap B \cap C}$

1 зад $\quad 2^2 + 5^2 + 8^2 \quad + (3n-1)^2 = \frac{1}{2} n \left(6n^2 + 3n - 1\right)$

Инд $\quad$ 1) $n = 1 \quad (3-1)^2 = 2^2 = 4$

$\frac{1}{2} \cdot 1 (6 + 3 - 1) = \frac{1}{2} \, 8 = 4$

✓

2) Стъпка – приемаме че тв. е вярно за някое $k \in N$

Ще док че зад е изпълнено за $k+1$

$2^2 + 5^2 + \cdot (3k-1)^2 + (3(k+1) - 1)^2 \stackrel{?}{=} -\shortparallel-$

$\frac{1}{2} k(6k^2 + 3k - 1) + (3(k+1) - 1)^2 \stackrel{?}{=} \frac{1}{2}(k+1)(6(k+1)^2 + 3(k+1) - 1)$

2 зад $\quad \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ (телескопична сума)

1 нач – индукция

2 нач – $\frac{1}{1 \cdot 3} \cdot 2 = \frac{1}{1} - \frac{1}{3} = \frac{2}{1 \cdot 3} \quad 2 \cdot \frac{1}{35} = \frac{1}{3} - \frac{1}{5} = \frac{2}{35} \Rightarrow \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} \cdot \frac{1}{2n-1} - \frac{1}{2n+1}\right]$

$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \frac{1}{2} \frac{2n}{2n+1} = \frac{n}{2n+1}$

2 зад Док че $F_n < 2^n$

$F_n = F_{n-1} + F_{n-2}$

База $n=1$ $F_1 = 1$ $1 < 2$

$n=2$ $F_2 = 1$ $1 < 4$

Стъпка приемаме че е изп за k
Ще док че е вярно за $k+1$

$F_{k+1} < 2^{k+1}$

$F_{k+1} = \underset{2^k}{\underset{\wedge}{F_k}} + \underset{2^{k-1}}{\underset{\wedge}{F_{k-1}}}$ Ако док че $2^k + 2^{k-1} < 2^{k+1}$, то ще е изп $F_{k+1} < 2^{k+1}$

$2^{k-1}(2+1) = 3.2^{k-1} < 4\,2^{k-1}$

$\Rightarrow$ е изп. $\Rightarrow$ $F_{k+1} < 2^{k+1}$

3 зад Док че $F_1 + F_3 + F_5 + F_{2n-1} = F_{2n}$

База. $n=1$ $F_1 \overset{?}{=} F_2$, $1=1$

$n=2$ $F_1 + F_3 = F_4$ $1+2=3$

Стъпка: изпълнено за k ще док за $k+1$

$$\underbrace{F_1 + \dots F_{2k-1}}_{= F_{2k}} + F_{2(k+1)-1} \overset{?}{=} F_{2(k+1)}$$

$$F_{2k} + F_{2(k+1)-1} \overset{?}{=} F_{2(k+1)}$$

$F_{2k} + F_{2k+1} \overset{?}{=} F_{2k+2}$, което е вярно от рекурентната връзка на числата

4 зад Док че $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$

База: $n=1$ $\frac{1}{2} \overset{?}{\leq} \frac{1}{\sqrt{4}} = \frac{1}{2}$

Стъпка - приемаме че е изп за k

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2k-1}{2k} \le \frac{1}{\sqrt{3k+1}}$$

Трябва да док че е вярно за k+1

$$\frac{1}{2}\cdots\frac{2k-1}{2k}\cdot\frac{2k+1}{2(k+1)} \overset{?}{\le} \frac{1}{\sqrt{3(k+1)+1}}$$

$$\le$$

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} \overset{?}{\le} \frac{1}{\sqrt{3k+3+1}}$$

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} \overset{?}{\le} \frac{1}{\sqrt{3k+4}} \quad \uparrow^2$$

$$\frac{(2k+1)^2}{(3k+1)(2k+2)^2} \overset{?}{\le} \frac{1}{3k+4}$$

$$(4k^2+4k+1)(3k+4) \le (3k+1)(4k^2+8k+4)$$

$$\cancel{12k^3}+\cancel{16k^2}+\cancel{12k^2}+16k+3k+\cancel{4} \le \cancel{12k^3}+\cancel{4k^2}+\cancel{24k^2}+8k+12k+\cancel{4}$$

$$0 \le k$$

5 зад Док че функ е обратима и намерете обратната ѝ

$g(x)=\frac{x}{1+|x|}$ $g:\mathbb{R}\to\mathbb{R}$

1сл.) $x\ge 0 \Rightarrow g(x)=\frac{x}{1+x}=y$

$x=y(1+x)=y+yx$ $\quad x(1-y)=y \quad \Rightarrow x=\frac{y}{1-y}$

$g^{-1}(y)=\frac{y}{1-y}$, $g^{-1}:[0,+\infty)\to[0,+\infty)$

$g^{-1}:[0,+\infty)\to[0,+\infty)$

* За да може една функ g^{-1} е обратима тя трябва да е инекция Нека $x_1 \neq x_2$ $\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|} \stackrel{?}{\Rightarrow} x_1 + x_1|x_2| = x_2 + x_2|x_1|$

1сл. $x_1, x_2 \geq 0$ $x_1 + \cancel{x_1 x_2} = x_2 + \cancel{x_2 x_1}$

но $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$

2сл $x_1, x_2 < 0$ $x_1 + x_1(-x_2) = x_2 + x_2(-x_1)$ но $x_1 \neq x_2$

$\Rightarrow \neq$

3сл $x_1 \geq 0$ $x_2 < 0$ $x_1 + x_1 \cdot x_2 = x_2 + x_2 x_1$ $x - x_2 \stackrel{?}{=} 2x_1x_2$

4сл $x_1 < 0$ $x_2 \geq 0$ $x_1 + x_1 x_2 \stackrel{?}{=} x_2 + x_2 - x_1$ $-x_2 > 0 \Rightarrow x_1 - x_2 > 0$

Ог 3 сл $\neq$ $2x_1 x_2 < 0$ защото $x_2 < 0$

$\Rightarrow x_1 - x_2 \neq 2x_1 x_2$

$\Rightarrow g$ е инекция $\Rightarrow$ е обратима

2 сл))

$x < 0 \Rightarrow g(x) \frac{x}{1-x} = y$

$x = (1-x)y = y - yx$

$x(1+y) = y \Rightarrow x = \frac{y}{1+y}$

$\Rightarrow$ За $x \in (-\infty, 0)$ $g^{-1}(y) = \frac{y}{1+y}$

$g: (-\infty, 0) \to (-\infty, 0)$

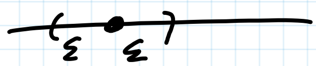
$g^{-1}(y) = \frac{y}{1-|y|}$???

Редици

Деф Всяко изображение $a: \mathbb{N} \to \mathbb{R}$ се нарича редица

Деф. ε-околност на точка:

Нека $a \in \mathbb{R}$. Множеството U се нарича околност на a



$U \subset \mathbb{R}$ ако $\exists\, \varepsilon > 0$ т.че $(a-\varepsilon, a+\varepsilon) \subset U$

дефиниция на Коши

Деф сходяща редица, граница на редица.

Казваме че ред е сходящ с граница a ако $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ т.че $\forall n \geq n_0$ $|a_n - a| < \varepsilon$

$\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$

$a_0 \quad a_1 \quad a_2 \ \dots \ a_{n_0} \quad a_{n_0+1} \ \dots$

Зад Док чрез Деф на Коши границата

$\lim\limits_{n \to \infty} \frac{n}{n+1} = 1$ фиксираме $\varepsilon > 0$

$\left\{\frac{n}{n+1}\right\}_{n=0}^{+\infty} \to 1$ Трябва да е изпълнено: $\left|\frac{n}{n+1} - 1\right| < \varepsilon$

$$\left|\frac{-1}{n+1}\right| < \varepsilon$$

$$\frac{1}{n+1} < \varepsilon$$

$\frac{1}{n+1} < \frac{1}{n}$ $\quad n_0 = \frac{1}{\varepsilon} \Rightarrow \forall n > n_0 \quad n > \frac{1}{\varepsilon} \Rightarrow \varepsilon > \frac{1}{n} > \frac{1}{n+1}$

Док. с коши $\lim_{n\to\infty} q^n = 0$, $q \in (0,1)$

Д-во Неравенство на Бернули $(1+a)^n \geq 1+n.a$, $a \in R^+$, $n \in Z^+$

фиксираме $\varepsilon > 0$

$\frac{1}{q} > 1$ полагаме $1+\alpha = \frac{1}{q}$

от Бернули $(1+\alpha)^n \geq 1+\alpha n$

Искаме да покажем, че

$\exists n_0$ т.ч. $\forall n > n_0$

$|q^n - 0| < \varepsilon$

Но $\frac{1}{q} = 1+\alpha \iff q = \frac{1}{1+\alpha}$

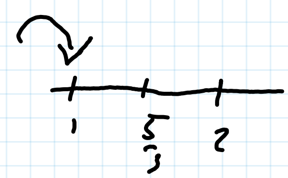
$\iff q^n = \left(\frac{1}{1+\alpha}\right)^n = \frac{1}{(1+\alpha)^n}$ Но $(1+\alpha)^n \geq 1+\alpha n \Rightarrow$

$\frac{1}{(1+\alpha)^n} \leq \frac{1}{1+\alpha n}$

$\Rightarrow q^n \leq \frac{1}{1+\alpha n}$ Нека $\frac{1}{1+\alpha n} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < 1+\alpha n \Rightarrow \frac{\frac{1}{\varepsilon}-1}{\alpha} < n$

$\Rightarrow$ ако $n_0 = \left[\frac{\frac{1}{\varepsilon}-1}{\alpha}\right]$, то $\forall n > n_0$ изпълнява условието

$\left[\frac{5}{3}\right]$



2 зад. Нека $\lim\limits_{n\to\infty} a_n = 2$

Да се изчисли

а) $\lim\limits_{n\to\infty} \dfrac{a_n^2+3a_n}{a_n-1} = \dfrac{\lim\limits_{n\to\infty} a_n^2+3a_n}{\lim\limits_{n\to\infty} a_n-1} = \dfrac{\lim\limits_{n\to\infty} a_n^2 + 3\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} 1} =$

$= \dfrac{4+3.2}{2-1} = 10$

б) $\lim \dfrac{a_n^3 - 6a_n^2 + 11a_n - 6}{a_n^2 - 8a_n - 12}$ $\quad a_n \neq 6$, $a_n \neq 2$

$= \lim$

3 зад. б) $\lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim\limits_{n\to\infty} \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\infty+\infty}$

$= \dfrac{1}{\infty} = 0$

$(2n-1+1)\ \dfrac{n}{2} = \dfrac{2n^2}{2} = n^2$

г) $\dfrac{(1+3+\dots+2n-1)}{\sqrt{n^4+n^2+1}} = \dfrac{n^2}{n^2\sqrt{1+\frac{1}{n^2}+\frac{1}{n^4}}} = \dfrac{1}{1} = 1$

$\frac{1}{n^2} \to 0$, $\frac{1}{n^4} \to 0$

* Lesno $\lim\left(\dfrac{1}{n}\cdot\sum\limits_{k=1}^{n}\left(1+\dfrac{k}{n}\right)\right)$ $\quad \sum\limits_{k=1}^{n}\left(1+\dfrac{k}{n}\right) = 1+\dfrac{1}{n}\dots\left(1+\dfrac{n}{n}\right) = n+\dfrac{1+2+\dots+n}{n} = \dfrac{n(n+1)}{2n}$

$= n+\dfrac{n+1}{2}$

$\lim\limits_{n\to\infty} \dfrac{1}{n}\left(n+\dfrac{n+1}{2}\right) = \lim\limits_{n\to\infty} 1+\dfrac{1+\frac{1}{n}}{2} = 1+\dfrac{1+0}{2} = \dfrac{3}{2}$

Основні границі

1. $\lim\limits_{n \to \infty} \frac{n}{a^n} = 0, \quad a > 0$

2. $\lim\limits_{n \to \infty} n q^n = 0 \quad |q| < 1$

3. $\lim\limits_{n \to \infty} \sqrt[n]{a} = 1 \quad a > 0$

   $a^{\frac{1}{n}}$

4. $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

4z

д) $\lim\limits_{n\to\infty} \dfrac{2^n+4^n}{3^n+(-5)^n} = \lim \dfrac{2^n+4^n}{3^n-5^n} = \dfrac{\cancel{5^n}\left(\frac{2}{5}^n+\frac{4}{5}^n\right)}{\cancel{5^n}\left(\frac{3}{5}^n-1\right)} = \dfrac{0}{-1} = 0$

4 зад $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{3^n+2n}-\sqrt{2^n+3n}}{\sqrt{n^2+2n}-n} = \lim\limits_{n\to\infty} \dfrac{n\left(\sqrt[3]{\frac{3^n}{n^3}+\frac{2}{n^2}}-\sqrt{\frac{2^n}{n^2}-\frac{3}{n}}\right)}{n\left(\sqrt{1+\frac{2}{n}}-1\right)} = \lim\limits_{n\to\infty} \dfrac{\sqrt[3]{\frac{3^n}{n^3}+\frac{2}{n^2}}-\sqrt{\frac{2^n}{n^2}-\frac{3}{n}}}{\sqrt{1+\frac{2}{n}}-1}$

∞

0

2 нач рацион $\lim\limits_{n\to\infty}$

5 зад. Док че $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Д-во Нека $a_n = \underset{\geq 0}{\sqrt[n]{n}-1} \Rightarrow n = (a_n+1)^n = 1 + n a_n + \underbrace{\binom{n}{2} a_n^2}_{\frac{n(n-1)}{2}} + \dots$

$\Rightarrow n > 1 + n a_n + \frac{n(n-1)}{2} a_n^2 > \frac{n(n-1)}{2} a_n^2$

$\Rightarrow a_n^2 < \frac{2}{n-1} \Leftrightarrow a_n < \frac{\sqrt{2}}{\sqrt{n-1}}$, но

по допускане

$a_n \geq 0$

$\Rightarrow$ по двете полиции $\lim a_n \geq 0$

и $\lim a_n \leq 0$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow \lim a_n = \lim \sqrt[n]{n} - 1 = 0$

$\Rightarrow \lim \sqrt[n]{n} = 1$

2) $\lim \dfrac{\sqrt{n^2+n}-\sqrt{n}}{n+\sqrt[3]{n^2+1}} = \dfrac{n\left(\sqrt{1+\frac{1}{n}}-\sqrt{\frac{1}{n}}\right)}{n\left(1+\sqrt[3]{\frac{1}{n}+\frac{1}{n^3}}\right)} = \dfrac{1}{1} =$

3) $\lim\limits_{n\to\infty} \sqrt[3]{1-8n^3}+2n = \lim\limits_{n\to\infty} 2n\left(\sqrt[3]{\frac{1}{8n^3}-1}\right)+2n = \lim\limits_{n\to\infty} -2n+2n = 0$

$\underset{0}{\downarrow}$

2ы

Основні границі

1) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+k} = e$

$\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right)^k = e \quad k\in N$

2) $\lim\left(1+\frac{k}{n}\right)^n = e^k \quad k\in Z$

$\left(\left(1+\frac{1}{\frac{n}{k}}\right)^{\frac{n}{k}}\right)^k = e^k$

3) $\lim\limits_{n\to\infty}\left(1+\frac{1}{kn}\right)^n = e^{\frac{1}{k}} \quad k\in N$

Нагадаємо що:

$e=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Ост

$e^0=1$

$e^x>1, \forall x>0$

$e^{x+y}=e^x e^y$

$\forall x_0 \; x>y \Rightarrow e^x>e^y$

4 зад. а) $\lim \left(\frac{n^2-1}{n^2-2n-3}\right)^n = \frac{(n-1)\cancel{(n+1)}}{(n-3)\cancel{(n+1)}} = \left(\frac{n-3+2}{n-3}\right)^n = \left(1+\frac{2}{n-3}\right)^{n-3+3} = \left(1+\frac{2}{n-3}\right)^{n-3} \cdot \left(1+\frac{2}{n-3}\right)^3$

$= e^2 \cdot 1 = e^2$

б) $\lim \left(\frac{2n^2+n-1}{2n^2+6n}\right)^n = \lim_{n\to\infty} \left(\frac{(n+1)(2n-1)}{2n(n+3)}\right)^n = \left(\frac{\cancel{2n^2}\left(1+\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)}{\cancel{2n^2}\left(1+\frac{3}{n}\right)}\right) =$

$= \frac{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{-2n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{e\, e^{-\frac{1}{2}}}{e^3} = e^{1-\frac{1}{2}-3} = e^{-\frac{5}{2}}$

Монотонни редици

ако $a_{n+1} - a_n < 0$, то $\{a_n\}_{n=0}^{k}$ е мон. намал.

. - .

5 зад. а) $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} \cdot \frac{1}{n^2}$ дали е сходяща

$= \sum_{k=1}^{n} \frac{1}{k^2}$

$1 + \frac{1}{2!} \dots + \frac{1}{n!}$

1) ... расящ

2) ограничен

б) $\lim\limits_{n\to\infty}\left(\dfrac{2^{\frac{1}{n}}+3^{\frac{1}{n}}}{2}\right)^n = \lim\limits_{n\to}\left(1+\dfrac{\overset{\sqrt[n]{2}}{2^{\frac{1}{n}}}+\overset{\sqrt[n]{3}}{3^{\frac{1}{n}}}-2}{2}\right)^n$

$\boxed{\begin{array}{l} \sqrt[n]{a} \xrightarrow[n\to\infty]{} 1 \Rightarrow \sqrt[n]{a}-1 \xrightarrow[n\to\infty]{} 0 \\ \sqrt[n]{n} \xrightarrow[n\to\infty]{} 1 \Rightarrow \sqrt[n]{n}-1 \xrightarrow[n\to\infty]{} 0 \end{array}}$

$= \lim\limits_{n\to\infty}\left(1+\dfrac{\sqrt[n]{2}-1+\sqrt[n]{3}-1}{2}\right)^n = -\|-\Big)^{\frac{2}{\sqrt[n]{2}-1+\sqrt[n]{3}-1}\cdot\frac{\sqrt[n]{2}-1+\sqrt[n]{3}-1}{2}\cdot n}$

$\downarrow$

0

$$= e^{\lim\limits_{n\to\infty} \frac{n(\sqrt[n]{2}+\sqrt[n]{3}-2)}{2}}$$

$$\lim \frac{1}{2} n(\sqrt[n]{2}+\sqrt[n]{3}-2) = \frac{1}{2} \lim n \cdot 3^{\frac{1}{n}}\left(\left(\frac{2}{3}\right)^n + 1 - \frac{2}{3^{\frac{1}{n}}}\right)$$

$$2^{\frac{1}{n}} \quad 3^{\frac{1}{n}}$$

$$\lim_{n\to\infty} n(a^{\frac{1}{n}}-1) = \ln a$$

$$e^{\lim n(a^{\frac{1}{n}}-1)} = e^{\ln a} = a$$

$$\lim_{n\to\infty} e^{n(a^{\frac{1}{n}}-1)} = a$$

$$\lim_{n\to\infty} (1 + a^{\frac{1}{n}} - 1)^{\frac{1}{a^{\frac{1}{n}}-1} \cdot n(a^{\frac{1}{n}}-1)}$$

$$= \lim_{n\to\infty} (a^{\frac{1}{n}})^n = a$$

Основни граници $\lim\limits_{n\to\infty} n(e^{\frac{1}{n}}-1) = 1 = \ln e$

* Теор. на Щолц

Нека $\exists$ редиците $\{a_n\}_{n=1}^{\infty}$ и $\{b_n\}_{n=1}^{\infty}$

са в сил. следните усл.

1) $\{b_n\}_{n=1}^{\infty}$ е строго монотонно растяща

(т.е. $b_n \xrightarrow[n\to\infty]{} \infty$)

2) Съществува границата

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = L$$

3 зад $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}\right)$

Нека $a_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}$

$$b_n = \sqrt{n}$$

1 $b_n \xrightarrow[n\to\infty]{} \infty$ Това щом е изп. за $b_n \xrightarrow[n\to\infty]{} \pm\infty$

2 $$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1}\right)}{\cancel{\sqrt{n}}\sqrt{1+\frac{1}{n}}} = 2$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 2$$

3 зад $\lim\limits_{n\to\infty} \frac{n^2}{2^n}$

I н. с бързина растене на ф-ии

II н. с Т-ма Щолц

Нека $a_n = n^2$, $b_n = 2^n$

$$b_n \xrightarrow[n\to\infty]{} \infty$$

$$2\ \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{(n+1)^2-n^2}{2^{n+1}-2^n} =$$

$$= -\text{״}- \frac{2n+1}{2^n(2-1)} = -\text{״}- \frac{2n+1}{2^n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\frac{2^n}{n}}$$

$$\frac{2^n}{n} \to \infty$$

$$= 0$$

4 зад $\lim\limits_{n\to\infty} \frac{n}{a^{n+1}}\left(a+\frac{a^2}{2}+\frac{a^3}{3}+\dots\frac{a^n}{n}\right)$, $a>0$

Нека $a_n = n\left(a+\frac{a^2}{2}+\dots\frac{a^n}{n}\right)$ $\quad b_n = a^{n+1}$

$$1\ b_n \xrightarrow[n\to\infty]{} \infty$$

$$2\ \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \text{—//—}\ \frac{(n+1)\left(a+\frac{a^2}{2}+\dots+\frac{a^n}{n}+\frac{a^{n+1}}{n+1}\right) - n\left(a+\frac{a^2}{2}+\frac{a^3}{3}+\dots\right)}{a^{n+2}-a^{n+1}}$$

$$= \text{—//—}\ \frac{n\frac{a^{n+1}}{n+1}+a+\frac{a^2}{2}+\dots\frac{a^n}{n}+\frac{a^{n+1}}{n+1}}{a^{n+2}+a^{n+1}} =$$

$$= \text{—//—}\ \frac{\cancel{a}\left(1+\frac{a}{2}+\dots\frac{a^{n-1}}{n-1}+\frac{(n+1)a^n}{\cancel{n+1}}\right)}{\cancel{a}\left(a^{n+1}-a^n\right)} =$$

$$= \text{—//—}\ \frac{1+\frac{a}{2}+\frac{a^2}{3}+\frac{a^{n-1}}{n}+a^n}{a^{n+1}-a^n}$$

Ще използваме отново теоремата на Щолц

$$a'_n = 1+\frac{a}{2}+\frac{a^2}{3}+\dots\frac{a^{n-1}}{n}+a^n \qquad b'_n = a^{n+1}-a^n$$

1) $\lim\limits_{n\to\infty} b'_n = \lim\limits_{n\to\infty} a^{n+1}-a^n = \lim\limits_{n\to\infty} a^n(a-1) = \infty$

2) $\lim\limits_{n\to\infty} \dfrac{a'_{n+1}-a'_n}{b'_{n+1}-b'_n} = \text{—//—}\ \dfrac{\frac{a^{n-1}}{n}+\frac{a^n}{n+1}+a^{n+1}-\left(\frac{a^{n-1}}{n}+a^n\right)}{a^{n+2}-a^{n+1}-\left(a^{n+1}-a^n\right)}$

$-\|- \frac{-n}{n+1} = -1$

$$= -\|- \frac{\left(\frac{1}{n+1}-1\right)\cancel{a^n} + a^{\cancel{n+1}}}{\cancel{a^n}(a^2-2a+1)} = -\|- \frac{\frac{-n}{n+1}+a}{(a-1)^2} =$$

$$= -\|- \frac{a-1}{(a-1)^2} = \frac{1}{a-1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a'_n}{b'_n} = \frac{1}{a-1} \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{1}{a-1} \Rightarrow$$

$$\Rightarrow -\|- \frac{a_n}{b_n} = \frac{1}{a-1}$$

def. (фундаментални редици) Казваме че $\{a_n\}_{n=1}^{\infty}$ е фунд. ако $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ и $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ $\forall n,m \geq n_0$ $|a_n - a_m| < \varepsilon$

Стинки

Сходящи и разходящи редове

$\sum_{n=1}^{\infty} u_n$ – безкраен числов ред

$S_n = \sum_{n=1}^{n} u_k$ - частична (парциална) сума

Казваме че $\sum_{n=1}^{\infty} u_n$ е сходящ ако редицата $\{S_n\}_{n=1}^{\infty}$ е сходяща

Казваме че $\sum_{n=1}^{\infty} u_n$ е разходящ ако $\{S_n\}_{n=1}^{\infty}$ е разходяща

$S_n \xrightarrow[n \to \infty]{} \pm\infty$

Пр $\sum_{n=1}^{\infty} q^n = \frac{1}{1-q}$, $|q| < 1$

Забележка Добавянето на краен брой ел не променя сходимостта на реда

Условие за сходимост $\sum_{n=1}^{\infty} a_n$ – сходящ, ако $a_n \xrightarrow[n \to \infty]{} 0$

Д-во $a_n = S_n - S_{n-1}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = 0$$

$$\lim_{n \to \infty} a_n = 0$$

Забележка В обратна посока невинаги е изпълнено

Пр: хармоничен ред

$\sum_{n=1}^{\infty} \frac{1}{n}$ не изпълнява (?) граничната но $\frac{1}{n} \xrightarrow[n \to \infty]{} 0$ $\quad a_{2n} - 1_n > \frac{1}{2}$

Зад. Да се испитат за сходност и да се најдат сумите на сходните

a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

$$\frac{1}{(2n-1)(2n+1)} \xrightarrow[n\to\infty]{} 0$$

$$\frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2}$$

$$\frac{1}{(2n-1)(2n+1)} \overset{?}{=} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(\frac{2n+1-2n-1}{(2n-1)(2n+1)}\right) =$$

$$= \frac{1}{2}\,\frac{2}{(2n-1)(2n+1)}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n}\left[\frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)\right] = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} \cdots \frac{1}{2n-1} - \frac{1}{2n+1}\right) =$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}\,\frac{2n}{2n+1} = \frac{n}{2n+1} \Rightarrow S_n = \frac{n}{2n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

б) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{n^2+2n+1-n^2}{n^2(n+1)^2} =$

n^2+2n+1

$= -\text{"}- \frac{\cancel{n^2+2n+1}}{n^2\cancel{(n+1)^2}} - \frac{\cancel{n^2}}{\cancel{n^2}(n+1)^2} = -\text{"}- \frac{1}{n^2} - \frac{1}{(n+1)^2}.$

$a_n = \frac{1}{n^2} - \frac{1}{(n+1)^2} \qquad S_n = \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{(k+1)^2} = 1 - \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}} + \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$

$= 1 - \frac{1}{(n+1)^2} = \frac{n^2+2n}{(n+1)^2}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{n\to\infty} S_n = \lim \frac{n^2+2n}{(n+1)^2} = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1$

в) $\sum_{v=1}^{\infty} \frac{1}{v(v+1)(v+2)}$

$\frac{1}{v(v+1)(v+2)} \xrightarrow[v\to\infty]{} 0 \qquad \frac{v+1-v}{v(v+1)(v+2)} = \frac{v+1}{v(v+1)(v+2)} - \frac{v}{v(v+1)(v+2)}$

$= \frac{1}{v(v+2)} - \frac{1}{(v+1)(v+2)}$

Нека $\frac{1}{v(v+1)(v+2)} = \frac{v+2-v}{v(v+1)(v+2)} \frac{1}{2} = \frac{1}{2}\left(\frac{v+2}{v(v+1)(v+2)} - \frac{v}{v(v+1)(v+2)}\right) =$

$= \frac{1}{2}\left(\frac{1}{v(v+1)} - \frac{1}{(v+1)(v+2)}\right) = \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+1} - \frac{1}{v+1} + \frac{1}{v+2}\right) = \frac{1}{2}\left(\frac{1}{v} - \frac{2}{v+1} + \frac{1}{v+2}\right)$

$$S_n = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{v} - \frac{2}{v+1} + \frac{1}{v+2}\right) = \frac{1}{2}\left(1 - \cancel{\frac{1}{2}} - \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \frac{1}{3}\right.$$

$$-\|- = \frac{1}{2}\left(\sum_{v=1}^{\infty} \frac{1}{v(v+1)} - \sum_{v=1}^{\infty} \frac{1}{(v+1)(v+2)}\right) \quad + \cancel{\frac{1}{4}} + \frac{1}{3} - \frac{1}{4} - \frac{1}{4} + \frac{1}{5} + \dots$$

$$\left. - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

Линейност $\sum_{n=1}^{\infty} \alpha a_n + \beta b_n = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$

нов метод

$$S'_n = 1 - \frac{1}{n+1}$$

$$S''_n = \frac{1}{2} - \frac{1}{n+2}$$

$$S_n = \sum_{v=1}^{\infty} \frac{1}{2}\left(S'_n - S''_n\right) = \frac{n^2+3n}{4n(n+1)(n+2)} \qquad \sum_{v=1}^{\infty} \frac{1}{v(v+1)(v+2)} = \lim_{n\to\infty} S_n =$$

$$= \lim_{n\to\infty} \frac{n^2+3n}{4(n+1)(n+2)} = \frac{1}{4}$$

г) $\sum_{v=1}^{\infty} (-1)^{v-1}$

д) $\sum_{v=1}^{\infty} \sin v$ — разходящи

2 зад. Да се нам. $\forall x \in \mathbb{R}$ т.ч. редовете да са сходящи и да се намерят стойностите

а) $\sum_{v=0}^{\infty} (-1)^v x^v = \sum_{v=1}^{\infty} (-x)^v$

Ако $x \in (-1, 1)$

в този случай $\sum_{v=1}^{\infty} (-x)^v = \frac{1}{1-(-x)} = \frac{1}{1+x}$

Ако $x \in (-\infty, -1] \cup [1, +\infty)$ редът е разходящ

б) $\sum_{v=0}^{\infty} 5^v x^{2v+1} \quad x^{2v} \cdot x = x \sum_{v=0}^{\infty} 5^v \cdot x^{2v} = x \sum_{v=0}^{\infty} (5x^2)^v$

За да може развит да е сходящ трябва $5x^2 \in (-1, 1)$

$x \in \left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$$x\sum_{\nu=0}^{\infty}(5x^2)^\nu = \frac{x}{1-5x^2}$$

Принцип за сравняване на редове

$$0 \le a_n \le b_n \Rightarrow \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$$

1) ако $\sum_{n=1}^{\infty} a_n$ е разх. то и $\sum_{n=1}^{\infty} b_n$ е разх.

2) ако $\sum_{n=1}^{\infty} b_n$ е сходящ то и $\sum_{n=1}^{\infty} a_n$ е сход.

3зад Да се изследва за сходимост

$$\sum_{\nu=2}^{\infty}\frac{\nu^2+\nu-1}{\nu(\nu^2-1)} = \sum_{\nu=2}^{\infty}\left(\frac{1}{\nu}+\frac{1}{\nu^2-1}\right)$$

$$\frac{1}{\nu} \le \frac{1}{\nu} + \frac{1}{\nu^2-2} \Rightarrow \sum_{\nu=2}^{\infty}\frac{1}{\nu} \le \sum_{\nu=2}^{\infty}\left(\frac{1}{\nu}+\frac{1}{\nu^2}\right)$$

харм. ред е разх.

$\Rightarrow$ от Принц. за срав. $\rightarrow$ че $\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}+\frac{1}{\nu^2-1}\right)$ е разх

4зад Да се изследват за сходимост

a) $\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}$ $\quad \frac{1}{\nu} + \frac{1}{\nu^2}$ $\quad \frac{1}{\nu^2} < \frac{1}{(\nu-1)(\nu+1)}$

$n^2 \overset{?}{>} n^2-1 \qquad \lim_{n\to\infty} \frac{1}{(n-1)(n+1)} = 0$

$$\frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$\frac{1}{2}\sum_{n=2}^{n}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \underline{1} - \cancel{\tfrac{1}{3}} + \underline{\tfrac{1}{2}} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}$$

$$+ \cancel{\tfrac{1}{5}} - \frac{1}{7} + \cancel{\tfrac{1}{6}} - \frac{1}{8}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = S_n \quad \ldots + \cancel{\tfrac{1}{n-2}} - \underline{\frac{1}{n}} + \cancel{\tfrac{1}{n-1}} - \underline{\frac{1}{n+1}}$$

$S_n \xrightarrow[n\to\infty]{} \frac{3}{4} \implies \sum \frac{1}{(n+1)(n-1)}$ – сходящ ~~ся~~ ред

$\implies \sum_{n=1}^{\infty} \frac{1}{n^2}$ – сходящ ~~ся~~ ~~ред~~

II н. $\frac{1}{n^2} < \frac{1}{n(n-1)}$

$\sum_{k=1}^{\infty} = \frac{1}{n^k}$ сходящ за $k > 1$

б) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

Критерии за сходимост на редове

. .

ако $k \in (0, +\infty) \Rightarrow a_n \sim b_n$

1 зад. изс за сходимост

а) $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{(n+1)^2\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{n^2\left(2\sqrt{\frac{1}{n^3}}+\frac{1}{n^2}\right)}{n^2\left[\left(1+\frac{1}{n}\right)^2\sqrt{n+1}\right]}$

$$\frac{\frac{2\sqrt{n}+1}{\cancel{n^2}\left(1+\frac{1}{n}\right)^2\sqrt{n+1}}}{\frac{1}{\cancel{n^2}}} \xrightarrow[n\to\infty]{} \frac{2\sqrt{n}+1}{(1+0)^2(n+1)}$$

$\xrightarrow{n\to\infty} 2$

$\Rightarrow \frac{2\sqrt{n}+1}{(n+1)^2\sqrt{n+1}} \sim \frac{1}{n^2}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ - сходящ

$\Rightarrow$ -"- е сходящ

б) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ $\quad \frac{(n!)^2}{(2n)!} = \frac{\cancel{n!}\,n!}{2^n(2n-1)!!\,\cancel{n!}} = \frac{n!}{2^n(2n-1)!!}$

= допуск ке не лесно

$$\frac{1\,2\,.\,3\,4\ldots(n-1)n}{1\,3\,.\,5\ldots(2n-1)} = \frac{2n\ldots n}{(n+1)(n+3)\ldots(2n-1)} < 1$$

$2 < n+1$

$n < n+3$

$\cdot$

$n < 2n-1$

$\Rightarrow \frac{2n!}{2^n(2n+1)!!} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \quad \sum_{n=1}^{\infty} \frac{1}{2^n}$ е сходящ ред

$\Rightarrow \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ~~сход~~ е сход.

Сходимост на числови редове

$$\sum_{n=1}^{\infty} \frac{\left((2n-1)!!\right)^2}{n!\, 3.7.11\ldots(4n-1)}$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = n\left[\frac{\dfrac{\left((2n-1)!!\right)^2}{n!\,(3.7.11\ldots 4n-1)}}{\dfrac{\left((2n+1)!!\right)^2}{(n+1)!\,(3.7.11\ldots(4n-1)(4n+3))}}-1\right]$$

$$= n.\left[\frac{(n+1)(4n+3)}{(2n+1)^2}-1\right] = n\,\frac{4n^2+7n+3-4n^2-4n-1}{4n^2+4n+1} = n\,\frac{3n+2}{4n^2+4n+1} = \frac{3n^2+2n}{4n^2+4n+1}$$

$$\xrightarrow[n\to\infty]{} \frac{3}{4} < 1 \Rightarrow \text{разходящ}$$

б) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\ldots(a+n)},\ a>0$

– изсл. за сход. спрямо a

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = n.\left(\frac{\dfrac{n!}{(a+1)(a+2)\ldots(a+n)}}{\dfrac{(n+1)!}{(a+1)(a+2)\ldots(a+n).(a+n+1)}}-1\right) = n\left(\frac{a+n+1}{n+1}-1\right) =$$

$$= n\left(\frac{a}{n+1}\right) = \frac{na}{n+1}$$

$\displaystyle\lim_{n\to\infty}\frac{na}{n+1} = a$

1) ако $a>1$ то е сх.

2) ако $a<1$ то е разх

3) ако $a=1$ $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{\underbrace{1.2.3.4\ldots n+1}_{(n+1)!}} = \sum_{n=1}^{\infty}\frac{1}{n+1}$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, но $\displaystyle\sum\frac{1}{n}$ е разх $\Rightarrow$ – ‖ – е разх

* Критерий на Лайбниц

$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ Ако редицата $\{a_n\}_{n=1}^{\infty}$ е монотонна и клони към 0 редът е сходящ

– алтернативен ред

* Критерий на Абел

Ако редицата $\{a_n\}$ е монотонна и ограничена, а редът $\sum_{n=1}^{\infty} b_n$ е сходящ то редът $\sum_{n=1}^{\infty} a_n . b_n$ също е сходящ

2 зад а) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$ $a_n = \frac{1}{2n-1} \xrightarrow[n\to\infty]{} \frac{1}{\infty} = 0$

$\frac{1}{2n-1} - \frac{1}{2n+1} = \frac{2}{(2n-1)(2n+1)} > 0 \; (1) \Rightarrow a_n > a_{n+1} \Rightarrow$ редицата е намаляваща (2)

от (1) и (2) следва че редът е сходящ

б) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n)!}{n^n}$ $a_n = \frac{(2n)!}{n^n}$

Забележка Ако $\sum_{n=1}^{\infty}|(-1)^{n-1}a_n|$ е разходящ то $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ също е разходящ

$\frac{a_{n+1}}{a_n}$ ще разгледам $\sum_{n=1}^{\infty}\frac{(2n)!}{n^n}$ – от кр. на Даламбер

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n)!} = \frac{(2n+1)(2n+2)}{n+1} \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{(2n+1)(2n+2)}{n+1}\left(1 + \frac{-1}{n+1}\right)^n = \frac{(2n+1)(2n+2)}{e.(n+1)} \xrightarrow[n\to\infty]{} \infty$$

от граничната форма на кр. на Даламбер $\Rightarrow$ че редът е разход.

! в) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{2^n}$

г) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2n-1)!!}{(2n)!!}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{\frac{(2n+1)!!}{(2n+2)!!}}{\frac{(2n-1)!!}{(2n)!!}} = \frac{2n+1}{2n+2}$

$(2a-1)(2a+1) = 4a^2-1 < 4a^2 = (2a)^2$ $\qquad \wedge$ 1 $\Downarrow$ монотонно намаляваща

$1 \cdot 3 < 2^2$

$3 \cdot 5 < 4^2$

$5 \cdot 7 < 6^2$

$$(2n-1)(2n+1) < (2n)^2$$

$$(1 \cdot 3 \cdot 5 \cdots (2n-1))^2 (2n+1) < (2 \cdot 4 \cdot 6 \cdots (2n))^2$$

$$\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 < \frac{1}{2n+1} \iff \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}}$$

$$\frac{1}{\sqrt{2n+1}} \xrightarrow[n\to\infty]{} 0 \Rightarrow \lim_{n\to\infty} \frac{(2n-1)!!}{(2n)!!} = 0$$

$\Rightarrow$ от кр. на Лайбниц редът е сходящ

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{2^n}$$

$$\frac{\frac{(n+1)^3}{2^{n+1}}}{\frac{(n)^3}{2^n}} = \frac{(n+1)(n+1)(n+1)}{2\cdot\cancel{2^n}} \cdot \frac{\cancel{2^n}}{n^3} =$$

n^3+3n^2+3n+1

$$= \frac{n^3+3n^2+3n+1}{2n^3} = \frac{\cancel{n^3}\left(\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}+1\right)}{2\cancel{n^3}} =$$

$$\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)\frac{1}{2} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

cxogsy

$n=1$

$$\frac{1}{\sqrt{1}} \overset{?}{\geq} \sqrt{1}$$

$$1 \geq 1$$

пуск n д/ n верно

$$\underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{n}}}_{\overset{\text{IV}}{\sqrt{n}}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$$

$$\frac{1}{\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} \quad \Big/ \frac{\sqrt{n+1}}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{n+1}} < \frac{\sqrt{n+1}\sqrt{n+1}}{\sqrt{n+1}} - \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n+1}}$$

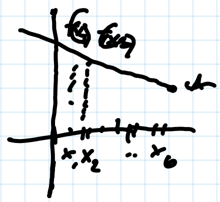
$$\sqrt{n+1} > \sqrt{n} \Rightarrow \sqrt{n+1}\sqrt{n+1} > \sqrt{n}\sqrt{n+1}$$

$1 <$

Граници на редица

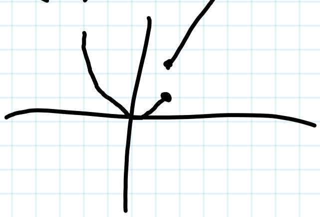
* def Хайне Ще казваме че ф-ята $f(x)$ има граница A при $x \to x_0$ ако за всяка редица от числа $x_n \in D$, която клони към x_0, $x_n \neq x_0$, съответната редица от функционални ст-ти клони към A

$x_1, x_2, x_3, \dots x_n$



def Хайне
- лява — лява граница $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x)$
- дясна — дясна граница $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x)$

Пр. $f(x) = \begin{cases} x^2, & x \in (-\infty, 1) \\ 2x, & x \in (1, +\infty) \end{cases}$



* def Коши Ще казваме че ф-ята $f(x)$ има граница A при $x \to x_0$ ако $\forall \varepsilon > 0$ т.че $|x - x_0| < \delta$ $\exists \delta > 0$ за което е изпълнено $|f(x) - A| < \varepsilon$

def Коши лява дясна граница Ще казваме че $f(x)$ има лява граница A при $x \to x_0$ ако $\forall \varepsilon > 0$ $\exists \delta > 0$ т.че
$\forall x \in D$ $x - \delta < x < x_0 \Rightarrow |f(x) - A| < \varepsilon$

-||- дясна граница A при $x \to x_0$, ако $\forall \varepsilon > 0$ $\exists \delta > 0$ т.ч.
$\forall x \in D$ $x_0 < x < x - \delta \Rightarrow |f(x) - A| < \varepsilon$

Т Ако функцията притежава лява и дясна граница при $x \to x_0$ и тези граница са равни то ф-ята притежава граница (двустранна) при $x \to x_0$

1 зад

а) $\lim\limits_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} = \frac{0}{0}$

$= \lim\limits_{x \to 1} \frac{(x^2 - \sqrt{x})(\sqrt{x} - 1)}{\sqrt{x} - 1} = \lim\limits_{x \to 1} \frac{(x^4 - x)(\sqrt{x} + 1)}{(x-1)(x^2 + \sqrt{x})} = -\text{"}- \frac{x\cancel{(x-1)}(x^2 + x + 1)(\sqrt{x} + 1)}{\cancel{(x-1)}(x^2 + \sqrt{x})} =$

$= \frac{1 \cdot 3 \cancel{2}}{\cancel{2}} = 3$

II н $\lim\limits_{x \to 1} \frac{\sqrt{x^4} - \sqrt{x}}{\sqrt{x} - 1} = \lim\limits_{x \to 1} \frac{\sqrt{x}(\sqrt{x^3} - 1)}{\sqrt{x} - 1} = \lim\limits_{x \to 1} \frac{\sqrt{x}(\sqrt{x} - 1)}{}$

б) $\lim\limits_{x \to 0} \frac{\sqrt[3]{x+1} - \sqrt[3]{1-x}}{x} = -\text{"}- \frac{(\sqrt[3]{1+x} - \sqrt[3]{1-x})(\sqrt[3]{x+1}^2 + \sqrt[3]{x+1}\sqrt[3]{1-x} + (\sqrt[3]{1-x})^2)}{x(\sqrt[3]{x+1}^2 + \sqrt[3]{x+1}\sqrt[3]{1-x} + (\sqrt[3]{1-x})^2)} =$

$2\cancel{x}$

$= -\text{"}- \frac{1 + x - (1 - x)}{\cancel{x}((\sqrt[3]{x+1})^2 + \sqrt[3]{x+1}\sqrt[3]{1-x} + (\sqrt[3]{1-x})^2)} = \frac{2}{3}$

в) $\lim\limits_{x \to \infty} \sqrt{x^2 - 2x - 1} - \sqrt{x^2 - 7x + 1} = -\text{"}- \frac{x^2 - 2x - 1 - x^2 + 7x - 1}{\sqrt{x^2 - 2x - 1} + \sqrt{x^2 - 7x + 1}} = -\text{"}- \frac{5x - 2}{-\text{"}-} =$

$= -\text{"}- \frac{\cancel{x}(5 - \frac{2}{x})}{\cancel{x}\left(\sqrt{1 - \frac{2}{x} - \frac{1}{x^2}} + \sqrt{1 - \frac{7}{x} + \frac{1}{x^2}}\right)} = -\text{"}- \frac{5 - \frac{2}{x}}{\sqrt{1} + \sqrt{1}} = \frac{5}{2}$

2 зад Пресметнете

а) $\lim\limits_{\substack{t \to 1 \\ t < 1}} \frac{2t}{1 - t^2} = \frac{2t}{\underset{>0}{(1 - t)}\underset{>0}{(1 + t)}} = \frac{2 \cdot 1}{+0 \cdot +0} = \infty$

б) $\lim\limits_{t\to 1} \dfrac{2t}{(1-t^2)^2} = \dfrac{2}{(\pm)^2} = \dfrac{2}{0} = +\infty$

в) $\lim\limits_{\substack{t\to 1\\ t<1}} \dfrac{2t^2+2}{t^2-3t+2} = \dfrac{2+2}{-0(-1)} = \dfrac{4}{+0} = +\infty$

$\underbrace{(t-1)}_{<0}(t-2)$

г) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ т.к. с радикандите ги полагаме $\sqrt[6]{x} = t \Rightarrow \sqrt{x} = t^3$, $\sqrt[3]{x} = t^2$

$\lim\limits_{t\to 1} \dfrac{t^3-1}{t^2-1} = \dfrac{\cancel{(t-1)}(t^2+t+1)}{\cancel{(t-1)}(t+1)} = \dfrac{3}{2}$

Първа основна граница

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ $\qquad$ $\lim\limits_{x\to 0} x \cot g\, x = 1$ $\qquad$ $\lim\limits_{x\to 0} \dfrac{x}{\operatorname{arctg} x} = 1$

$\lim\limits_{x\to 0} \dfrac{x}{\sin x} = 1$ $\qquad$ $\lim\limits_{x\to 0} \dfrac{\operatorname{arctg} x}{x} = 1$ $\qquad$ $\lim\limits_{x\to 0} \dfrac{x}{\arcsin x} = 1$

$\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x}{x} = 1$ $\qquad$ $\operatorname{tg} 45^\circ = 1$

$\operatorname{arctg} 1 = 45^\circ$

$\lim\limits_{x\to 0} \dfrac{x}{\operatorname{tg} x} = 1$

Д-во $\lim\limits_{x\to 0} \dfrac{\arcsin x}{x} = 1$

Нека $\arcsin x = \alpha \Rightarrow \sin\alpha = x$

$\alpha \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$x \to 0 \Leftrightarrow \alpha \to 0$

$\lim\limits_{x\to 0} \dfrac{\arcsin x}{x} = \lim\limits_{\alpha\to 0} \dfrac{\alpha}{\sin\alpha} = 1$