Suppose we have the continuous-time transition equation

$$dX_t = TX_t\,dt + R\,dW_t,$$

where $X_t \in \mathbb{R}^n$, $T \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times m}$, and dW_t is a m-dimensional standard Brownian motion. In expectation, we have that this system behaves according to

$$dX_t = TX_t\,dt,$$

which corresponds to the linear ODE system[1]

$$\frac{dX_t}{dt} = TX_t,$$

yielding the standard solution

$$X_t = \exp((t-\tau)T)X_\tau,$$

where X_τ is some initial condition (though τ need not be zero).

Our problem is to find a matrix Z which convert flows into stocks. Our states, X_t, represents the instantaneous state variable, so, for example, if output is a state, then X_t tracks the *rate* of output. Therefore, to acquire the total output procued, a *stock* variable, we must integrate the path of X_t. In the case that T is invertible, we have the analytical solution[2]

$$\left|\int_\tau^S X_t\,dt\right| = \left|T^{-1}\exp((t-\tau)T)X_\tau - T^{-1}\right|.$$

Note that we use absolute values here to allow for $\tau > S$ in the case that we want to determine output by integrating backwards in time (i.e. I observe my current state and guess the stock of output by extrapolating backward rather than observing my past state and guessing its evolution forward from there).

However, T is not always invertible, and in test trials with HANK models, T won't be invertible. Therefore, we approximate it using the definition of exponential matrices. Given any matrix T, we have that

$$\exp((t-\tau)T) = \sum_{n=0}^{\infty}\frac{((t-\tau)T)^n}{n!},$$

[1] Formally, any stochastic differential equation (SDE) is an *integral* equation. In our case, our SDE is actually short hand for

$$X_t - X_0 = \int_0^t TX_s\,ds + \int_0^t R\,dW_s.$$

The second integral is an Ito integral, which is a martingale. With standard Brownian motion, it has mean zero, so in expectation, our equation becomes

$$\mathbb{E}[X_t - X_0] = \int_0^t TX_s\,ds,$$

which is the *integral* equivalent of a linear ODE (i.e. every ODE can be represented with integrals).

[2] To develop intuition for this, it may be useful to study the scalar case, where the invertibility of T does not matter, as it will just be some real.

where $((t-\tau)T)^0 = I$. We can integrate term by term by the dominated convergence theorem[3], yielding

$$
\begin{aligned}
&\int_\tau^S \exp((t-\tau)T)\,dt = \sum_{n=0}^{\infty}\int_\tau^S \frac{((t-\tau)T)^n}{n!} \\
&= (S-\tau) + \frac{(S-\tau)^2 T}{2!} + \frac{(S-\tau)^3 T^2}{3!} + \frac{(S-\tau)^4 T^3}{4!} + \cdots \\
&= (S-\tau)\sum_{n=0}^{\infty}\frac{((S-\tau)T)^n}{n!}.
\end{aligned}
$$

We can therefore write

$$
\int_\tau^S \exp((t-\tau)T)X_\tau\,ds = \left(\int_\tau^S \exp((t-\tau)T)\,ds\right)X_\tau = \left((S-\tau)\sum_{n=0}^{\infty}\frac{((S-\tau)T)^n}{n!}\right)X_\tau.
$$

[3]Since we know the exponential matrix converges, we can bound the tail above by some constant and integrate the resulting *finite* sum of powers of T, plus some constant. Since this function is integrable and dominates $\exp((t-\tau)T)$, we can apply the Lebesgue dominated convergence theorem and integrate term by term.