HANK Theory

William Chen and Emily Martell

DSGE Interns 2018

August 8, 2018

Outline

Steady State Theory

Agent Problem

Consumers face the problem

$$\rho v_t = \max_c u(c) + \frac{1}{dt}\mathbb{E}[dv_t] \tag{1}$$

subject to the dynamic budget constraint

$$da_t = (w_t z_t + r a_t - c_t)\, dt. \tag{2}$$

Simplified HJB

$$\rho v_t = \max_c u(c) + \mathcal{A} v_t + \frac{1}{dt}\mathbb{E}_t[dv_t], \tag{3}$$

- $\mathcal{A}$: generator of idiosyncratic states' stochastic process

Kolmogorov Forward Equation and Market-Clearing

$$\frac{dg_t}{dt} = \mathcal{A}^* g_t, \tag{4}$$

- g_t: cross-sectional distribution of agents
- $\mathcal{A}^*$: adjoint of $\mathcal{A}$

$$\mathbf{p}_t = \mathcal{F}(g_t), \tag{5}$$

- $\mathbf{p}_t$: price vector
- $\mathcal{F}$: functional mapping distribution to prices

Discretized Steady State System

$$
\begin{aligned}
\rho \mathbf{v} &= \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v};\mathbf{p})\mathbf{v} \\
\mathbf{0} &= \mathbf{A}(\mathbf{v};\mathbf{p})^T \mathbf{g} \\
\mathbf{p} &= \mathbf{F}(\mathbf{g}),
\end{aligned} \tag{6}
$$

- In steady state, $\mathbb{E}_t[dv_t] = 0$
- $\mathbf{A}$: matrix approximation of $\mathcal{A}$
- $\mathbf{A}^T$: adjoint becomes the transpose

Finite Differences

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be some differentiable function and $\{x_0, x_1, \ldots, x_n\}$ a grid of points. A first-order forward finite difference is

$$\frac{df(x_i)}{dx} \approx \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)}. \tag{7}$$

and a second-order forward finite difference is

$$\frac{df(x_i)}{dx} \approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(x_{i+2} - x_{i+1})(x_{i+1} - x_i)}. \tag{8}$$

Upwind Scheme

- Use forward difference when drift in state variable is positive
- Use backward difference when drift in state variable is negative
- Example: Wealth a_t is a state variable. When an agent is saving and accumulating wealth, $\dot{a}_t > 0 \Rightarrow$ positive drift $\Rightarrow$ use forward difference to approximate value function derivative

Discretized Dynamic System

Now we add aggregate shocks back in *around* the steady state:

$$
\begin{aligned}
\rho \mathbf{v}_t &= \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t] \\
\frac{d\mathbf{g}_t}{dt} &= \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^T \mathbf{g}_t \\
\mathbf{p}_t &= \mathbf{F}(\mathbf{g}_t), \\
dZ_t &= \eta Z_t \, dt + \sigma \, dW_t,
\end{aligned}
\tag{9}
$$

- Z_t: vector of aggregate state shocks

Linearizing

$$\begin{aligned}
\mathbb{E}_t[d\mathbf{v}_t] &= (\rho\mathbf{v}_t - \mathbf{u}(\mathbf{v}_t) - \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)\mathbf{v}_t)\ dt \\
d\mathbf{g}_t &= \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)^T\mathbf{g}_t\,dt \\
\mathbf{p}_t &= \mathbf{F}(\mathbf{g}_t), \\
dZ_t &= \eta Z_t\,dt + \sigma\,dW_t,
\end{aligned} \tag{10}$$

which is the same as

$$\begin{bmatrix} \mathbb{E}[d\mathbf{v}_t] \\ d\mathbf{g}_t \\ dZ_t \end{bmatrix} = \mathbf{f}(\mathbf{v}_t;\mathbf{g}_t;\mathbf{p}_t;Z_t), \tag{11}$$

where $\mathbf{f}(\cdot) = \mathbf{0}$ in the steady state.

Linearized System

$$\begin{bmatrix} \mathbb{E}[d\hat{\mathbf{v}}_t] \\ d\hat{\mathbf{g}}_t \\ dZ_t \end{bmatrix} = \Gamma_1 \begin{bmatrix} \hat{\mathbf{v}}_t \\ \hat{\mathbf{g}}_t \\ Z_t \end{bmatrix} + \Psi \, dW_t + \Pi \, \eta_t + C \tag{12}$$

- η_t: generally, expectational errors of value functions

Some Terminology

- Jump/control variable: Any variable that some agent directly controls, e.g. value function, inflation target
- Aggregate/endogenous state variable: Any variable agents take as given, e.g. cross-sectional distribution, aggregate productivity
- Static conditions: Any conditions statically solved, e.g. market-clearing for capital
- When solving the code, reduction steps require a specific ordering of jump and aggregate state variables:
  1. Value function variables first, then any other jump variables, followed by ...
  2. Cross-sectional distribution variables, then any other aggregegage state variables

Krylov Subspace Methods and Aggregate State Variable Reduction

Goal

Find a basis $\mathbf{X}_\gamma$ for a lower dimensional subspace S such that

$$\begin{bmatrix} \hat{\mathbf{g}}_t \\ Z_t \end{bmatrix} \approx \mathbf{X}_\gamma \gamma_t,$$

where $\gamma_t \in S$.

Observability Matrix

We can re-write the linearized system as

$$\mathbb{E}\begin{bmatrix}\mathbb{E}[d\hat{\mathbf{v}}_t] \\ d\hat{\mathbf{g}}_t\end{bmatrix} = \begin{bmatrix}\mathbf{B}_{vv} & \mathbf{B}_{vp}\mathbf{B}_{pg} & \mathbf{B}_{vp}\mathbf{B}_{pz} \\ \mathbf{B}_{gb} & \mathbf{B}_{gg} + \mathbf{B}_{gp}\mathbf{B}_{pg} & \mathbf{B}_{gp}\mathbf{B}_{pZ}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{v}}_t \\ \hat{\mathbf{g}}_t \\ Z_t\end{bmatrix}.$$

Our desired observability matrix is $\mathcal{O}(\mathbf{B}_{pg}, \mathbf{B}_{gg} + \mathbf{B}_{gp}\mathbf{B}_{pg})^T$. For reasons described in Ahn et al. (2017), we project aggregate state variables onto the subspace generated by the observability matrix.

Krylov Subspaces

Definition

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^{n}$. Then the order-$k$ Krylov subspace is

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}(\{\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \ldots, \mathbf{A}^{k-1}\mathbf{b}\}),$$

Inspection of this definition should indicate that the observability matrix is an order-k Krylov subspace, namely

$$\mathcal{K}_k(\mathbf{B}_{gg}^T + \mathbf{B}_{gp}^T\mathbf{B}_{pg}^T, \mathbf{B}_{pg}^T).$$

Algorithm

- Partition Γ_1 matrix in linearized system (12) into blocks used to create an order-k Krylov subspace.
- Use deflated block Arnoldi iteration to find a basis for this subspae
  1. Stable method for orthogonalizing basis
  2. Handles multicollinearity that arises from the polynomial-esque definition of Krylov subspaces via deflation

Spline Basis Projection and Value Function Reduction

Spline Theory

Assuming that $\hat{\mathbf{v}}_t$ is smooth, splines provide a sufficient approximation. We reduce $\hat{\mathbf{v}}_t$ by projecting into a spline basis:

$$\hat{\mathbf{v}}_t \approx \mathbf{X}_\nu \nu_t. \tag{13}$$

- $\mathbf{X}_\nu$: creates linear combination of spline knot points
- ν_t: coefficients at knot points in spline basis

Constructing Basis

- Use quadratic splines: maintains monotoncity and concavity b/n knot points
- Non-uniform grid: better approximate complicated regions of value function
- Create $\mathbf{X}_\nu$ by mapping actual state space to knot points in spline basis

$$\mathbf{x} = \mathbf{X}_\nu \mathbf{k}_t,$$

where $\mathbf{x}$ are the actual state space grid points and $\mathbf{k}_t$ are the knot points.

Description of KrusellSmith and OneAssetHANK

KrusellSmith

Basic RBC model with the following features:

- Bewley-Aiyagari structure for the consumption-savings problem $\Rightarrow$ cross-sectional distribution in income/wealth matters
- Incomplete markets (consequence of Bewley-Aiyagari structure)
- Perfectly inelastic labor supply
- Wealth is held in capital
- Aggregate productivity shocks

Consumption-Savings Problem

Consumers solve the following problem

$$
\begin{aligned}
&\max_{c_{jt}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{c_{jt}^{1-\gamma}}{1-\gamma} \, dt \right], \\
&\dot{a}_{jt} = w_t z_{jt} + r_t a_{jt} - c_{jt}, \\
&z_{jt} \in \{z_L, z_H\}, \; z_L < z_H, \quad \text{with Poisson arrival rates} \quad \lambda_L, \lambda_H \\
&a_{jt} \geq 0.
\end{aligned}
$$

Production Side

Perfectly competitive firms $\Rightarrow$ representative firm produces according to

$$Y_t = e^{Z_t} K_t^{\alpha} N_t^{1-\alpha}$$

- Z_t is logarithm of aggregate productivity and follows mean-reverting Ornstein-Uhlenbeck process
- K_t is aggregate capital
- N_t is aggregate labor

OneAssetHANK

New Keynesian model in continuous time with

- Bewley-Aiyagari structure for the consumption-savings problem $\Rightarrow$ cross-sectional distribution in income/wealth matters
- Incomplete markets (consequence of Bewley-Aiyagari structure)
- Fiscal policy and government bonds for saving
- Monetary policy due to nominal rigidities from Rotemberg pricing and monopolistic competition in intermediate goods

Consumption-Savings Problem

Consumers solve the following problem

$$\max_{c_{jt}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{c_{jt}^{1-\gamma}}{1-\gamma} - \phi_0 \frac{l_{jt}^{1+1/\phi_1}}{1+1/\phi_1} \right) dt \right],$$

$$da_{jt} = \left(r_t \cdot a_{jt} + (1-\tau) \cdot w_t \cdot z_{jt} \cdot l_{jt} + T_t + \Pi_t - c_{jt} \right) dt$$

$$z_{jt} \in \{z_L, z_H\}, \; z_L < z_H, \quad \text{with Poisson arrival rates} \quad \lambda_L, \lambda_H$$

$$a_{jt} \geq \underline{a}.$$

Production Side

- Final goods created from a variety of intermediates according to a CES aggregator
- Rotemberg pricing: quadratic adjustment costs to price re-setting
- Labor is the only input into intermediates

Government Policy

Fiscal policy satisfies

$$\dot{B}_t^g + G_t + T_t = \tau_t \int w_t z l_t(a,z) g(a,z)\, da\, dz + r_t B_t^g .$$

Monetary policy follows

$$\begin{aligned} i_t &= \bar{r}_t + \phi_\pi \pi_t + \phi_y (y - \bar{y}) + \varepsilon_{MP,t}, \\ d\varepsilon_t &= -\theta_{MP} \varepsilon_t \, dt + \sigma_t \cdot dW_t . \end{aligned}$$

Estimation

Kalman Filter w/ Continuous-Time State & Discrete Observations

- Transition and measurement equations:

$$\begin{aligned} ds_t &= Ts_t\,dt + R\,dW_t \\ y_\tau &= Zs_\tau + D + u_\tau, \end{aligned}$$

  where $t \in [0, \infty)$ and $\tau \in \mathbb{N}$.
- Update step the same as in discrete Kalman Filter.
- Prediction step changes to solving deterministic ODE

$$\frac{\mathbb{E}[ds_t]}{dt} = Ts_t.$$

- Intuition: States driven by Brownian motion $\Rightarrow$ shock can be treated as if it all occurs at once in period τ $\Rightarrow$ predict between periods as if deterministic motion and update as if we had an instantenous normal shock in τ

Kalman Filter w/Simulated Subintervals

Let τ be the length of time in between discrete observations. Approximate path of s_t by subdividing $[0, \tau]$ into n subintervals and using Euler-Maruyama scheme to guess how s_t moves:

$$s_{t,i+1} = Ts_{t,i} \cdot \frac{\tau}{n} + R\epsilon_{t,i+1}, \quad \text{for } i = 0, 1, 2, \ldots, n$$

where $\epsilon_{t,i+1} \sim N(0, \frac{\tau}{n} I)$, $s_{t,0} = s_t$, and $s_{t,n} = s_{t+\tau}$. Our new "state" vector is

$$S_t = \begin{bmatrix} s_{t,1} \\ s_{t,2} \\ \cdot \\ \cdot \\ \cdot \\ s_{t,n} \end{bmatrix}$$

Transformed Matrices for Simulated Subintervals KF

Let I be the identity matrix. The new transition equation

$$S_{t+\tau} = TT\, S_t + RR\, \vec{\epsilon}_t,$$

where

$$TT = \begin{bmatrix} 0 & 0 & \cdots & 0 & (T+I) \\ 0 & 0 & \cdots & 0 & (T+I)^2 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 0 & (T+I)^n \end{bmatrix} + I$$

$$RR = \begin{bmatrix} R & 0 & 0 & \cdots & 0 & 0 \\ (T+I)R & R & 0 & \cdots & 0 & 0 \\ (T+I)^2 R & (T+I)R & R & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ (T+I)^{n-1}R & (T+I)^{n-2}R & (T+I)^{n-3}R & \cdots & (T+I)R & R \end{bmatrix}.$$

Data Generation

1. Initialize $s_0 = \mathbf{0}$, i.e. we start at the steady state.
2. Set $dt = 1/90$ so that 1 period is a quarter, and each time step dt is equivalent to a day.
3. To generate N years of data, perform the following calculation $360 \cdot N$ times:

$$s_{t+1} = (I + T\,dt)s_t + R\varepsilon_{t+1},$$

where T and R are the transition matrices from the true state transition equation

$$ds_t = Ts_t\,dt + R\,dW_t,$$

and $\varepsilon_{t+1} \sim N(0, dt * I)$, i.e. ε_{t+1} is a mean-zero normal shock with variance dt, per shock.

4. Save the path of s_t, which generates $360 \cdot N + 1$ data points.

Example Data Generation

- Our state vector, when unreduced, is 406×1, with flow output in the 404th entry
- Z matrix is 1×406 matrix and is all zeros except for a 1 in the 404 entry
- Zs_t returns the flow output whenever data is available
- No measurement noise added
- Resulting system:

$$\begin{aligned} ds_t &= Ts_t \, dt + R \, dW_t \\ y_\tau &= Zs_\tau, \end{aligned}$$

- Computing likelihoods on this system is the same as evaluating the Kalman filter's behavior when we can only see the true state vector every 90th data point.

Timing

Method	Time (ms)	Allocations	Memory (MiB)
ODE - Euler	418.813	440419	634.9
ODE - Tsitouras 5/4	421.565	440019	634.89
Subintervals - 2	50.181	15071	67.84
Subintervals - 3	126.252	15071	149.88
Subintervals - 12	4601	16214	2300

Accuracy

Method	Mean absolute distance	Std(error)	Ma
ODE Integration - Tsitouras 5/4	.00122	.0004855	
Subintervals - 2	.00157	.000438	
Subintervals - 3	.00158	.000436	
Subintervals - 12	.00159	.000423	

- max_lik_σ: σ predicted by MLE when using Kalman filter and varying the estimated parameter
- true_σ: σ that generated the data, which is .007 for KrusellSmith
- mean abs dist: mean(max_lik_σ - true_σ)
- std(error): std(max_lik_σ - true_σ)
- max abs err: maximum(abs.(max_lik_σ - true_σ))

Other Kalman Filters

We made some notes and found some references for block kalman filters, which use block multiplication to speed up Kalman Filter, and ensemble Kalman Filters. See the "working paper" documentation for these notes.

Measurement Equation: Set Up

Suppose we have the continuous-time transition equation

$$dX_t = TX_t\,dt + R\,dW_t,$$

where $X_t \in \mathbb{R}^n$, $T \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times m}$, and dW_t is an m-dimensional standard Brownian motion. In expectation, we have

$$dX_t = TX_t\,dt \Rightarrow \frac{dX_t}{dt} = TX_t \Rightarrow X_t = \exp((t-\tau)T)X_\tau.$$

From Flows to Stocks

- X_t represents flows like output per second
- Data is on stock variables $\Rightarrow$
- Need to construct Z so that ZX_t yields stock variables
- Time is continuous $\Rightarrow$ integrate

Integrating Exponential Matrix

We have $X_t = \exp((t-\tau)T)X_\tau$. T is not invertible usually, so we integrate power series definition:

$$\int_\tau^S \exp((t-\tau)T)X_\tau \, dt$$

$$= \int_\tau^S \left(\sum_{n=0}^{\infty} \frac{((t-\tau)T)^n}{n!} \right) dt \, X_\tau$$

$$= \left(\sum_{n=0}^{\infty} \int_\tau^S \frac{((t-\tau)T)^n}{n!} dt \right) X_\tau$$

$$= \left((S-\tau) + \frac{(S-\tau)^2 T}{2!} + \frac{(S-\tau)^3 T^2}{3!} + \frac{(S-\tau)^4 T^3}{4!} + \cdots \right) X_\tau$$

$$= \left((S-\tau) \sum_{n=0}^{\infty} \frac{((S-\tau)T)^n}{n!} \right) X_\tau$$

We make Z by choosing rows of the parenthetical term in last line.

track_lag and Subinterval Kalman Filter

- Since the subinterval KF has the state vector

$$S_t = (s_{t,1}, \ldots, s_{t,n}),$$

  we want

$$ZS_t = \sum_{i=1}^{n} \int \exp((\Delta t)T) s_{t,i-1}\, d(\Delta t).$$

- But S_t doesn't have $s_{t,0} = s_{t-1,n}$
- track_lag adjusts our TT and RR matrix in

$$S_{t+\tau} = TT\, S_t + RR\, \vec{\epsilon}_t$$

  so that

$$S_t = (s_{t,0}, s_{t,1}, \ldots, s_{t,n}) = (s_{t-1,n}, s_{t,1}, \ldots, s_{t,n}).$$

Why track_lag rather than augment states?

- solve returns T, C, R matrices, but with augment states, we would have
$$T = \begin{bmatrix} 0 & I \\ 0 & T_{orig} \end{bmatrix}, \qquad R = \begin{bmatrix} 0 \\ R_{orig} \end{bmatrix}$$
where T_{orig} is the unaugmented T matrix, and R_{orig} is the unaugmented R matrix.
- Not all measurement equations need to integrate flows (e.g. asset prices) $\Rightarrow$ user would have to slice T to get T_{orig}.
- Users may want to conduct theoretical experiments, e.g. impulse responses

Future Work

Immediately Doable Work

1. Internal consistency check to allow for endogenous decision rules. Currently, the methods ignore feedback from changes in endogenous decision rules (from the value function) and get around it by simply raising k, the dimension of Krylov subspace, to a sufficiently high power. Experimentation by Ahn et al. (2017) suggests that this is fine for many models, but it may be desirable to have this method available.
2. Translate two-asset HANK, as described in the Ahn et al. (2017) paper and "Monetary Policy according to HANK".

Larger Extensions for Heterogeneous-Agent Models

1. Translate Smets-Wouters into a HANK format (e.g. idiosyncratic labor)
2. Create a HANK model with heterogeneity among firms. For example, we could investigate heterogeneity in firms' balance sheets/funding constraints.
3. Ben Moll developed a method for choosing optimal policies in the steady state of HACT models. We may wish to implement this as part of the toolkit to allow users to conduct welfare analysis. See "working paper" for link.
4. Fernandez-Villaverde extends Krusell-Smith's original methods by using neural network instead of linear regression to approximate perceived law of motion. This allows the inclusion of a Brunnermeier-Sannikov style financial friction and its nonlinearities while maintaining heterogeneity among consumers. See "working paper" for link.