Without any aggregate shocks and after applying Ito's lemma, the steady state in a HANK model can be represented as the system

$$\begin{aligned}
\rho v_t &= \max_c u(c) + \frac{1}{dt}\mathbb{E}_t[\mathcal{A} v_t], \\
\frac{dg_t}{dt} &= -\mathcal{A}^* g_t, \\
p_t &= F(g_t),
\end{aligned}$$

where $\mathcal{A}$ is the generator of an Ito diffusion (or jump diffusion or semi-martingale), $\mathcal{A}^*$ is the adjoint, etc.

In the steady state, where we set aggregate shocks to zero, we can represent the discretized system as

$$\begin{aligned}
\rho \mathbf{v}_t &= \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\mathbf{v}_t \\
\mathbf{0} &= \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^T \mathbf{g}_t \\
\mathbf{p}_t &= \mathbf{F}(\mathbf{g}_t).
\end{aligned}$$

In this system, $\mathbf{v}, \mathbf{g}, \mathbf{p}$ are all now vectors at points in a discrete grid, with $\mathbf{A}$ performing a variety of finite difference operations. To arrive at the system with aggregate shocks Z_t, we have

$$\begin{aligned}
\rho \mathbf{v}_t &= \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t] \\
\frac{d\mathbf{g}_t}{dt} &= \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^T \mathbf{g}_t \\
dZ_t &= (\mu_1 Z_t + \mu_2)\, dt + \sigma\, dW_t \\
\mathbf{p}_t &= \mathbf{F}(\mathbf{g}_t; Z_t).
\end{aligned}$$

The first equation is just the HJB, where the RHS reflects the fact that technically, with Ito calculus, $d\mathbf{v}$ represents a stochastic integral, so "technically", you can't take the time derivative. The second line is the Fokker-Planck/KFE. Since we are not looking for a stationary density, the zeros become the time derivative of $\mathbf{g}$. The third equation is just some generic linear SDE, where μ_1, μ_2 are assumed to not depend on $\mathbf{v}, \mathbf{g}, Z$. Finally, the last equation is market-clearing.

We now re-arrange time derivatives onto the same side. Note $\mathbf{p}$ is statically solved, so technically we don't need to track it as a state. After plugging $\mathbf{p}$ directly into our equations, we have

$$\begin{aligned}
\mathbb{E}_t[d\mathbf{v}_t] &= [\rho \mathbf{v}_t - \mathbf{u}(\mathbf{v}_t) - \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\mathbf{v}_t]\ dt \\
d\mathbf{g}_t &= \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^T \mathbf{g}_t\, dt \\
dZ_t &= (\mu_1 Z_t + \mu_2)\, dt + \sigma\, dW_t
\end{aligned}$$

We may re-write this nonlinear system as

$$d\mathbf{s}_t = \mathbf{f}(\mathbf{v}_t, \mathbf{g}_t, p_t, Z_t),$$

with $d\mathbf{s}_t = 0$ at the steady-state. We now linearize the model by taking the first-order Taylor expansions w.r.t. $\mathbf{v}, \mathbf{g}, Z, \mathbf{p}$. Since all the arguments of $\mathbf{f}$ are just dependent on these *levels* of state variables, differentiation should not pop out any time derivatives/stochastic differentials. Let $d\hat{\mathbf{g}}_t$ represent the linear approximation of the time differential of the distribution (i.e. an approximation of $d\mathbf{g}_t$. Similar notation is used for the other state variables. Noting that $d\mathbf{s}_t = 0$ at the steady state, we can write, in gensys form,

$$\begin{bmatrix} \mathbb{E}[d\hat{\mathbf{v}}_t] \\ d\hat{\mathbf{g}}_t \\ dZ_t \end{bmatrix} = \mathbf{B} \begin{bmatrix} \hat{\mathbf{v}}_t \\ \hat{\mathbf{g}}_t \\ Z_t + \sigma dW_t. \end{bmatrix}$$

In Sims's notation, we have

$$\dot{s} = \Gamma_1 s + \Psi dW + \Pi \eta$$

since we can completely summarize the time derivatives/differentials with their current levels at time t.