CENTER FOLIATION RIGIDITY FOR PARTIALLY HYPERBOLIC TORAL DIFFEOMORPHISMS

ANDREY GOGOLEV*, BORIS KALININ**, VICTORIA SADOVSKAYA***

ABSTRACT. We study perturbations of a partially hyperbolic toral automorphism L which is diagonalizable over $\mathbb{C}$ and has a dense center foliation. For a small perturbation of L with a smooth center foliation we establish existence of a smooth leaf conjugacy to L. We also show that if a small perturbation of an ergodic irreducible L has smooth center foliation and is bi-Hölder conjugate to L, then the conjugacy is smooth. As a corollary, we show that for any symplectic perturbation of such an L any bi-Hölder conjugacy must be smooth. For a totally irreducible L with two-dimensional center, we establish a number of equivalent conditions on the perturbation that ensure smooth conjugacy to L.

1. INTRODUCTION AND STATEMENTS OF RESULTS

Partially hyperbolic ergodic toral automorphisms, which are sometimes called quasi-hyperbolic, form an important class of algebraic partially hyperbolic systems. They have been extensively studied and shown to have strong stochastic and other properties, often similar to those of hyperbolic systems: Bernoulli property [Kz71], uniqueness of the measure of maximal entropy [B67], exponential mixing [L82], density of periodic measures [M80] and their asymptotic equidistribution [L82], and cohomological properties similar to Livšic periodic point theorem and measurable Livšic theorem [V86].

Perturbations of partially hyperbolic ergodic toral automorphisms give a natural class of partially hyperbolic systems. In contrast to linear models, the properties of such perturbations are much less understood. Some of the difficulties presented by these nonlinear systems are due to multidimensional non-compact center leaves. For totally irreducible ergodic toral automorphisms with two-dimensional center foliation, stable ergodicity was established by Rodriguez Hertz in [RH05]. Further properties for this case, including the stable Bernoulli property for symplectic perturbations, were obtained by Avila and Viana in [AV10].

In this paper we study rigidity properties for perturbations of partially hyperbolic toral automorphisms related to the smoothness of their center foliation. In

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particular, we obtain smoothness of the leaf conjugacy to the linear system, and smoothness of the conjugacy when one exists. Our main results hold for systems with dense center foliation of any dimension, but have no analogs in the hyperbolic case. Further results are then deduced for systems with two-dimensional center foliation using [RH05, AV10].

We consider a linear map $L \in SL(d, \mathbb{Z})$ and use the same notation for the corresponding toral automorphism $L : \mathbb{T}^d \to \mathbb{T}^d$. The map L is called *irreducible* if it has no rational invariant subspaces, or equivalently if its characteristic polynomial is irreducible over $\mathbb{Q}$. The automorphism L is ergodic with respect to the Lebesgue measure if and only if no root of unity is its eigenvalue. We define the stable, unstable, and center subspaces E^s, E^u, and E^c for L as those corresponding to eigenvalues of modulus less than 1, greater than 1, and equal to 1, respectively. We denote by W^s, W^u, and W^c the corresponding linear foliations. An irreducible ergodic automorphism L is always *partially hyperbolic*, that is, it has non-trivial E^s and E^u. We will consider partially hyperbolic automorphisms L with non-trivial center E^c.

We consider a C^∞ diffeomorphism f which is C^1 close to L. Such f is *partially hyperbolic*, more precisely, there exist a nontrivial Df-invariant splitting $\mathcal{E}^s \oplus \mathcal{E}^c \oplus \mathcal{E}^u$ of the tangent bundle of $\mathbb{T}^d$, a continuous Riemannian metric on $\mathbb{T}^d$, and constants $\nu < 1$, $\hat\nu > 1$, γ, $\hat\gamma$ such that for any $x \in \mathcal{M}$ and any unit vectors $v^s \in \mathcal{E}^s(x)$, $v^c \in \mathcal{E}^c(x)$, and $v^u \in \mathcal{E}^u(x)$,

$$\|D_x f(v^s)\| < \nu < \gamma < \|D_x f(v^c)\| < \hat\gamma < \hat\nu < \|D_x f(v^u)\|.$$

The sub-bundles $\mathcal{E}^s$, $\mathcal{E}^u$, and $\mathcal{E}^c$ are called *stable, unstable, and center*. The stable and unstable sub-bundles are tangent to the stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively. The leaves of these foliations are C^∞. By structural stability of partially hyperbolic systems [HPS77], f is *dynamically coherent*, that is, the bundles $\mathcal{E}^c$, $\mathcal{E}^{cu} = \mathcal{E}^u \oplus \mathcal{E}^c$, and $\mathcal{E}^{cs} = \mathcal{E}^s \oplus \mathcal{E}^c$ are tangent to foliations $\mathcal{W}^c$, $\mathcal{W}^{cu}$, and $\mathcal{W}^{cs}$ with C^r leaves, where $r > 1$ is determined by expansion/contraction in $\mathcal{E}^c$ relative to the rates for $\mathcal{E}^u$ and $\mathcal{E}^s$. Moreover, f is leaf conjugate to L by a bi-Hölder homeomorphism h close to the identity. *A leaf conjugacy* is a homeomorphism $h \colon \mathbb{T}^d \to \mathbb{T}^d$ mapping the leaves of $\mathcal{W}^c$ homeomorphically to the leaves of W^c such that

$$h(f(\mathcal{W}^c(x))) = W^c(L(h(x))) \quad \text{for every } x \in \mathbb{T}^d.$$

Now we formulate our main results. First we establish existence of a smooth leaf conjugacy for a perturbation with a smooth center foliation.

Theorem 1.1 (Smooth leaf conjugacy)**.** *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a partially hyperbolic automorphism which is diagonalizable over $\mathbb{C}$ and has dense center foliation W^c. Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a C^∞ diffeomorphism which is C^1 close to L. If $\mathcal{W}^c$ is a C^∞ foliation, then f is C^∞ leaf-conjugate to L.*

We note that the theorem applies, in particular, to all irreducible partially hyperbolic automorphisms. Also, if the theorem applies to automorphisms L_1 and L_2, then it applies to $L_1 \times L_2$ and $L_1 \times \mathrm{Id}_{\mathbb{T}^k}$ as well.

Remark 1.2. *Theorem 1.1 has a finite regularity version: if $\mathcal{W}^c$ is a C^r foliation with $r > r(L)$ from (1.1) below, then f is C^q leaf-conjugate to L where $q = r$ if $r \notin \mathbb{N}$ and $q = r - \varepsilon$ for any $\varepsilon > 0$ if $r \in \mathbb{N}$. This can be obtained by the same argument using C^r normal form coordinates and Journé's lemma* [J88]*.*

Next we consider the case when f is bi-Hölder conjugate to L. That is, we assume that there exists a Hölder continuous conjugacy h with a Hölder continuous inverse. We obtain C^∞ smoothness of this conjugacy if $\mathcal{W}^c$ has sufficient regularity defined as follows. Let $1 < \rho_{\min}^u \le \rho_{\max}^u$ be the smallest and largest moduli of unstable eigenvalues of L, and let $0 < \rho_{\min}^s \le \rho_{\max}^s < 1$ be the smallest and largest moduli of its stable eigenvalues. We set

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\begin{aligned}
r^u(L) &= (\log \rho_{\max}^u)/(\log \rho_{\min}^u) \ge 1, \\
r^s(L) &= (\log \rho_{\min}^s)/(\log \rho_{\max}^s) \ge 1, \\
r(L) &= \max\{r^u(L),\, r^s(L)\}.
\end{aligned}
$$

(1.1)

Theorem 1.3 (Smoothness of bi-Hölder conjugacy)**.** *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an irreducible ergodic automorphism and let $r > r(L)$. Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a volume-preserving C^∞ diffeomorphism that is sufficiently C^1 close to L. If f has C^r center foliation and is conjugate to L by a bi-Hölder homeomorphism h, then h is C^∞.*

Remark 1.4. *If E^u and E^s are one-dimensional it suffices to take $r = 1$ rather than $r > r(L) = 1$, that is, to assume that the center foliation is C^1. Indeed, for one-dimensional leaves the analog of the centralizer part of Theorems 2.1 and 2.3 was proved in* [KtL91] *in C^1 regularity.*

For a symplectic perturbation f we obtain the following corollary.

Corollary 1.5. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a symplectic irreducible ergodic automorphism and let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a C^∞ symplectic diffeomorphism which is C^1-close to L. If f is bi-Hölder conjugate to L, then f is C^∞ conjugate to L.*

This result is a rare example of rigidity in smooth dynamics, in the sense of "weak equivalence implies strong equivalence", that holds for a single system rather than an action of higher rank group. It relies on coexistence of hyperbolic and elliptic behavior in one system, and thus is also a rare example of a result for partially hyperbolic systems that does not cover hyperbolic systems as a particular case.

Remark 1.6. *Theorem 1.3 and Corollary 1.5 hold more generally for any automorphism L which is partially hyperbolic, ergodic, diagonalizable over $\mathbb{C}$ and has a dense center foliation. This class of automorphisms includes products of irreducible ergodic automorphisms. The proof is essentially the same utilizing Proposition 5.1. See Remark 3.3.*

Now we consider the case of L with two-dimensional center. We call a toral automorphism L *totally irreducible* if L^n is irreducible for every $n \in \mathbb{N}$. Such an L is always ergodic. For a totally irreducible automorphism L with exactly two eigenvalues of absolute value one, that is $\dim E^c = 2$, Rodriguez Hertz proved in [RH05] that it is stably ergodic, more precisely, any sufficiently C^{22}-small volume-preserving perturbation of L is also ergodic. For such L we use some results from [RH05] and [AV10] to obtain further corollaries of Theorem 1.3.

We recall definitions accessibility and Lyapunov exponents before stating further results. A partially hyperbolic diffeomorphism f of $\mathbb{T}^d$ is called *accessible* if any two points in $\mathbb{T}^d$ can be connected by an *su*-path, that is, by a concatenation of finitely many subpaths each lying in a single leaf of $\mathcal{W}^s$ or $\mathcal{W}^u$.

Let μ be an ergodic f-invariant measure. Then by Oseledets Multiplicative Ergodic Theorem [O68] there exist numbers $\lambda_1 < \cdots < \lambda_m$, called the *Lyapunov exponents* of f with respect to μ, an f-invariant set Λ with $\mu(\Lambda) = 1$, and a Df-invariant Lyapunov splitting $\mathbb{R}^d = T_x\mathbb{T}^d = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^m$ for $x \in \Lambda$ such that

$$\lim_{n \to \pm\infty} n^{-1} \log \|D_x f^n(v)\| = \lambda_i \ \text{ for any } i = 1, \ldots, m \text{ and any } 0 \neq v \in \mathcal{E}_x^i.$$

Clearly, the Lyapunov splitting refines the partially hyperbolic one.

In the next theorem and corollary we set $N = 5$ if $d > 4$ and $N = 22$ if $d = 4$.

Theorem 1.7 (Rigidity for two-dimensional center). *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a totally irreducible automorphism with exactly two eigenvalues of absolute value one and let $r > r(L)$. Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a volume-preserving C^∞ diffeomorphism which is sufficiently C^N close to L and has C^r center foliation. Then any of the following equivalent conditions implies that f is C^∞ conjugate to L.*

  (1) *Lyapunov exponents of f with respect to the volume on $\mathcal{E}^c$ are all 0;*
  (2) *Lyapunov exponents of f with respect to the volume on $\mathcal{E}^c$ are equal;*
  (3) *f is not accessible;*
  (4) *f is topologically conjugate to L;*
  (5) *$\mathcal{W}^s$ and $\mathcal{W}^u$ are jointly integrable, that is, there exists a continuous foliation of dimension $\dim \mathcal{W}^s + \dim \mathcal{W}^u$ sub-foliated by $\mathcal{W}^s$ and $\mathcal{W}^u$.*

Corollary 1.8. *Let L be as in Theorem 1.7 and symplectic, and let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a C^∞ symplectic diffeomorphism which is sufficiently C^N-close to L. Then any of the following equivalent conditions implies that f is C^∞ conjugate to L.*

  (0) *f has at least one zero Lyapunov exponent with respect to the volume;*
(1-5) *as in Theorem 1.7;*
  (6) *$\mathcal{E}^s \oplus \mathcal{E}^u$ is C^1.*

Remark 1.9. *Thus for perturbations as in Corollary 1.8 we have a dichotomy: either f is non-uniformly hyperbolic or f is smoothly conjugate to L.*

This strengthens the earlier result in the same setting [AV10, Theorem I] which showed that either f is non-uniformly hyperbolic or f is conjugate to L via a volume-preserving homeomorphism. Smoothness of the conjugacy was only known for the case of $\mathbb{T}^4$ [AV10].

Our results are somewhat similar to some of the recent rigidity results for partially hyperbolic systems related to absolute continuity of the center foliation [AVW15, AVW, SY18, DX17]. For example, our Theorem 1.1 can be compared to a theorem of Avila-Viana-Wilkinson [AVW] on $\mathbb{T}^3$. Namely, they consider volume preserving perturbations f of a partially hyperbolic automorphism $L(x, y) = (Ax, y)$ of the 3-torus $\mathbb{T}^3$. Then, by applying the invariance principle [AV10], they show that the center foliation is absolutely continuous if and only if it is smooth. Consequently, f is smoothly conjugate to a diffeomorphism of the form $(x, y) \mapsto (g(x), y + \varphi(x))$. This result was generalized to the case of higher dimensional compact center foliation by Damjanovic and Xu [DX17, Theorem 6].

We note that papers [AVW15, AVW, SY18, DX17] consider diffeomorphisms whose center foliation either has compact leaves or comes from the orbit foliation of a hyperbolic flow. Further, they also strongly rely on one-dimensionality of stable and unstable foliations (or a replacement assumption such as quasi-conformality or splitting into one-dimensional subbundles). In contrast our methods treat all dimensions in a uniform way and primarily rely on denseness of center leaves and the theory of normal forms [GuKt98, Gu02, KS17, K19].

Structure of the paper. In Section 2 we summarize results on normal forms that play an important part in our arguments. Then we prove Theorem 1.3 in Section 3. The existence of the conjugacy in this case allows us to present one of the main arguments, smoothness along stable/unstable foliations via normal forms and holonomies, in a simplified form. We deduce Corollary 1.5, Theorem 1.7, and Corollary 1.8 in Section 4. In section Section 5 we prove Theorem 1.1, giving modifications needed to carry out the normal forms and holonomies arguments in the case of leaf conjugacy.

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2. Normal forms for contractions

In this section we give preliminaries on non-stationary normal forms for contractions. To make the presentation less technical, we formulate the results only for perturbations of linear maps. This is sufficient for our purposes.

Let f be a homeomorphism of a compact connected manifold (or a compact metric space) $\mathcal{M}$. Let $\mathcal{E} = \mathcal{M} \times \mathbb{R}^k$ be a vector bundle and let $U \subset \mathcal{E}$ be a neighborhood of the zero section. We will consider a C^r extension F of f, that is, a map $F : U \to \mathcal{E}$ that projects to f, preserves the zero section, and such that the corresponding fiber

maps $F_x : U_x \to \mathcal{E}_{f(x)}$ are C^r and depend continuously on x in C^r topology. We will assume that the derivative of F at the zero section is sufficiently C^0 close on $\mathcal{M}$ to a constant linear contraction, that is, $D_0 F_x$ is close uniformly in x to a fixed linear map $A \in GL(k, \mathbb{R})$ with $\|A\| < 1$.

For any such matrix A there exists a finite dimensional Lie group $\mathcal{P}_A$ with respect to composition which consists of certain polynomial maps $P : \mathbb{R}^k \to \mathbb{R}^k$ with $P(0) = 0$ and invertible derivative at 0. The elements of $\mathcal{P}_A$ are so called *sub-resonance generated polynomials*. This group is determined by the (ratios of) absolute values $\chi_1 < \cdots < \chi_\ell < 0$ of eigenvalues of A and by the corresponding invariant subspaces. The degrees of these polynomials are bounded above by $d(A) = \chi_1/\chi_\ell$, which yields that this group is finite dimensional. A precise definition of $\mathcal{P}_A$ can be found in [GuKt98, Gu02], but it does not play a role in this paper.

The following theorem was established in [GuKt98, Gu02] for $r \in \mathbb{N} \cup \{\infty\}$, in [KS17] for any r in nonuniformly hyperbolic setting, and in [K19] for this setting.

Theorem 2.1 (Normal forms for contracting extensions). *Let $A \in GL(k, \mathbb{R})$ with $\|A\| < 1$, let $\varepsilon > 0$ and $r \in [d(A) + \varepsilon, \infty]$. Let $F : U \to \mathcal{E}$ be a C^r extension of f whose derivative at the zero section is sufficiently C^0-close to A.*

Then there exist a neighborhood V of the zero section and a family $\{\Phi_x\}_{x \in \mathcal{M}}$ of C^r diffeomorphisms $\Phi_x : V_x \to \mathcal{E}_x$, satisfying $\Phi_x(0) = 0$ and $D_0 \Phi_x = Id$ and depending continuously on x in the C^r topology, which conjugate F to a polynomial extension P, i.e., for all $x \in \mathcal{M}$,

$$(2.1) \qquad \Phi_{f(x)} \circ F_x = P_x \circ \Phi_x, \quad \text{where } P_x \in \mathcal{P}_A.$$

Moreover, let $g : \mathcal{M} \to \mathcal{M}$ be a homeomorphism commuting with f and let $G : U \to \mathcal{E}$ be a $C^{d(A)+\varepsilon}$ extension of g preserving the zero section and commuting with F. Then for all $x \in \mathcal{M}$,

$$(2.2) \qquad \Phi_{g(x)} \circ G_x \circ \Phi_x^{-1} \in \mathcal{P}_A.$$

Remark 2.2 (Global version). *Suppose that $F : \mathcal{E} \to \mathcal{E}$ is a globally defined extension which satisfies the assumptions of Theorem 2.1 and either contracts fibers or, more generally, satisfies the property that for any compact set $K \subset \mathcal{E}$ and any neighborhood V of the zero section we have $F^n(K) \subset V$ for all sufficiently large n. Then the family $\{\Phi_x\}_{x \in \mathcal{M}}$ can be uniquely extended "by invariance" $\Phi_x = (P_x^n)^{-1} \circ \Phi_{f^n(x)} \circ F_x^n$ to the family of global C^r diffeomorphisms $\Phi_x : \mathcal{E}_x \to \mathcal{E}_x$ satisfying (2.1). Moreover, if G is another extension which commutes with F, then it satisfies (2.2) globally.*

These results can be applied in the context of foliations as follows. Let f be a diffeomorphism of a compact connected manifold $\mathcal{M}$, and let $\mathcal{W}$ be an f-invariant continuous foliation of $\mathcal{M}$ with uniformly C^∞ leaves. The latter means that all leaves are C^∞ submanifolds and all their derivatives are continuous on $\mathcal{M}$.

Suppose that f contracts $\mathcal{W}$ and that the derivative $Df|_{T\mathcal{W}}$, as a linear extension on $\mathcal{E} = T\mathcal{W}$, is close to a constant A. Restricting f to the leaves of $\mathcal{W}$ and identifying locally $\mathcal{W}_x = \mathcal{W}(x)$ with $T_x\mathcal{W}$, we obtain a corresponding non-linear extension F as in Theorem 2.1 and hence a family $\{\Phi_x\}_{x\in\mathcal{M}}$ of local normal form coordinates, Then, as in Remark 2.2, they can be extended to global diffeomorphisms $\Phi_x : \mathcal{W}_x \to \mathcal{E}_x$ satisfying (2.1). The important new statements in this setting describing dependence along the leaves, parts (2) and (3) in the next theorem, were established in [KS16].

Theorem 2.3 (Normal forms for contracting foliations, [KS16])**.** *Let f be a C^∞ diffeomorphism of a smooth compact connected manifold $\mathcal{M}$, and let $\mathcal{W}$ be an f-invariant topological foliation of $\mathcal{M}$ with uniformly C^∞ leaves. Suppose that $\mathcal{W}$ is contracted by f, and that the linear extension $Df|_{T\mathcal{W}}$ is close to a constant A as in Theorem 2.1. Then there exists a family $\{\Phi_x\}_{x\in\mathcal{M}}$ of C^∞ diffeomorphisms $\Phi_x : \mathcal{W}_x \to \mathcal{E}_x = T_x\mathcal{W}$ such that for each $x \in \mathcal{M}$,*

$$P_x = \Phi_{f(x)} \circ f \circ \Phi_x^{-1} : \mathcal{E}_x \to \mathcal{E}_{f(x)} \text{ is in } \mathcal{P}_A.$$

The family $\{\Phi_x\}_{x\in\mathcal{M}}$ has the following properties:
   (1) $\Phi_x(x) = 0$ and $D_x\Phi_x$ is the identity map for each $x \in \mathcal{M}$;
   (2) Φ_x depends continuously on $x \in \mathcal{M}$ in C^∞ topology and smoothly on x along the leaves of $\mathcal{W}$;
   (3) For any $x \in \mathcal{M}$ and $y \in \mathcal{W}_x$, the map $\Phi_y \circ \Phi_x^{-1} : \mathcal{E}_x \to \mathcal{E}_y$ is a composition of a sub-resonance generated polynomial in $\mathcal{P}_A$ with a translation;
   (4) If g is a homeomorphism of $\mathcal{M}$ which commutes with f, preserves $\mathcal{W}$, and is $C^{d(A)+\varepsilon}$ along the leaves of $\mathcal{W}$, then for each $x \in \mathcal{M}$

$$Q_x = \Phi_{f(x)} \circ g \circ \Phi_x^{-1} : \mathcal{E}_x \to \mathcal{E}_{g(x)} \text{ is in } \mathcal{P}_A.$$

Another way to interpret (3) is to view Φ_x as a coordinate chart on $\mathcal{W}_x$, identifying it with $\mathcal{E}_x$, and in particular identifying $\mathcal{E}_y = T_y\mathcal{W}_x$ with $T_{\Phi_x(y)}\mathcal{E}_x$ by $D_y\Phi_x$. In this coordinate chart, (3) yields that all transition maps $\Phi_y \circ \Phi_x^{-1}$ for $y \in \mathcal{W}_x$ are in the group generated by the translations of $\mathcal{E}_x$ and the sub-resonance generated polynomials, which is isomorphic to the Lie group $\bar{\mathcal{P}}_A$ generated by $\mathcal{P}_A$ and the translations of $\mathbb{R}^k$. Clearly, this group is also finite dimensional.

3. Proof of Theorem 1.3

By standard considerations we may assume that h is homotopic to $id_{\mathbb{T}^d}$. Indeed, the induced linear map $h_* : \mathbb{T}^d \to \mathbb{T}^d$ is in the centralizer of L. Hence, by compositing with h_*^{-1}, we may assume that $h_* = id$, i.e., h is homotopic to the identity map. Note that h does not have to be C^0 close to identity.

3.1. **Outline of the proof.** We denote the stable, unstable, and center sub-bundles for L by E^s, E^u, E^c, and the ones for f by $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$. Similarly, we use W and $\mathcal{W}$ for the corresponding foliations for L and for f. Lemma 3.1 below shows that the conjugacy h respects the foliations, so essentially we study its smoothness by

restricting it to $\mathcal{W}^s$, $\mathcal{W}^u$, and $\mathcal{W}^c$. The first part of the proof, Section 3.2, is showing smoothness along the stable and unstable foliations using normal forms and center holonomies.

The second part of the proof is to establish uniform smoothness of h along the center foliation. We first do it for the stable and unstable components of h in Section 3.4 and then global smoothness of the stable and unstable components follows by the standard application of the Journé's Lemma [J88]. Finally, wee use a different argument to establish global smoothness of the center component in Section 3.5.

The following lemma has a rather standard proof and we include it for the sake of completeness.

Lemma 3.1. *Let L be a partially hyperbolic toral automorphism and let f be a dynamically coherent partially hyperbolic toral diffeomorphism topologically conjugate to L by a homeomorphism h. Then $h(\mathcal{W}^*) = W^*$ for $* = s, u, c, cs, cu$.*

Proof. We show that center unstable leaves for f are mapped to those for L. With respect to a suitable metric, L does not increase distances along W^{cs}, that is, $\mathrm{dist}(L^n x, L^n y) \le \mathrm{dist}(x, y)$ for any $y \in W^{cs}(x)$ and $n \in \mathbb{N}$. Then $h^{-1}(y)$ will remain close to $h^{-1}(x)$ under forward iterates of f. More precisely, for any $\varepsilon > 0$ there is $\delta > 0$ such that $\mathrm{dist}(f^n(h^{-1}(x)), f^n(h^{-1}(y))) \le \varepsilon$ for any $n \in \mathbb{N}$ and $y \in W^{cs}(x)$ with $\mathrm{dist}(x, y) < \delta$. If ε is sufficiently small, this implies that $h^{-1}(y) \in \mathcal{W}^{cs}(h^{-1}(x))$, as otherwise they would separate exponentially along the unstable direction until reaching a "moderate" distance $> \varepsilon$. By connectedness, all points of $W^{cs}(x)$ must be mapped to the same center stable leaf of f, so we get $h^{-1}(W^{cs}(x)) \subset \mathcal{W}^{cs}(h^{-1}(x))$. The equality follows from h being a homeomorphism. Applying the Invariance of Domain Theorem to h^{-1} from a small ball in $W^{cs}(x)$ to $\mathcal{W}^{cs}(h^{-1}(x))$, we conclude that h is a local homeomorphism between center stable leaves of f and L on small balls of fixed size. By connectedness, all points of $\mathcal{W}^{cs}(h^{-1}(x))$ must come from the same center stable leaf of L.

Similarly, $\mathcal{W}^{cs}$ is mapped to W^{cs}, and it follows that $\mathcal{W}^c$ is mapped to W^c as the intersection of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

We also have $\mathcal{W}^s$ is mapped to W^s, and similarly for $\mathcal{W}^u$ and W^u. Indeed if $y \in \mathcal{W}^s(x)$ then $d(f^n x, f^n y) \to 0$ and hence $d(L^n h(x), L^n h(y)) \to 0$. It follows that $h(y) \in W^s(h(x))$ and so $h(\mathcal{W}^s(x)) \subset W^s(h(x))$. The equality again follows since h is a homeomorphism. $\square$

3.2. **Smoothness of the conjugacy along the stable leaves.** Since f is a small perturbation of L, Theorem 2.3 applies and yields existence of the normal forms on $\mathcal{W}^s$ and $\mathcal{W}^u$ corresponding to the groups of sub-resonance generated polynomials $\mathcal{P}_s = \mathcal{P}_{L|E^s}$ and $\mathcal{P}_u = \mathcal{P}_{L|E^u}$, respectively.

Now we consider the holonomies $\mathcal{H} = \mathcal{H}^c$ of $\mathcal{W}^c$ inside $\mathcal{W}^{cs}$, that is, the maps

$$\mathcal{H}_{x,y} : \mathcal{W}^s(x) \to \mathcal{W}^s(y) \quad \text{given by} \quad \mathcal{H}_{x,y}(z) = \mathcal{W}^c(z) \cap \mathcal{W}^s(y).$$

The corresponding linear holonomies H for L are the translations along W^c: if $y \in W^c(x)$ then $H_{x,y}(z) = z + (y - x)$. Because h maps stable leaves to stable leaves and center leaves to center leaves, the topological conjugacy h intertwines the holonomies $\mathcal{H}$ and H, that is,

$$H_{h(x),h(y)} = h \circ \mathcal{H}_{x,y} \circ h^{-1}.$$

It follows that $\mathcal{H}$ is globally defined on the leaves of $\mathcal{W}^s$ and is as smooth as $\mathcal{W}^c$.

We fix a pair of complex conjugate eigenvalues of L of absolute value 1 and denote by V the corresponding invariant 2-dimensional subspace of E^c. We will now show that holonomies $\mathcal{H}$ which are conjugate to translations in V preserve normal forms on $\mathcal{W}^s$.

Proposition 3.2. *For each $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^c(x)$ with $h(x) - h(y) \in V$, the center holonomy $\mathcal{H}_{x,y} : \mathcal{W}^s(x) \to \mathcal{W}^s(y)$ preserves normal forms on $\mathcal{W}^s$, that is, $\Phi_y \circ \mathcal{H}_{x,y} \circ \Phi_x^{-1} : \mathcal{E}_x^s \to \mathcal{E}_y^s$ is a sub-resonance generated polynomial in $\mathcal{P}_s$.*

Remark 3.3. *We will later extend this result in Proposition 5.1, which implies that all center holonomies preserve normal forms on $\mathcal{W}^s$. Using this we can replace the assumption that L is irreducibility by the assumption that L is diagonalizable over $\mathbb{C}$ and has dense center foliation. This is yield a somewhat more general result given in Remark 1.6*

Proof. For any vector $v \in V$, the translation $H_v(x) = x + v$, $x \in \mathbb{T}^d$, is a globally defined map whose restriction to any stable leaf is a center holonomy for L. While L and H_v do not commute, we have $L(H_v(x)) = Lx + Rv$, where the restriction $R = L|_V$ is a linear map conjugate to the rotation by some angle $2\pi\theta$. We will denote by R^t the corresponding conjugate of the rotation by the angle $2\pi\theta t$, for which $R^{1/\theta} = \text{Id}$. Therefore, in order to apply Theorem 2.3, we pass to the suspension flow and use time-$1/\theta$ map. The argument below is inspired by the one in [FKSp11].

We consider the mapping tori

$$M_f = \mathbb{T}^d \times [0,1] \,/\, (x,1) \sim (f(x),0) \quad \text{and} \quad M_L = \mathbb{T}^d \times [0,1] \,/\, (x,1) \sim (L(x),0)$$

and the corresponding suspension flows $\{f^s\}$ and $\{L^s\}$ given by $(x,t) \mapsto (x, t+s)$. Then h induces the conjugacy $\tilde{h} : M_f \to M_L$ between the suspension flows $\tilde{h}(x,t) = (h(x), t)$. We also consider the map $T_v : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}$ given by

$$T_v(x,t) = (x + R^{-t}v, \, t)$$

and its projection $\tilde{H}_v : M_L \to M_L$ to M_L. The translation H_v embeds as $t = 0$ level of the map $\tilde{H}_v$. The projection $\tilde{H}_v$ is well-defined since

$$T_v(x,1) = (x + R^{-1}v, \, 1) \text{ is identified with}$$
$$T_v(Lx, 0) = (Lx + v, \, 0) = (Lx + LR^{-1}v), \, 0) = (L(x + R^{-1}v), \, 0).$$

We note that T_v commutes with the map $(x, t) \mapsto (x, t + 1/\theta)$ on $\mathbb{T}^d \times \mathbb{R}$. Indeed, as $R^{1/\theta} = \mathrm{Id}$ we obtain

$$T_v(x,\, t + 1/\theta) = (x + R^{-t+1/\theta}v,\, t + 1/\theta) = (x + R^{-t}v,\, t + 1/\theta).$$

It follows that $\tilde{H}_v$ commutes with time-$1/\theta$ map $L^{1/\theta}$, as the projections to M_L.

Since f is a small perturbation of L, Theorem 2.3 applies to the time-$1/\theta$ map of the suspension flow $\{f^s\}$ and yields existence of the normal form coordinates $\{\Phi_x\}$ on its stable foliation $\tilde{\mathcal{W}}^s$ in M_f. In fact, the corresponding groups of sub-resonance generated polynomials $\mathcal{P}_s = \mathcal{P}_{L|E^s}$ are the same for all t.

For any $v \in V$, we have that the map $g = \tilde{h}^{-1} \circ \tilde{H}_v \circ \tilde{h} : M_f \to M_f$ is a holonomy map of the lifted center foliation $\tilde{\mathcal{W}}^c$, and hence is C^r along the leaves. Since $\tilde{H}_v$ commutes with $L^{1/\theta}$ we obtain that g commutes with $f^{1/\theta}$. Thus part (4) of Theorem 2.3 applies and we conclude that $\Phi_{g(x)} \circ g \circ \Phi_x^{-1} : \tilde{\mathcal{E}}_x^s \to \tilde{\mathcal{E}}_{g(x)}^s$ is a sub-resonance generated polynomial. In particular, this holds at the level $t = 0$ of M_f where g coincides with a holonomy map of $\mathcal{W}^c$ on $\mathbb{T}^d$. Moreover, any holonomy map $\mathcal{H}_{x,y}$ as in the statement is given by $\tilde{h}^{-1} \circ \tilde{H}_v \circ \tilde{h}$ for some $v \in V$. Thus we conclude that $\Phi_y \circ \mathcal{H}_{x,y} \circ \Phi_x^{-1} : \mathcal{E}_x^s \to \mathcal{E}_y^s$ is a sub-resonance generated polynomial map. $\square$

We fix arbitrary $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^s(x)$. Since L is irreducible and V is L-invariant, the linear foliation of planes parallel to V has dense leaves in $\mathbb{T}^d$. Hence there exists a sequence of vectors $v_n \in V$ such that $h(x) + v_n$ converges to $h(y)$. Denoting $y_n = h^{-1}(h(x) + v_n)$ we obtain a sequence of points $y_n \in \mathcal{W}^c(x)$ converging to y so that Proposition 3.2 applies to holonomies $\mathcal{H}_{x,y_n} : \mathcal{W}^s(x) \to \mathcal{W}^s(y_n)$. The corresponding linear holonomies $H_{h(x), h(y_n)} = H_{v_n}$ for L converge in C^0 to the translation H_v in $W^s(h(x))$ by the vector $v = h(y) - h(x)$. Hence the holonomies $\mathcal{H}_{x,y_n}$ converge in C^0 norm to some map $\mathcal{H}_{x,y} : \mathcal{W}^s(x) \to \mathcal{W}^s(y)$, which is the conjugate by h of this linear translation.

By Proposition 3.2, $\mathcal{H}_{x,y_n}$ is a sub-resonance generated polynomial map P_n in normal form coordinates, i.e.

$$P_n = \Phi_{y_n} \circ \mathcal{H}_{x,y_n} \circ \Phi_x^{-1} : \mathcal{E}_x^s \to \mathcal{E}_{y_n}^s.$$

Since the normal form coordinates Φ_y depend continuously on y, the maps P_n converge in C^0 to the map

$$P = \Phi_y \circ \mathcal{H}_{x,y} \circ \Phi_x^{-1} : \mathcal{E}_x^s \to \mathcal{E}_y^s,$$

which is also a sub-resonance generated polynomial. Using (3) of Theorem 2.3 and identifying $\mathcal{W}^s(x)$ with $\mathcal{E}_x^s$ by the C^∞ coordinate map Φ_x, we see as in the remark after Theorem 2.3 that P is in the Lie group $\bar{\mathcal{P}}_x$ generated by the translations of $\mathcal{E}_x^s$ and the sub-resonance generated polynomials, which is isomorphic to the Lie group $\bar{\mathcal{P}}_A$ generated by $\mathcal{P}_A$ and the translations of $\mathbb{R}^k$.

Thus h conjugates the action of $E^s = \mathbb{R}^k$ by translations of $W^s(h(x))$ with the corresponding continuous action of $\mathbb{R}^k$ by elements of the Lie group $\bar{\mathcal{P}}_x$ of C^∞ polynomial diffeomorphisms of $\mathcal{W}^s(x)$. This conjugacy defines the continuous homomorphism

$$\eta_x : E^s \to \bar{\mathcal{P}}_x \quad \text{given by} \quad \eta_x(v) = h^{-1} \circ H_v \circ h.$$

It is a classical result that η_x is automatically a C^∞ homomorphism, see for example [Ha, Corollary 3.50]. Since η_x determines the conjugacy along the leaf by

$$h^{-1}(h(x) + v) = \eta_x(v)(x),$$

we conclude that h^{-1} is a C^∞ diffeomorphism between $W^s(h(x))$ and $\mathcal{W}^s(x)$, and hence h is also C^∞ along $\mathcal{W}^s(x)$.

Since the normal form coordinates Φ_x, as well as holonomies and their limits, depend continuously on x, the constructed continuous action on $\mathcal{W}^s(x)$ and the corresponding homomorphism η_x also depend continuously on x. This implies that η_x depend continuously on x in C^∞ topology, for example because it is determined by the corresponding linear homomorphism of the Lie algebras. So we conclude that h is uniformly C^∞ along $\mathcal{W}^s$.

A similar argument shows that h is uniformly C^∞ along $\mathcal{W}^u$.

Remark 3.4. *The last part of the proof is similar to an argument pioneered by Katok and Spatzier in* [KtSp97] *and used in other papers on higher rank actions. In these arguments a continuous action by C^∞ diffeomorphisms of $\mathcal{W}^s(x)$ is obtained. The smoothness of this action, and hence of h, follows then from a more difficult result* [MZ74, Section 5.1, Corollary]. *This argument, however, does not immediately yield that h is* uniformly C^∞ *along $\mathcal{W}^s$. Our argument relies on the advanced results on normal forms from* [KS16] *to show that all maps P are contained in a single Lie group $\bar{\mathcal{P}}_x$.*

3.3. **The conjugacy h is volume-preserving.** We denote the Lebesgue measure on $\mathbb{T}^d$ by m and the f-invariant volume by μ. We will show that $h_*(\mu) = m$.

We denote the Lyapunov exponents of f with respect to μ by λ^f, and the Lyapunov exponents of L by λ^L. Since $Df|_{\mathcal{E}^u}$ is conjugate to $L|_{E^u}$ by the derivative of h along $\mathcal{W}^u$, the Lyapunov exponents of f along $\mathcal{E}^u$ with respect to μ are equal to the unstable Lyapunov exponents of L. Since f and L are topologically conjugate, they have the same topological entropy. Combining these observations with Pesin's formula for the metric entropy, we obtain

$$\mathbf{h}_{top}(f) \geq \mathbf{h}_\mu(f) = \sum_{\lambda^f > 0} \lambda^f \geq \sum_{\lambda^f \text{on } \mathcal{E}^u} \lambda^f = \sum_{\lambda^L > 0} \lambda^L = \mathbf{h}_m(L) = \mathbf{h}_{top}(L) = \mathbf{h}_{top}(f).$$

Therefore, $\mathbf{h}_{top}(f) = \mathbf{h}_\mu(f)$, that is, μ is the measure of maximal entropy for f, and so is the measure $h_*^{-1}(m)$ since $\mathbf{h}_{h_*^{-1}(m)}(f) = \mathbf{h}_m(L)$ by isomorphism. We conclude that $h_*(\mu) = m$ by uniqueness of the measure of maximal entropy [B67].

3.4. **Global smoothness of the stable and unstable components.** We already proved that h is uniformly C^∞ along $\mathcal{W}^s$ and $\mathcal{W}^u$. To show global smoothness of h we now study its regularity along $\mathcal{W}^c$. For this we will decompose h into stable, unstable, and center components and consider them separately using their series representations. In this section we will obtain uniform smoothness along $\mathcal{W}^c$ of the stable and unstable components and thus establish their global smoothness by Journé's Lemma. For the center component, in the next section, we will use a different argument based on exponential mixing and a regularity result from [FKSp13].

Recall that $h \circ f = L \circ h$. We denote by $\bar{f}$ and $\bar{h}$ the lifts of f and h to $\mathbb{R}^d$ which are compatible with the standard lift of L so that we have $\bar{h} \circ \bar{f} = L \circ \bar{h}$. Also recall that h is homotopic to the identity and f is homotopic to L. Hence we can write

$$\bar{h} = \mathrm{Id} + \bar{H} \quad \text{and} \quad \bar{f} = L + \bar{F},$$

where $\bar{H}, \bar{F} : \mathbb{R}^d \to \mathbb{R}^d$ are $\mathbb{Z}^d$-periodic, and hence can be viewed as functions H and F from $\mathbb{T}^d$ to $\mathbb{R}^d$. Then the commutation relation

$$(\mathrm{Id} + \bar{H}) \circ (L + \bar{F}) = L \circ (\mathrm{Id} + \bar{H}) \quad \text{yields} \quad \bar{H} = L^{-1}(\bar{H} \circ \bar{f}) + L^{-1}\bar{F}.$$

It is easy to check that the latter projects to the torus as the following equation for $\mathbb{R}^d$-valued functions on $\mathbb{T}^d$

$$H = L^{-1}(H \circ f) + G, \quad \text{where } G = L^{-1}F.$$

Using the L-invariant splitting $\mathbb{R}^d = E^u \oplus E^c \oplus E^s$ we define the projections H_* and G_* of H and G to E^*, where $* = s, u, c$, and obtain

$$(3.1) \qquad H_* = L_*^{-1}(H_* \circ f) + G_*, \quad \text{where } L_* = L|_{E^*}.$$

Thus H_* is a fixed point of the affine operator

$$(3.2) \qquad T_*(\psi) = L_*^{-1}(\psi \circ f) + G_*$$

with the inverse $T_*^{-1}(\phi) = L_* (\phi \circ f^{-1}) - L_* (G_* \circ f^{-1})$.

Since $\|L_u^{-1}\| < 1$, the operator T_u is a contraction on the space $C^0(\mathbb{T}^d, E^u)$, and thus H_u is its unique fixed point

$$(3.3) \qquad H_u = \lim_{k \to \infty} T_u^k(0) = \sum_{k=0}^{\infty} L_u^{-k}(G_u \circ f^k).$$

Similarly, T_s^{-1} is a contraction on $C^0(\mathbb{T}^d, E^s)$ and H_s is its unique fixed point

$$(3.4) \qquad H_s = \lim_{k \to \infty} T_s^{-k}(0) = -\sum_{k=1}^{\infty} L_s^k(G_s \circ f^{-k}).$$

Our goal now is to show that H_c, H_u, and H_s are C^∞, which would yield that h is C^∞. We already know that h is uniformly C^∞ along $\mathcal{W}^s$ and $\mathcal{W}^u$, and hence so are H_c, H_u, and H_s. Thus it remains to study the derivatives for each of these maps along $\mathcal{W}^c$.

We will now prove that the derivatives of H_u of any order along $\mathcal{W}^c$ exist and are continuous functions on $\mathbb{T}^d$ by term-wise differentiation of (3.3), and thus we will show that H_u is uniformly C^∞ along $\mathcal{W}^c$.

First we observe that the Lyapunov exponents of $\|Df|_{\mathcal{E}^c}\|$ are zero with respect to any f-invariant measure. Indeed, a non-zero Lyapunov exponent implies exponential expansion/contraction by f inside the leaves of $\mathcal{W}^c$, more precisely, existence of $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^c(x)$ such that $\mathrm{dist}(f^n x, f^n y)$ decays exponentially as n goes to ∞ or $-\infty$. Since conjugacy h is Hölder this yields similar exponential decay of $\mathrm{dist}(L^n h(x), L^n h(y))$, which is impossible as $h(y) \in W^c(h(x))$.

The fact that the Lyapunov exponents of $\|Df|_{\mathcal{E}^c}\|$ are zero with respect to any f-invariant measure is well-known to imply that $\|Df^n|_{\mathcal{E}^c}\|$ grows sub-exponentially, that is, for any $\varepsilon > 0$ there is C_ε such that

$$(3.5) \qquad \|Df^n|_{\mathcal{E}^c}\| \leq C_\varepsilon \, e^{\varepsilon n} \quad \text{for all } n \in \mathbb{N},$$

see e.g. [Schr98]. It follows that the norms of all higher derivatives also grow sub-exponentially, see e.g. [dlLW10, Lemma 5.5]: for each m and $\delta > 0$ there exists a constant $K_{\delta,\ell}$ such that

$$(3.6) \qquad \|f^n\|_{C^\ell_{\mathcal{W}^c}} \leq K_{\delta,\ell} \, e^{n\delta} \quad \text{for all } n \in \mathbb{N},$$

where $\|g\|_{C^\ell_{\mathcal{W}^c}}$ denotes the supremum of all derivatives of g of orders up to ℓ along the foliation $\mathcal{W}^c$.

Since $\|L_u^{-1}\| < 1$, the above estimate yields that term-wise differentiation of any order of (3.3) gives an exponentially converging series. Hence the derivatives of H_u of any order along $\mathcal{W}^c$ are continuous functions on $\mathbb{T}^d$, that is H_u is uniformly C^∞ along $\mathcal{W}^c$. We have already established that H_u is uniformly C^∞ along $\mathcal{W}^u$ and $\mathcal{W}^s$, and so we conclude that H_u is C^∞ on $\mathbb{T}^d$ by Journé's lemma [J88]. A similar argument using differentiation of (3.4) shows that H_s is C^∞ on $\mathbb{T}^d$.

We remark that term-wise differentiation can be used to establish smoothness of H_u and H_c along $\mathcal{W}^s$ and of H_s and H_c along $\mathcal{W}^u$, but not of H_u along $\mathcal{W}^u$ or H_s along $\mathcal{W}^s$.

3.5. **Global smoothness of the center component.** In this section we complete the proof of Theorem 1.3 by establishing global smoothness of H_c. While H_c is a fixed point of the operator T_c given by (3.2), T_c is not a contraction on $C^0(\mathbb{T}^d, E^c)$. We will show, however, that H_c can be expressed by series similarly to H_u and H_s in the sense of distributions

$$(3.7) \qquad H_c = \sum_{k=0}^{\infty} L_c^{-k}(G_c \circ f^k).$$

More precisely, we consider the distribution space $\mathcal{D}$ of E^c-valued functionals ω on the space of C^∞ test functions $\eta : \mathbb{T}^d \to \mathbb{R}$ with zero average with the vector-valued

pairing

$$\langle \omega, \eta \rangle = \int_{\mathbb{T}^d} \eta(x)\omega(x) \, d\mu(x).$$

We also fix a norm $|.|$ on E^c to estimate the magnitude. We need the space of C^∞ test functions only for the formal definition of distributional derivatives. All estimates in the proof will be done for a Hölder continuous η and all distributions will be shown to be defined on the space of Hölder continuous test function.

To verify (3.7) we iterate equation (3.1), $H_c = L_c^{-1}(H_c \circ f) + G_c$, and get that for any $j \in \mathbb{N}$,

$$(3.8) \qquad H_c = \sum_{k=0}^{j-1} L_c^{-k}(G_c \circ f^k) + L_c^{-j}(H_c \circ f^j).$$

Since L_c is conjugate to an orthogonal matrix, $\|L_c^{-j}\|$ is bounded uniformly in j. Since (f,μ) is mixing, as isomorphic to (L,m), we can estimate the last term in (3.8) as

$$|\langle L_c^{-j}(H_c \circ f^j), \eta \rangle| = |L_c^{-j} \langle H_c \circ f^j, \eta \rangle| \le \|L_c^{-j}\| \cdot |\langle H_c \circ f^j, \eta \rangle| \to 0$$

as $j \to \infty$ for any Hölder or L^2 function η with 0 average, and we conclude that

$$(3.9) \qquad \langle H_c, \eta \rangle = \langle \sum_{k=0}^{\infty} L_c^{-k}(G_c \circ f^k), \eta \rangle.$$

Now we will prove that H_c is C^∞ on $\mathbb{T}^d$ using a regularity result from [FKSp13, Corollary 8.5], which yields that it suffices to show that the derivatives of H_c of any order along $\mathcal{W}^c$, $\mathcal{W}^s$, and $\mathcal{W}^u$ are distributions dual to the space of Hölder functions, i.e., their norms can be estimated by the Hölder norm of a test function. Recall that the derivatives of H_c of any order along $\mathcal{W}^s$ and $\mathcal{W}^u$ are continuous functions by uniform smoothness of h along $\mathcal{W}^s$ and $\mathcal{W}^u$ established in Section 3.2. This can also be seen by term-wise differentiation of the series for H_c. To complete the proof of smoothness of H_c, we will now show that the derivatives of H_c of any order along $\mathcal{W}^c$ are distributions dual to Hölder functions. We use the following result which says that L has exponential mixing on Hölder functions.

[L82, Theorem 6], [GoSp14, Theorem 1.1] *Let L be an ergodic automorphism of a torus, or more generally of a compact nilmanifold X. Then for any $\theta \in (0,1]$ there exists $\rho = \rho(\theta) \in (0,1)$ such that for all $g_0, g_1 \in C^\theta(X)$ and $n \in \mathbb{N}$,*

$$\int_X g_0(x)g_1(L^n(x)) \, dm(x) = \left(\int_X g_0 \, dm \right) \left(\int_X g_1 \, dm \right) + O(\rho^n \|g_0\|_{C^\theta} \|g_1\|_{C^\theta}).$$

Since the bi-Hölder conjugacy h maps the volume μ to the Lebesgue measure m and preserves the class of Hölder functions, the same holds for (f,μ) in place of (L,m).

We fix $\ell \in \mathbb{N}$ and for a smooth function g on $\mathbb{T}^d$ consider its partial derivative $D_c^\ell g$ of order ℓ along $\mathcal{W}^c$. We will use the same notation for distributional derivatives along $\mathcal{W}^c$ (see [FKSp13, Section 8] for detailed description of distributional derivatives in the context of foliations). Using equation (3.9) we obtain the formula for distributional derivative of H_c,

$$(3.10) \qquad \langle D_c^\ell H_c, \eta \rangle = \langle \sum_{k=0}^\infty D_c^\ell (L_c^{-k}(G_c \circ f^k)), \eta \rangle.$$

Since G_c and f are smooth, the terms $D_c^\ell(L_c^{-k}(G_c \circ f^k))$ are continuous functions. Now we estimate these pairings in terms of the Hölder norm of η.

We will use smooth approximations of η by convolutions with a smooth kernel $\eta_\varepsilon = \eta * \phi_\varepsilon$. More precisely, we fix a smooth bump function ϕ supported on the unit ball and define $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$, so that we have

$$\phi_\varepsilon \geq 0, \quad \int_{\mathbb{T}^d} \phi_\varepsilon = 1, \quad \|\phi_\varepsilon\|_{C^\ell} = \varepsilon^{-(d+\ell)}\|\phi\|_{C^\ell}.$$

Then for any $0 < \theta \leq 1$ and $\ell \in \mathbb{N}$ we have the standard estimates of the norms for any θ-Hölder function η,

$$(3.11) \qquad \begin{aligned} &\|\eta_\varepsilon - \eta\|_{C^0} \leq \varepsilon^\theta \|\eta\|_\theta \quad \text{and} \\ &\|\eta_\varepsilon\|_{C^\ell} \leq c_\ell\, \varepsilon^{-d-\ell}\|\eta\|_0 \ \text{ for } \ell \in \mathbb{N}, \end{aligned}$$

where c_ℓ is a constant depending only on ℓ.

We split η as $\eta_\varepsilon + (\eta - \eta_\varepsilon)$ and estimate the corresponding pairings.

$$\begin{aligned} |\langle D_c^\ell L_c^{-k}(G_c \circ f^k), \eta_\varepsilon \rangle| &\leq \|L_c^{-k}\| \cdot |\langle D_c^\ell(G_c \circ f^k), \eta_\varepsilon \rangle| = \\ &= \|L_c^{-k}\| \cdot |\langle G_c \circ f^k, D_c^\ell \eta_\varepsilon \rangle|. \end{aligned}$$

Since $\|D_c^\ell \eta_\varepsilon\|_\theta \leq \|D_c^\ell \eta_\varepsilon\|_1 \leq \|\eta_\varepsilon\|_{C^{\ell+1}}$, using the exponential mixing and (3.11) we can estimate

$$|\langle G_c \circ f^k, \eta_\varepsilon^{\ell,c} \rangle| \leq K_1\, \rho^k\, \|G_c\|_\theta\, \|D_c^\ell \eta_\varepsilon\|_\theta \leq K_2\, \rho^k \varepsilon^{-(d+\ell+1)}\|G_c\|_\theta\, \|\eta\|_0\,.$$

Since $\|L_c^{-k}\|$ is bounded we conclude that

$$(3.12) \qquad |\langle L_c^{-k}(G_c \circ f^k)^{\ell,c}, \eta_\varepsilon \rangle| \leq K_3\, \rho^k \varepsilon^{-(d+\ell+1)}\|G_c\|_\theta\, \|\eta\|_0\,.$$

Now we estimate the pairings in (3.10) with $\eta - \eta_\varepsilon$. We use an estimate on norms of compositions of C^ℓ functions

$$\|h \circ g\|_{C^\ell} \leq M_\ell\, \|h\|_{C^\ell}(1 + \|g\|_{C^\ell})^\ell,$$

which follows, for example, from Proposition 5.5 in [dlLO98]. Thus we have

$$\begin{aligned} |\langle D_c^\ell L_c^{-k}(G_c \circ f^k), (\eta - \eta_\varepsilon) \rangle| &\leq \|L_c^{-k} D_c^\ell(G_c \circ f^k)\|_0 \cdot \|(\eta - \eta_\varepsilon)\|_0 \leq \\ &\leq \|L_c^{-k}\| \cdot \|G_c \circ f^k\|_{C_{\mathcal{W}^c}^\ell} \cdot \varepsilon^\theta \|\eta\|_\theta \leq K_5\, \|G_c\|_{C^\ell}\, (1 + \|f^k\|_{C_{\mathcal{W}^c}^\ell})^\ell \cdot \varepsilon^\theta \|\eta\|_\theta. \end{aligned}$$

Now using (3.6) we obtain

$$(3.13) \quad \begin{aligned} |\langle L_c^{-k}(G_c \circ f^k)^{\ell,c}, (\eta - \eta_\varepsilon)\rangle| &\le K_6 \, e^{\ell k \delta} \cdot \varepsilon^\theta \cdot \|G_c\|_{C^\ell} \cdot \|\eta\|_\theta = \\ &= K_6 \, \xi^k \, \|G_c\|_{C^\ell} \cdot \|\eta\|_\theta, \quad \text{where } \xi = e^{\ell\delta} \varepsilon^{\theta/k}. \end{aligned}$$

We choose $\varepsilon = \varepsilon(k) = \rho^{k/(2(d+\ell+1))}$ so that $\rho^k \varepsilon^{-(d+\ell+1)} = \rho^{k/2}$ to obtain exponential decay in (3.12). Then we take $\delta > 0$ sufficiently small so that

$$\xi = e^{\ell\delta} \rho^{\theta/(2(d+\ell+1))} < 1,$$

which ensures exponential decay in (3.13). Noting that $\rho^{1/2} < \xi < 1$, we combine (3.12) and (3.13) to get

$$|\langle L_c^{-k}(G_c \circ f^k)^{\ell,c}, \eta\rangle| \le K_7 \, \xi^k \cdot \|G_c\|_{C^\ell} \, \|\eta\|_\theta.$$

Thus, for any θ and derivative D_c^ℓ, we obtain exponential convergence in (3.10) and conclude that $|\langle D_c^\ell H_c, \eta\rangle| \le C\|\eta\|_\theta$. Therefore $D_c^\ell H_c$ extends to a functional on the space of θ-Hölder functions. This concludes the argument that H_c is C^∞ and completes the proof of Theorem 1.3.

4. Proofs of Corollary 1.5, Theorem 1.7, and Corollary 1.8

4.1. **Proof of Corollary 1.5.** We will verify that $\mathcal{W}^c$ is sufficiently smooth, in fact that $\mathcal{E}^c$ is C^∞. The latter is equivalent to $\mathcal{E}^s \oplus \mathcal{E}^u$ being C^∞ since $\mathcal{E}^c$ is the symplectic orthogonal to $\mathcal{E}^s \oplus \mathcal{E}^u$. Indeed, if $u \in \mathcal{E}^c$ and $v \in \mathcal{E}^s$ then by invariance of the symplectic form ω we have that

$$|\omega_x(v, u)| = |\omega_{f^n x}(D_x f^n(v), D_x f^n(u))| \le C\|D_x f^n(v)\| \cdot \|D_x f^n(u)\| \to 0$$

as $n \to \infty$, and so $\omega_x(v, u) = 0$. Similarly $\omega_x(v, u) = 0$ for any $u \in \mathcal{E}^c$ and $v \in \mathcal{E}^u$.

Now we show that $\mathcal{E}^s \oplus \mathcal{E}^u$ is C^∞. Since f is topologically conjugate to L, the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ are topologically jointly integrable in the sense that there is a continuous foliation $\mathcal{W} = \mathcal{W}^{s+u}$ of dimension $\dim \mathcal{W}^s + \dim \mathcal{W}^u$ which is sub-foliated by $\mathcal{W}^u$ and $\mathcal{W}^s$. First we note that the leaves of $\mathcal{W}^{s+u}$ are uniformly C^∞ by the following lemma.

Lemma 4.1. [KS06, Lemma 4.1] *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be foliations with uniformly C^∞ leaves. Suppose that $\mathcal{W}_1$ and $\mathcal{W}_2$ are topologically jointly integrable to a continuous foliation $\mathcal{W}$. Then $\mathcal{W}$ has uniformly C^∞ leaves.*

Now to prove that $\mathcal{W}^{s+u}$, and hence $\mathcal{E}^s \oplus \mathcal{E}^u$, is C^∞ it suffices to show that the holonomies of $\mathcal{W}^{s+u}$ between leaves of $\mathcal{W}^c$ are C^∞. By the dynamical coherence of f, the holonomy of $\mathcal{W}^{s+u}$ between the center leaves is smooth as a composition of holonomies of $\mathcal{W}^u$ inside $\mathcal{W}^{cu}$ and of $\mathcal{W}^s$ inside $\mathcal{W}^{cs}$. We claim that the latter, and similarly the former, holonomies are C^∞, since $\mathcal{W}^s$ is a C^∞ foliation inside the leaves of $\mathcal{W}^{cs}$. For this we note that, as we already observed in the proof of Theorem 1.3, $Df|_{\mathcal{E}^c}$ has sub-exponential growth, as the exponents of f along $\mathcal{E}^c$ are all zero. This implies that f is so called *strongly r-bunched* for any r and thus the leaves of

$\mathcal{W}^{cs}$ are C^∞ [PSW97]. It also implies that $Df|_{\mathcal{E}^{cs}}$ has sub-exponential growth and so applying the C^r Section Theorem [HPS77] as for example in [KS07, Theorem 3.7 and Proposition 3.9] we obtain that $\mathcal{W}^s$ is C^∞ along the leaves of $\mathcal{W}^{cs}$.

4.2. **Proof of Theorem 1.7.**

It is clear that smooth conjugacy implies (1)-(5). Then it suffices to show that all other items imply (4) and that the topological conjugacy in (4) is bi-Hölder, so that Theorem 1.3 applies and yields smoothness.

The implication (1) $\Longrightarrow$ (2) is clear and the implication (2) $\Longrightarrow$ (3) follows from the next result by Avila and Viana:

[AV10, Theorem 8.1] *Let L be as in Theorem 1.7. Then there exists a neighborhood U of L in the space of C^N volume preserving diffeomorphisms of $\mathbb{T}^d$ such that if $f \in U$ is accessible then its center Lyapunov exponents are distinct.*

The (topological) joint integrability of $\mathcal{W}^s \oplus \mathcal{W}^u$ implies that accessibility classes are the leaves of $\mathcal{W}^{s+u}$, thus (5) $\Longrightarrow$ (3). The implication (3) $\Longrightarrow$ (4) was established in [RH05]. For a perturbation f which is not accessible, it was proved in [RH05, Section 6] (cf. [AV10, Remark 8.3]) that f and L are conjugate by a bi-Hölder homeomorphism h. This completes the proof of Theorem 1.7.

4.3. **Proof of Corollary 1.8.**

Combining the above proof of Theorem 1.7 with Corollary 1.5 we conclude that smooth conjugacy in this case is equivalent to (1)-(5). A smooth conjugacy also clearly implies (0) and (6).

For a symplectic f the Lyapunov spectrum is a symmetric subset of $\mathbb{R}$, that is, the Lyapunov exponents come in pairs $\lambda, -\lambda$. Indeed, let ω be the invariant symplectic form. Since ω is non-degenerate, each Lyapunov space $\mathcal{E}^i$ is not symplectic orthogonal to at least one Lyapunov space $\mathcal{E}^j$. Then for suitable vectors v_i and v_j in these spaces we have by invariance that

$$0 \neq \omega_x(v_i, v_j) = \omega_{f^n x}(D_x f^n(v_i), D_x f^n(v_j)).$$

This implies $\lambda_i^f + \lambda_j^f = 0$ as otherwise the right hand side must go to 0 under forward or backward iterates.

Since $\mathcal{E}^c$ is symplectic orthogonal to $\mathcal{E}^s \oplus \mathcal{E}^u$, the argument above also shows that the center exponents are of the form $\lambda, -\lambda$, and thus (0) $\Longrightarrow$ (1).

Finally, (6) implies (5) or (0) by a result of Hammerlindl [H11, Theorem 1.1]: if $\mathcal{E}^s \oplus \mathcal{E}^u$ is C^1 and not integrable, then a center exponent must be qual to the sum of a stable one and an unstable one. We let

$$2\varepsilon = \min_{\lambda_i \neq \pm \lambda_j} ||\lambda_i^L| - |\lambda_j^L|| > 0.$$

If f is sufficiently C^1 close to L then the similar minimum for f is at least ε while the center exponents satisfy $|\lambda_c^f| < \varepsilon$. Then, by the symmetry of Lyapunov spectrum, the equation $\lambda_c^f = \lambda_i^f + \lambda_j^f$ can only hold in the case $\lambda_i + \lambda_j = 0$, yielding (0).

This completes the proof of Corollary 1.8.

5. Proof of Theorem 1.1

5.1. **Outline of the proof.** The main part of the proof is establishing smoothness of the leaf conjugacy transversely to the center foliation. This is similar in spirit to proving smoothness of the conjugacy along the stable and unstable foliations in Section 3.2. However, in absence of a true conjugacy, the argument with holonomies and normal forms becomes more difficult.

As before, we denote the stable, unstable, and center sub-bundles for L by E^s, E^u, E^c, and the ones for f by $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$. Similarly, we use W and $\mathcal{W}$ for the corresponding foliations for L and for f.

We recall that there exists a leaf conjugacy h, that is, a homeomorphism close to the identity which maps center leaves to center leaves and conjugates f to L modulo the center foliation [HPS77]. Further, it maps center-stable leaves to center-stables leaves and center-unstable leaves to center-unstable leaves. Such a leaf conjugacy can always be chosen to be smooth along the center foliation $\mathcal{W}^c$. In fact, this is true in general, for partially hyperbolic diffeomorphisms with C^1 center foliation. We can make a specific choice of h as follows. Denote by $\bar{h} \colon \mathbb{R}^d \to \mathbb{R}^d$ the lift of h that we would like to define. On the universal cover we have a direct splitting $\mathbb{R}^d = E^s \oplus E^c \oplus E^u$ and hence we can use (s, c, u)-coordinates

$$\bar{h}(x) = \bar{h}(x_s, x_c, x_u) = (\bar{h}_s(x), \bar{h}_c(x), \bar{h}_u(x)).$$

In the notations $\bar{h}(x) = x + \bar{H}(x)$ of Section 3.4 this corresponds to $\bar{h}_*(x) = x_* + H_*(x)$. Then $\bar{h}_s$ and $\bar{h}_u$ are uniquely determined by (3.3) and (3.4). We take $\bar{h}_c(x) = x_c$, which corresponds to setting $\bar{H}^c = 0$. Then $\bar{h}_c$ is obviously smooth. Because $\bar{h}$ sends center leaves to center leaves, if x varies in $\mathcal{W}^c(x_0)$ then $\bar{h}(x)$ varies in $W^c(\bar{h}(x_0))$ and, hence, the coordinates $\bar{h}_s$ and $\bar{h}_u$ do not change. In the same way, if x varies in $\mathcal{W}^s(x_0)$ then $\bar{h}(x)$ varies in $W^{cs}(\bar{h}(x_0))$ and, hence, the coordinate $\bar{h}_u$ does not change. And when x varies in $\mathcal{W}^u(x_0)$ the coordinate $\bar{h}_s$ does not change. Hence, to prove that h is C^∞ it suffices to show that $\bar{h}_s$ is uniformly C^∞ along $\mathcal{W}^s$. This is done in Section 5.2 below. Similarly, $\bar{h}_u$ is uniformly C^∞ along $\mathcal{W}^u$, which completes the proof.

5.2. **Smoothness of h_s along $\mathcal{W}^s$.** In this section we give modifications needed to carry out the arguments from Section 3.2 in the case of leaf conjugacy. The main part is to establish the following generalization of Proposition 3.2.

Proposition 5.1. *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a partially hyperbolic automorphism which is diagonalizable over $\mathbb{C}$. Let $f \colon \mathbb{T}^d \to \mathbb{T}^d$ be a sufficiently C^1-small perturbation of L. Let $\{\Phi_x\}_{x \in \mathbb{T}^d}$ be normal form coordinates for f on $\mathcal{W}^s$, as in Theorem 2.3. For any $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^c(x)$ the center holonomy $\mathcal{H}_{x,y} : \mathcal{W}^s(x) \to \mathcal{W}^s(y)$ preserves normal forms, that is, the map*

$$\Phi_y \circ \mathcal{H}_{x,y} \circ \Phi_x^{-1} : \mathcal{E}_x^s \to \mathcal{E}_y^s \ \text{ is in } \ \mathcal{P}_{L_s},$$

the group of sub-resonance generated polynomial map defined by $L_s = L|_{E^s}$.

Proof. As in the proof of Proposition 3.2, we consider the mapping tori and the corresponding suspension flows f^t and L^t. Then the leaf conjugacy h, which was chosen in the previous subsection, induces the leaf conjugacy $\tilde{h} : M_f \to M_L$ between the suspension flows given by $\tilde{h}(x,t) = (h(x),t)$.

We recall that L is diagonalizable and all its eigenvalues on E^c have modulus 1. Hence we can decompose $E^c = \oplus V_j$ as the direct sum of eigenspaces corresponding to eigenvalues 1 and -1 and of L-invariant subspaces corresponding to pairs of complex eigenvalues $e^{\pm 2\pi i \theta_j}$. We will consider center holonomies corresponding to each of these subspaces separately. We fix one of the subspaces corresponding to a complex pair and write $V = V_j$ and $\theta = \theta_j$. The case when V is an eigenspace of 1 or -1 can be considered similarly using L^2 in place of $L^{1/\theta}$.

For any $v \in V$ we again consider the translation $H_v(x) = x + v$, $x \in \mathbb{T}^d$, which embeds as $t = 0$ level of the map $\tilde{H}_v : M_L \to M_L$. The map $\tilde{H}_v$ commutes with $L^{1/\theta}$, the time $1/\theta$ map of the suspension flow. Since h is not a conjugacy, we will first consider the normal forms for a different dynamics on M_f. Denoting

$$\phi^t = \tilde{h}^{-1} \circ L^t \circ \tilde{h} \quad \text{for } t \in \mathbb{R},$$

we obtain a continuous flow on M_f. We fix $v \in E^c$ and define the homeomorphism

$$g = g_v = \tilde{h}^{-1} \circ \tilde{H}_v \circ \tilde{h}$$

which again commutes with $\phi^{1/\theta}$. However, ϕ^t and g may not preserve the foliation $\tilde{\mathcal{W}}^s$. Since h is a leaf conjugacy, the homeomorphisms ϕ^t and g preserve foliations $\tilde{\mathcal{W}}^c$ and $\tilde{\mathcal{W}}^{cs}$, and they differ from f^t and from a center holonomy between strong leaves respectively, by "adjusting along the center". More precisely,

$$\phi^t(x) \in \tilde{\mathcal{W}}^c(f^t(x)) \quad \text{and} \quad g(x) \in \tilde{\mathcal{W}}^c(x).$$

Now we define smooth extensions F^t and G of ϕ^t and g. They reflect the behavior of f^t and of the center holonomies between the corresponding strong stable leaves. We fix $x \in M_f$ and for each $t \in \mathbb{R}$ we define

$$F_x^t : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(\phi(x)) \quad \text{as} \quad F_x^t = \mathcal{H}_{\phi(x)} \circ \phi^t|_{\tilde{\mathcal{W}}^s(x)},$$

where $\mathcal{H}_{\phi(x)}$ is "holonomy projection" along $\tilde{\mathcal{W}}^c$ inside the leaf of $\tilde{\mathcal{W}}^{cs}$, that is

$$\mathcal{H}_x = \mathcal{H}_x^{cs} : \tilde{\mathcal{W}}^{cs}(x) \to \tilde{\mathcal{W}}^s(x) \quad \text{given by} \quad \mathcal{H}_x(z) = \tilde{\mathcal{W}}^c(z) \cap \tilde{\mathcal{W}}^s(x).$$

Note that $\mathcal{H}_x$ is globally defined on $\tilde{\mathcal{W}}^{cs}$ since the leaf conjugacy h maps the leaves of $\tilde{\mathcal{W}}^c$ and $\tilde{\mathcal{W}}^{cs}$ to those of $\tilde{W}^c$ and $\tilde{W}^{cs}$. Also, since $\phi^t(y)$ and $f^t(y)$ are on the same leaf of $\tilde{\mathcal{W}}^c$, we can also express F_x^t as

(5.1) $$F_x^t = \mathcal{H}_{f^t(x),\phi^t(x)} \circ f^t|_{\tilde{\mathcal{W}}^s(x)} : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(\phi(x)),$$

where $\mathcal{H}_{x,y} : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(y)$ denotes the usual $\tilde{\mathcal{W}}^c$ holonomy. Similarly, for any $x \in M_f$, we define

$$G_x : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(g(x)) \quad \text{as} \quad G_x = \mathcal{H}_{g(x)} \circ g|_{\tilde{\mathcal{W}}^s(x)}.$$

Since $\phi^{1/\theta}$ and g commute, it is clear from the definitions that the extensions also commute: $G_{\phi(x)} \circ F_x^{1/\theta} = F_{g(x)}^{1/\theta} \circ G_x$. Again, as $g(y) \in \tilde{\mathcal{W}}^c(y)$ we see that G_x coincides with the center holonomy

(5.2) $$G_x = \mathcal{H}_{x,g(x)} : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(g(x)).$$

Since $\tilde{\mathcal{W}}^c$ is a C^∞ foliation, $\mathcal{H}_{\phi(x)}$ and $\mathcal{H}_{x,g(x)}$ are C^∞, and thus both F_x^t and G_x are C^∞ diffeomorphisms.

Now we construct normal forms for the extension F^t and show that G preserves them. To apply Theorem 2.1, we locally identify $\tilde{\mathcal{W}}^s(x)$ and $\tilde{\mathcal{E}}_x^s$ and obtain the corresponding smooth extensions $\bar{F}^t$ and $\bar{G}$ of ϕ^t and g, respectively, defined on a neighborhood of the zero section in $\mathcal{E} = \tilde{\mathcal{E}}^s$. We claim that the derivative of $\bar{F}^t$ at the zero section is a contraction which is close to the linear flow L^t, provided that f is sufficiently C^1 close to L. Indeed, differentiating (5.1) at x we obtain

$$D_0 \bar{F}_x^t = D_x F_x^t = D_{f^t(x)} \mathcal{H}_{f^t(x), \phi^t(x)} \circ Df^t|_{\tilde{\mathcal{E}}^s(x)}.$$

If f is sufficiently C^1 close to L then h is C^0 close to the identity, and hence ϕ^t is C^0 close to f^t. Thus $\phi^t(x)$ is close to $f^t(x)$ and hence the derivative of the holonomy $\mathcal{H}_{f^t(x), \phi^t(x)}$ is close to the identity. Thus $D_0 \bar{F}_x^t$ is close to $Df^t|_{\tilde{\mathcal{E}}^s(x)}$, which is close to L^t. In particular, $D_0 \bar{F}^1$ is close to L and, as $\bar{F}^t$ is C^∞, we can now apply Theorem 2.1 with $F = F^1$ and $A = L_s = L|_{E^s}$ to obtain a family of local normal form coordinates $\bar{\Phi}_x$ for $\bar{F}^1$ on $\tilde{\mathcal{E}}^s$.

Since all maps $\bar{F}^t$ commute, the second part of Theorem 2.1 implies that $\bar{\Phi}_x$ are also normal form coordinates for the whole one-parameter group $\{\bar{F}^t\}$. Hence by the identification we obtain local normal form coordinates Φ_x for F^t on $\tilde{\mathcal{W}}^s$. Then we can extend Φ_x, as in the Remark 2.2, to get global normal form coordinates on the whole leaves $\Phi_x : \tilde{\mathcal{W}}_x^s \to \tilde{\mathcal{E}}_x^s$. Indeed, while F_x^t may not be a global contraction, for any bounded set $B \subset \tilde{\mathcal{W}}_x^s$, the set $F_x^n(B)$ will be in a small neighborhood of $\phi^n(x)$ for all sufficiently large n, and hence we can define Φ_x on B by

$$\Phi_x = (\mathcal{P}_x^n)^{-1} \circ \Phi_{\phi^n(x)} \circ F_x^n.$$

Since the extension G is also C^∞ and commutes with $F^{1/\theta}$, the second part of Theorem 2.1 implies that G preserves the normal form coordinates for F^t on $\tilde{\mathcal{W}}^s$, i.e., $\Phi_{g(x)} \circ G_x \circ \Phi_x^{-1} \in \mathcal{P}_{L_s}$, the sub-resonance group given by $A = L_s$. By (5.2), G_x is the holonomy $\mathcal{H}_{x,g(x)} : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(g(x))$ and we conclude that

$$\Phi_{g(x)} \circ \mathcal{H}_{x,g(x)} \circ \Phi_x^{-1} \in \mathcal{P}_{L_s}.$$

Recall that $E^c = \oplus V_j$. The above conclusion holds for $g_v = h^{-1} \circ \tilde{H}_v \circ h$, where v is any vector in any V_j. We decompose any vector $w \in E^c$ as the sum $w = \sum v_j$ and note that the holonomy $\tilde{H}_w$ is the composition of the holonomies $\tilde{H}_{v_j}$. Therefore, $g_w = h^{-1} \circ \tilde{H}_w \circ h$ preserves normal forms as the corresponding composition of the maps g_{v_j}. Since for any $x \in M_f$ and any $y \in \mathcal{W}^c(x)$ we can take $w = h(y) - h(x)$ so that $g_w(x) = y$, we conclude that any center holonomy map $\mathcal{H}_{x,y} : \tilde{\mathcal{W}}^s(x) \to \tilde{\mathcal{W}}^s(y)$ preserves normal forms.

Considering $t = 0$ level of the suspension M_f we obtain this result for $\mathbb{T}^d$: for any $x \in \mathbb{T}^d$ and any $y \in \mathcal{W}^c(x)$

$$\Phi_y \circ \mathcal{H}_{x,y} \circ \Phi_x^{-1} : \mathcal{E}_x^s \to \mathcal{E}_y^s \quad \text{is in } \mathcal{P}_{L_s}.$$

Finally, we note that by (5.1) we have $F_x^1 = \mathcal{H}_{f(x),\phi(x)} \circ f|_{\mathcal{W}^s(x)}$. Since both F_x^1 and the holonomy are in $\mathcal{P}_{L_s}$, we conclude that so is $f|_{\mathcal{W}^s(x)}$. Therefore, $\{\Phi_x\}_{x \in \mathbb{T}^d}$ are normal form coordinates for f on $\mathcal{W}^s$, as in Theorem 2.3. This completes the proof of Proposition 5.1. $\qquad\qquad\square$

Now we show that h_s is uniformly C^∞ along $\mathcal{W}^s$. We fix a point $x \in \mathbb{T}^d$ and consider the map

$$\hat{h}_x : \mathcal{W}^s(x) \to W^s(h(x)) \quad \text{given by} \quad \hat{h}_x = H_{h(x)}^c \circ h|_{\mathcal{W}^s(x)},$$

where $H_{h(x)}^c$ is the linear projection inside $W^{cs}(h(x))$ to $W^s(h(x))$ along W^c. We will prove that $\hat{h}_x$ is uniformly C^∞. This will show that the component h_s is uniformly C^∞ along the leaves of $\mathcal{W}^s$, as it is easy to see that $\hat{h}_x = h_s|_{\mathcal{W}^s(x)}$ under the natural identification of $W^s(h(x))$ with E^c.

We fix $y \in \mathcal{W}^s(x)$ and take a sequence of points $y_n \in \mathcal{W}^c(x)$ converging to y. This can be done since the leaves of the linear foliation W^c are dense in $\mathbb{T}^d$ and the fact that the leaf conjugacy h is a homeomorphism which sends $\mathcal{W}^c$ to W^c. We consider holonomies $\mathcal{H}_{x,y_n} : \mathcal{W}^s(x) \to \mathcal{W}^s(y_n)$ of $\mathcal{W}^c$ inside $\mathcal{W}^{cs}$. We claim that the holonomy maps $\mathcal{H}_{x,y_n}$ converge in C^0 to the map $\mathcal{H}_{x,y} : \mathcal{W}^s(x) \to \mathcal{W}^s(y)$, which is conjugate by $\hat{h}_x$ to linear translation $H_{\hat{v}}$ in $W^s(h(x))$ by the vector

$$\hat{v} = \hat{h}_x(y) - \hat{h}_x(x) = \hat{h}_x(y) - h(x).$$

Indeed, since $y_n \in \mathcal{W}^c(x)$ converge to y, $h(y_n) \in \mathcal{W}^c(h(x))$ converge to $h(y)$. The corresponding linear center holonomies $H_{h(x),h(y_n)}$ for L are translations H_{v_n} by the vectors $v_n = h(y_n) - h(x)$ and thus converge in C^0 to the translation $H_v : W^s(h(x)) \to W^s(h(y))$ by the vector $v = h(y) - h(x)$. Composing with the translation $H_{\hat{v}-v}$, which is also a linear center holonomy, we see that

$$H_{\hat{v}-v} \circ H_{h(x),h(y_n)} \quad \text{converges to} \quad H_{\hat{v}} : W^s(h(x)) \to W^s(h(x)) = W^s(\hat{h}_x(y)),$$

and that the map

$$(\hat{h}_x)^{-1} \circ H_{\hat{v}} \circ \hat{h}_x : \mathcal{W}^s(x) \to \mathcal{W}^s(x) = \mathcal{W}^s(y)$$

is the limit $\mathcal{H}_{x,y}$ of the holonomies $\mathcal{H}_{x,y_n} : \mathcal{W}^s(x) \to \mathcal{W}^s(y_n)$.

Once we have this convergence of $\mathcal{H}_{x,y_n}$ to $\mathcal{H}_{x,y}$ and Proposition 5.1, we can use the same normal form argument as in Section 3.2. Indeed, we again obtain that $P_n = \Phi_{y_n} \circ \mathcal{H}_{x,y_n} \circ \Phi_x^{-1}$ and their C^0 limit $P = \Phi_y \circ \mathcal{H}_{x,y} \circ \Phi_x^{-1}$ are sub-resonance generated polynomials. Identifying $\mathcal{W}^s(x)$ with $\mathcal{E}_x^s$ by Φ_x we obtain that P is in the Lie group $\bar{\mathcal{P}}_x$ generated by the translations of $\mathcal{E}_x^s$ and the sub-resonance generated polynomials. Then $\hat{h}$ defines the continuous homomorphism

$$\eta_x : E^s \to \bar{\mathcal{P}}_x \quad \text{given by} \quad \eta_x(\hat{v}) = (\hat{h}_x)^{-1} \circ H_{\hat{v}} \circ \hat{h}_x,$$

which are C^∞. This yields that $\hat{h}_x^{-1}$ and $\hat{h}_x$ are C^∞ diffeomorphisms that depend continuously on x in C^∞ topology.

This shows that h_s is uniformly C^∞ along $\mathcal{W}^s$ and completes the proof of Theorem 1.1.

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Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

E-mail address: gogolyev.1@osu.edu

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: kalinin@psu.edu, sadovskaya@psu.edu