Rigidity Properties of Higher Rank Abelian Actions

Boris Kalinin

Dynamical Systems and Group Actions

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Equivalence: a conjugacy between α and α' is a homeomorphism or diffeomorphism $h : M \to M'$ such that $h \circ \alpha(g) = \alpha'(g) \circ h$ for all $g \in G$.

$$\begin{array}{ccc} M & \xrightarrow{\alpha(g)} & M \\ \text{h}\downarrow & & \downarrow\text{h} \\ M' & \xrightarrow{\alpha'(g)} & M' \end{array}$$

Automorphisms of tori and nilmanifolds.

$A \in SL(n, \mathbb{Z})$ gives an automorphism of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
A is **hyperbolic** or **Anosov** if for any eigenvalue $|\lambda| \neq 1$.

$$\text{e.g.} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \ \mathbb{T}^2 \to \mathbb{T}^2$$

Hyperbolic $\mathbb{Z}$ actions, algebraic case

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Γ is a cocompact lattice in N;
$A : N \to N$ is an automorphism such that $A(\Gamma) = \Gamma$.

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Then A projects to an automorphism of the nilmanifold N/Γ.

A is **Anosov** if $D_eA : T_eN \to T_eN$ is a hyperbolic automorphism.

Hyperbolic $\mathbb{Z}$ actions, smooth case

A diffeomorphism f of a compact manifold M is called **Anosov** if the tangent bundle has a continuous Df-invariant splitting $TM = E^s \oplus E^u$ such that for all $n > 0$

$$\|Df^n(v)\| \leq Ce^{-\lambda n}\|v\| \quad \text{for all } v \in E^s,$$
$$\|Df^{-n}(v)\| \leq Ce^{-\lambda n}\|v\| \quad \text{for all } v \in E^u.$$

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E^s and E^u are called the **stable** and **unstable** distributions. They are tangent to the stable and unstable foliations W^s and W^u.

The leaves of W^s are C^∞ injectively immersed Euclidean spaces, but E^s usually varies only Hölder continuously transversally to W^s. Similarly for W^u and E^u.

Topological rigidity problem for hyperbolic systems

Structural stability (Local topological rigidity):
Any C^1-small perturbation g of an Anosov diffeomorphism f is Anosov and is *topologically* conjugate to f, i.e. $g \circ h = h \circ f$.

The conjugacy h is a bi-Hölder homeomorphism, but rarely smooth.

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Conjecture: Any Anosov diffeomorphism is topologically conjugate to a finite factor of an Anosov automorphism of a nilmanifold.

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Flows: No classification conjecture ("anomalous" examples).

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$\alpha : \mathbb{R}^k \to Diff(M)$ is called a *hyperbolic action* if for some $b \in \mathbb{Z}^k$ $\alpha(b)$ is normally hyperbolic to the orbit: $TM = E^s_b \oplus E^o \oplus E^u_b$.

Local smooth rigidity of algebraic actions:

Any standard higher rank hyperbolic algebraic action is C^∞ conjugate to any C^1-small perturbation. (Katok and Spatzier, for most types).

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Rigidity properties of higher rank actions

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Under mild dynamical assumptions.

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Under strong dynamical assumptions.

Measure rigidity:
Scarcity of invariant measures.

Algebraic examples: non-invertible case

Expanding endomorphisms of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:
$E_m : \mathbb{T} \to \mathbb{T}, \qquad E_m x = mx \,(\mathrm{mod}\ 1), \;\; m \geq 2.$

Abundance of invariant measures for a single map E_m.

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Two (commuting) endomorphisms of the circle
E_2 and E_3 generate $\mathbb{Z}_+^2$-action on $\mathbb{T}$.

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Theorem (D. Rudolph) If an invariant measures for this action has positive entropy for some element then it is Lebesgue.

Open problem. Are there nontrivial zero entropy measures?

Algebraic Measure Rigidity

Conjecture (Margulis, Katok and Spatzier): Scarcity of invariant measures for hyperbolic algebraic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$.

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Example: $\mathbb{R}^2 \cong A = \{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s-t} \end{pmatrix} \mid t, s \in \mathbb{R}\} \subset SL(3, \mathbb{R})$

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Littlewood Conjecture: $\liminf \left(n \, \|na\| \, \|nb\|\right) = 0$ for any $a, b \in \mathbb{R}$

Corollary The set of exceptional pairs has Hausdorff dimension 0.

Algebraic Measure Rigidity

Hope for more general $\mathbb{R}^k$ actions by left translations on G/Γ:

Any invariant measure is *algebraic*, i.e. Haar measure on a compact homogeneous space $Hx \subset G/\Gamma$ for a subgroup $H \subset G$.

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Follows by applying the theorem to the invariant measure on the graph of the measure-preserving isomorphism.

Non-algebraic actions.

Let α_0 be a Cartan action of $\mathbb{Z}^k$ on $\mathbb{T}^{k+1}$:
full centralizer in $SL(k+1, \mathbb{Z})$ of an irreducible hyperbolic matrix with real eigenvalues.

We say that a $\mathbb{Z}^k$-action α has *Cartan homotopy data* if its elements are homotopic to the corresponding ones of α_0.

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Theorem (B.K. and A. Katok)

Any $\mathbb{Z}^k$-action α by $C^{1+\varepsilon}$ diffeomorphisms of $\mathbb{T}^{k+1}$, $k \geq 2$, with Cartan homotopy data preserves an ergodic absolutely continuous invariant measure μ.

Such measure μ is unique and (α,μ) is isomorphic to (α_0,λ) as a measure preserving action.

Lyapunov exponents

Two commuting hyperbolic automorphisms $A, B : \mathbb{T}^3 \to \mathbb{T}^3$ generate a $\mathbb{Z}^2$ action on $\mathbb{T}^3$: (m, n) acts by $A^m B^n$.

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$\chi_i(m, n) = \ln|i^{th}$ eigenvalue of $A^m B^n|$, $\quad i = 1, 2, 3$
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The kernels $\ker(\chi_i) \subset \mathbb{R}^k$ are called *Lyapunov hyperplanes*.
Weyl chambers are the connected components of $\mathbb{R}^k \setminus \cup \ker(\chi)$.

Nonuniform measure rigidity.

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Theorem (B.K., A. Katok, F. Rodriguez Hertz)

Let μ be an ergodic invariant measure for a $C^{1+\varepsilon}$ action α of $\mathbb{Z}^k$, $k \geq 2$, on a $(k+1)$-dimensional manifold M. Suppose that the Lyapunov exponents of μ are in general position and that at least one element in $\mathbb{Z}^k$ has positive entropy with respect to μ. Then μ is absolutely continuous.

Smooth rigidity for $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions

Two cases:

1. $\mathbb{Z}^k$ actions on tori and nilmanifolds.
2. $\mathbb{Z}^k$ actions on an arbitrary manifold or $\mathbb{R}^k$ actions.

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If action α is of type 1, then it is topologically conjugate to the

algebraic action α_0 : $\begin{array}{ccc} M & \xrightarrow{\alpha} & M \\ \text{h}\downarrow & & \downarrow\text{h} \\ M & \xrightarrow{\alpha_0} & M \end{array}$ Need to show: h is smooth.

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Building blocks: Intersections of stable foliations of different elements of the action give dynamically natural invariant foliations. The finest such intersections are called *coarse Lyapunov foliations*.

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Let α be a TNS action of $\mathbb{Z}^k$ by Anosov diffeomorphisms.
For each coarse Lyapunov distribution E of dimension more than 1 we assume that some elements of α contracting E are 1/2-pinched:

there exist $0 < \mu < \lambda < 2\mu$ and $K > 0$ such that for any $v \in E$,

$$K^{-1} e^{-n\lambda} \|v\| \le \|Df^n(v)\| \le K e^{-n\mu} \|v\|.$$

Classification results for $\mathbb{Z}^k$ actions on arbitrary manifold

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If any two coarse Lyapunov foliations are (topologically) jointly integrable, then a finite cover of α is C^∞ conjugate to a $\mathbb{Z}^k$ action by affine automorphisms of $\mathbb{T}^n$.

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For actions with stronger pinching (uniform quasiconformality), joint integrability can be replace by certain *non-resonance* assumption on Lyapunov exponents.

Main result for $\mathbb{Z}^k$ actions on tori and nilmanifolds

Theorem (D. Fisher, B.K., R. Spatzier)

Let α be a C^∞-action of $\mathbb{Z}^k$, $k \geq 2$, on a nilmanifold N/Γ. Let α_0 be the corresponding algebraic action.

(1) there is an Anosov element for α in each Weyl chamber,
(2) there is $\mathbb{Z}^2 \subset \mathbb{Z}^k$ such that $\alpha_0(b)$ is ergodic for all $0 \neq b \in \mathbb{Z}^2$.

Then α is C^∞-conjugate to α_0.

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Main new ingredients:

- Exponential decay of correlations for Hölder functions.
- New regularity result.

Regularity result

We consider Hölder foliations with smooth leaves on a manifold M. We assume that they are strongly absolutely continuous, i.e. the volume has absolutely continuous conditional measures on the leaves with densities that are smooth along the leaves and Hölder continuous transversally.

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Theorem (D. Fisher, B.K., R. Spatzier)

Suppose that tangent spaces of foliations $\mathcal{F}_1, \dots, \mathcal{F}_r$ span TM. Let D be a distribution defined by an L^1 function ϕ on M. If the derivatives of D along each $\mathcal{F}_i$ extend to functionals on C^θ for any $\theta > 0$, then ϕ is C^∞.

Conjugacy equation

Fix $\mathbf{n} \in \mathbb{Z}^k$ and denote $a = \alpha(\mathbf{n})$ and $A = \alpha_0(\mathbf{n})$.
We lift a and A to $\mathbb{R}^n$ and write the lift of conjugacy as $\mathrm{Id} + h$.

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The conjugacy equation yields functional fixed point equation for h

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Derivatives as distributions

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For $a = \alpha(\mathbf{n})$ with $\mathbf{n}$ close to the Lyapunov hyperplane of $\mathcal{V}'$ we obtain uniform slow exponential estimates on the growth of the derivatives of a^n along $\mathcal{V}'$. We use exponential mixing to show the pairing can be estimated by the Hölder norm of f.

Exponential mixing for Hölder functions.

Theorem (D. Fisher, B.K., R. Spatzier)

Let α be a $\mathbb{Z}^k$ action on $\mathbb{T}^n$ by affine automorphisms such that $\alpha(\mathbf{n})$ is ergodic for every nonzero $\mathbf{n} \in \mathbb{Z}^k$.
Then there exist $0 < r < 1$ and $C = C(\theta)$ such that for any θ-Hölder functions f and g on $\mathbb{T}^n$ with $\int_{\mathbb{T}^n} f = \int_{\mathbb{T}^n} g = 0$ and for any $\mathbf{n} \in \mathbb{Z}^k$

$$\langle \alpha(\mathbf{n}) f, g \rangle = \int_{\mathbb{T}^n} f \circ \alpha(\mathbf{n}) \cdot g < C(\|f\|_\theta \|g\|_2 + \|g\|_\theta \|f\|_2)\, r^{\theta \|\mathbf{n}\|}$$

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The proof uses Fejér kernel approximations for f and g, Katznelson lemma for the distance of integer lattice points from an irrational invariant subspace of an integral matrix, and uniform lower bound for maximal expansion of $\alpha(\mathbf{n})$.