

# NONUNIFORM MEASURE RIGIDITY

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*Dedicated to the memory of Bill Parry (1935–2006)*

ABSTRACT. We consider an ergodic invariant measure  $\mu$  for a smooth action  $\alpha$  of  $\mathbb{Z}^k$ ,  $k \geq 2$ , on a  $(k+1)$ -dimensional manifold or for a locally free smooth action of  $\mathbb{R}^k$ ,  $k \geq 2$  on a  $(2k+1)$ -dimensional manifold. We prove that if  $\mu$  is hyperbolic with the Lyapunov hyperplanes in general position and if one element in  $\mathbb{Z}^k$  has positive entropy, then  $\mu$  is absolutely continuous. The main ingredient is absolute continuity of conditional measures on Lyapunov foliations which holds for a more general class of smooth actions of higher rank abelian groups.

## 1. INTRODUCTION

In this paper we continue and significantly advance the line of development started in [7] and [19]. The general program is to show that actions of higher rank abelian groups, i.e.  $\mathbb{Z}^k \times \mathbb{R}^l$ ,  $k + l \geq 2$ , by diffeomorphisms of compact manifolds must preserve a geometric structure, such as an absolutely continuous invariant measure, under global conditions of topological or dynamical nature which ensure both infinitesimal hyperbolic behavior and sufficient global complexity of the orbit structure.

In [7] and [19] we considered  $\mathbb{Z}^k$  actions on the torus  $\mathbb{T}^{k+1}$ ,  $k \geq 2$ , that induce on the first homology group the action of a maximal abelian subgroup of  $SL(k+1, \mathbb{Z})$  diagonalizable over  $\mathbb{R}$ . We say that such an action has *Cartan homotopy data*.<sup>1</sup> The central feature of that situation is existence of a semi-conjugacy  $h$  between the action, which we denote by  $\alpha$ , and the corresponding Cartan action  $\alpha_0$  by automorphisms of the torus, i.e. a unique surjective continuous map  $h : \mathbb{T}^{k+1} \rightarrow \mathbb{T}^{k+1}$  homotopic to identity such that

$$h \circ \alpha = \alpha_0 \circ h.$$

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<sup>1</sup>In the case of the torus it may look more natural to speak about homology data, but we wanted to emphasize that what mattered was the homotopy types on individual elements; this notion can be generalized while homological information in general is clearly insufficient.

This gives desired global complexity right away and allows to produce nonuniform hyperbolicity (non-vanishing of the Lyapunov exponents) with little effort (see [7, Lemma 2.3]). Existence of the semi-conjugacy allows to use specific properties of the linear action  $\alpha_0$  and reduces the proofs to showing that the semi-conjugacy is absolutely continuous and bijective on an invariant set of positive Lebesgue measure.<sup>2</sup> Thus, this may be considered as a version in the setting of global measure rigidity of the *a priori* regularity method developed for the study of local differentiable rigidity in [21] (see also an earlier paper [16]) and successfully applied to the global conjugacy problem on the torus in [30].

In the present paper we consider an essentially different and more general situation. We make no assumptions on the topology of the ambient manifold or the action under consideration and instead assume directly that the action preserves a measure with non-vanishing Lyapunov exponents whose behavior is similar to that of the exponents for a Cartan action. Namely, we consider a  $\mathbb{Z}^k$ ,  $k \geq 2$ , action on a  $(k+1)$ -dimensional manifold or an  $\mathbb{R}^k$ ,  $k \geq 2$ , action on a  $(2k+1)$ -dimensional manifold with an ergodic invariant measure for which the kernels of the Lyapunov exponents are in general position (see the Definition below). Dynamical complexity is provided by the assumption that at least one element of the action has positive entropy. In fact our results for  $\mathbb{Z}^k$  actions are direct corollaries of those for  $\mathbb{R}^k$  actions via suspension construction.

To formulate our results precisely recall that the Lyapunov characteristic exponents with respect to an ergodic invariant measure for a smooth  $\mathbb{R}^k$  action are linear functionals on  $\mathbb{R}^k$ . For a smooth  $\mathbb{Z}^k$  action they are linear functionals on  $\mathbb{Z}^k$  which are extended to  $\mathbb{R}^k$  by linearity. The kernels of these functionals are called the *Lyapunov hyperplanes*. A Lyapunov exponent is called *simple* if the corresponding Lyapunov space is one-dimensional. See Section 2 for more details.

**Definition.** We will say that  $m$  hyperplanes (containing 0) in  $\mathbb{R}^k$  are *in general position* if the dimension of the intersection of any  $l$  of them is the minimal possible, i.e. is equal to  $\max\{k-l, 0\}$ .

We will say that the Lyapunov exponents of an ergodic invariant measure for a  $\mathbb{Z}^k$  action are *in general position* if they are all simple and nonzero, and if the Lyapunov hyperplanes are distinct hyperplanes in general position.

Similarly for an  $\mathbb{R}^k$  action the Lyapunov exponents of an ergodic invariant measure are *in general position* if the zero exponent has multiplicity  $k$  and the remaining exponents are all simple and nonzero, and if the Lyapunov hyperplanes are distinct hyperplanes in general position.

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<sup>2</sup>And in fact smooth in the sense of Whitney on smaller non-invariant sets of positive Lebesgue measure.

**Main Theorem.**

- (1) *Let  $\mu$  be an ergodic invariant measure for a  $C^{1+\theta}$ ,  $\theta > 0$ , action  $\alpha$  of  $\mathbb{Z}^k$ ,  $k \geq 2$ , on a  $(k+1)$ -dimensional manifold  $M$ . Suppose that the Lyapunov exponents of  $\mu$  are in general position and that at least one element in  $\mathbb{Z}^k$  has positive entropy with respect to  $\mu$ . Then  $\mu$  is absolutely continuous.*
- (2) *Let  $\mu$  be an ergodic invariant measure for a locally free  $C^{1+\theta}$ ,  $\theta > 0$ , action  $\alpha$  of  $\mathbb{R}^k$ ,  $k \geq 2$ , on a  $2k+1$ -dimensional manifold  $M$ . Suppose that Lyapunov exponents of  $\mu$  are in general position and that at least one element in  $\mathbb{R}^k$  has positive entropy with respect to  $\mu$ . Then  $\mu$  is absolutely continuous.*

As we already mentioned the statement (1) is a direct corollary of (2) applied to the suspension of the  $\mathbb{Z}^k$  action  $\alpha$ . We are not aware of any examples of  $\mathbb{R}^k$  actions satisfying assumptions of (2) other than time changes of suspensions of  $\mathbb{Z}^k$  actions satisfying (1).

Thus, what we prove is the first case of existence of an absolutely continuous invariant measure for actions of abelian groups whose orbits have codimension two or higher which is derived from general purely dynamical assumptions. Nothing of that sort takes places in the classical dynamics for actions of orbit codimension two or higher.<sup>3</sup> Only for codimension one actions (diffeomorphisms of the circle and fixed point free flows on the torus) of sufficient smoothness Diophantine condition on the rotation number (which is of dynamical nature) guarantees existence of a smooth invariant measure [5, 32]. One can point out though that even in those cases existence of topological conjugacy (for the circle) or orbit equivalence (for the torus) to an algebraic system follows from the classical Denjoy theorem (see e.g. [14, Theorem 12.1.1]) and the work goes into proving smoothness. Thus this falls under the general umbrella of *a priori* regularity methods, albeit substantively very different from the hyperbolic situations, and should be more appropriately compared with results of [7] and [19].

In order to prove measure rigidity we develop principal elements of the basic geometric approach of [20] in this general non-uniform setting. This has been done partially already in [7] and we will rely on those results and constructions of that paper which do not depend on existence of the semi-conjugacy.

The main technical problem which we face is showing recurrence for elements within the Lyapunov hyperplanes. For the actions on the torus the semi-conjugacy was used in a critical way. One main innovation here is a construction of a particular time change which is smooth along the orbits of the action but only measurable transversally which “straightens out” the expansion and contractions coefficients. This is somewhat similar to the

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<sup>3</sup>In our situation the codimension of orbits is at least three. When codimension of orbits equals two there is not enough space for nontrivial behavior of higher rank actions involving any kind of hyperbolicity, see [12].

“synchronization” time change for Anosov flows introduced by Bill Parry in [27]. The main technical difficulty lies in the fact that we need the new action to possess certain properties as if it were smooth. Section 6 where this time change is defined and its properties are studied is the heart and the main technical part of the present paper.

## 2. PRELIMINARIES

**2.1. Lyapunov exponents and suspension.** In this section we briefly recall the definitions of Lyapunov characteristic exponents and related notions for  $\mathbb{Z}^k$  and  $\mathbb{R}^k$  actions by measure preserving diffeomorphisms of smooth manifolds. We refer to [6, Sections 5.1 and 5.2] for more details on general theory in the discrete case and to [7] for further development in a more specialized setting. We will use those notions without special references.

Let  $\alpha$  be a smooth  $\mathbb{Z}^k$  action on a manifold  $M$  with an ergodic invariant measure  $\mu$ . According to Multiplicative Ergodic Theorem for  $\mathbb{Z}^k$  actions (see [6]) the Lyapunov decompositions for individual elements of  $\alpha$  have a common refinement  $TM = \bigoplus E_\chi$  called the *Lyapunov decomposition* for  $\alpha$ . For each Lyapunov distribution  $E_\chi$  the corresponding Lyapunov exponent, viewed as a function of an element in  $\mathbb{Z}^k$ , is a linear functional  $\chi : \mathbb{Z}^k \rightarrow \mathbb{R}$  which is called a *Lyapunov exponent* of  $\alpha$ . The Lyapunov exponents of  $\alpha$  are extended by linearity to functionals on  $\mathbb{R}^k$ . The hyperplanes  $\ker \chi \subset \mathbb{R}^k$  are called the *Lyapunov hyperplanes* and the connected components of  $\mathbb{R}^k \setminus \bigcup_\chi \ker \chi$  are called the *Weyl chambers* of  $\alpha$ . The elements in the union of the Lyapunov hyperplanes are called *singular*, and the elements in the union of the Weyl chambers are called *regular*. The corresponding notions for a smooth  $\mathbb{R}^k$  action are defined similarly (see Proposition 2.1 below for more details). We note that any  $\mathbb{R}^k$  action has  $k$  identically zero Lyapunov exponents corresponding to the orbit directions. These Lyapunov exponents are called *trivial* and the other ones are called *nontrivial*. For the rest of the paper a Lyapunov exponent of an  $\mathbb{R}^k$  action will mean a nontrivial one.

One of the reasons for extending the Lyapunov exponents for a  $\mathbb{Z}^k$  action to  $\mathbb{R}^k$  is that the Lyapunov hyperplanes may be irrational and hence “invisible” within  $\mathbb{Z}^k$ . It is also natural to construct an  $\mathbb{R}^k$  action for which the extensions of the exponents from  $\mathbb{Z}^k$  will provide the nontrivial exponents. This is given by the suspension construction which associates to a given  $\mathbb{Z}^k$  action on a manifold  $M$  an  $\mathbb{R}^k$  action on the suspension manifold  $S$ , which is a bundle over  $\mathbb{T}^k$  with fiber  $M$ . Namely, let  $\mathbb{Z}^k$  act on  $\mathbb{R}^k \times M$  by  $z(x, m) = (x - z, z m)$  and form the quotient space

$$S = \mathbb{R}^k \times M / \mathbb{Z}^k.$$

Note that the action of  $\mathbb{R}^k$  on  $\mathbb{R}^k \times M$  by  $x(y, n) = (x + y, n)$  commutes with the  $\mathbb{Z}^k$ -action and therefore descends to  $S$ . This  $\mathbb{R}^k$ -action is called the *suspension* of the  $\mathbb{Z}^k$ -action. There is a natural correspondence between

the invariant measures, nontrivial Lyapunov exponents, Lyapunov distributions, stable and unstable manifolds, etc. for the original  $\mathbb{Z}^k$  action and its suspension.

Since most of the arguments will be for the  $\mathbb{R}^k$  case, we summarize in the next proposition important properties of smooth  $\mathbb{R}^k$  actions given by the nonuniformly hyperbolic theory (see [6, 2]). For a smooth  $\mathbb{R}^k$  action  $\alpha$  on a manifold  $M$  and an element  $\mathbf{t} \in \mathbb{R}^k$  we denote the corresponding diffeomorphism of  $M$  by  $\alpha(\mathbf{t})$ . Sometimes we will omit  $\alpha$  and write, for example,  $\mathbf{t}x$  in place of  $\alpha(\mathbf{t})x$  and  $D\mathbf{t}$  in place of  $D\alpha(\mathbf{t})$  for the derivative of  $\alpha(\mathbf{t})x$ .

**Proposition 2.1.** *Let  $\alpha$  be a locally free  $C^{1+\theta}$ ,  $\theta > 0$ , action of  $\mathbb{R}^k$  on a manifold  $M$  preserving an ergodic invariant measure  $\mu$ . There are linear functionals  $\chi_i$ ,  $i = 1, \dots, l$ , on  $\mathbb{R}^k$  and an  $\alpha$ -invariant measurable splitting, called the Lyapunov decomposition, of the tangent bundle of  $M$*

$$TM = T\mathcal{O} \oplus \bigoplus_{i=1}^l E_i$$

over a set of full measure  $\mathcal{R}$ , where  $T\mathcal{O}$  is the distribution tangent to the  $\mathbb{R}^k$  orbits, such that for any  $\mathbf{t} \in \mathbb{R}^k$  and any nonzero vector  $v \in E_i$  the Lyapunov exponent of  $v$  is equal to  $\chi_i(\mathbf{t})$ , i.e.

$$\lim_{n \rightarrow \pm\infty} n^{-1} \log \|D(n\mathbf{t})v\| = \chi_i(\mathbf{t}),$$

where  $\|\cdot\|$  is any continuous norm on  $TM$ . Any point  $x \in \mathcal{R}$  is called a regular point.

Furthermore, for any  $\varepsilon > 0$  there exist positive measurable functions  $C_\varepsilon(x)$  and  $K_\varepsilon(x)$  such that for all  $x \in \mathcal{R}$ ,  $v \in E_i(x)$ ,  $\mathbf{t} \in \mathbb{R}^k$ , and  $i = 1, \dots, l$ ,

- (1)  $C_\varepsilon^{-1}(x)e^{\chi_i(\mathbf{t}) - \frac{1}{2}\varepsilon\|\mathbf{t}\|}\|v\| \leq \|D\mathbf{t}v\| \leq C_\varepsilon(x)e^{\chi_i(\mathbf{t}) + \frac{1}{2}\varepsilon\|\mathbf{t}\|}\|v\|$ ;
- (2)  $\angle(E_i(x), T\mathcal{O}) \geq K_\varepsilon(x)$  and  $\angle(E_i(x), E_j(x)) \geq K_\varepsilon(x)$ ,  $i \neq j$ ;
- (3)  $C_\varepsilon(\mathbf{t}x) \leq C_\varepsilon(x)e^{\varepsilon\|\mathbf{t}\|}$  and  $K_\varepsilon(\mathbf{t}x) \geq K_\varepsilon(x)e^{-\varepsilon\|\mathbf{t}\|}$ ;

The stable and unstable distributions of an element  $\alpha(\mathbf{t})$  are defined as the sums of the Lyapunov distributions corresponding to the negative and the positive Lyapunov exponents for  $\alpha(\mathbf{t})$  respectively:

$$E_{\alpha(\mathbf{t})}^- = \bigoplus_{\chi_i(\mathbf{t}) < 0} E_i, \quad E_{\alpha(\mathbf{t})}^+ = \bigoplus_{\chi_i(\mathbf{t}) > 0} E_i.$$

**2.2. Actions with Lyapunov exponents in general position.** Let  $\alpha$  be an  $\mathbb{R}^k$  action as in the Main Theorem. Since  $(k+1)$  nontrivial Lyapunov exponents of  $\alpha$  with respect to  $\mu$  are nonzero functionals and the Lyapunov hyperplanes are in general position, the total number of Weyl chambers is equal to  $2^{k+1} - 2$ . Each Weyl chamber corresponds to a different combination of signs for the Lyapunov exponents. In fact,  $2^{k+1} - 2$  Weyl chambers correspond to all possible combinations of signs except for all pluses and all minuses. The fact that these two combinations are impossible can be seen as

follows. First we note that  $\mu$  is non-atomic since it is ergodic for  $\alpha$  and the entropy for some element is positive. Now assume that there is an element  $\mathbf{t} \in \mathbb{R}^k$  such that all exponents for  $\alpha(\mathbf{t})$  are negative. Then every ergodic component for  $\alpha(\mathbf{t})$  is an isolated contracting periodic orbit [12, Proposition 1.3] and hence the measure  $\mu$  must be atomic. In particular, we obtain the following property which will play an important role in our considerations. Let  $\chi_i$ ,  $i = 1, \dots, k+1$ , be the Lyapunov exponents of the action  $\alpha$  and let  $E_i$ ,  $i = 1, \dots, k+1$ , be the corresponding Lyapunov distributions.

(C) *For every  $i \in \{1, \dots, k+1\}$  there exists a Weyl chamber  $\mathcal{C}_i$  such that for every  $\mathbf{t} \in \mathbb{R}^k \cap \mathcal{C}_i$  the signs of the Lyapunov exponents are*

$$\chi_i(\mathbf{t}) < 0 \text{ and } \chi_j(\mathbf{t}) > 0 \text{ for all } j \neq i.$$

In other words, property (C) implies that each Lyapunov distribution  $E_i$  is the full stable distribution for any  $\mathbf{t} \in \mathcal{C}_i$ .

Recall that stable distributions are always Hölder continuous (see, for example, [2]). Therefore, property (C) implies, in particular, that all Lyapunov distributions for such actions inherit the Hölder continuity of stable distributions. More generally, we have the following.

**Proposition 2.2.** *Let  $\alpha$  be a  $C^{1+\theta}$ ,  $\theta > 0$  action  $\mathbb{R}^k$  as in Proposition 2.1. Suppose that a Lyapunov distributions  $E$  is the intersection of the stable distributions of some elements of the action. Then  $E$  is Hölder continuous on any Pesin set*

$$(2.1) \quad \mathcal{R}_\varepsilon^l = \{x \in \mathcal{R} : C_\varepsilon(x) \leq l, K_\varepsilon(x) \geq l^{-1}\}$$

*with Hölder constant which depends on  $l$  and Hölder exponent  $\delta > 0$  which depends on the action  $\alpha$  only.*

**2.3. Invariant manifolds.** We will use the standard material on invariant manifolds corresponding to the negative and positive Lyapunov exponents (stable and unstable manifolds) for  $C^{1+\theta}$  measure preserving diffeomorphisms of compact manifolds, see for example [1, Chapter 4].

We will denote by  $W_{\alpha(\mathbf{t})}^-(x)$  and  $\mathcal{W}_{\alpha(\mathbf{t})}^-(x)$  correspondingly the local and global stable manifolds for the diffeomorphism  $\alpha(\mathbf{t})$  at a regular point  $x$ . Those manifolds are tangent to the stable distribution  $E_{\alpha(\mathbf{t})}^-$ . The global manifold is an immersed Euclidean space and is defined uniquely. Any local manifold is a  $C^{1+\theta}$  embedded open disc in a Euclidean space. Its germ at  $x$  is uniquely defined and for any two choices their intersection is an open neighborhood of the point  $x$  in each of them. On any Pesin set  $\mathcal{R}_\varepsilon^l$  the local stable manifolds can be chosen of a uniform size and changing continuously in the  $C^{1+\theta}$  topology. The local and global unstable manifolds  $W_{\alpha(\mathbf{t})}^+(x)$  and  $\mathcal{W}_{\alpha(\mathbf{t})}^+(x)$  are defined as the stable manifolds for the inverse map  $\alpha(-\mathbf{t})$  and thus have similar properties.

It is customary to use words “distributions” and “foliations” in this setting although in fact the objects we are dealing with are correspondingly

measurable families of tangent spaces defined almost everywhere with respect to an invariant measure, and measurable families of smooth manifolds which fill a set of full measure.

Let  $\alpha$  be an  $\mathbb{R}^k$  action as in the Main Theorem. Then property (C) gives that each Lyapunov distribution  $E$  coincides with the full stable distribution of some element of the action. Therefore, we have the corresponding local and global manifolds  $W(x)$  and  $\mathcal{W}(x)$  tangent to  $E$ . More generally, these local and global manifolds are defined for any Lyapunov distribution  $E$  as in Proposition 2.2. We will refer to these manifolds as local and global leaves of the Lyapunov foliation  $\mathcal{W}$ .

### 3. PROOF OF MAIN THEOREM

As we mentioned before, part (1) of the Main Theorem follows immediately from part (2) by passing to the suspension. In this section we deduce part (2) from the technical Theorem 4.1. First we show that the existence of an element with positive entropy implies that all regular elements have positive entropy and that the conditional measures on every Lyapunov foliation are non-atomic almost everywhere. This is done in Section 3.1. Applying Theorem 4.1 we obtain that for every Lyapunov foliation  $\mathcal{W}$  the conditional measures on  $\mathcal{W}$  are absolutely continuous. We conclude the proof of the Main Theorem in Section 3.2 by showing, as in [7], that this implies absolute continuity of  $\mu$  itself.

#### 3.1. Conditional measures on all Lyapunov foliations are nonatomic.

We recall that a diffeomorphism has positive entropy with respect to an ergodic invariant measure  $\mu$  if and only if the conditional measures of  $\mu$  on its stable (unstable) foliation are non-atomic a.e. This follows for example from [25]. Thus if the entropy  $h_\mu(\mathbf{t})$  is positive for some element  $\mathbf{t} \in \mathbb{R}^k$  then the conditional measures of  $\mu$  on  $\mathcal{W}_{\alpha(\mathbf{t})}^+$  are non-atomic. Then there exists an element  $\mathbf{s}$  in a Weyl chamber  $\mathcal{C}_i$  such that the one-dimensional distribution  $E_i = E_{\alpha(\mathbf{s})}^-$  is not contained in  $E_{\alpha(\mathbf{t})}^+$  and thus  $E_{\alpha(\mathbf{t})}^+ \subset E_{\alpha(\mathbf{s})}^+ = \bigoplus_{j \neq i} E_j$ . Hence the conditional measures on  $\mathcal{W}_{\alpha(\mathbf{s})}^+$  are also non-atomic. This gives  $h_\mu(\mathbf{s}) > 0$  which implies that the conditional measures on  $\mathcal{W}_i = \mathcal{W}_{\alpha(\mathbf{s})}^-$  must be also non-atomic. Now for any  $j \neq i$  consider the codimension one distribution  $E'_j = \bigoplus_{k \neq j} E_k = E_{\alpha(\mathbf{t}_j)}^+$  for any  $\mathbf{t}_j$  in the Weyl chamber  $\mathcal{C}_j$ . Since  $E_i \subset E'_j$  we see that the conditional measures on the corresponding foliation  $\mathcal{W}'_j$  are non-atomic. Hence  $h_\mu(\mathbf{t}_j) > 0$  and the conditional measures on  $\mathcal{W}_j = \mathcal{W}_{\alpha(\mathbf{t}_j)}^-$  are non-atomic too. We conclude that the conditional measures on every Lyapunov foliation  $\mathcal{W}_i$ ,  $i = 1, \dots, k+1$ , are non-atomic. This implies, in particular, that the entropy is positive for any non-zero element of the action.

**3.2. The absolute continuity of  $\mu$ .** The remaining argument is similar to that in [7]. In order to prove that  $\mu$  is an absolutely continuous measure

we shall use the following theorem that is essentially the flow analogue of what is done in Section 5 of [22] (see particularly [22, Theorem (5.5)] and also [25, Corollary H]).

**Theorem 3.1.** *Let  $f : M \rightarrow M$  be a  $C^{1+\alpha}$  diffeomorphism with invariant measure  $\mu$  and assume that  $h_\mu(f)$  is equal both to the sum of the positive Lyapunov exponents and to the absolute value of the sum of the negative Lyapunov exponents. If the directions corresponding to zero Lyapunov exponents integrates to a smooth foliation and the conditional measures with respect to this central foliation are absolutely continuous, then  $\mu$  is absolutely continuous with respect to Lebesgue measure.*

To use Theorem 3.1 recall that there are  $2^{k+1} - 2$  Weyl chambers for  $\alpha$  and any combination of positive and negative signs for the Lyapunov exponents, except for all positive or all negative, appears in one of the Weyl chambers. We use notations of Section 2.2 and consider an element  $\mathbf{t}$  in the Weyl chamber  $-C_i$ . Then the Lyapunov exponents of  $\mathbf{t}$  have the following signs:  $\chi_i(\mathbf{t}) > 0$  and  $\chi_j(\mathbf{t}) < 0$  for all  $j \neq i$ . Since the conditional measures on  $\mathcal{W}_{\alpha(\mathbf{t})}^+$  are absolutely continuous by Lemma 7.4, we obtain that

$$h_\mu(\alpha(\mathbf{t})) = \chi_i(\mathbf{t})$$

for any  $\mathbf{t}$  in  $-C_i$ . By the Ruelle entropy inequality  $h_\mu(\alpha(\mathbf{t})) \leq -\sum_{j \neq i} \chi_j(\mathbf{t})$  and hence

$$\sum_{j=1}^{k+1} \chi_j(\mathbf{t}) \leq 0.$$

If  $\sum_{j=1}^{k+1} \chi_j(\mathbf{t}) = 0$  then Theorem 3.1 applies and the proof is finished.

Thus we have to consider the case when  $\sum_{j=1}^{k+1} \chi_j(\mathbf{t}) < 0$  for all  $\mathbf{t}$  in all Weyl chambers  $-C_i$ ,  $i = 1, \dots, k+1$ . This implies that  $\bigcup_{i=1}^{k+1} C_i$  lies in the positive half space of the linear functional  $\sum_{j=1}^{k+1} \chi_j$ . But this is impossible since there exist elements  $\mathbf{t}_i \in C_i$ ,  $i = 1, \dots, k+1$  such that  $\sum_{i=1}^{k+1} \mathbf{t}_i = 0$ .  $\square$

#### 4. THE TECHNICAL THEOREM

In the notations of Proposition 2.1, an ergodic invariant measure  $\mu$  for a smooth locally free  $\mathbb{R}^k$  action  $\alpha$  is called *hyperbolic* if all nontrivial Lyapunov exponents  $\chi_i$ ,  $i = 1, \dots, l$ , are nonzero linear functionals on  $\mathbb{R}^k$ .

**Theorem 4.1.** *Let  $\mu$  be a hyperbolic ergodic invariant measure for a locally free  $C^{1+\theta}$ ,  $\theta > 0$ , action  $\alpha$  of  $\mathbb{R}^k$ ,  $k \geq 2$ , on a compact smooth manifold  $M$ . Suppose that a Lyapunov exponent  $\chi$  is simple and there are no other exponents proportional to  $\chi$ . Let  $E$  be the one-dimensional Lyapunov distribution corresponding to the exponent  $\chi$ .*

*Then  $E$  is tangent  $\mu$ -a.e. to a Lyapunov foliation  $\mathcal{W}$  and the conditional measures of  $\mu$  on  $\mathcal{W}$  are either atomic a.e. or absolutely continuous a.e.*



The assumptions on the Lyapunov exponents in Theorem 4.1 are considerably more general than in the Main Theorem. In particular they may be satisfied for all exponents of a hyperbolic measure for an action on any rank greater than one on a manifold of arbitrary large dimension. As an example one can take restriction of an action satisfying the assumption of part (1) of the Main Theorem to any lattice  $L \subset \mathbb{Z}^k$  of rank at least two which has trivial intersection with all Lyapunov hyperplanes. For this reason Theorem 4.1 has applications beyond the maximal rank case considered in the Main Theorem. Those application will be discussed in a subsequent paper.

On the other hand, positivity of entropy for some or even all non-zero elements is not sufficient to exclude atomic measures on some of the Lyapunov foliations. Thus application to more general actions may include stronger assumptions on ergodic properties of the measure.

**4.1. Outline of the proof of Theorem 4.1.** We note that the Lyapunov distribution  $E$  may not coincide with the full stable distribution of any element of  $\alpha$ . First we will show that  $E$  is an intersection of some stable distributions of  $\alpha$ .

An element  $\mathbf{t} \in \mathbb{R}^k$  is called *generic singular* if it belongs to exactly one Lyapunov hyperplane. We consider a generic singular element  $\mathbf{t}$  in  $L$ , i.e.  $\chi$  is the only non-trivial Lyapunov exponent that vanishes on  $\mathbf{t}$ . Thus

$$TM = T\mathcal{O} \oplus E_{\alpha(\mathbf{t})}^- \oplus E \oplus E_{\alpha(\mathbf{t})}^+$$

We can take a regular element  $\mathbf{s}$  close to  $\mathbf{t}$  for which  $\chi(\mathbf{s}) > 0$  and all other non-trivial exponents have the same signs as for  $\mathbf{t}$ . Thus  $E_{\alpha(\mathbf{s})}^- = E_{\alpha(\mathbf{t})}^-$  and  $E_{\alpha(\mathbf{s})}^+ = E_{\alpha(\mathbf{t})}^+ \oplus E$ . Similarly, we can take a regular element  $\mathbf{s}'$  close to  $\mathbf{t}$  on the other side of  $L$  for which  $\chi(\mathbf{s}') < 0$  and  $E_{\alpha(\mathbf{s}')}^+ = E_{\alpha(\mathbf{t})}^+$  and  $E_{\alpha(\mathbf{s}')}^- = E_{\alpha(\mathbf{t})}^- \oplus E$ . Therefore,

$$E = E_{\alpha(\mathbf{s})}^+ \cap E_{\alpha(\mathbf{s}')}^- = E_{\alpha(-\mathbf{s})}^- \cap E_{\alpha(\mathbf{s}')}^-.$$

We conclude that the Lyapunov distribution  $E$  is an intersection of stable distributions and, as in Proposition 2.2, is Hölder continuous on Pesin sets. As in Section 2.3,  $E$  is tangent  $\mu$ -a.e. to the corresponding Lyapunov foliation  $\mathcal{W} = \mathcal{W}_{\alpha(-\mathbf{s})}^- \cap \mathcal{W}_{\alpha(\mathbf{s}')}^-$ .

We denote by  $\mu_x^{\mathcal{W}}$  the system of conditional measures of  $\mu$  on  $\mathcal{W}$ . By ergodicity of  $\mu$  these conditional measures are either non-atomic or have atoms for  $\mu$ -a.e.  $x$ . Since  $\mathcal{W}$  is an invariant foliation contracted by some elements of the action, it is easy to see that in the latter case the conditional measures are atomic with a single atom for  $\mu$ -a.e.  $x$  (see, for example, [20, Proposition 4.1]). The main part of the proof is to show that if the conditional measures  $\mu_x^{\mathcal{W}}$  are non-atomic for  $\mu$ -a.e.  $x$ , then they are absolutely continuous  $\mu$ -a.e.

To prove absolute continuity of the conditional measures on  $\mathcal{W}$  we show in Section 7 that they are Haar with respect to the invariant family of smooth affine parameters on the leaves of  $\mathcal{W}$ . As in [7], this uses affine maps of the

leaves which preserve the conditional measures up to a scalar multiple. Such affine maps are obtained in [7] as certain limits of actions along  $\mathcal{W}$  by some elements of the action. It is essential that derivatives of these elements along  $\mathcal{W}$  are uniformly bounded. In [7] it was possible to choose such elements within the Lyapunov hyperplane  $L$ . We note that in general Multiplicative Ergodic Theorem only guarantees that the elements in  $L$  expand or contract leaves of  $\mathcal{W}$  at a subexponential rate.

The main part of the proof is to produce a sequence of elements of the action with uniformly bounded derivatives along  $\mathcal{W}$  and with enough recurrence. In Section 5 we define a special Lyapunov metric on distribution  $E$  and show that it is Hölder continuous on Pesin sets. Then in Section 6.1 we construct a *measurable* time change for which the expansion or contraction in  $E$  with respect to this Lyapunov metric is given exactly by the Lyapunov exponent  $\chi$ . This gives sufficient control for the derivatives along  $\mathcal{W}$ .

To produce enough recurrence we study properties of this measurable time change in Section 6.2. We prove that it is differentiable along regular orbits and Hölder continuous when restricted to any Pesin set (2.1). This allows us to show in Section 6.3 that the time change has some structure similar to that of the original action. First, it preserves a measure equivalent to  $\mu$ . Second, it preserves certain “foliations” whose restrictions to Pesin sets are Hölder graphs over corresponding foliations of the original action  $\alpha$ . More precisely, the leaves for  $\alpha$  are tilted along the orbits to produce invariant sets for the time change action  $\beta$  and the tilt is a Hölder function when restricted to a set of large measure (the intersection with such a set has large conditional measure for a typical leaf). Of course, the Hölder constants (but not the exponents) deteriorate when one increases the Pesin set but in the end one gets a measurable function defined almost everywhere.

Using these properties, we show in Section 6.4 that for a typical element in the Lyapunov hyperplane  $L$  the time change acts sufficiently transitively along the leaves of  $\mathcal{W}$ . For this we use the “ $\pi$ -partition trick” first introduced in [20] for the study of invariant measure of actions by automorphisms of a torus and adapted to the general nonuniform situation in [7]. We use this argument for the time change action  $\beta$  and the main technical difficulty is in showing that the weird “foliations” described above can still be used in essentially the same way as for smooth actions.

## 5. LYAPUNOV METRIC

In this section we use notations of Theorem 4.1. We define a Lyapunov metric on the Lyapunov distribution  $E$  and establish its properties.

We fix a smooth Riemannian metric  $\langle \cdot, \cdot \rangle$  on  $M$ . Given  $\varepsilon > 0$  and a regular point  $x \in M$  we define the *standard  $\varepsilon$ -Lyapunov scalar product (or metric)*  $\langle \cdot, \cdot \rangle_{x,\varepsilon}$  as follows. For any  $u, v \in E(x)$  we define

$$(5.1) \quad \langle u, v \rangle_{x,\varepsilon} = \int_{\mathbb{R}^k} \langle (Ds)u, (Ds)v \rangle \exp(-2\chi(s) - 2\varepsilon\|s\|) ds$$

We observe using (1) of Proposition 2.1 that for any  $\varepsilon > 0$  the integral above converges exponentially for any regular point  $x$ .

We will usually omit the word “standard” and will call this scalar product  $\varepsilon$ -Lyapunov metric or, if  $\varepsilon$  has been fixed and no confusion may appear, simply *Lyapunov metric*. The norm generated by this scalar product will be called the (standard  $\varepsilon$ -) *Lyapunov norm* and denoted by  $\|\cdot\|_{x,\varepsilon}$  or  $\|\cdot\|_\varepsilon$ .

**Remark.** The definition above gives a measurable scalar product on any Lyapunov distribution  $E$  of an arbitrary nonuniformly hyperbolic  $\mathbb{R}^k$  action (and similarly for a  $\mathbb{Z}^k$  action), without any assumption on Lyapunov exponents, such as multiplicity, or on geometry of Lyapunov hyperplanes. One can also define the Lyapunov scalar product on the whole tangent space  $T_x M$  by declaring the Lyapunov distributions to be pairwise orthogonal and orthogonal to the distribution  $T\mathcal{O}$  tangent to the orbits of the  $\mathbb{R}^k$  action (on  $T\mathcal{O}$  one can take a canonical Euclidean metric given by the action). Proposition 5.1 as well as estimates (5.4) and (5.5) hold for such general case. Also, continuity of the Lyapunov scalar product on sets of large measure follows simply from measurability by Luzin’s theorem. However, Hölder continuity on Pesin sets for the Lyapunov scalar product on a given Lyapunov distribution  $E$  requires similar Hölder continuity of  $E$ . The latter is not necessarily true for an arbitrary Lyapunov distribution.

We denote by  $D_x^E$  the restriction of the derivative to the Lyapunov distribution  $E$ . The main motivation for introducing Lyapunov metric is the following estimate for the norm of this restriction with respect to the Lyapunov norm.

**Proposition 5.1.** *For any regular point  $x$  and any  $\mathbf{t} \in \mathbb{R}^k$*

$$(5.2) \quad \exp(\chi(\mathbf{t}) - \varepsilon\|\mathbf{t}\|) \leq \|D_x^E \mathbf{t}\|_\varepsilon \leq \exp(\chi(\mathbf{t}) + \varepsilon\|\mathbf{t}\|).$$

**Proof.** Fix a nonzero  $u \in E(x)$ . By the definition of the standard  $\varepsilon$ -Lyapunov norm we obtain

$$(5.3) \quad \begin{aligned} \|(D_x \mathbf{t})u\|_{\mathbf{t}x,\varepsilon}^2 &= \int_{\mathbb{R}^k} \|(D_{\mathbf{t}x} \mathbf{s})(D_x \mathbf{t})u\|^2 \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|) d\mathbf{s} = \\ &= \int_{\mathbb{R}^k} \|(D_x(\mathbf{s} + \mathbf{t}))u\|^2 \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|) d\mathbf{s} = \\ &= \int_{\mathbb{R}^k} \|(D_x \mathbf{s}')u\|^2 \exp(-2\chi(\mathbf{s}' - \mathbf{t}) - 2\varepsilon\|\mathbf{s}' - \mathbf{t}\|) d\mathbf{s}' \end{aligned}$$

We note that the exponent can be estimated above and below as follows

$$\begin{aligned} -2\chi(\mathbf{s}' - \mathbf{t}) - 2\varepsilon\|\mathbf{s}' - \mathbf{t}\| &\leq (-2\chi(\mathbf{s}') - 2\varepsilon\|\mathbf{s}'\|) + 2(\chi(\mathbf{t}) + \varepsilon\|\mathbf{t}\|), \\ -2\chi(\mathbf{s}' - \mathbf{t}) - 2\varepsilon\|\mathbf{s}' - \mathbf{t}\| &\geq (-2\chi(\mathbf{s}') - 2\varepsilon\|\mathbf{s}'\|) + 2(\chi(\mathbf{t}) - \varepsilon\|\mathbf{t}\|). \end{aligned}$$

These inequalities together with the definition

$$\|u\|_{x,\varepsilon}^2 = \int_{\mathbb{R}^k} \|(D_x \mathbf{s}')u\|^2 \exp(-2\chi(\mathbf{s}') - 2\varepsilon\|\mathbf{s}'\|) d\mathbf{s}'$$

give the following estimate

$$e^{2(\chi(\mathbf{t}) - \varepsilon \|\mathbf{t}\|)} \|u\|_{x,\varepsilon}^2 \leq \|(D_x \mathbf{t})u\|_{\mathbf{t}x,\varepsilon}^2 \leq e^{2(\chi(\mathbf{t}) + \varepsilon \|\mathbf{t}\|)} \|u\|_{x,\varepsilon}^2$$

which concludes the proof of the proposition.  $\square$

Now we establish some important properties of the Lyapunov metric. First we note that the original smooth metric gives a uniform below estimate for the Lyapunov metric, i.e. there exists positive constant  $C$  such that for all regular  $x \in M$  and all  $u \in E$

$$(5.4) \quad \|u\|_{x,\varepsilon} \geq C \|u\|$$

The next proposition establishes the opposite inequality as well as continuity of the Lyapunov metric on a given Pesin set. We note that similarly to the proof of Lemma 6.1 below one can show that  $\varepsilon$ -Lyapunov metric is actually smooth along the orbits.

**Proposition 5.2.** *The  $\varepsilon$ -Lyapunov metric is continuous along any regular orbit and on any Pesin set  $\mathcal{R}_\varepsilon^l$ . Furthermore, there exists  $C(l, \varepsilon) > 0$  such that for all  $x \in \mathcal{R}_\varepsilon^l$  and all  $u \in E$*

$$(5.5) \quad \|u\|_{x,\varepsilon} \leq C(l, \varepsilon) \|u\|.$$

**Proof.** Let us fix  $u \in E(x)$  with  $\|u\| = 1$ . The integrand in equation (5.1)

$$f(x, \mathbf{s}) = \langle (D_x \mathbf{s})u, (D_x \mathbf{s})u \rangle \exp(-2\chi(\mathbf{s}) - 2\varepsilon \|\mathbf{s}\|)$$

is continuous with respect to  $x$  on  $\mathcal{R}_\varepsilon^l$  by Proposition 2.2. Also, by (1) of Proposition 2.1 we have  $|f(x, \mathbf{s})| \leq C_\varepsilon(x) \exp(-\varepsilon \|\mathbf{s}\|)$  and hence for  $x \in \mathcal{R}_\varepsilon^l$

$$\int_{\mathbb{R}^k} f(x, \mathbf{s}) d\mathbf{s} \leq \int_{\mathbb{R}^k} C_\varepsilon(x) \exp(-\varepsilon \|\mathbf{s}\|) d\mathbf{s} \leq l \int_{\mathbb{R}^k} \exp(-\varepsilon \|\mathbf{s}\|) d\mathbf{s}.$$

This implies the estimate (5.5) and the continuity of the metric on the Pesin set  $\mathcal{R}_\varepsilon^l$ . The continuity along orbits follows since for any regular point  $x$  and any bounded set  $B \subset \mathbb{R}^k$  there is  $l$  such that  $Bx \subset \mathcal{R}_\varepsilon^l$ .  $\square$

Next we obtain Hölder continuity of the Lyapunov metric which will be crucial for deducing properties of the time change in Section 6.

**Proposition 5.3.** *There exists  $\gamma > 0$  which depends only on  $\varepsilon$  and on the action and  $K(l, \varepsilon) > 0$  which, in addition, depends on the Pesin set  $\mathcal{R}_\varepsilon^l$  such that the  $\varepsilon$ -Lyapunov metric is Hölder continuous on  $\mathcal{R}_\varepsilon^l$  with exponent  $\gamma$  and constant  $K(l, \varepsilon)$ .*

**Remark.** We note the dependence of the constant in (5.5) and Hölder constants in Propositions 2.2 and 5.3, as well as below in Propositions 6.4 and 6.7 on the Pesin set  $\mathcal{R}_\varepsilon^l$ . For a fixed  $\varepsilon$  these constants depend only on  $l$  and can be estimated by  $Cl^p$  for some power  $p$ . This holds in Proposition 2.2 and can be observed in the proofs of the other propositions. By (3) of Proposition 2.1 for any  $\mathbf{t} \in \mathbb{R}^k$  we have  $\alpha(\mathbf{t})(\mathcal{R}_\varepsilon^l) \subset \mathcal{R}_\varepsilon^{l'}$  with  $l' = \exp(\varepsilon \|\mathbf{t}\|)l$ . Therefore, we can say that these constants may grow in  $\mathbf{t}$  with a slow exponential rate, more precisely, by a factor at most  $\exp(p\varepsilon \|\mathbf{t}\|)$ .

**Proof.** We consider two nearby points  $x$  and  $y$  in a Pesin set  $\mathcal{R}_\varepsilon^l$ . By Proposition 2.2 we can take vectors  $u \in E(x)$  and  $v \in E(y)$  with  $\|u\| = \|v\| = 1$  for which the distance in  $TM$  can be estimated as  $\text{dist}(u, v) \leq K_1 \rho^\delta$ , where  $\rho = \text{dist}(x, y)$ . Since  $E$  is one-dimensional it suffices to show that  $\|u\|_{x,\varepsilon}$  and  $\|v\|_{y,\varepsilon}$  are Hölder close in  $\rho$ . We will now estimate

$$\|u\|_{x,\varepsilon}^2 - \|v\|_{y,\varepsilon}^2 = \int_{\mathbb{R}^k} (\|(D\mathbf{s})u\|^2 - \|(D\mathbf{s})v\|^2) \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|) d\mathbf{s}$$

Using spherical coordinates  $\mathbf{s} = \mathbf{s}(r, \theta)$  where  $r = \|\mathbf{s}\|$  and denoting

$$\psi(\mathbf{s}) = (\|(D\mathbf{s})u\|^2 - \|(D\mathbf{s})v\|^2) \exp(-2\chi(\mathbf{s}) - 2\varepsilon r)$$

we can write

$$\|u\|_{x,\varepsilon}^2 - \|v\|_{y,\varepsilon}^2 = \int_0^\infty f(r) dr, \quad \text{where } f(r) = r^{k-1} \int_{S^{k-1}} \psi(\mathbf{s}) d\theta.$$

We will estimate the difference  $\|(D\mathbf{s})u\|^2 - \|(D\mathbf{s})v\|^2$  inside a large ball using closeness of  $u$  and  $v$  and outside it by estimating each of the two terms. Since the action  $\alpha$  is smooth we observe that

$$\begin{aligned} | \|(D\mathbf{s})u\|^2 - \|(D\mathbf{s})v\|^2 | &\leq \text{dist}((D\mathbf{s})u, (D\mathbf{s})v) \cdot (\|(D\mathbf{s})u\| + \|(D\mathbf{s})v\|) \\ &\leq C_1 \exp(L\|\mathbf{s}\|) \cdot \text{dist}(u, v) \leq C_1 \exp(Lr) K_1 \rho^\delta \end{aligned}$$

for some  $L > 0$ . Hence we obtain  $|\psi(\mathbf{s})| \leq K_1 C_2 \exp(M'r) \rho^\delta$ , where  $M' = L + 2(\|\chi\| + \varepsilon)$ . Then for sufficiently large  $a$  we have

$$\int_0^a |f(r)| dr \leq \int_0^a r^{k-1} K_1 C_3 \exp((M')r) \rho^\delta dr \leq K_1 C_4 \exp(Ma) \rho^\delta,$$

where for simplicity we absorbed the polynomial factor appearing in the estimates into the exponent. Then

$$(5.6) \quad \text{for } a = \frac{\delta}{2M} \log \frac{1}{\rho} \quad \text{we have } \int_0^a |f(r)| dr \leq K_2 \rho^{\delta/2}.$$

Now we consider  $\int_a^\infty |f(r)| dr$ . Since  $x, y \in \mathcal{R}_\varepsilon^l$ , using (1) of Proposition 2.1 we obtain  $|\psi(\mathbf{s})| \leq 2l^2 \exp(-\varepsilon r)$ . Hence

$$\int_a^\infty |f(r)| dr \leq \int_a^\infty r^{k-1} C_5 l^2 \exp(-\varepsilon r) dr \leq K_3 \exp(-\varepsilon a/2)$$

where we again absorbed the polynomial factor into the exponent. For  $a$  defined in (5.6) this gives us

$$\int_a^\infty |f(r)| dr \leq K_3 \exp(-\varepsilon a/2) \leq K_3 \rho^\gamma,$$

where  $\gamma = \frac{\varepsilon \delta}{4M}$ . Combining this with the estimate (5.6) for  $\int_0^a |f(r)| dr$  we obtain

$$| \|u\|_{x,\varepsilon}^2 - \|v\|_{y,\varepsilon}^2 | \leq \int_0^\infty |f(r)| dr \leq K_4 \rho^\gamma.$$

According to (5.4), the Lyapunov norm is bounded below by the usual norm, so that

$$\begin{aligned} |||u||_{x,\varepsilon} - |||v||_{y,\varepsilon} | &\leq |||u||_{x,\varepsilon}^2 - |||v||_{y,\varepsilon}^2| / (|||u||_{x,\varepsilon} + |||v||_{y,\varepsilon}) \leq \\ &\leq |||u||_{x,\varepsilon}^2 - |||v||_{y,\varepsilon}^2| / 2K \leq K_5 \rho^\gamma \end{aligned}$$

which completes the desired Hölder estimate.  $\square$

## 6. MEASURABLE TIME CHANGE AND ITS PROPERTIES

**6.1. Construction of a measurable time change.** In this section we use notations of Theorem 4.1. We fix small  $\varepsilon > 0$  and consider the Lyapunov metric  $||\cdot||_\varepsilon$  on the Lyapunov distribution  $E$ . We first study the behavior of the derivative restricted to  $E$  along the  $\mathbb{R}^k$ -orbits. For a regular point  $x$  we consider the function  $f_x : \mathbb{R}^k \rightarrow \mathbb{R}$  given by

$$(6.1) \quad f_x(\mathbf{t}) = \log \|D_x^E \mathbf{t}\|_\varepsilon$$

According to Proposition 5.1 the function  $f$  satisfies inequalities

$$(6.2) \quad \chi(\mathbf{t}) - \varepsilon \|\mathbf{t}\| \leq f_x(\mathbf{t}) \leq \chi(\mathbf{t}) + \varepsilon \|\mathbf{t}\|.$$

Also, since  $E$  is one-dimensional,  $f$  satisfies the cocycle identity

$$(6.3) \quad f_x(\mathbf{t} + \mathbf{s}) = f_x(\mathbf{t}) + f_{\mathbf{t}x}(\mathbf{s}).$$

We will now establish smoothness of the function  $f_x$  in  $\mathbf{t}$ .

**Lemma 6.1.** *For any regular point  $x$  the function  $f_x(\mathbf{t})$  is  $C^1$ . More precisely, for any  $\mathbf{t}, \mathbf{e} \in \mathbb{R}^k$  we have*

$$(D_{\mathbf{t}} f_x) \mathbf{e} = \chi(\mathbf{e}) + \varepsilon \psi_{\mathbf{t}x}(\mathbf{e}),$$

where  $|\psi_{\mathbf{t}x}(\mathbf{e})| \leq \frac{1}{2} \|\mathbf{e}\|$  and  $\psi_{\mathbf{t}x}(\mathbf{e})$  is continuous in  $\mathbf{t}$  and  $\mathbf{e}$ .

**Proof.** Fix a regular point  $x$  and consider the function

$$F(\mathbf{t}) = \exp(f_x(\mathbf{t})) = \|D_x^E \mathbf{t}\|_\varepsilon.$$

Fix a vector  $u \in E(x)$  with  $\|u\|_{x,\varepsilon} = 1$ . Since  $E(x)$  is one-dimensional,

$$F(\mathbf{t}) = \|(D_x \mathbf{t})u\|_{\mathbf{t}x,\varepsilon}.$$

Using the definition of the Lyapunov metric we obtain as in (5.3) that

$$F^2(\mathbf{t}) = \int_{\mathbb{R}^k} \|(D_x \mathbf{s})u\|^2 \exp(-2\chi(\mathbf{s}) + 2\chi(\mathbf{t}) - 2\varepsilon \|\mathbf{s} - \mathbf{t}\|) d\mathbf{s}$$

Differentiating at  $\mathbf{t}$  we obtain

$$\begin{aligned} (D_{\mathbf{t}} F^2) \mathbf{e} &= \int_{\mathbb{R}^k} \|(D_x \mathbf{s})u\|^2 \exp(-2\chi(\mathbf{s}) + 2\chi(\mathbf{t}) - 2\varepsilon \|\mathbf{s} - \mathbf{t}\|) \times \\ &\quad \left( 2\chi(\mathbf{e}) + \varepsilon \frac{\langle \mathbf{s} - \mathbf{t}, \mathbf{e} \rangle}{\|\mathbf{s} - \mathbf{t}\|} \right) d\mathbf{s} = \\ &= \int_{\mathbb{R}^k} \|(D_x(\mathbf{s} + \mathbf{t}))u\|^2 \exp(-2\chi(\mathbf{s}) - 2\varepsilon \|\mathbf{s}\|) \cdot \left( 2\chi(\mathbf{e}) + \varepsilon \frac{\langle \mathbf{s}, \mathbf{e} \rangle}{\|\mathbf{s}\|} \right) d\mathbf{s} = \end{aligned}$$

$$= 2\chi(\mathbf{e})F^2(\mathbf{t}) + \varepsilon\tilde{\psi}(\mathbf{t}, \mathbf{e}),$$

where

$$\tilde{\psi}(\mathbf{t}, \mathbf{e}) = \int_{\mathbb{R}^k} \|(D_x(\mathbf{s} + \mathbf{t}))u\|^2 \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|) \frac{\langle \mathbf{s}, \mathbf{e} \rangle}{\|\mathbf{s}\|} d\mathbf{s}$$

Then for the function  $f_x$  we obtain

$$(D_{\mathbf{t}}f_x)\mathbf{e} = \frac{1}{2}D_{\mathbf{t}}(\log F^2)\mathbf{e} = \frac{(D_{\mathbf{t}}F^2)\mathbf{e}}{2F^2(\mathbf{t})} = \chi(\mathbf{e}) + \varepsilon\psi_x(\mathbf{t}, \mathbf{e}),$$

where  $\psi(\mathbf{t}, \mathbf{e}) = \tilde{\psi}(\mathbf{t}, \mathbf{e})/2F^2(\mathbf{t})$ . We observe that  $\tilde{\psi}(\mathbf{t}, \mathbf{e})$  is continuous in  $\mathbf{t}$  and  $\varepsilon$  and hence so is  $\psi_x(\mathbf{t}, \mathbf{e})$ . We conclude that  $f_x(\mathbf{t})$  is  $C^1$ . Since  $|\langle \mathbf{s}, \mathbf{e} \rangle \cdot \|\mathbf{s}\|^{-1}| \leq \|\mathbf{e}\|$  we obtain  $|\tilde{\psi}(\mathbf{t}, \mathbf{e})| \leq F^2(\mathbf{t})\|\mathbf{e}\|$  and hence

$$|\psi_x(\mathbf{t}, \mathbf{e})| \leq \|\mathbf{e}\|/2.$$

We also note that  $\psi_x(\mathbf{t}, \mathbf{e}) = \psi_{\mathbf{t}x}(\mathbf{0}, \mathbf{e})$ , which follows for example from the cocycle relation (6.3). Denoting  $\psi_x(\mathbf{e}) = \psi(\mathbf{0}, \mathbf{e})$  we obtain the desired formula for  $(D_{\mathbf{t}}f_x)\mathbf{e}$  with function  $\psi_x(\mathbf{e})$  which is continuous in  $\mathbf{e} \in \mathbb{R}^k$  and satisfies  $|\psi_{\mathbf{t}x}(\mathbf{e})| \leq \frac{1}{2}\|\mathbf{e}\|$ .  $\square$

Now we proceed to constructing the time change. We fix a vector  $\mathbf{w}$  in  $\mathbb{R}^k$  normal to  $L$  with  $\chi(\mathbf{w}) = 1$ . We will assume that  $\varepsilon$  and  $\varepsilon\|\mathbf{w}\|$  are both small, in particular  $\varepsilon\|\mathbf{w}\| < 1/2$ .

**Proposition 6.2.** *For  $\mu$ -a.e.  $x \in M$  and any  $\mathbf{t} \in \mathbb{R}^k$  there exists a unique real number  $g(x, \mathbf{t})$  such that the function  $\mathbf{g}(x, \mathbf{t}) = \mathbf{t} + g(x, \mathbf{t})\mathbf{w}$  satisfies the equality*

$$\|D_x^E \alpha(\mathbf{g}(x, \mathbf{t}))\|_{\varepsilon} = e^{\chi(\mathbf{t})}.$$

*The function  $g(x, \mathbf{t})$  is measurable and is continuous in  $x$  on Pesin sets (2.1) and along the orbits of  $\alpha$ . It satisfies the inequality  $|g(x, \mathbf{t})| \leq 2\varepsilon\|\mathbf{t}\|$ .*

In Section 6.2 we will show that  $g(x, \mathbf{t})$  is actually Hölder continuous in  $x$  on Pesin sets and is  $C^1$  in  $\mathbf{t}$ .

**Proof.** Recall that by Proposition 5.1 for any regular point  $z$  we have

$$(6.4) \quad \exp(\chi(\mathbf{t}) - \varepsilon\|\mathbf{t}\|) \leq \|D_z^E \alpha(\mathbf{t})\|_{\varepsilon} \leq \exp(\chi(\mathbf{t}) + \varepsilon\|\mathbf{t}\|);$$

thus in particular

$$(6.5) \quad \exp(s - \varepsilon s\|\mathbf{w}\|) \leq \|D_z^E \alpha(s\mathbf{w})\|_{\varepsilon} \leq \exp(s + \varepsilon s\|\mathbf{w}\|).$$

We fix a regular point  $x$  and define

$$\phi(s) = \log \|D_{\alpha(\mathbf{t})x}^E \alpha(s\mathbf{w})\|_{\varepsilon} = f_{\alpha(\mathbf{t})x}(s\mathbf{w}).$$

Using Lemma 6.1 we obtain

$$(6.6) \quad \begin{aligned} \phi'(s) &= \frac{d}{ds} f_{\alpha(\mathbf{t})x}(s\mathbf{w}) = (D_{s\mathbf{w}} f_{\alpha(\mathbf{t})x})\mathbf{w} = \\ &= \chi(\mathbf{w}) + \varepsilon \psi_{(s\mathbf{w} + \alpha(\mathbf{t}))x}(\mathbf{w}) \geq 1 - \varepsilon\|\mathbf{w}\|/2 > 0 \end{aligned}$$

This implies that  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$  bijection. Hence there exists a unique number  $s_0$  such that  $\phi(s_0) = \chi(\mathbf{t}) - \log \|D_x^E \alpha(\mathbf{t})\|_\varepsilon$  and thus  $g(x, \mathbf{t}) = s_0$  satisfies the equation in the lemma. We observe that (6.4) implies

$$-\varepsilon \|\mathbf{t}\| \leq \chi(\mathbf{t}) - \log \|D_x^E \alpha(\mathbf{t})\|_\varepsilon^{-1} \leq \varepsilon \|\mathbf{t}\|.$$

Also, (6.5) implies

$$s - \varepsilon s \|\mathbf{w}\| \leq \phi(s) \leq s + \varepsilon s \|\mathbf{w}\|.$$

Hence  $|\phi(s)| \geq \frac{1}{2}|s|$  and we conclude that  $|g(x, \mathbf{t})| = |s_0| \leq 2\varepsilon \|\mathbf{t}\|$

The continuity of  $g(x, \mathbf{t})$  in  $x$  on Pesin sets and along the orbits of  $\alpha$  follows from the corresponding continuity of the Lyapunov norm.  $\square$

**Proposition 6.3.** *The formula  $\beta(\mathbf{t}, x) = \alpha(\mathbf{g}(x, \mathbf{t}))x$  defines an  $\mathbb{R}^k$  action  $\beta$  on  $M$  which is a measurable time change of  $\alpha$ , i.e.*

$$(6.7) \quad \beta(\mathbf{s} + \mathbf{t}, x) = \beta(\mathbf{s}, \beta(\mathbf{t}, x)), \quad \text{or}$$

$$(6.8) \quad \mathbf{g}(x, \mathbf{s} + \mathbf{t}) = \mathbf{g}(x, \mathbf{t}) + \mathbf{g}(\alpha(\mathbf{g}(x, \mathbf{t}))x, \mathbf{s}).$$

The action  $\beta$  is measurable and is continuous on any Pesin set for  $\alpha$ .

**Remark.** The time change is defined using a condition on the derivative of the *original* action restricted to  $E$ . The new action is not necessarily smooth and typically does not preserve the Lyapunov foliation  $\mathcal{W}$ . However, it does preserve the sum of the distribution  $E$  with the orbit distribution as well as the corresponding *orbit-Lyapunov* foliation.

**Proof.** We will verify (6.7). This relies on the uniqueness part of the previous proposition. If we denote

$$y = \beta(\mathbf{t}, x) = \alpha(\mathbf{g}(x, \mathbf{t}))x$$

we can rewrite the right side of (6.7) as

$$(6.9) \quad \begin{aligned} \beta(\mathbf{s}, \beta(\mathbf{t}, x)) &= \beta(\mathbf{s}, y) = \alpha(\mathbf{g}(y, \mathbf{s}))y = \\ \alpha(\mathbf{g}(y, \mathbf{s})) \circ \alpha(\mathbf{g}(x, \mathbf{t}))x &= \alpha(\mathbf{g}(y, \mathbf{s}) + \mathbf{g}(x, \mathbf{t}))x = \\ \alpha(\mathbf{s} + \mathbf{t} + (g(y, \mathbf{s}) + g(x, \mathbf{t}))\mathbf{w})x. \end{aligned}$$

From this equation we see that the point  $\beta(\mathbf{s}, \beta(\mathbf{t}, x))$  belongs to the  $\{t\mathbf{w}\}$ -orbit of  $\alpha(\mathbf{s} + \mathbf{t})x$ . By definition, the point  $\beta(\mathbf{s} + \mathbf{t}, x)$  also belongs to this orbit, moreover, it is the unique point of the form  $\alpha(\mathbf{s} + \mathbf{t} + g\mathbf{w})x$  for which

$$\|D_x^E \alpha(\mathbf{s} + \mathbf{t} + g\mathbf{w})\|_\varepsilon = e^{\chi(\mathbf{s} + \mathbf{t})}.$$

On the other hand, equation (6.9) and the definition of  $\mathbf{g}(y, \mathbf{s})$  and  $\mathbf{g}(x, \mathbf{t})$  imply that

$$\begin{aligned} \|D_x^E \alpha(\mathbf{s} + \mathbf{t} + (g(y, \mathbf{s}) + g(x, \mathbf{t}))\mathbf{w})\|_\varepsilon &= \\ \|D_y^E \alpha(\mathbf{g}(y, \mathbf{s}))\|_\varepsilon \cdot \|D_x^E \alpha(\mathbf{g}(x, \mathbf{t}))\|_\varepsilon &= e^{\chi(\mathbf{s})} \cdot e^{\chi(\mathbf{t})}. \end{aligned}$$



Thus we conclude that the points  $\beta(\mathbf{s}, \beta(\mathbf{t}, x))$  and  $\beta(\mathbf{s} + \mathbf{t}, x)$  coincide, i.e. (6.7). In particular, we obtain

$$g(x, \mathbf{s} + \mathbf{t}) = g(x, \mathbf{t}) + g(\beta(\mathbf{t}, x), \mathbf{s})$$

which gives equation (6.8).  $\square$

## 6.2. Properties of the time change.

**Proposition 6.4.** *The time change  $\mathbf{g}(x, \mathbf{t})$  is Hölder continuous in  $x$  with Hölder exponent  $\gamma$  on any Pesin set  $\mathcal{R}_\varepsilon^l$ . The Hölder constant depends on the Pesin set and can be chosen uniform in  $\mathbf{t}$  for any compact subset of  $\mathbb{R}^k$ .*

**Proof.** We fix  $\mathbf{t} \in \mathbb{R}^k$  and two nearby points  $x$  and  $y$  in a Pesin set  $\mathcal{R}_\varepsilon^{l'}$ . We take  $l = l' \exp(\varepsilon \|\mathbf{t}\|)$  and note that by Proposition 2.1 (3) we have  $\alpha(\mathbf{t})(\mathcal{R}_\varepsilon^{l'}) \subset \mathcal{R}_\varepsilon^l$ . Hence the points  $x$ ,  $y$ ,  $\alpha(\mathbf{t})x$ , and  $\alpha(\mathbf{t})y$  are all in the Pesin set  $\mathcal{R}_\varepsilon^l$ . To prove the Hölder continuity of  $\mathbf{g}(x, \mathbf{t})$  we need to show that  $|g(x, \mathbf{t}) - g(y, \mathbf{t})|$  are Hölder close with respect to the distance between  $x$  and  $y$ , i.e. can be estimated from above by a constant multiple of a power of  $\rho = \text{dist}(x, y)$ .

First we show that  $\|D_x^E \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})\|_\varepsilon$  and  $\|D_y^E \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})\|_\varepsilon$  are Hölder close in  $\rho$ . This can be seen as follows. Since the action  $\alpha$  is smooth, the points  $\alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w}, x)$  and  $\alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w}, y)$  as well as the derivatives  $D_x \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})$  and  $D_y \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})$  are Hölder close in  $\rho$  with constant depending only on the action and  $\|\mathbf{t}\|$ . Also, by Proposition 2.2 the distribution  $E$  is Hölder continuous in  $\rho$  on the Pesin set  $\mathcal{R}_\varepsilon^l$ . Finally, the Lyapunov metric is Hölder continuous in  $\rho$  on  $\mathcal{R}_\varepsilon^l$  with the Hölder exponent  $\gamma$  by Proposition 5.3 and its ratio to a smooth metric is uniformly bounded above and below on  $\mathcal{R}_\varepsilon^l$  by (5.4) and (5.5). We conclude that

$$(6.10) \quad \left| \|D_x^E \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})\|_\varepsilon - \|D_y^E \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})\|_\varepsilon \right| \leq K_1 \rho^\gamma$$

By the definition of  $\mathbf{g}(x, \mathbf{t}) = \mathbf{t} + g(x, \mathbf{t})\mathbf{w}$  we have

$$(6.11) \quad \|D_x^E \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})\|_\varepsilon = e^{\chi(\mathbf{t})} = \|D_y^E \alpha(\mathbf{t} + g(y, \mathbf{t})\mathbf{w})\|_\varepsilon.$$

Then (6.10) and the first equality in (6.11) imply that

$$\left| \|D_y^E \alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w})\|_\varepsilon - e^{\chi(\mathbf{t})} \right| \leq K_1 \rho^\gamma.$$

We note that the points  $\alpha(\mathbf{t} + g(x, \mathbf{t})\mathbf{w}, y)$  and  $\alpha(\mathbf{t} + g(y, \mathbf{t})\mathbf{w}, y)$  are on  $\{t\mathbf{w}\}$ -orbit of point  $\alpha(\mathbf{t}, y)$ , and that the value  $g(y, \mathbf{t})$  is determined by the second equality in (6.11). Therefore, the difference  $g(y, \mathbf{t}) - g(x, \mathbf{t})$  represents the time adjustment in  $\mathbf{w}$  direction required to bring the norm  $\|D_y^E \alpha(\mathbf{t} + s\mathbf{w})\|_\varepsilon$  from being  $K_1 \rho^\gamma$ -close to  $e^{\chi(\mathbf{t})}$  to being exactly  $e^{\chi(\mathbf{t})}$ . Recall that by Lemma 6.1 the norm  $\|D_y^E \alpha(\mathbf{t} + s\mathbf{w})\|_\varepsilon$  varies smoothly with  $s$  (see equation (6.6) in the proof of Proposition 6.2). Thus we conclude that  $|g(y, \mathbf{t}) - g(x, \mathbf{t})| \leq K_2 \rho^\gamma$ .  $\square$

**Proposition 6.5.** *The time change  $\mathbf{g}(x, \mathbf{t}) = \mathbf{t} + g(x, \mathbf{t})\mathbf{w}$  is differentiable and  $C^1$  close to identity in  $\mathbf{t}$ . More precisely, for a.e.  $x$*

$$\left\| \frac{\partial g}{\partial \mathbf{t}}(x, \mathbf{t}) \right\| \leq \varepsilon.$$

**Proof.** We fix a regular point  $x$  and vectors  $\mathbf{t}, \mathbf{e}$  in  $\mathbb{R}^k$  and consider a function of two real variables

$$(6.12) \quad \Phi(s, g) = f_x(\mathbf{t} + s\mathbf{e} + g\mathbf{w}) - \chi(\mathbf{t} + s\mathbf{e}).$$

We note that, by Proposition 6.2,  $g(s) = g(x, \mathbf{t} + s\mathbf{e})$  is the unique solution for the implicit function equation  $\Phi(s, g(s)) = 0$ .

Using Lemma 6.1 we obtain

$$\frac{\partial \Phi}{\partial g}(s, g) = (D_{\mathbf{t} + s\mathbf{e} + g\mathbf{w}} f_x) \mathbf{w} = \chi(\mathbf{w}) + \varepsilon \psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{w}),$$

where  $|\psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{w})| \leq \frac{1}{2}\|\mathbf{w}\|$  and is continuous in  $s$  and  $g$ . Similarly,

$$\frac{\partial \Phi}{\partial s}(s, g) = \chi(\mathbf{e}) + \varepsilon \psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{e}) - \chi(\mathbf{e}) = \varepsilon \psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{e}).$$

We conclude that  $\Phi$  is a  $C^1$  function of  $(s, g)$ . Moreover, since  $\chi(\mathbf{w}) = 1$  and  $\varepsilon$  is small we obtain

$$\frac{\partial \Phi}{\partial g}(s, g) = 1 + \varepsilon \psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{w}) \geq 1 - \varepsilon\|\mathbf{w}\|/2 > 0,$$

Therefore, by implicit function theorem,  $g(s)$  is differentiable and

$$g'(s) = - \left( \frac{\partial \Phi}{\partial s}(s, g(s)) \right) \left( \frac{\partial \Phi}{\partial g}(s, g(s)) \right)^{-1} = \frac{-\varepsilon \psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{e})}{1 + \varepsilon \psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\mathbf{w})}.$$

Moreover, since  $|\psi_{(\mathbf{t} + s\mathbf{e} + g\mathbf{w})x}(\cdot)| \leq \frac{1}{2}\|\cdot\|$  we obtain

$$|g'(s)| \leq \frac{\varepsilon\|\mathbf{e}\|}{2 - \varepsilon\|\mathbf{w}\|} \leq \varepsilon\|\mathbf{e}\|$$

provided that  $\varepsilon\|\mathbf{w}\| < 1$ . Since  $g(s) = g(x, \mathbf{t} + s\mathbf{e})$  we have

$$\left( \frac{\partial g}{\partial \mathbf{t}}(x, \mathbf{t}) \right) \mathbf{e} = g'(s)$$

and thus the partial derivatives of  $g(x, \mathbf{t})$  in the second variable exist and are continuous in  $\mathbf{t}$ . We conclude that  $g(x, \mathbf{t})$  is  $C^1$  in  $\mathbf{t}$  with  $\left\| \frac{\partial g}{\partial \mathbf{t}}(x, \mathbf{t}) \right\| \leq \varepsilon$ .  $\square$

**6.3. Properties of the action  $\beta$ .** We note that the new action  $\beta$  is not smooth. Hence the notions and results of nonuniformly hyperbolic theory do not apply to  $\beta$  formally. In particular, such objects as derivatives, Lyapunov distributions, Lyapunov exponents, and Lyapunov hyperplanes will always refer to the ones of the original action  $\alpha$ . However, the new action  $\beta$  inherits some structures of  $\alpha$  such as invariant measure and invariant “foliations” which are close to those of  $\alpha$ . This is described in the following two statements. We will use these structures in the next section to obtain some important transitivity properties of  $\beta$ .

**Proposition 6.6.** *The determinant of the time change  $\mathbf{g}(x, \mathbf{t})$*

$$\Delta(x) = \det \left( \frac{\partial \mathbf{g}}{\partial \mathbf{t}}(x, \mathbf{0}) \right)$$

*is a measurable function which is  $L^\infty$  close to constant 1 on  $M$ . Therefore, the new action  $\beta$  preserves an invariant measure  $\nu$  which is absolutely continuous with respect to  $\mu$  (and equivalent to  $\mu$ ) with density  $\frac{d\nu}{d\mu} = \Delta(x)^{-1}$ .*

**Proof.** The  $L^\infty$  estimate for the determinant follows immediately from the fact that by Proposition 6.5 for a.e.  $x$

$$\left\| \frac{\partial \mathbf{g}(x, \mathbf{t})}{\partial \mathbf{t}} - \text{Id} \right\| \leq \varepsilon.$$

Then the existence of the invariant measure  $\nu$  for  $\beta$  follows from [11].  $\square$

We will denote by  $\mathcal{N}$  the orbit foliation of the one-parameter subgroup  $\{t\mathbf{w}\}$ .

**Proposition 6.7.** *For any element  $\mathbf{s} \in \mathbb{R}^k$  there exists stable “foliation”  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-$  which is contracted by  $\beta(\mathbf{s})$  and invariant under the new action  $\beta$ . It consists of “leaves”  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)$  defined for almost every  $x$ . The “leaf”  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)$  is a measurable subset of the leaf  $(\mathcal{N} \oplus \mathcal{W}_{\alpha(\mathbf{s})}^-)(x)$  of the form*

$$\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x) = \{\alpha(\varphi_x(y)\mathbf{w})y : y \in \mathcal{W}_{\alpha(\mathbf{s})}^-(x)\},$$

where  $\varphi_x : \mathcal{W}_{\alpha(\mathbf{s})}^-(x) \rightarrow \mathbb{R}$  is an almost everywhere defined measurable function. For  $x$  in a Pesin set the  $\varphi_x$  is Hölder continuous on the intersection of this Pesin set with any ball of fixed radius in  $\mathcal{W}_{\alpha(\mathbf{s})}^-(x)$  with Hölder exponent  $\gamma$  and constant which depends on the Pesin set and radius.

**Proof.** We will give an explicit formula for the function  $\varphi_x$  in terms of the time change so that its graph is contracted by  $\beta(\mathbf{s})$ . The calculation is similar to finding stable manifolds for a time change of a flow. The Hölder continuity of  $\varphi_x$  will follow from the formula and the Hölder continuity of the time change. Since  $\mathcal{W}_{\alpha(\mathbf{s})}^-$  is invariant under  $\alpha$ , we note that  $\mathcal{N} \oplus \mathcal{W}_{\alpha(\mathbf{s})}^-$  is invariant under  $\beta$  by the construction of the time change. Since  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)$  is clearly characterized within  $\mathcal{N} \oplus \mathcal{W}_{\alpha(\mathbf{s})}^-(x)$  by the contraction property and since  $\beta(\mathbf{t})$  is continuous on Pesin sets, the usual argument yields that for  $\mu$ -a.e. regular point  $x$  we have  $\beta(\mathbf{t})(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)) = \tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(\beta(\mathbf{t})x) \pmod{0}$ . Thus we obtain the invariance of  $\tilde{\mathcal{W}}$  under the whole action  $\beta$ .

Let  $x$  and  $y \in \mathcal{W}_{\alpha(\mathbf{s})}^-(x)$  be in a Pesin set  $\mathcal{R}_\varepsilon^l$ . We denote  $x_0 = x$  and

$$x_n = \beta(\mathbf{s}, x_{n-1}) = \beta(n\mathbf{s}, x) = \alpha(\mathbf{s}_n)x, \quad \text{where } \mathbf{s}_n = n\mathbf{s} + g(x, n\mathbf{s})\mathbf{w}. \quad (1)$$

Since points  $y$  and  $x_n$ ,  $n \geq 1$ , are in the same orbit-stable leaf  $\mathcal{O} \oplus \mathcal{W}_{\alpha(\mathbf{s})}^-(x)$  we can define  $y_n$  to be the intersection of the orbit of  $y$  with  $\mathcal{W}_{\alpha(\mathbf{s})}^-(x_n)$ .

Since all points  $y_n$ ,  $n \geq 1$ , are on the orbit of  $y$ , we can represent  $y_{n+1}$  as  $\beta(\mathbf{s} + \mathbf{t}_n, y_n)$  for some  $\mathbf{t}_n \in \mathbb{R}^k$  and write

$$y_n = \beta(\mathbf{s} + \mathbf{t}_{n-1}, y_{n-1}) = \dots = \beta(n\mathbf{s} + (\mathbf{t}_0 + \dots + \mathbf{t}_{n-1}), y) = \alpha(\mathbf{s}_n)y. \quad (2)$$

The last equality follows from invariance of  $\mathcal{W}_{\alpha(\mathbf{s})}^-$  under  $\alpha$  which gives that  $\alpha(\mathbf{s}_n)y$  is on  $\mathcal{W}_{\alpha(\mathbf{s})}^-$  leaf of  $x_n = \alpha(\mathbf{s}_n)x$  and thus coincides with  $y_n$  by definition. Recall that by Proposition 6.2 the function  $g$  satisfies

$$|g(z, \mathbf{t})| \leq 2\varepsilon \|\mathbf{t}\| \quad (3)$$

for any regular point  $z$  and  $\mathbf{t} \in \mathbb{R}^k$ . Hence the sequence  $\mathbf{s}_n = n\mathbf{s} + g(x, n\mathbf{s})\mathbf{w}$  remains in a narrow cone around the direction of  $\mathbf{s}$ . We conclude that diffeomorphisms  $\alpha(\mathbf{s}_n)$  contract the stable manifold  $\mathcal{W}_{\alpha(\mathbf{s})}^-(x)$  exponentially and thus

$$\text{dist}(x_n, y_n) = \text{dist}(\alpha(\mathbf{s}_n)x, \alpha(\mathbf{s}_n)y) \leq K_1 e^{-n\chi} \text{dist}(x, y) \quad (4)$$

for some  $\chi > 0$  which can be chosen close to the slowest contraction rate for  $\alpha(\mathbf{s})$ .

Next we will show that the series  $\mathbf{t} = \sum_{i=0}^{\infty} \mathbf{t}_i$  converges exponentially so that according to (2) we have  $\text{dist}(y_n, \beta(n\mathbf{s} + \mathbf{t}, y)) \rightarrow 0$  exponentially. Combining this with (4) we obtain that for  $\tilde{y} = \beta(\mathbf{t}, y)$

$$\text{dist}(\beta(n\mathbf{s}, x), \beta(n\mathbf{s}, \tilde{y})) = \text{dist}(x_n, \beta(n\mathbf{s} + \mathbf{t}, y)) \rightarrow 0 \quad (5)$$

exponentially and thus  $\tilde{y}$  belongs to the stable “leaf”  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)$ .

To show that the series  $\mathbf{t} = \sum_{i=0}^{\infty} \mathbf{t}_i$  converges we estimate  $\mathbf{t}_n$  as follows. Similarly to the last equalities in (1) and (2) we can write

$$x_{n+1} = \alpha(\mathbf{s} + g(x_n, \mathbf{s})\mathbf{w})x_n \quad \text{and} \quad y_{n+1} = \alpha(\mathbf{s} + g(y_n, \mathbf{s})\mathbf{w})y_n. \quad (6)$$

Denoting  $\mathbf{t}'_n = (g(x_n, \mathbf{s}) - g(y_n, \mathbf{s}))\mathbf{w}$  we obtain using (2) and (6) that

$$\begin{aligned} \beta(\mathbf{t}_n)\beta(\mathbf{s}, y_n) &= \beta(\mathbf{s} + \mathbf{t}_n, y_n) = y_{n+1} = \alpha(\mathbf{s} + g(y_n, \mathbf{s})\mathbf{w})y_n = \\ \alpha(\mathbf{t}'_n + \mathbf{s} + g(y_n, \mathbf{s})\mathbf{w})y_n &= \alpha(\mathbf{t}'_n)\alpha(\mathbf{s} + g(y_n, \mathbf{s})\mathbf{w})y_n = \alpha(\mathbf{t}'_n)\beta(\mathbf{s}, y_n) \end{aligned}$$

This shows that  $\mathbf{t}_n$  is uniquely determined by the following equations

$$\alpha(\mathbf{t}'_n)z_n = \beta(\mathbf{t}_n)z_n \quad \text{or} \quad \mathbf{t}_n + g(z_n, \mathbf{t}_n)\mathbf{w} = \mathbf{t}'_n = (g(x_n, \mathbf{s}) - g(y_n, \mathbf{s}))\mathbf{w},$$

where  $z_n = \beta(\mathbf{s}, y_n)$ . Using (3) we conclude that  $\mathbf{t}_n$  is a vector parallel to  $\mathbf{w}$  whose length satisfies

$$c t'_n \leq \|\mathbf{t}_n\|_{\mathbb{R}^k} \leq C t'_n, \quad \text{where} \quad t'_n = |g(x_n, \mathbf{s}) - g(y_n, \mathbf{s})|.$$

Thus we need to investigate the convergence of the series

$$t' = \sum_{n=0}^{\infty} t'_n = \sum_{n=0}^{\infty} |g(x_n, \mathbf{s}) - g(y_n, \mathbf{s})|.$$

By Proposition 6.4 the function  $g$  is Hölder continuous with exponent  $\gamma$  and constant depending on the Pesin set. For  $x$  and  $y$  in the Pesin set  $\mathcal{R}_\varepsilon^l$ ,  $x_n$  and  $y_n$  are in another Pesin set  $\mathcal{R}_\varepsilon^{l'}$  for which the Hölder constant deteriorates from that of  $\mathcal{R}_\varepsilon^l$  by a factor at most  $\exp(p\varepsilon\|\mathbf{s}_n\|) \leq \exp(2p\varepsilon n)$

(see Remark after Proposition 5.3). Replacing  $\mathbf{s}$  by its multiple if necessary we may assume without loss of generality that  $\chi > 2p\varepsilon$ . Then (4) implies that the series converges exponentially and its sum satisfies

$$t' \leq K_2 \text{dist}(x, y)^\gamma.$$

This completes the proof of (5) and shows that  $\tilde{y} \in \tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)$ , where  $\tilde{y} = \beta(\mathbf{t}, y)$  with  $\mathbf{t} = t\mathbf{w}$  and  $\|\mathbf{t}\| \leq Ct' \leq K_3 \text{dist}(x, y)^\gamma$ . Hence  $\tilde{y}$  can be represented as

$$\tilde{y} = \alpha(\varphi_x(y)\mathbf{w})y, \quad \text{where } |\varphi_x(y)| \leq K_4 \text{dist}(x, y)^\gamma.$$

We conclude that  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-(x)$  is of the form stated in the proposition. The Hölder continuity of  $\varphi_x$  with Hölder exponent  $\gamma$  can be obtained similarly to the Hölder estimate for  $\varphi_x$  in the previous equation. The constant  $K_4$  depends on the Pesin set  $\mathcal{R}_\varepsilon^t$ .  $\square$

The corresponding unstable “foliation”  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+$  can be obtained as  $\tilde{\mathcal{W}}_{\beta(-\mathbf{s})}^-$ . Since the foliation  $\mathcal{W}$  corresponding to the Lyapunov distribution  $E$  is an intersection of stable foliations we obtain the following corollary.

**Corollary 6.8.** *For the Lyapunov foliation  $\mathcal{W}$  corresponding to the distribution  $E$  of the original action  $\alpha$  there exists “foliation”  $\tilde{\mathcal{W}}$  invariant under the new action  $\beta$  of the form described in Proposition 6.7.*

The foliation  $\tilde{\mathcal{W}}$  will be referred to as the Lyapunov foliation of  $\beta$  corresponding to the Lyapunov distribution  $E$ .

**6.4. Recurrence argument for the time change.** Recall that an element  $\mathbf{t} \in \mathbb{R}^k$  is generic singular if it belongs to exactly one Lyapunov hyperplane. We consider a generic singular element  $\mathbf{t}$  in the Lyapunov hyperplane  $L$ . Our goal is to show that for a typical point  $x$  the limit points of the orbit  $\beta(n\mathbf{t})$ ,  $n \in \mathbb{N}$ , contain the support of the conditional measure of  $\nu$  on the leaf  $\tilde{\mathcal{W}}(x)$ . More precisely, we prove the following lemma which is an adaptation of an argument from [20] for the current setting.

We say that partition  $\xi_1$  is *coarser* than  $\xi_2$  (or that  $\xi_2$  *refines*  $\xi_1$ ) and write  $\xi_1 < \xi_2$  if  $\xi_2(x) \subset \xi_1(x)$  for a.e.  $x$ .

**Proposition 6.9.** *For any generic singular element  $\mathbf{t} \in L$  the partition  $\xi_{\beta(\mathbf{t})}$  into ergodic components of  $\nu$  with respect to  $\beta(\mathbf{t})$  is coarser than the measurable hull  $\xi(\tilde{\mathcal{W}})$  of the foliation  $\tilde{\mathcal{W}}$ .*

**Proof.** For a generic singular element  $\mathbf{t}$  in  $L$ ,  $\chi$  is the only non-trivial Lyapunov exponent that vanishes on  $\mathbf{t}$ . As in Section 4.1 we take a regular element  $\mathbf{s}$  close to  $\mathbf{t}$  for which  $\chi(\mathbf{t}) > 0$  and all other non-trivial exponents have the same signs as for  $\mathbf{t}$ . Then  $E_{\alpha(\mathbf{s})}^- = E_{\alpha(\mathbf{t})}^-$  and  $E_{\alpha(\mathbf{s})}^+ = E_{\alpha(\mathbf{t})}^+ \oplus E$ . Consequently, for the action  $\beta$  we have  $\xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-) = \xi(\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^-)$  and  $\xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+) < \xi(\tilde{\mathcal{W}}_{\alpha(\mathbf{t})}^+)$ . Birkhoff averages with respect to  $\beta(\mathbf{t})$  of any continuous function

are constant on the leaves of  $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^-$ . Since such averages generate the algebra of  $\beta(\mathbf{t})$ -invariant functions, we conclude that the partition  $\xi_{\beta(\mathbf{t})}$  into the ergodic components of  $\beta(\mathbf{t})$  is coarser than  $\xi(\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^-)$ , the measurable hull of the foliation  $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^-$ . The equality  $\xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+) = \xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-)$  is proved in the next proposition, so we conclude that

$$(6.13) \quad \xi_{\beta(\mathbf{t})} < \xi(\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^-) = \xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-) = \xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+) < \xi(\tilde{\mathcal{W}}).$$

□

**Remark.** Equalities in (6.13) represent the “ $\pi$ -partition trick” which first appeared in [20] in the setting of actions by automorphisms of a torus. Absence of Lyapunov exponents *negatively* proportional to  $\chi$  is necessary for this argument to work. If this condition holds for all exponents (other than the trivial one corresponding to the orbit directions) the action is called *totally non-symplectic (TNS)*. On the other hand, presence of exponents *positively* proportional to  $\chi$ , e.g. non-simplicity of  $\chi$  itself, forces considering multidimensional coarse Lyapunov foliations, corresponding to all exponents positively proportional to  $\chi$ . Naturally one cannot hope any more to have the dichotomy of atomic vs. absolutely continuous but nevertheless under additional assumptions the  $\pi$ -partition trick still works and allows to make conclusions about conditional measures.

The (long) remainder of this section is dedicated to the justification of the “ $\pi$ -partition trick” in our setting.

**Proposition 6.10.**  $\xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+) = \xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^-) = \pi(\beta(\mathbf{s}))$ , the  $\pi$ -partition of  $\beta(\mathbf{s})$ .

**Proof.** We will show that  $\xi(\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+) = \pi(\beta(\mathbf{s}))$ . The other equality is obtained in the same way. We note that in the case of diffeomorphisms this result is given by Theorem B in [24]. Also, for the case of a hyperbolic measure this result was established earlier in [22, Theorem 4.6]. In our case, although some zero Lyapunov exponents appear, they correspond to the orbit direction, so that the central direction will be easier to control than in [24]. We will follow [22] and [24], so let us give a sketch of the proof in their case.

6.4.1. *Sketch of the proof in [22, 24].* In what follows,  $f$  will be a diffeomorphism preserving a measure  $\mu$  with unstable foliation  $\mathcal{W}^+$  and local unstable manifolds  $W^+$ . The idea is to use the following criterium due to Rokhlin [29] (Theorems 12.1 and 12.3). Given a partition  $\xi$  we denote by  $\mathcal{M}_\xi$  the  $\sigma$ -algebra generated by  $\xi$ .

**Theorem 6.11.** *Let  $f$  be a measure preserving transformation and assume  $\xi$  is an increasing partition, i.e  $\xi > f\xi$ , satisfying:*

- (1)  $\bigvee_{n=0}^{\infty} f^{-n}\xi$  is the partition into points,
- (2)  $h(f) = h(f, \xi) < \infty$ .

*Then the Pinsker  $\sigma$ -algebra coincides mod 0 with the  $\sigma$ -algebra  $\bigcap_{n=0}^{\infty} \mathcal{M}_{f^n\xi}$ .*

For two partitions  $\eta$  and  $\xi$ ,

$$H(\eta|\xi) = - \int \log \mu_x^\xi(\eta(x)) d\mu(x),$$

where  $\mu^\xi$  are the conditional measures associated to the measurable partition  $\xi$ . And for an increasing partition  $\xi$ ,  $h(f, \xi) = H(f^{-1}\xi|\xi)$ . Also,  $H(\eta) = H(\eta|\tau)$  where  $\tau$  is the trivial partition. We shall make use of the following known formulas of the conditional entropy. Given partitions  $\eta, \xi, \zeta$ ,

- (1)  $H(\zeta \vee \xi|\eta) = H(\zeta|\eta) + H(\xi|\zeta \vee \eta)$ ,
- (2) If  $\eta > \xi$  then for any  $\zeta$ ,  $H(\zeta|\eta) \leq H(\zeta|\xi)$ ,
- (3)  $H(\zeta|\xi) \geq H(\eta|\xi) - H(\eta|\zeta)$ .

Also we shall make use of the following Lemma whose proof is left to the reader.

**Lemma 6.12.** *Let  $\xi$  and  $\eta_n$  be measurable partitions and assume that if  $D_n = \{x : \xi(x) \subset \eta_n(x)\}$  then  $\mu(D_n) \rightarrow 1$ . Then for any  $\zeta$ ,*

$$\lim H(\zeta|\xi \vee \eta_n) = H(\zeta|\xi)$$

We say that a partition  $\xi$  is subordinated to  $W^+$  if  $U_x \subset \xi(x) \subset W^+(x)$  for a.e.  $x$ , where  $U_x$  is some open neighborhood of  $x$  in  $W^+(x)$ . In [22, 24] first a partition  $\xi$  is constructed as follows,

**Lemma 6.13.** [24, Lemma 3.1.1.] *There exists a measurable partition  $\xi$  with the following properties,*

- (1)  $\xi$  is an increasing partition subordinated to  $W^u$ ,
- (2)  $\bigvee_{n=0}^{\infty} f^{-n}\xi$  is the partition into points,
- (3)  $\bigcap_{n=0}^{\infty} \mathcal{M}_{f^n\xi} = \mathcal{M}_{\xi(W^u)}$ .

Partitions of this type were used by Sinai [31] to study uniformly hyperbolic systems and were built in the general context in [23, Proposition 3.1.], (see also [22, Proposition 3.1.]). Then it is proven that hypothesis (2) of Theorem 6.11 is satisfied by any such partition.

In [22] and [24] the proof that hypothesis (2) of Theorem 6.11 is satisfied by any of these partitions is in various steps. On one hand, the following lemma is proven.

**Lemma 6.14.** [24, Lemma 3.1.2.] *For any two partitions  $\xi_1$  and  $\xi_2$  built in Lemma 6.13,  $h(f, \xi_1) = h(f, \xi_2)$ .*

On the other hand, it is proven that the entropy of the partitions  $\xi$  built in Lemma 6.13 approach  $h(f)$ . To this end, it is built a countable partition  $\mathcal{P}$ , with finite entropy, i.e.,  $H(\mathcal{P}) < \infty$  and such that  $h(f, \mathcal{P})$  is close to  $h(f)$ . Then  $h(f, \mathcal{P})$  and  $h(f, \xi)$  are compared where  $\xi$  is a partition built in Lemma 6.13. It is in comparing this two entropies where the proof in [22] and [24] differ. In [22] the comparison follows from the properties of  $\xi$  and that  $\mathcal{P}^+ := \bigvee_{n=0}^{\infty} f^n\mathcal{P}$  refines  $\xi$  (this is done in the proof of [22, Proposition 4.5.]) while in [24] more work is needed because  $\mathcal{P}^+$  does not a priori refines  $\xi$  due

to the presence of zero exponents. However, in our case, although some zero exponents appear, they correspond to the orbit direction, so that  $\mathcal{P}^+$  will essentially refine  $\xi$  because of the properties of the partition  $\mathcal{P}$ . That is why we are somehow closer to the proof in [22]. Finally, the proof ends because we can take the partition  $\mathcal{P}$  with entropy as close to  $h(f)$  as wanted.

**6.4.2. Proof in our case.** Let us go now to the proof in our case. We will follow the above sketch, but now  $f := \beta(\mathbf{s})$  will not be a diffeomorphism, so we need to take some care. Let  $\tilde{\mathcal{W}} := \tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+$  and  $\tilde{W} := \tilde{W}_{\beta(\mathbf{s})}^+$  be the global and local unstable "manifolds" built in Proposition 6.7. As the global "manifold"  $\tilde{\mathcal{W}}(x)$  is a graph over  $\mathcal{W}_{\alpha(\mathbf{s})}^+(x)$ , the local "manifold"  $\tilde{W}(x)$  is the restriction of this graph to  $\mathcal{W}_{\alpha(\mathbf{s})}^+(x)$ .

As the main contraction/expansion properties of  $\tilde{\mathcal{W}}$  comes from the contraction/expansion properties of  $\mathcal{W}_{\alpha(\mathbf{s})}^+(x)$ , we will mostly measure the distances between points in  $\tilde{\mathcal{W}}$  projecting them into  $\mathcal{W}_{\alpha(\mathbf{s})}^+(x)$ . Thus, let us define  $\pi_x : \tilde{\mathcal{W}}(x) \rightarrow \mathcal{W}_{\alpha(\mathbf{s})}^+(x)$  the projection and observe that the restriction of  $\pi_x$  to  $\tilde{W}(x)$  is Lipschitz continuous, with Lipschitz constant depending only on the Pesin set  $x$  belongs to. Observe also that the inverse of  $\pi_x$  is not Lipschitz. Let us define for  $z, y \in \mathcal{W}(x)$ ,  $\tilde{d}_x(y, z) = d(\pi_x(y), \pi_x(z))$ .

Let us begin now with a useful Lemma,

**Lemma 6.15.** *If  $\eta$  is an increasing partition and  $\eta(x) \subset \tilde{W}(x)$  for a.e.  $x$  then the sequence of partitions  $\{f^{-n}\eta\}$  generates, i.e.  $\bigvee_{n=0}^{\infty} f^{-n}\eta$  is the partition into points.*

**Proof.** Let us see that for a.e.  $x$ , if  $y \in (f^{-n}\eta)(x)$  for every  $n$  then  $x = y$ . We have that a.e.  $x$  belongs infinitely many times to some fixed Pesin set, say  $R$ . Take  $n_i$  the sequence of integers such that  $f^{n_i}(x) \in R$ . So we have that  $y \in f^{-n_i}(\eta(f^{n_i}(x))) \subset f^{-n_i}(\tilde{W}(f^{n_i}(x)))$  for every  $i$ . Since  $f^{n_i}(x)$  is in  $R$ , we have that the projected diameter of  $\tilde{W}(f^{n_i}(x))$  is uniformly bounded and that the projected diameter of  $f^{-n_i}(\tilde{W}(f^{n_i}(x)))$  tends to 0. Hence, since  $\pi_x(y) \in \pi_x(f^{-n_i}(\tilde{W}(f^{n_i}(x))))$ , the distance between the projection of  $y$  into  $\mathcal{W}_{\alpha(\mathbf{s})}^+(x)$  with  $x$  is 0, this means that in fact  $y$  is in the orbit of  $x$ . But this is only possible if  $x = y$  since  $y \in \tilde{W}(x)$ .  $\square$

Recall that by Proposition 6.7 we have that  $\tilde{\mathcal{W}}(x) = \{\alpha(\varphi_x(y)\mathbf{w})y : y \in \mathcal{W}_{\alpha(\mathbf{s})}^+(x)\}$ . Following the above philosophy, let us say that a partition  $\xi$  is subordinated to  $\tilde{W}$  if  $\xi(x) \subset \tilde{W}(x)$  for a.e.  $x$ , and there is an open neighborhood  $U_x \subset \mathcal{W}_{\alpha(\mathbf{s})}^+(x)$  such that  $\{\alpha(\varphi_x(y)\mathbf{w})y : y \in U_x\} \subset \xi(x)$ .

Here again, we will use the criterium in Theorem 6.11 to prove Proposition 6.10, that is to prove that the Pinsker  $\sigma$ -algebra coincides with the  $\sigma$ -algebra generated by the "foliation"  $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^+$ . So that let us build partitions like the ones in Lemma 6.13.



**Lemma 6.16.** *There exists a measurable partition  $\xi$  with the following properties:*

- (1)  $\xi$  is an increasing partition subordinated to  $\tilde{W}$ ,
- (2)  $\bigvee_{n=0}^{\infty} f^{-n}\xi$  is the partition into points,
- (3)  $\bigcap_{n=0}^{\infty} \mathcal{M}_{f^n\xi} = \mathcal{M}_{\xi(\tilde{W})}$ .

**Proof.** Let us take  $\hat{\xi}$ , the measurable partition built in Lemma 6.13 for  $\alpha(s)$ , and define the partition  $\xi$  as the graph over  $\hat{\xi}(x)$ :

$$\xi(x) = \{\alpha(\varphi_x(y)\mathbf{w})y : y \in \hat{\xi}(x)\}.$$

Let us see that this is a partition that satisfies the three properties. Property (1) follows by definition and because  $\hat{\xi}$  satisfies also property (1). Property (2) follows from Lemma 6.15. Observe that property (3) is the same as proving that  $\bigwedge_{n=0}^{\infty} f^n\xi = \xi(\tilde{W})$ . Notice that  $\bigwedge_{n=0}^{\infty} f^n\xi$  is the graph over  $\bigwedge_{n=0}^{\infty} f^n\hat{\xi}$  which equals  $\xi(\mathcal{W}_{\alpha(s)}^+)$  by property (3) of Lemma 6.13. So, since  $\xi(\tilde{W})$  is the graph over  $\xi(\mathcal{W}_{\alpha(s)}^+)$  we get property (3).  $\square$

The next step is to prove the analog of Lemma 6.14, in fact we prove a more general result. We follow the proof in [24, Lemma 3.1.1.].

**Lemma 6.17.** *If  $\xi$  is a partition built in Lemma 6.16, and  $\zeta$  is an increasing partition such that  $\zeta(x) \subset \tilde{W}(x)$ , then  $h(f, \zeta \vee \xi) = h(f, \zeta)$ .*

**Proof.** For  $n \geq 1$  we have

$$\begin{aligned} h(f, \zeta \vee \xi) &= h(f, \zeta \vee f^n\xi) = H(\zeta \vee f^n\xi | f\zeta \vee f^{n+1}\xi) \\ &= H(\zeta | f\zeta \vee f^{n+1}\xi) + H(\xi | f\xi \vee f^{-n}\zeta) \end{aligned}$$

As  $n \rightarrow \infty$ , the second term goes to 0 since  $\{f^{-n}\zeta\}$  generates by Lemma 6.15. So we want to show that  $H(\zeta | f\zeta \vee f^{n+1}\xi) \rightarrow H(\zeta | f\zeta)$ . To this end we shall make use of Lemma 6.12. So let  $D_n = \{x : (f\zeta)(x) \subset (f^{n+1}\xi)(x)\}$ . Since  $\zeta(x) \subset \tilde{W}(x)$ , and the projected diameter of  $\tilde{W}(x)$  into  $W_{\alpha(s)}^+(x)$  is finite a.e., we have that the projected diameter of  $(f^{-n}\zeta)(x)$  goes to 0. Hence, since  $\xi(x)$  contains a graph over an open neighborhood of  $x$  in  $W_{\alpha(s)}^+(x)$  we have that  $(f^{-n}\zeta)(x) \subset \xi(x)$  if  $n$  is big enough and hence  $\mu(D_n) \rightarrow 1$ . Now the lemma follows from Lemma 6.12 and the fact that  $h(f, \zeta) = H(\zeta | f\zeta)$  since  $\zeta$  is an increasing partition.  $\square$

**Corollary 6.18.** *For any two partitions  $\xi_1$  and  $\xi_2$  built in Lemma 6.16,  $h(f, \xi_1) = h(f, \xi_2)$ .*

It remains to show that the entropy of a partition built in Lemma 6.16 equals the entropy of  $f$ . We shall build a countable partition  $\mathcal{P}$  with finite entropy to compare  $h(f, \mathcal{P})$  with  $h(f, \xi)$  as in the sketch. To this end we shall use the following Lemma due to Mañé,

**Lemma 6.19.** [26, Lemma 2] *If  $\mu$  is a probability measure and  $0 < \psi < 1$  is such that  $\log \psi$  is  $\mu$  integrable, then there exists a countable partition  $\mathcal{P}$  with entropy  $H(\mathcal{P}) < +\infty$  such that  $\mathcal{P}(x) \subset B(x, \psi(x))$  for a.e.  $x$ .*

So let us construct a suitable function  $\psi$ . For a set  $A \subset M$  let us define  $\mathcal{O}_\varepsilon A = \{\alpha(\mathbf{t})(a) : a \in A; \|\mathbf{t}\| < \varepsilon\}$ . Let us fix a Pesin set  $R_1$  of positive measure and take  $R_0$  another Pesin set such that  $\mathcal{O}_\varepsilon R_1 \subset R_0$ . Arguing similar to Lemma 2.4.2. of [24] let us define a measurable function  $\psi : S \rightarrow \mathbb{R}^+$  by

$$\psi(x) = \begin{cases} \delta & \text{if } x \notin R_0 \\ \delta l_0^{-1} e^{-\lambda r(x)} & \text{if } x \in R_0 \end{cases}$$

Where  $r(x)$  is the smallest positive integer  $k > 0$  such that  $f^k(x) \in R_0$ ,  $\lambda$  and  $l_0 = l_{R_0}$  are the constants in Lemma 6.21 below and  $\delta$  is such that if  $x, y \in R_0$  and  $\text{dist}(x, y) < \delta$  then  $\mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^+(x) \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^-(y) \neq \emptyset$  and vice versa interchanging  $x$  and  $y$  for some  $\varepsilon > 0$  small that depends on the Pesin set (such  $\delta$  and  $\varepsilon$  exists by transversality and uniformity over Pesin sets). We will require other properties for that  $\delta$  later (see Lemma 6.20). Since  $\int_{R_0} r d\mu = 1$ , we get that  $\log \psi$  is integrable. We may assume also, by an appropriate choice of  $R_0$ , that  $\inf_{n \geq 0} \psi(f^{-n}(x)) = 0$  for a.e.  $x$ .

Hence, by Lemma 6.19, there is a partition  $\tilde{\mathcal{P}}$  such that  $H(\tilde{\mathcal{P}}) < \infty$  and  $\tilde{\mathcal{P}}(x) \subset B(x, \psi(x))$  for a.e.  $x$ . Take  $\hat{R}_1 \subset R_1$  such that if  $x, y \in \hat{R}_1$  and  $\text{dist}(x, y) < \delta$  then there is a point  $z \in R_1 \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^+(x) \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^-(y)$ . If  $R_1$  is taken of big enough measure, then there is such a set  $\hat{R}_1$  of positive measure. Let us define  $\mathcal{P} = \tilde{\mathcal{P}} \vee \{\hat{R}_1, S \setminus \hat{R}_1\}$  and recall that  $\mathcal{P}^+ = \bigvee_{n=0}^\infty f^n \mathcal{P}$ .

**Lemma 6.20.** *For some  $\delta > 0$  we have that  $\mathcal{P}^+(x) \subset \tilde{W}(x)$ ,  $x$  a.e.*

Before the proof of this Lemma, let us begin with a property of the  $\tilde{W}$  “manifolds” that, apart from invariance and uniformity over Pesin sets, simply reflects the Lipschitz property of the original map  $\alpha(\mathbf{s})$ .

**Lemma 6.21.** *There is  $\lambda > 0$  that depends on  $\mathbf{s}$ ,  $l = l_R > 0$  that depends on the Pesin set  $R$  such that for  $n > 0$ , and  $0 < \delta \leq l^{-1}$ , if  $z \in \tilde{W}(x)$ ,  $x \in R$  are such that  $\tilde{d}_x(x, z) < \delta e^{-n\lambda}$  then*

$$\tilde{d}_{f^n(x)}(f^n(x), f^n(z)) < \delta$$

and  $f^n(z) \in \tilde{W}(f^n(x))$ .

Let us go now into the proof of Lemma 6.20.

**Proof.** of Lemma 6.20. Let us see first that  $\mathcal{P}^+(x) \subset \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^+(x)$ ,  $x$  a.e. Let  $y \in \mathcal{P}^+(x)$ . Take the sequence of negative integers  $-n_i < -n_{i-1}$  such that  $x_{n_i} = f^{-n_i}(x) \in \hat{R}_1$ . Since  $y \in \mathcal{P}^+(x) = \bigvee_{n=0}^\infty f^n \mathcal{P}$  and  $\mathcal{P} = \tilde{\mathcal{P}} \vee \{\hat{R}_1, S \setminus \hat{R}_1\}$  we have that  $y_{n_i} = f^{-n_i}(y) \in \hat{R}_1$  and hence, since  $\text{dist}(x_{n_i}, y_{n_i}) < \delta$ , we get that

$$R_1 \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^+(x_{n_i}) \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^-(y_{n_i}) \neq \emptyset.$$

So, the whole piece of orbit  $\mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^+(x_{n_i}) \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^-(y_{n_i})$  is in  $R_0$ . Call  $z_{n_i}^1 = \tilde{W}_{\beta(\mathbf{s})}^+(x_{n_i}) \cap \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^-(y_{n_i})$  and  $z_{n_i}^2 = \mathcal{O}_\varepsilon W_{\alpha(\mathbf{s})}^+(x_{n_i}) \cap \tilde{W}_{\beta(\mathbf{s})}^-(y_{n_i})$ .

We claim that  $f^{n_i-n_{i-1}}(z_{n_i}^1) = z_{n_{i-1}}^1$  for every  $i$  and hence  $f^{n_i-n_j}(z_{n_i}^1) = z_{n_j}^1$ . The same happens for the sequence  $z_{n_i}^2$ . Let us proof the claim.

Since  $z_{n_i}^1 = \tilde{W}_{\beta(s)}^+(x_{n_i}) \cap \mathcal{O}_\varepsilon W_{\alpha(s)}^-(y_{n_i})$  we have that

$$f^{n_i-n_{i-1}}(z_{n_i}^1) = f^{n_i-n_{i-1}}(\tilde{W}_{\beta(s)}^+(x_{n_i})) \cap f^{n_i-n_{i-1}}(\mathcal{O}_\varepsilon W_{\alpha(s)}^-(y_{n_i})).$$

Hence to prove the claim it is enough to show that  $f^{n_i-n_{i-1}}(z_{n_i}^1) \in \tilde{W}_{\beta(s)}^+(x_{n_{i-1}})$  and to this end we shall use Lemma 6.21. Take the sequence of positive integers  $k_j, j = 0, \dots, l$  such that  $f^{k_j}(x_{n_i})$  enters in  $R_0$ ,  $k_0 = 0$ ,  $k_l = n_i - n_{i-1}$ . By definition we have that  $k_j - k_{j-1} = r(f^{k_{j-1}}(x_{n_i})) = r_{j-1}$ . Let us see that

$$f^{r_{j-1}}(f^{k_{j-1}}(z_{n_i}^1)) \in \tilde{W}_{\beta(s)}^+(f^{r_{j-1}}(f^{k_{j-1}}(x_{n_i}))).$$

By Lemma 6.21 it is enough to see that

$$\tilde{d}_{f^{k_{j-1}}(x_{n_i})}(f^{k_{j-1}}(x_{n_i}), f^{k_{j-1}}(z_{n_i}^1)) < l_0^{-1} e^{-r_{j-1}\lambda}.$$

Let us assume by induction that

$$f^{k_{j-1}}(z_{n_i}^1) = \tilde{W}_{\beta(s)}^+(f^{k_{j-1}}(x_{n_i})) \cap \mathcal{O}_\varepsilon W_{\alpha(s)}^-(f^{k_{j-1}}(y_{n_i})).$$

Since  $f^{k_{j-1}}(x_{n_i}) \in R_0$  we know that

$$\text{dist}(f^{k_{j-1}}(x_{n_i}), f^{k_{j-1}}(y_{n_i})) < \delta e^{-r_{j-1}\lambda}.$$

Now, by the uniformity of the invariant stable and unstable manifolds for points in a given Pesin set and by the uniform transversality of the invariant distribution, there is a constant  $C_0$  that depends on the Pesin set such that

$$\tilde{d}_{f^{k_{j-1}}(x_{n_i})}(f^{k_{j-1}}(x_{n_i}), f^{k_{j-1}}(z_{n_i}^1)) \leq C_0 \text{dist}(f^{k_{j-1}}(x_{n_i}), f^{k_{j-1}}(y_{n_i})).$$

So that taking  $\delta$  small enough we get that

$$f^{r_{j-1}}(f^{k_{j-1}}(z_{n_i}^1)) \in \tilde{W}_{\beta(s)}^+(f^{r_{j-1}}(f^{k_{j-1}}(x_{n_i}))),$$

hence

$$f^{k_j}(z_{n_i}^1) = \tilde{W}_{\beta(s)}^+(f^{k_j}(x_{n_i})) \cap \mathcal{O}_\varepsilon W_{\alpha(s)}^-(f^{k_j}(y_{n_i})).$$

So the claim is proved for  $z_{n_i}^1$ .

For the case of  $z_{n_i}^2$  observe that  $z_{n_i}^2 = \mathcal{O}_\varepsilon(z_{n_i}^1) \cap \tilde{W}_{\beta(s)}^-(y_{n_i})$  for every  $i$ . On the other hand,  $\tilde{W}_{\beta(s)}^-$  is  $f$ -invariant and since the derivative of  $f^n$  restricted to any orbit  $\mathcal{O}$  is uniformly bounded from below and from above we get that

$$\begin{aligned} f^{n_i-n_{i-1}}(z_{n_{i-1}}^2) &= f^{n_i-n_{i-1}}(\mathcal{O}_\varepsilon(z_{n_{i-1}}^1) \cap \tilde{W}_{\beta(s)}^-(y_{n_{i-1}})) \\ &\subset \mathcal{O}_{C\varepsilon}(f^{n_i-n_{i-1}}(z_{n_{i-1}}^1)) \cap \tilde{W}_{\beta(s)}^-(f^{n_i-n_{i-1}}(y_{n_{i-1}})) \\ &= \mathcal{O}_{C\varepsilon}(z_{n_i}^1) \cap \tilde{W}_{\beta(s)}^-(y_{n_i}) \end{aligned}$$

for some fixed constant  $C$ . So, if  $\varepsilon$  is small enough we get that the last term equals  $z_{n_i}^2$ .

So finally, since  $f^{n_i-n_0}(z_{n_i}^2) = z_{n_0}^2$ , we get that

$$\text{dist}(z_{n_0}^2, y_{n_0}) = \text{dist}(f^{n_i-n_0}(z_{n_i}^2), f^{n_i-n_0}(y_{n_i}))$$

and hence, since the right hand side tends to zero because  $z_{n_i}^2 \in \tilde{W}_{\beta(s)}^-(y_{n_i})$  we get that  $z_{n_0}^2 = y_{n_0}$ . Hence,  $f^{-n_0}(y) \in \mathcal{O}_\varepsilon W_{\alpha(s)}^+(f^{-n_0}(x))$  and since  $n_0 \leq r(f^{-n_0}(x))$  we get that  $\mathcal{P}^+(x) \subset \mathcal{O}_\varepsilon W_{\alpha(s)}^+(x)$  by using Lemma 6.21 and the fact that the derivative of  $f^n$  restricted to any orbit  $\mathcal{O}$  is uniformly bounded from below and from above.

So we get that  $\mathcal{P}^+(x) \subset \mathcal{O}_\varepsilon W_{\alpha(s)}^+(x)$ ,  $x$  a.e. Let us see now that in fact  $\mathcal{P}^+(x) \subset \tilde{W}_{\beta(s)}^+(x)$ ,  $x$  a.e. We will use the same notations as above. Take  $y \in \mathcal{P}^+(x)$ , we already get that  $y_{n_0} = z_{n_0}^2$  and hence that it is in the  $\varepsilon$ -orbit of  $z_{n_0}^1$ , where  $z_{n_0}^1 = \tilde{W}_{\beta(s)}^+(x_{n_0}) \cap \mathcal{O}_\varepsilon W_{\alpha(s)}^-(y_{n_0})$ . Let us see that  $z_{n_0}^1 = y_{n_0}$ . In fact,  $\text{dist}(f^{-n}(z_{n_0}^1), f^{-n}(x_{n_0})) \rightarrow 0$  since  $z_{n_0}^1 \in \tilde{W}_{\beta(s)}^+(x_{n_0})$ . On the other hand, since  $\inf_{n \geq 0} \psi(f^{-n}(x)) = 0$ , we get that  $\liminf \text{dist}(f^{-n}(y_{n_0}), f^{-n}(x_{n_0})) = 0$ . So

$$\liminf \text{dist}(f^{-n}(y_{n_0}), f^{-n}(z_{n_0}^1)) = 0.$$

But the derivative of  $f^n$  restricted to any orbit  $\mathcal{O}$  is uniformly bounded from below and from above, that is:  $\|D_z f^n|_{T\mathcal{O}}\| \leq C$  for every  $n \in \mathbb{Z}$ . So we have that

$$C^{-1} \text{dist}(y_{n_0}, z_{n_0}^1) \leq \text{dist}(f^{-n}(y_{n_0}), f^{-n}(z_{n_0}^1))$$

for every  $n$  and hence  $\text{dist}(y_{n_0}, z_{n_0}^1) = 0$ . Hence,  $f^{-n_0}(y) \in \tilde{W}_{\beta(s)}^+(f^{-n_0}(x))$  and since  $n_0 \leq r(f^{-n_0}(x))$  we get the Lemma by using Lemma 6.21.  $\square$

So we can now begin the comparison of the entropies  $h(f, \mathcal{P})$  and  $h(f, \xi)$  for the partition  $\mathcal{P}$  built just before Lemma 6.20 and the partition  $\xi$  built in Lemma 6.16. But first let us state the following corollary of Lemma 6.17.

**Corollary 6.22.** *Let  $\mathcal{P}$  be the partition in Lemma 6.20 and  $\mathcal{Q}$  be any finite entropy partition. Then, for  $\mathcal{P}_0 = \mathcal{P} \vee \mathcal{Q}$  and  $\xi$  a partition built in Lemma 6.16 we have that  $h(f, \mathcal{P}_0) = h(f, \mathcal{P}_0^+) = h(f, \xi \vee \mathcal{P}_0^+)$ .*

**Proof.** The result follows since  $\mathcal{P}_0^+(x) \subset \tilde{W}(x)$ .  $\square$

Finally we get:

**Lemma 6.23.**  $h(f, \xi \vee \mathcal{P}_0^+) = h(f, \xi)$ .

**Proof.** As in the argument in the proof of Lemma 6.17, see also [24, Lemma 3.2.1.], we have,

$$\begin{aligned} h(f, \xi \vee \mathcal{P}_0^+) &= h(f, \xi \vee f^n \mathcal{P}_0^+) \\ &= H(\xi | f\xi \vee f^{n+1} \mathcal{P}_0^+) + H(\mathcal{P}_0^+ | f^{-n} \xi \vee f \mathcal{P}_0^+) \end{aligned}$$

where the first term is  $\leq H(\xi | f\xi)$  and the second term goes to 0 since  $\{f^{-n}\xi\}$  generates. Hence  $h(f, \xi \vee \mathcal{P}_0^+) \leq H(\xi | f\xi) = h(f, \xi)$ . Finally, since  $H(\mathcal{P}_0) < \infty$  we have that  $h(f, \xi \vee \mathcal{P}_0^+) \geq h(f, \xi)$  and hence we are done.  $\square$

Finally, combining the above Lemma with Corollary 6.22 we get that  $h(f, \mathcal{P}_0) = h(f, \xi)$ . So that taking finite partitions  $\mathcal{Q}_n$  such that  $h(f, \mathcal{Q}_n) \rightarrow$

$h(f)$  we get that

$$h(f, \xi) = h(\mathcal{P} \vee \mathcal{Q}_n) \geq h(f, \mathcal{Q}_n)$$

and hence  $h(f, \xi) \geq h(f)$ . The other inequality follows since  $\xi$  is a measurable partition.

## 7. CONCLUSION OF THE PROOF OF THEOREM 4.1.

We will use the properties of the time change and the transitivity property of the action  $\beta$  to produce elements of the action  $\alpha$  with recurrence and uniformly bounded derivatives along  $\mathcal{W}$ .

We denote by  $\mu_x^{\mathcal{W}}$  the conditional measure of  $\mu$  on  $\mathcal{W}(x)$  and by  $B_r^{\mathcal{W}}(x)$  the ball in  $\mathcal{W}(x)$  of radius  $r$  with respect to the induced smooth metric.

**Lemma 7.1.** *For any Pesin set  $\mathcal{R}_\varepsilon^l$  there exist positive constants  $K$  and  $l'$  so that for  $\mu$ - a.e.  $x \in \mathcal{R}_\varepsilon^l$  and for  $\mu_x^{\mathcal{W}}$ - a.e.  $y \in \mathcal{R}_\varepsilon^l \cap B_r^{\mathcal{W}}(x)$  there exists a sequence of elements  $\mathbf{t}_j \in \mathbb{R}^k$  with*

- (1)  $x_j = \alpha(\mathbf{t}_j)x \in \mathcal{R}_\varepsilon^{l'}$ ,
- (2)  $x_j \rightarrow y$ ,
- (3)  $K^{-1} \leq \|D_x^E \alpha(\mathbf{t}_j)\| \leq K$ .

**Proof.** Consider typical points  $x \in \mathcal{R}_\varepsilon^l$  and  $y \in \mathcal{R}_\varepsilon^l \cap B_r^{\mathcal{W}}(x)$  and let  $\tilde{y} = \alpha(\varphi_x(y)\mathbf{w})y$  be the point on  $\tilde{\mathcal{W}}(x)$  corresponding to  $y$ . We denote  $\mathbf{s} = \varphi_x(y)\mathbf{w}$  and observe that  $\tilde{y} \in \mathcal{R}_\varepsilon^{l''}$  with  $l'' = l \exp(\|\mathbf{s}\|)$ . By Proposition 6.7 the function  $\varphi_x(y)$  is Hölder on  $\mathcal{R}_\varepsilon^l \cap B_r^{\mathcal{W}}(x)$  and hence  $\mathbf{s}$  is uniformly bounded, so that the constant  $l''$  can be chosen the same for all  $x$  and  $y$  in the lemma.

Since the invariant measure  $\nu$  for  $\beta$  is equivalent to  $\mu$  we may assume that  $x$  and  $\tilde{y}$  are typical points for  $\nu$ . Then by Proposition 6.9 there exists a sequence  $n_j \rightarrow \infty$  such that  $\tilde{x}_j = \beta(n_j \mathbf{t}, x) = \alpha(\mathbf{g}(x, n_j \mathbf{t}))x \rightarrow \tilde{y}$ . Since both  $x$  and  $\tilde{y}$  are in  $\mathcal{R}_\varepsilon^{l''}$  then the iterates  $\tilde{x}_j$  can also be taken in this set. Denoting  $\mathbf{t}_j = \mathbf{g}(x, n_j \mathbf{t}) - \mathbf{s}$  we conclude that  $x_j = \alpha(\mathbf{t}_j)x \rightarrow y$ . Again, all points  $x_j$  are in a Pesin set  $\mathcal{R}_\varepsilon^{l'}$  with  $l'$  the same for all  $x$  and  $y$  in the lemma. Thus the sequence  $\mathbf{t}_j$  satisfies (1) and (2).

To obtain (3) we note that by the definition of the time change

$$\|D_x^E \alpha(\mathbf{g}(x, n_j \mathbf{t}))\|_\varepsilon = 1.$$

Then the estimates in (3) follow from the uniform boundedness of the correction  $\mathbf{s}$  for all  $x$  and  $y$  in the lemma and from the uniform estimates (5.4), (5.5) for the ratio of the Lyapunov and smooth norms on the Pesin set  $\mathcal{R}_\varepsilon^{l'}$ .  $\square$

We will use the notion of an affine map on a leaf of a Lyapunov foliation. These are the maps which are affine with respect to the atlas given by affine parameters on these leaves. The notion of affine parameters is similar to that of nonstationary linearization. The following proposition provides  $\alpha$ -invariant affine parameters on the leaves of any Lyapunov foliation  $\mathcal{W}$ .

**Proposition 7.2.** [7, Proposition 3.1, Remark 5] *There exists a unique measurable family of  $C^{1+\theta}$  smooth  $\alpha$ -invariant affine parameters on the leaves  $\mathcal{W}(x)$ . Moreover, they depend uniformly continuously in  $C^{1+\theta}$  topology on  $x$  in any Pesin set.*

Now we can apply Lemma 7.1 to obtain the following invariance property for the conditional measures of  $\mu$  on  $\mathcal{W}$ . We note that if the conditional measures  $\mu_x^{\mathcal{W}}$  are atomic this invariance property degenerates into trivial.

**Lemma 7.3.** [7, Lemma 3.9] *For  $\mu$ - a.e.  $x \in \mathcal{R}_\varepsilon^l$  and for  $\mu_x^{\mathcal{W}}$ - a.e.  $y \in \mathcal{R}_\varepsilon^l \cap B_r^{\mathcal{W}}(x)$  there exists an affine map  $g : \mathcal{W}(x) \rightarrow \mathcal{W}(x)$  with  $g(x) = y$  which preserves the conditional measure  $\mu_x^{\mathcal{W}}$  up to a positive scalar multiple.*

This lemma is proved by finding a limit for the restrictions of maps  $\alpha(\mathbf{t}_j)$  to  $\mathcal{W}(x)$ . The proof is identical to the one of Lemma 3.9 in [7]. It relies only on the conclusions of Lemmas 3.7 and 3.8 in [7], which are now given by Lemma 7.1.

Assuming that the conditional measures  $\mu_x^{\mathcal{W}}$  are non-atomic for  $\mu$ -a.e.  $x$ , the following lemma from [7] establishes the absolute continuity of these conditional measures and completes the proof of Theorem 4.1. Its proof in [7] relies only on the conclusion of Lemma 7.3 (Lemma 3.9 in [7]).

**Lemma 7.4.** [7, Lemma 3.10] *The conditional measures  $\mu_x^{\mathcal{W}}$  are absolutely continuous for  $\mu$ - a.e.  $x$ . (In fact,  $\mu_x^{\mathcal{W}}$  is Haar with respect to the affine parameter on  $\mathcal{W}(x)$ .)*

## 8. CONCLUDING REMARKS AND SOME OPEN PROBLEMS

**8.1. Further properties of maximal rank actions.** For a  $\mathbb{Z}^k$ ,  $k \geq 2$  action  $\alpha$  on the torus  $\mathbb{T}^{k+1}$  with Cartan homotopy data there is a unique invariant measure  $\mu$  which is projected to Lebesgue measure  $\lambda$  by the semi-conjugacy with the corresponding linear Cartan action  $\alpha_0$ ; this measure is absolutely continuous and the semi-conjugacy is bijective and measure preserving between certain sets of full  $\mu$ -measure and full Lebesgue measure. Thus  $(\alpha, \mu)$  and  $(\alpha_0, \lambda)$  are isomorphic as measure preserving actions ([19, Corollary 2.2.]); furthermore, the measurable conjugacy is smooth on almost every local (and hence global) stable manifold for any element of  $\alpha$ , in particular, along the Lyapunov foliations, [19, Proposition 2.9.]. This implies that the Jacobians along those foliations are rigid, i.e. multiplicatively cohomologous to the corresponding eigenvalues of elements of  $\alpha_0$  ([7, Lemma 4.4])<sup>4</sup>. Another consequence is that the metric entropy of  $\alpha$  with respect to  $\mu$  is the logarithm of an algebraic integer of degree at most  $k + 1$ .

<sup>4</sup>In fact, we can prove in this setting rigidity of general cocycles which are Hölder with respect to the Lyapunov metric. Although the proof is not very difficult it uses the semi-conjugacy and its regularity properties very heavily and thus it would not fit well with the program of the present paper which aims at deriving geometric/rigidity properties from purely dynamical assumptions irrespective of any model.

It is natural to ask whether in our more general setting similar properties of the expansion coefficients and for entropy hold. In a general setting Jacobian along a Lyapunov foliation is called *rigid* if its logarithm is cohomologous (with a measurable transfer function) to the corresponding Lyapunov exponent. Notice that our proof of the key recurrence property is based on rigidity of Jacobians for the special time changes constructed in Section 6.1 which is true essentially by definition. Notice however that different Lyapunov foliations may require different time changes.

**Conjecture 1.** *Jacobians along Lyapunov foliations for an action  $\alpha$  satisfying assumptions of Main Theorem are rigid.*

**Problem 1.** [13] What are possible values of entropy for elements of an action satisfying assumptions of Main Theorem?

The following conjecture represents a cautiously optimistic view which presumes existence of a certain underlying arithmetic structure.

**Conjecture 2.** *The values of Lyapunov exponents and hence of entropy are logarithms of algebraic integers.*

Notice that this is true for all known examples on a variety of manifolds as described in the introduction and in those cases the algebraic integers have degree at most  $k + 1$ .

Another result of [19] (Theorem 3.1.) establishes existence of a set of periodic points dense in the support of the measure  $\mu$  whose eigenvalues are equal to the corresponding powers of the eigenvalues of  $\alpha_0$ . This implies that the Lyapunov exponents of atomic measures concentrated on the corresponding periodic orbits are equal to those of  $\mu$ . Again the latter property is also true for all known examples on manifolds other than tori. Let us call such periodic points *proper*.

**Problem 2.** Under the assumptions of Main Theorem (1) is there any proper periodic point for  $\alpha$ ? Are proper periodic points dense in the support of the measure  $\mu$ ?

Another circle of questions concerns relations between the  $\mathbb{Z}^k$  actions satisfying assumptions of the Main Theorem (1) and  $\mathbb{R}^k$  actions satisfying assumptions of Main Theorem (2). Any suspension of the action of the first kind is an action of the second kind. One can also make time changes for the suspension. A trivial type of time change is given by a linear automorphism of  $\mathbb{R}^k$ . Any time change is given by an  $\mathbb{R}^k$  cocycle over the action and Lyapunov exponents are transformed according to the cocycle averages and hence assumptions of Main Theorem (2) are preserved under a smooth time change. Thus existence of non-trivial time changes is closely related to the problem of cocycle rigidity.

**Problem 3.** Is any smooth  $\mathbb{R}$ -valued cocycle over an action satisfying assumptions of Main Theorem (1) (or (2)) cohomologous to a constant cocycle?

The answer may depend on the regularity of cohomology. In particular, it is more likely to be positive if the cohomology in question is only measurable, rather than smooth.

**Problem 4.** Are there  $\mathbb{R}^k$  actions satisfying assumptions of Main Theorem (2) which do not appear from time changes of suspensions of  $\mathbb{Z}^k$  actions satisfying assumptions of Main Theorem (1)?

Notice that on the torus for an action with Cartan homotopy data the unique “large” invariant measure, ie. the measure which projects to Lebesgue measure for the linear Cartan action under the semi-conjugacy (see[19]) changes continuously in weak\* topology. Thus there is not only global but also local rigidity for such a measure. While global rigidity is problematic in the setting of Main Theorem, the local version is plausible.

**Conjecture 3.** *Given a  $C^2$  action  $\alpha$  with an invariant measure  $\mu$  satisfying assumptions of the Main Theorem (1) or (2), any action  $\alpha'$  close to  $\alpha$  in  $C^2$  topology has an ergodic invariant measure  $\mu'$  satisfying the same assumptions. One can choose  $\mu'$  in such a way that when  $\alpha'$  converges to  $\alpha$  in  $C^2$  topology,  $\mu'$  converges to  $\mu$  in weak\* topology.*

*Furthermore, in the  $\mathbb{Z}^k$  case Lyapunov exponents of  $\mu'$  are equal to those of  $\mu$ .*

**8.2. High rank and low dimension.** As explained in [19, Section 4] many examples of manifolds with actions satisfying assumptions of Main Theorem (1) can be obtained by starting from the torus and applying two procedures:

- blowing up points and glueing in copies of the projective space  $\mathbb{R}P(k)$ , and
- cutting pairs of holes and attaching handles  $S^k \times \mathbb{D}^1$ .

**Conjecture 4.** *An action satisfying assumptions of Main Theorem (1) exists on any compact manifold of dimension three or higher.*

The sphere  $S^3$  looks as a good open test case.

**Definition.** [12] An ergodic invariant measure of a  $\mathbb{Z}^k$  action with non-vanishing Lyapunov exponents is called *strongly hyperbolic* if intersection of all Lyapunov hyperplanes is the origin.

Obviously the rank of a strongly hyperbolic action does not exceed the dimension of the manifold. Furthermore, any ergodic measure for a strongly hyperbolic action of  $\mathbb{Z}^k$  on a  $k$ -dimensional manifold is atomic and is supported by a single closed orbit, [12, Proposition 1.3]. Thus the maximal rank for a strongly hyperbolic action on a manifold  $M$  with a non-atomic ergodic measure, in particular measure with positive entropy, is  $\dim M - 1$ , exactly the case considered in the present paper.

Let us consider the lowest dimension compatible with the higher rank assumption, namely strongly hyperbolic  $\mathbb{Z}^2$  actions on three-dimensional manifolds. Lyapunov hyperplanes are lines in this case and the general



position condition is equivalent to three Lyapunov lines being different. In this case our theorem applies and any ergodic invariant measure either has zero entropy for all elements of  $\mathbb{Z}^2$  or is absolutely continuous.

Let us consider other possible configurations of Lyapunov lines:

- (1) Two Lyapunov exponents proportional with negative proportionality coefficient; two Lyapunov lines.
- (2) Two Lyapunov exponents proportional with positive proportionality coefficient; two Lyapunov lines.
- (3) All three Lyapunov exponents proportional; one Lyapunov line.

(1) First notice that such a measure cannot be absolutely continuous. For an absolutely continuous invariant measure sum of the Lyapunov exponents is identically equal to zero. But in this case two exponents are zero along the common kernel of two proportional exponents while the third one is not zero there.

Now we construct an example of an action with a singular positive entropy measure of this type. Consider the following action on  $\mathbb{T}^3$ : Cartesian product of the action generated by a diffeomorphism  $f$  of  $S^1$  with one contracting fixed point  $p$  with positive eigenvalue  $\beta < 1$  and one expanding fixed point, with the action generated by a hyperbolic automorphism  $F$  of  $\mathbb{T}^2$  with an eigenvalue  $\rho > 1$ . The measure  $\mu = \delta_p \times \lambda_{\mathbb{T}^2}$  is invariant under the Cartesian product and is not absolutely continuous. Lyapunov exponents are  $x \log \beta$ ,  $y \log \rho$  and  $-y \log \rho$ , the entropy is  $h_\mu(f^m F^n) = |n| \log \rho$ .

(2) There are four Weyl chambers and in one of those all three Lyapunov exponents are negative; hence by [12, Proposition 1.3] any ergodic invariant measure is atomic. Notice that this includes the case of a multiple exponent.

(3) On the Lyapunov line all three Lyapunov exponents vanish; hence the action is not strongly hyperbolic.

Thus we obtain the following necessary and sufficient condition for a configuration of Lyapunov lines.

**Corollary 8.1.** *A strongly hyperbolic ergodic invariant measure for a  $\mathbb{Z}^2$  action on a three-dimensional compact manifold with positive entropy for some element is absolutely continuous if and only if Lyapunov lines for three exponents are different.*

There is an open question related to the case (1). In our examples both Lyapunov lines are rational. One can modify this example to make the “single” Lyapunov line (the kernel of a single exponent) irrational. It is conceivable, although not very likely, that situation when the “double” line, i.e. the kernel of two exponents, or both lines, are irrational may be different.

**Problem 5.** Construct an example of a smooth  $\mathbb{Z}^2$  action on a compact three-dimensional manifold with a singular ergodic invariant measure with

positive entropy with respect to some element of the action, such that two Lyapunov exponents are negatively proportional and their common kernel is an irrational line.

**8.3. Low rank and high dimension.** Essentially all known rigidity results for *algebraic actions* (hyperbolic or partially hyperbolic), including cocycle, measurable, and local differentiable rigidity, assume only some sort of “genuine higher ( $\geq 2$ ) rank” assumption, see e.g. [21, 15, 3] and references thereof. By contrast, global rigidity for Anosov actions on a torus [30] and nonuniform measure rigidity on a torus [7, 19], as well as results of this paper, deal with maximal rank actions. Notice, however, that global rigidity results for Anosov actions on an arbitrary manifold satisfying stronger dynamical assumptions only require rank  $\geq 3$  [10] or rank  $\geq 2$  [8, 9]

We expect that global measure rigidity results, both on the torus and in the general setting similar to those of the present paper, hold at a greater generality although we do not see a realistic approach for the most general “genuine higher rank” situation even on the torus. There is still an intermediate class which is compatible with the lowest admissible rank (i.e rank two) on manifolds of arbitrary dimension. We already mentioned it in the remark in Section 6.4.

**Definition.** An ergodic invariant measure for a  $\mathbb{Z}^k$  action is called *totally non-symplectic (TNS)* if for any two Lyapunov exponents there exists an element of  $\mathbb{Z}^k$  for which both those exponents are negative.

Equivalently all Lyapunov exponents are non-zero and there are no proportional exponents with negative coefficient of proportionality<sup>5</sup>.

The TNS condition is the most general one for the “ $\pi$ -partition trick” to work. It also greatly helps in the geometric treatment of cocycle rigidity, see e.g. [18]. It has a nice property that it is inherited by a restriction of the action to a subgroup of rank  $\geq 2$  if it is in general position. While it is possible that our result generalizes to the TNS measures, i.e. assuming only existence of some elements with positive entropy, we prefer to be more conservative and formulate a conjecture under a stronger entropy assumption.

**Conjecture 5.** *Let  $\mu$  be an ergodic invariant totally non-symplectic measure for a smooth action  $\alpha$  of  $\mathbb{Z}^k$ ,  $k \geq 2$ . Assume that every element other than identity has positive entropy. Then  $\mu$  is absolutely continuous.*

A serious difficulty even in TNS case may appear in the presence of multiple exponents. Recall that even for linear actions multiple eigenvalues lead to Jordan blocks so that when the eigenvalue has absolute value one the action is not isometric. More generally, positively proportional eigenvalues also lead to complications. Thus a more tractable case would be that with simple Lyapunov exponents and no proportional ones. In this case the Lyapunov distributions are one-dimensional and coincide with coarse Lyapunov

<sup>5</sup>The latter is exactly what happens for symplectic actions; hence the name.

ones. For the suspension every Lyapunov exponent satisfies assumptions of Theorem 4.1. This is a nonuniform counterpart of Cartan actions in the sense of [10].

**Conjecture 6.** *Let  $\mu$  be an ergodic invariant measure for a smooth action  $\alpha$  of  $\mathbb{Z}^k$ ,  $k \geq 2$ , such that all Lyapunov exponents are simple and all Lyapunov hyperplanes different. Assume that some element of the action has positive entropy. Then  $\mu$  is absolutely continuous.*

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