LYAPUNOV EXPONENTS OF COCYCLES OVER NON-UNIFORMLY HYPERBOLIC SYSTEMS

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ABSTRACT. We consider linear cocycles over non-uniformly hyperbolic dynamical systems. The base system is a diffeomorphism f of a compact manifold X preserving a hyperbolic ergodic probability measure μ. The cocycle $\mathcal{A}$ over f is Hölder continuous and takes values in $GL(d,\mathbb{R})$ or, more generally, in the group of invertible bounded linear operators on a Banach space. For a $GL(d,\mathbb{R})$-valued cocycle $\mathcal{A}$ we prove that the Lyapunov exponents of $\mathcal{A}$ with respect to μ can be approximated by the Lyapunov exponents of $\mathcal{A}$ with respect to measures on hyperbolic periodic orbits of f. In the infinite-dimensional setting one can define the upper and lower Lyapunov exponents of $\mathcal{A}$ with respect to μ, but they cannot always be approximated by the exponents of $\mathcal{A}$ on periodic orbits. We prove that they can be approximated in terms of the norms of the return values of $\mathcal{A}$ on hyperbolic periodic orbits of f.

1. Introduction and statements of the results

The theory of non-uniformly hyperbolic dynamical systems was pioneered by Ya. Pesin in [P1, P2] as a generalization of uniform hyperbolicity. It has become one of the central areas in smooth dynamics with numerous applications, see [BP, Po]. Periodic points play a major role in the study of both uniformly and non-uniformly hyperbolic systems. In the non-uniformly hyperbolic case, the existence of hyperbolic periodic orbits and their relations to dynamical and ergodic properties of the system were established by A. Katok in a seminal paper [Kt]. In fact, any hyperbolic invariant measure can be approximated in weak* topology by invariant measures supported on hyperbolic periodic points of the system [BP]. A further advance in this direction was obtained by Z. Wang and W. Sun who showed that Lyapunov exponents of any hyperbolic measure can be approximated by Lyapunov exponents of periodic points [WS]. This does not follow from weak* approximation as Lyapunov exponents in general do not depend continuously on the measure in weak* topology.

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The Lyapunov exponents above correspond to the derivative cocycle Df of the base system (X, f), which is a particular case of a linear cocycle, that is an automorphism of a vector bundle over X that projects to f. Linear cocycles are the prime examples of non-commutative cocycles over dynamical systems. For uniformly hyperbolic systems, where every invariant measure is hyperbolic, a periodic approximation of Lyapunov exponents of linear cocycles was established by the first author in [K]. The results and techniques in [K] proved useful in various areas such as cohomology of non-commutative cocycles and the study of random and Markovian matrices and operators. More recently, approximation results were obtained by the authors for cocycles with values in the group of invertible bounded linear operators on a Banach space [KS2] and by L. Backes for semi-invertible matrix cocycles [B].

In this paper we extend the periodic approximation results to linear cocycles over non-uniformly hyperbolic systems. In the base we consider a diffeomorphism f of a compact manifold X preserving a hyperbolic ergodic probability measure μ. The cocycles over (X, f) will take values in $GL(d, \mathbb{R})$ or, more generally, the group $GL(V)$ of invertible bounded linear operators on a Banach space V. The space $L(V)$ of bounded linear operators on V is a Banach space equipped with the operator norm $\|A\| = \sup\{\|Av\| : v \in V, \|v\| \leq 1\}$. The open set $GL(V)$ of invertible elements in $L(V)$ is a topological group and a complete metric space with respect to the metric

$$d(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|.$$

Definition 1.1. *Let f be a homeomorphism of a compact metric space X and let A be a function from X to $GL(V)$. The $GL(V)$-valued* cocycle over f generated by A *is the map $\mathcal{A} : X \times \mathbb{Z} \to GL(V)$ defined by $\mathcal{A}(x, 0) = Id$ and for $n \in \mathbb{N}$,*

$$\mathcal{A}(x, n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x), \quad \mathcal{A}(x, -n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

In the finite-dimensional case of $V = \mathbb{R}^d$ we will call $\mathcal{A}$ a $GL(d, \mathbb{R})$-valued cocycle.

*We say that the cocycle $\mathcal{A}$ is α-*Hölder*, $0 < \alpha \leq 1$, if there exists $M > 0$ such that*

$$d(\mathcal{A}_x, \mathcal{A}_y) \leq M\, dist(x, y)^\alpha \quad \textit{for all } x, y \in X. \tag{1.1}$$

Clearly, $\mathcal{A}$ satisfies the cocycle equation $\mathcal{A}_x^{n+k} = \mathcal{A}_{f^k x}^n \circ \mathcal{A}_x^k$. Since X is compact, the Hölder condition (1.1) is equivalent to $\|\mathcal{A}_x - \mathcal{A}_y\| \leq M' \mathrm{dist}(x, y)^\alpha$ for all $x, y \in X$. Hölder continuity of a cocycle is natural in our setting as the lowest regularity allowing development of a meaningful theory beyond the measurable case. It covers the case of the derivative cocycles of $C^{1+\text{Hölder}}$ diffeomorphisms and their restrictions to Hölder continuous sub-bundles of TX, which play an important role in hyperbolic systems.

Any $GL(V)$-valued cocycle $\mathcal{A}$ can be viewed as an automorphism of the trivial vector bundle $\mathcal{E} = X \times V$, $\mathcal{A}(x, v) = (fx, \mathcal{A}_x(v))$. More generally, we can consider a linear cocycle $\mathcal{A}$, i.e. an automorphism of any vector bundle $\mathcal{E}$ over X that projects to f. This setting covers the case of the derivative cocycle Df of a diffeomorphism f of X with nontrivial tangent bundle. For any measure μ on X, any vector bundle $\mathcal{E}$

over X is trivial on a set of full measure [BP, Proposition 2.1.2] and hence any linear cocycle $\mathcal{A}$ can be viewed as a $GL(V)$-valued cocycle on a set of full measure.

First we consider the finite dimensional case where the Lyapunov exponents and Lyapunov decomposition for $\mathcal{A}$ with respect to an ergodic f-invariant measure μ are given by Oseledets Multiplicative Ergodic Theorem. We note that both are defined μ-a.e. and depend on the choice of μ.

Oseledets Multiplicative Ergodic Theorem. [O, BP] *Let f be an invertible ergodic measure-preserving transformation of a Lebesgue probability space (X, μ). Let $\mathcal{A}$ be a measurable $GL(d, \mathbb{R})$-valued cocycle over f satisfying $\log \|\mathcal{A}_x\| \in L^1(X, \mu)$ and $\log \|\mathcal{A}_x^{-1}\| \in L^1(X, \mu)$. Then there exist numbers $\lambda_1 < \cdots < \lambda_m$, an f-invariant set Λ with $\mu(\Lambda) = 1$, and an $\mathcal{A}$-invariant Lyapunov decomposition*

$$\mathbb{R}^d = \mathcal{E}_x = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^m \quad \textit{for } x \in \Lambda \textit{ such that}$$

(i) $\lim\limits_{n \to \pm\infty} n^{-1} \log \|\mathcal{A}_x^n v\| = \lambda_i$ *for any* $i = 1, \dots, m$ *and any* $0 \neq v \in \mathcal{E}_x^i$, *and*

(ii) $\lim\limits_{n \to \pm\infty} n^{-1} \log |\det \mathcal{A}_x^n| = \sum_{i=1}^m d_i \lambda_i$, *where* $d_i = \dim \mathcal{E}_x^i$.

Definition 1.2. *The numbers $\lambda_1, \dots, \lambda_m$ are called the Lyapunov exponents of $\mathcal{A}$ with respect to μ and the integers $d_1, \dots, d_m$ are called their multiplicities.*

Definition 1.3. *Let μ be an ergodic invariant Borel probability measure for a diffeomorphism f of a compact manifold X. The measure is called* hyperbolic *if all the Lyapunov exponents of the derivative cocycle Df with respect to μ are non-zero.*

By Lyapunov exponents of $\mathcal{A}$ at a periodic point $p = f^k p$ we mean the Lyapunov exponents of $\mathcal{A}$ with respect to the invariant measure μ_p on the orbit of p. They equal $(1/k)$ of the logarithms of the absolute values of the eigenvalues of $\mathcal{A}_p^k$. A periodic point p is called *hyperbolic* if Df has no zero exponents at p, that is $D_p f^k$ has no eigenvalues of absolute value 1.

The following theorem extends the periodic approximation results in [WS] and [K] to linear cocycles over non-uniformly hyperbolic systems.

Theorem 1.4. *Let f be a $C^{1+\text{Hölder}}$ diffeomorphism of a compact manifold X, let μ be a hyperbolic ergodic f-invariant Borel probability measure on X, and let $\mathcal{A}$ be a $GL(d, \mathbb{R})$-valued Hölder continuous cocycle over f.*

Then the Lyapunov exponents $\lambda_1 \leq \cdots \leq \lambda_d$ of $\mathcal{A}$ with respect to μ, listed with multiplicities, can be approximated by the Lyapunov exponents of $\mathcal{A}$ at periodic points. More precisely, for each $\epsilon > 0$ there exists a hyperbolic periodic point $p \in X$ for which the Lyapunov exponents $\lambda_1^{(p)} \leq \cdots \leq \lambda_d^{(p)}$ of $\mathcal{A}$ satisfy

$$|\lambda_i - \lambda_i^{(p)}| < \epsilon \quad \textit{for } i = 1, \dots, d. \tag{1.2}$$

The largest and smallest Lyapunov exponents $\lambda_+(\mathcal{A},\mu) = \lambda_m$ and $\lambda_-(\mathcal{A},\mu) = \lambda_1$ can be expressed as follows:

$$\begin{aligned} \lambda_+(\mathcal{A},\mu) &= \lim_{n\to\infty} n^{-1} \log \|\mathcal{A}_x^n\| \quad \text{for } \mu\text{-a.e. } x \in X, \\ \lambda_-(\mathcal{A},\mu) &= \lim_{n\to\infty} n^{-1} \log \|(\mathcal{A}_x^n)^{-1}\|^{-1} \quad \text{for } \mu\text{-a.e. } x \in X. \end{aligned} \tag{1.3}$$

While there is no Multiplicative Ergodic Theorem in the infinite-dimensional case in general, the upper and lower Lyapunov exponents λ_+ and λ_- of $\mathcal{A}$ can still be defined by (1.3), see Section 4.1. For the invariant measure μ_p on the orbit of $p = f^k p$ we have

$$\lambda_+(\mathcal{A},\mu_p) = k^{-1} \log (\text{spectral radius of } \mathcal{A}_p^k) \;\le\; k^{-1} \log \|\mathcal{A}_p^k\|.$$

In the infinite-dimensional setting, it is not always possible to approximate $\lambda_+(\mathcal{A},\mu)$ by $\lambda_+(\mathcal{A},\mu_p)$, even for cocycles over uniformly hyperbolic systems [KS2, Proposition 1.5]. However, an approximation of $\lambda_+(\mathcal{A},\mu)$ by $k^{-1}\log\|\mathcal{A}_p^k\|$ was obtained in [KS2] for cocycles over uniformly hyperbolic systems. The next theorem establishes such an approximation in the non-uniformly hyperbolic setting.

Theorem 1.5. *Let f be a $C^{1+\text{Hölder}}$ diffeomorphism of compact manifold X, let μ be a hyperbolic ergodic f-invariant Borel probability measure on X, and let $\mathcal{A}$ be a Hölder continuous $GL(V)$-valued cocycle over f.*

Then for each $\epsilon > 0$ there exists a hyperbolic periodic point $p = f^k p$ in X such that

$$\Big|\, \lambda_+(\mathcal{A},\mu) - k^{-1}\log\|\mathcal{A}_p^k\| \,\Big| < \epsilon \quad \textit{and} \quad \Big|\, \lambda_-(\mathcal{A},\mu) - k^{-1}\log\|(\mathcal{A}_p^k)^{-1}\|^{-1} \,\Big| < \epsilon. \tag{1.4}$$

Moreover, for any $N \in \mathbb{N}$ there exists such $p = f^k p$ with $k > N$.

The proof of the finite dimensional approximation in Theorem 1.4 relies on Multiplicative Ergodic Theorem, which yields that the cocycle has finitely many Lyapunov exponents and, in particular, the largest one is isolated. As this may not be the case in infinite dimensional setting even for a single operator, in Theorem 1.5 we use a different approach which relies on norm estimates. In particular we use a suitable version of Lyapunov norm and results on subadditive cocycles [KaM].

Remark 1.6. *Theorems 1.4 and 1.5 can be strengthened to conclude the existence of a hyperbolic periodic point $p = f^k p$ which gives simultaneous approximation as in* (1.2) *and* (1.4) *for finitely many cocycles $\mathcal{A}^{(j)}$, $i = 1, \dots, m$, over f with values in $GL(V_i)$. We describe the modifications for this case in the proofs.*

Remark 1.7. *Theorems 1.4 and 1.5 hold if we replace $X \times V$ by a Hölder continuous vector bundle $\mathcal{E}$ over X with fiber V and the cocycle $\mathcal{A}$ by an automorphism $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ covering f. This setting is described in detail in Section 2.2 of* [KS1] *and the proofs work without any significant modifications.*

2. Preliminaries

For a $GL(V)$-valued cocycle $\mathcal{A}$ over (X, f) we consider the trivial bundle $\mathcal{E} = X \times V$ and view $\mathcal{A}_x^n$ as a fiber map from $\mathcal{E}_x$ to $\mathcal{E}_{f^n x}$. This makes notations and arguments more intuitive and the extension to non-trivial bundles more transparent. In fact, all our arguments are written for the bundle setting, except we sometimes identify fibers $\mathcal{E}_x$ and $\mathcal{E}_y$ at nearby points x and y. This is automatic for a trivial bundle, and a detailed description of a suitable identification for a non-trivial bundle is given in [KS1, Section 2.2].

2.1. **Lyapunov metric.** We consider a $GL(d, \mathbb{R})$-valued cocycle $\mathcal{A}$ over (X, f, μ) as in the Oseledets Multiplicative Ergodic Theorem and denote the standard scalar product in $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$. We fix $\epsilon > 0$ and for any point $x \in \Lambda$ define the *Lyapunov scalar product* $\langle \cdot, \cdot \rangle_{x,\epsilon}$ on $\mathbb{R}^d$ as follows.

For $u \in \mathcal{E}_x^i,\ v \in \mathcal{E}_x^j,\ i \neq j$, we set $\langle u, v \rangle_{x,\epsilon} = 0$.

For $u, v \in \mathcal{E}_x^i,\ i = 1, \dots, m$, we define $\langle u, v \rangle_{x,\epsilon} = d \sum_{n \in \mathbb{Z}} \langle \mathcal{A}_x^n u, \mathcal{A}_x^n v \rangle\, e^{-2\lambda_i n - \epsilon |n|}$.

The series converges exponentially for any $x \in \Lambda$. The constant d in the formula allows a more convenient comparison with the standard scalar product. The norm generated by this scalar product is called the *Lyapunov norm* and is denoted by $\|.\|_{x,\epsilon}$. When ϵ is fixed we will denote the scalar product by $\langle \cdot, \cdot \rangle_x$ and the norm by $\|.\|_x$.

We summarize the main properties of the Lyapunov scalar product and norm, see [BP, Sections 3.5.1-3.5.3] for more details. A direct calculation shows [BP, Theorem 3.5.5] that for any $x \in \Lambda$ and any $u \in \mathcal{E}_x^i$,

$$e^{n\lambda_i - \epsilon|n|} \cdot \|u\|_{x,\epsilon} \leq \|\mathcal{A}_x^n u\|_{f^n x,\epsilon} \leq e^{n\lambda_i + \epsilon|n|} \cdot \|u\|_{x,\epsilon} \quad \text{for all } n \in \mathbb{Z}, \tag{2.1}$$

$$e^{n\lambda_+ - \epsilon n} \leq \|\mathcal{A}_x^n\|_{f^n x \leftarrow x} \leq e^{n\lambda_+ + \epsilon n} \quad \text{for all } n \in \mathbb{N}, \tag{2.2}$$

where $\lambda_+ = \lambda_m$ is the largest Lyapunov exponent and $\|.\|_{f^n x \leftarrow x}$ is the operator norm with respect to the Lyapunov norms defined as

$$\|A\|_{y \leftarrow x} = \sup \{ \|Au\|_{y,\epsilon} \cdot \|u\|_{x,\epsilon}^{-1} :\ 0 \neq u \in \mathbb{R}^d \}.$$

The Lyapunov scalar product and norm are defined only on Λ and, in general, depend only measurably on the point even if the cocycle is Hölder, so comparison with the standard norm is important. The lower bound follows easily from the definition: $\|u\|_{x,\epsilon} \geq \|u\|$. An upper bound is not uniform, but can be chosen to change slowly along the orbits [BP, Proposition 3.5.8]: there exists a measurable function $K_\epsilon(x)$ on Λ such that

$$\|u\| \leq \|u\|_{x,\epsilon} \leq K_\epsilon(x)\|u\| \qquad \text{for all } x \in \Lambda \text{ and } u \in \mathbb{R}^d, \quad \text{and} \tag{2.3}$$

$$K_\epsilon(x)e^{-\epsilon|n|} \leq K_\epsilon(f^n x) \leq K_\epsilon(x)e^{\epsilon|n|} \qquad \text{for all } x \in \Lambda \text{ and } n \in \mathbb{Z}. \tag{2.4}$$

For any matrix A and any points $x, y \in \Lambda$ inequalities (2.3) and (2.4) yield

$$K_\epsilon(x)^{-1}\|A\| \le \|A\|_{y\leftarrow x} \le K_\epsilon(y)\|A\| \,. \tag{2.5}$$

For any $\ell > 1$ we define the sets

$$\Lambda_{\epsilon,\ell} = \{x \in \Lambda : \quad K_\epsilon(x) \le \ell\}. \tag{2.6}$$

and note that $\mu(\Lambda_{\epsilon,\ell}) \to 1$ as $\ell \to \infty$. Without loss of generality we can assume that the set $\Lambda_{\epsilon,\ell}$ is compact and that Lyapunov splitting and Lyapunov scalar product are continuous on $\Lambda_{\epsilon,\ell}$. Indeed, by Luzin theorem we can always find a subset of $\Lambda_{\epsilon,\ell}$ satisfying these properties with arbitrarily small loss of measure.

2.2. **Pesin sets and Closing lemma.** Let f be a diffeomorphism of a compact manifold X and μ be an ergodic f-invariant Borel probability measure. We apply the Multiplicative Ergodic Theorem and construct the Lyapunov metric as above for the derivative cocycle $\mathcal{A}_x = D_x f$. For this cocycle, we will denote the corresponding set $\Lambda_{\epsilon,\ell}$ defined in the previous section by $\mathcal{R}_{\epsilon,\ell}$, which is often called a Pesin set.

Suppose now that the measure μ is hyperbolic, i.e. all Lyapunov exponents of the derivative cocycle Df with respect to μ are non-zero. We assume that there are both positive and negative such exponents. Otherwise μ is an atomic measure on a single periodic orbit [BP, Lemma 15.4.2], in which case our results are trivial. We denote by $\chi > 0$ the smallest absolute value for these exponents. We will fix $\epsilon > 0$ sufficiently small compared to χ and $\ell \in \mathbb{N}$ large enough so that the corresponding Pesin set $\mathcal{R}_{\epsilon,\ell}$ has positive measure. We will apply the following closing lemma. It does not use the splitting into individual Lyapunov sub-bundles $\mathcal{E}^i$, only the stable/unstable sub-ones, which are the sums of all Lyapunov sub-bundles corresponding to negative/positive Lyapunov exponents, respectively. Consequently, a cruder version of a Pesin set can be used instead of $\mathcal{R}_{\epsilon,\ell}$.

Lemma 2.1 (Closing Lemma)**.** [Kt], [BP, Lemma 15.1.2] *Let $f : X \to X$ be a $C^{1+Hölder}$ diffeomorphism preserving a hyperbolic Borel probability measure μ and let be $\chi > 0$ the smallest absolute value of its Lyapunov exponents. Then for any sufficiently large $\ell \in \mathbb{N}$, any sufficiently small $\epsilon > 0$, and any $\delta > 0$ there exist $\gamma = \gamma(\epsilon, \ell) \in (\epsilon, \chi - 2\epsilon)$ and $\beta = \beta(\delta, \epsilon, \ell) > 0$ such that if*

$$x \in \mathcal{R}_{\epsilon,\ell}, \;\; f^k x \in \mathcal{R}_{\epsilon,\ell} \;\; \textit{and} \;\; dist(x, f^k x) < \beta \;\; \textit{for some } k \in \mathbb{N},$$

then there exists a hyperbolic periodic point $p = f^k p$ such that

$$dist(f^i x, f^i p) \le \delta\, e^{-\gamma \min\{\,i,\, k-i\,\}} \quad \textit{for every } i = 0, \dots, k. \tag{2.7}$$

While the lemma is usually stated with a constant on the right hand side of (2.7), this constant can be absorbed using the choice of β. We will not use hyperbolicity of the periodic point p in the proof. In fact, for sufficiently large k the hyperbolicity of p can be recovered by applying our argument to the derivative cocycle Df.

3. Proof of Theorem 1.4

Let $\lambda_1 < \cdots < \lambda_m$ be the Lyapunov exponents of $\mathcal{A}$ with respect to μ, listed without multiplicities. We will denote the largest exponent λ_m by λ and second largest λ_{m-1} by λ'. Similarly, for any periodic point p we denote by $\lambda^{(p)}$ the largest Lyapunov exponent of $\mathcal{A}$ at p.

We fix $\epsilon' > 0$ sufficiently small compared to χ and $\ell' \in \mathbb{N}$ large enough so that the corresponding Pesin set $\mathcal{R}_{\epsilon',\ell'}$ for the derivative cocycle Df of the base system has positive measure. We apply the Closing Lemma 2.1 and get $\gamma = \gamma(\epsilon', \ell') > 0$.

If λ is not the only Lyapunov exponent of $\mathcal{A}$ with respect to μ, we define

$$\epsilon_0 = \min\{\, \alpha\gamma,\ (\lambda - \lambda')/4,\ \epsilon' \,\}, \tag{3.1}$$

and otherwise we set $\epsilon_0 = \min\{\, \alpha\gamma,\ \epsilon' \,\}$.

We fix $0 < \epsilon < \epsilon_0$ and consider the sets $\Lambda_{\ell,\epsilon}$ for the cocycle $\mathcal{A}$. We denote

$$P = \Lambda_{\ell,\epsilon} \cap \mathcal{R}_{\ell',\epsilon'}.$$

and fix ℓ sufficiently large so that $\mu(P) > 0$.

We take a point $x \in P$ which is in the support of μ restricted to P. Then for any $\beta > 0$ we have $\mu(P \cap B_{\beta/2}(x)) > 0$, where $B_{\beta/2}(x)$ is the open ball of radius $\beta/2$ centered at x. By Poincare recurrence there are infinitely many $k \in \mathbb{N}$ such that $f^k x \in P \cap B_{\beta/2}(x)$. For any such k we have: $x, f^k x \in P$ and $\mathrm{dist}(x, f^k x) < \beta$. Taking $\beta = \beta(\delta, \epsilon', \ell') > 0$ from the Closing Lemma 2.1 we obtain a hyperbolic periodic point satisfying (2.7). We can assume that $\beta \le \delta$, so that when δ is small so is β.

We conclude that for each $\delta > 0$ there exist arbitrarily large k such that $x, f^k x \in P$ and there is a hyperbolic periodic point $p = f^k p$ satisfying (2.7). Now we show that for such a point p with a sufficiently large k and a sufficiently small δ we have

$$|\lambda - \lambda^{(p)}| \le 3\epsilon. \tag{3.2}$$

To estimate $\lambda^{(p)}$ from above we use the fact [K, Lemma 3.1] that for such a point p

$$\|\mathcal{A}_p^k\| \le \ell\, e^{c\,\ell\delta^\alpha} e^{k(\lambda+\epsilon)}, \tag{3.3}$$

where the constant c depends only on $\mathcal{A}$ and on the number $(\alpha\gamma - \epsilon)$, which also follows from Lemma 4.3 below. Since we chose $\epsilon < \alpha\gamma$ we obtain

$$\lambda^{(p)} \le k^{-1} \log \|\mathcal{A}_p^k\| \le \lambda + \epsilon + k^{-1} \log(\ell\, e^{c\,\ell\delta^\alpha}) \le \lambda + 2\epsilon$$

provided that $\delta < 1$ and k is large enough compared to ℓ.

Now we estimate $\lambda^{(p)}$ from below. We denote $x_i = f^i x$ and $p_i = f^i p$, and we write $\|.\|_i$ for the Lyapunov norm at x_i. Since the Lyapunov norm may not exist at points p_i we will use the Lyapunov norms at the corresponding points x_i for the estimates. For each i we have the orthogonal splitting $\mathbb{R}^d = \mathcal{E}_i^{(1)} \oplus \mathcal{E}_i^{(2)}$, where $\mathcal{E}_i^{(1)} = \mathcal{E}_{x_i}^m$ is the Lyapunov space at x_i corresponding to the largest Lyapunov exponent $\lambda = \lambda_m$, and $\mathcal{E}_i^{(2)}$ is the direct sum of the other Lyapunov spaces at x_i. We will assume that λ is *not* the only Lyapunov exponent of $\mathcal{A}$, as otherwise $\mathcal{E}_i^{(2)} = \{0\}$ and the argument is simpler. For a vector $u \in \mathbb{R}^d$ we write $u = u_1 + u_2$, where $u_1 \in \mathcal{E}_i^{(1)}$ and $u_2 \in \mathcal{E}_i^{(2)}$.

We take $\theta = e^{\lambda' - \lambda + 4\epsilon} < 1$ by the choice of ϵ. For $i = 0, \dots, k$ we consider the cones

$$K_i = \{u \in \mathbb{R}^d : \|u_2\|_i \le \|u_1\|_i\} \quad \text{and} \quad K_i^\theta = \{u \in \mathbb{R}^d : \|u_2\|_i \le \theta \, \|u_1\|_i \}.$$

Now we show that there exist $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and all $i = 0, \dots, k-1$

$$\mathcal{A}_{p_i}(K_i) \subset K_{i+1}^\theta \quad \text{and} \quad \|\left(\mathcal{A}_{p_i} u\right)_1\|_{i+1} \ge e^{\lambda - 2\epsilon} \|u_1\|_i \;\text{ for each } u \in K_i. \tag{3.4}$$

We fix $0 \le i < k$ and a vector $u \in K_i$. Denoting $\mathcal{A}_{x_i} u = v = v_1 + v_2$ we get by (2.1)

$$e^{\lambda - \epsilon} \|u_1\|_i \le \|v_1\|_{i+1} \le e^{\lambda + \epsilon} \|u\|_i \quad \text{and} \quad \|v_2\|_{i+1} \le e^{\lambda' + \epsilon} \|u_2\|_i \,. \tag{3.5}$$

We write $\mathcal{A}_{p_i} = (\mathrm{Id} + \Delta_i)\, \mathcal{A}_{x_i}$, where $\Delta_i = \mathcal{A}_{p_i}(\mathcal{A}_{x_i})^{-1} - \mathrm{Id} = (\mathcal{A}_{p_i} - \mathcal{A}_{x_i})(\mathcal{A}_{x_i})^{-1}$.

Since both x_0 and x_k are in Λ_ℓ and $\mathrm{dist}(x_i, p_i) \le \delta e^{-\gamma \min\{i,\, k-i\}}$, using (2.5) we can estimate

$$\begin{aligned} \|\Delta_i\|_{x_{i+1} \leftarrow x_{i+1}} \;&\le\; K(x_{i+1}) \, \|\Delta_i\| \le K(x_{i+1}) \, \|\mathcal{A}_{p_i} - \mathcal{A}_{x_i}\| \cdot \|(\mathcal{A}_{x_i})^{-1}\| \;\le \\ &\le K(x_{i+1}) \cdot c_1 \, \mathrm{dist}(x_i, p_i)^\alpha \;\le\; \ell e^{\epsilon \min\{i+1,\, k-i-1\}} \cdot c_1 \delta^\alpha e^{-\alpha\gamma \min\{i,\, k-i\}} \;\le \\ &\le \ell c_1 \delta^\alpha e^\epsilon e^{(-\gamma\alpha + \epsilon) \min\{i,\, k-i\}} \le c_2 \ell \delta^\alpha \quad \text{since } -\gamma\alpha + \epsilon < 0. \end{aligned} \tag{3.6}$$

Since $\|u\|_i \le \sqrt{2}\, \|u_1\|_i$ we conclude using (3.5) that

$$\|\Delta_i \, v\|_{i+1} \le \|\Delta_i\|_{x_{i+1} \leftarrow x_{i+1}} \|v\|_{i+1} \le c_2 \ell \delta^\alpha e^{\lambda + \epsilon} \|u\|_i \le c_3 \ell \, \delta^\alpha \|u_1\|_i. \tag{3.7}$$

Setting $w = \mathcal{A}_{p_i} u = (\mathrm{Id} + \Delta_i) \mathcal{A}_{x_i} u = (\mathrm{Id} + \Delta_i) v$ we observe that

$$w_1 = v_1 + (\Delta_i v)_1 \quad \text{and} \quad w_2 = v_2 + (\Delta_i v)_2 \tag{3.8}$$

and hence using (3.5) and (3.7) we obtain that for small enough δ

$$\|w_1\|_{i+1} \ge \|v_1\|_{i+1} - \|\Delta_i v\|_{i+1} \ge e^{\lambda - \epsilon} \|u_1\|_i - c_3 \ell \, \delta^\alpha \|u_1\|_i \ge e^{\lambda - 2\epsilon} \|u_1\|_i \,,$$

which gives the inequality in (3.4). Similarly, using $\|u_2\|_i \le \|u_1\|_i$, we get

$$\begin{aligned} \|w_2\|_{i+1} \le \|v_2\|_{i+1} + \|\Delta_i v\|_{i+1} &\le e^{\lambda' + \epsilon} \|u_2\|_i + c_3 \ell \, \delta^\alpha \|u_1\|_i \le \\ &\le (e^{\lambda' + \epsilon} + c_3 \ell \, \delta^\alpha) \|u_1\|_i \le e^{\lambda' + 2\epsilon} \|u_1\|_i \end{aligned}$$

for all sufficiently small δ. Finally, if $u \neq 0$ we get that

$$\|w_2\|_{i+1} \,/\, \|w_1\|_{i+1} \le e^{\lambda' + 2\epsilon} / e^{\lambda - 2\epsilon} = e^{\lambda' - \lambda + 4\epsilon} = \theta.$$

This shows that $w \in K_{i+1}^\theta$ and the inclusion $\mathcal{A}_{p_i}(K_i) \subset K_{i+1}^\theta$ in (3.4) follows.

We conclude that (3.4) holds for each $i = 0, \dots, k-1$ and hence $\mathcal{A}_p^k(K_0) \subset K_k^\theta$. Since Λ_ℓ is chosen compact and so that the Lyapunov splitting and Lyapunov metric are continuous on it, the cones K_0^θ and K_k^θ are close if $\mathrm{dist}(x, f^k x) < \beta$ is small. Thus we can ensure that $K_k^\theta \subset K_0$ if β small enough and hence $\mathcal{A}_p^k(K_0) \subset K_0$. Finally, using the inequality in (3.4) for each $i = 0, \dots, k-1$ we obtain that for any $u \in K_0$

$$\|\mathcal{A}_p^k \, u\|_k \ge \|(\mathcal{A}_p^k \, u)_1\|_k \ge e^{k(\lambda - 2\epsilon)} \|u_1\|_0 \ge e^{k(\lambda - 2\epsilon)} \|u\|_0 / \sqrt{2} \ge e^{k(\lambda - 2\epsilon)} \|u\|_k / 2$$

since Lyapunov norms at x and $f^k x$ are close if δ and hence β is small enough. Since $\mathcal{A}_p^k u \in K_0$ for any $u \in K_0$, we can iterate $\mathcal{A}_p^k$ and use the inequality above to estimate $\lambda^{(p)}$ from below by the exponent of any $u \in K_0$:

$$\lambda^{(p)} \geq \lim_{n\to\infty} (nk)^{-1} \log \|\mathcal{A}_p^{nk}\, u\|_k \geq k^{-1} \cdot \lim_{n\to\infty} n^{-1} \log \left((e^{k(\lambda-2\epsilon)}/2)^n \, \|u\|_k \right) \geq$$

$$\geq k^{-1} \left[k(\lambda - 2\epsilon) - \log 2 \right] + k^{-1} \lim_{n\to\infty} n^{-1} \log \|u\|_k \geq (\lambda - 2\epsilon) - k^{-1} \log 2 \geq \lambda - 3\epsilon$$

provided that k is large enough. This completes the proof of the approximation of the largest exponent (3.2).

To approximate all Lyapunov exponents of $\mathcal{A}$ we consider cocycles $\wedge^i \mathcal{A}$ induced by $\mathcal{A}$ on the i-fold exterior powers $\wedge^i \mathbb{R}^d$, for $i = 1, \dots, d$. The largest Lyapunov exponent of $\wedge^i \mathcal{A}$ is $(\lambda_d + \dots + \lambda_{d-i+1})$, where $\lambda_1 \leq \dots \leq \lambda_d$ are the Lyapunov exponents of $\mathcal{A}$ *listed with multiplicities.* If a periodic point $p = f^k p$ satisfies

$$|(\lambda_d + \dots + \lambda_{d-i+1}) - (\lambda_d^{(p)} + \dots + \lambda_{d-i+1}^{(p)})| \leq 3\epsilon \quad \text{for } i = 1, \dots, d,$$

then we obtain the approximation $|\lambda_i - \lambda_i^{(p)}| \leq 3d\epsilon$ for all $i = 1, \dots, d$, completing the proof of Theorem 1.4. Such a periodic point exists since we can take a set

$$P = \mathcal{R}_{\epsilon',\ell'} \cap \Lambda^1_{\epsilon,\ell} \cap \dots \cap \Lambda^d_{\epsilon,\ell}$$

with $\mu(P) > 0$, where Λ^i_ℓ are the corresponding sets for all cocycles $\wedge^i \mathcal{A}$, $i = 1, \dots, d$. Then the previous argument applies yielding (3.2) for each $\wedge^i \mathcal{A}$.

A similar argument shows that one can obtain a simultaneous approximation of all Lyapunov exponents for several cocycles.

4. Subadditive cocycles and infinite-dimensional Lyapunov norm

4.1. **Subadditive cocycles and their exponents.** A *subadditive cocycle* over a dynamical system (X, f) is a sequence of functions $a_n : X \to \mathbb{R}$ such that

$$a_{n+k}(x) \leq a_k(x) + a_n(f^k x) \quad \text{for all } x \in X \text{ and } k, n \in \mathbb{N}.$$

For any ergodic measure-preserving transformation f of a probability space (X, μ) and any subadditive cocycle over f with integrable a_n, the Subadditive Ergodic Theorem yields that for μ almost all x

$$\lim_{n\to\infty} \frac{a_n(x)}{n} = \lim_{n\to\infty} \frac{a_n(\mu)}{n} = \inf_{n\in\mathbb{N}} \frac{a_n(\mu)}{n} =: \nu(a, \mu), \quad \text{where } a_n(\mu) = \int_X a_n(x) d\mu.$$

The limit $\nu(a, \mu) \geq -\infty$ is called the *exponent* of the cocycle a_n with respect to μ.

For a $GL(V)$-valued cocycle $\mathcal{A}$ over (X, f) it is easy to see that

$$a_n(x) = \log \|\mathcal{A}_x^n\| \quad \text{and} \quad \tilde{a}_n(x) = \log \|(\mathcal{A}_x^n)^{-1}\|. \tag{4.1}$$

are subadditive cocycles over f. For these continuous cocycles the Subadditive Ergodic Theorem gives the existence of the limits in (1.3): for μ almost all x

$$\lambda_+(\mathcal{A}, \mu) = \lim_{n\to\infty} n^{-1} \log \|\mathcal{A}_x^n\| = \lim_{n\to\infty} n^{-1} a_n(x) = \nu(a, \mu), \tag{4.2}$$

$$-\lambda_-(\mathcal{A},\mu) = \lim_{n\to\infty} n^{-1}\log\|(\mathcal{A}_x^n)^{-1}\| = \lim_{n\to\infty} n^{-1}\,\tilde a_n(x) = \nu(\tilde a,\mu). \tag{4.3}$$

It is easy to see that $\lambda_- \le \lambda_+$ and both are finite. Also, for μ almost all x

$$-\lambda_-(\mathcal{A},\mu) = \lim_{n\to\infty} n^{-1}\log\|\mathcal{A}_x^{-n}\| \tag{4.4}$$

since the integrals of $\log\|\mathcal{A}_x^{-n}\|$ and $\log\|(\mathcal{A}_x^n)^{-1}\|$ are equal. We denote

$$\Lambda = \Lambda(\mathcal{A},\mu) = \{x\in X:\ \text{equations (4.2), (4.3), and (4.4) hold}\}, \tag{4.5}$$

which implies that both equalities in (1.3) hold for $x \in \Lambda$. Clearly, $\mu(\Lambda) = 1$.

We will also use the following more detailed result on the behavior of subadditive cocycles established by A. Karlsson and G. Margulis.

Proposition 4.1. [KaM, Proposition 4.2] *Let $a_n(x)$ be an integrable subadditive cocycle with exponent $\lambda > -\infty$ over an ergodic measure-preserving system (X, f, μ). Then there exists a set $E \subset X$ with $\mu(E) = 1$ such that for each $x \in E$ and each $\epsilon > 0$ there exists an integer $L = L(x,\epsilon)$ so that the set $S = S(x,\epsilon,L)$ of integers n satisfying*

$$a_n(x) - a_{n-i}(f^i x) \ge (\lambda-\epsilon)i \quad \textit{for all } i \textit{ with } L \le i \le n \tag{4.6}$$

is infinite. (In fact, S has positive asymptotic upper density [GoKa]*).*

We will use the following corollary of this result.

Corollary 4.2. *Let $a_n^{(1)}(x), \dots, a_n^{(m)}(x)$ be integrable subadditive cocycles with exponents $\lambda^{(1)} > -\infty, \dots, \lambda^{(m)} > -\infty$, respectively, over an ergodic measure-preserving system (X, f, μ). Then there exists a set $E \subset X$ with $\mu(E) = 1$ such that for each $x \in E$ and each $\epsilon > 0$ there exists an integer $L = L(x,\epsilon)$ so that the set $S = S(x,\epsilon,L)$ of integers n satisfying the following condition is infinite:*

$$a_n^{(j)}(x) - a_{n-i}^{(j)}(f^i x) \ge (\lambda^{(j)}-\epsilon)i \quad \textit{for all } L \le i \le n \textit{ and } 1 \le j \le m. \tag{4.7}$$

Proof. We apply Proposition 4.1 to the subadditive cocycle

$$a_n(x) = a_n^{(1)}(x) + \cdots + a_n^{(m)}(x)$$

with exponent $\lambda = \lambda^{(1)} + \cdots + \lambda^{(m)} > -\infty$ and obtain the set E' for this cocycle. Then for each $x \in E'$ and each $\epsilon > 0$ there exists $L' = L'(x,\epsilon)$ and an infinite set $S = S(x,\epsilon,L')$ such that for all $n \in S$ we have

$$\sum_{j=1}^m \left(a_n^{(j)}(x) - a_{n-i}^{(j)}(f^i x)\right) \ge \left(\sum_{j=1}^m \lambda^{(j)} - \epsilon\right) i \quad \text{for all } L' \le i \le n. \tag{4.8}$$

By the Subadditive Ergodic Theorem there exists a set G of full measure such that for each $x \in G$ and each $j = 1, \dots, m$ we have $\lim_{n\to\infty} n^{-1} a_n^{(j)}(x) = \lambda^{(j)}$ and hence there exists $M = M(x,\epsilon)$ such that for

$$a_i^{(j)}(x) \le (\lambda^{(j)}+\epsilon)i \quad \text{for all } i \ge M \text{ and } 1 \le j \le m.$$

It follows by subadditivity that for all j and all $n \geq M$

$$a_n^{(j)}(x) - a_{n-i}^{(j)}(f^i x) \leq a_i^{(j)}(x) \leq (\lambda^{(j)} + \epsilon)i \quad \text{for all } \ M \leq i \leq n.$$

We set $L = L(x, \epsilon) = \max\{L', M\}$ and $E = G \cap E'$, with $\mu(E) = 1$. Subtracting the inequalities above for $j = 2, \dots, m$ from (4.8) we get that for all $x \in E$ and $n \in S$

$$a_n^{(1)}(x) - a_{n-i}^{(1)}(f^i x) \geq (\lambda^{(1)} - m\epsilon)i \quad \text{for all } \ i \text{ with } L \leq i \leq n.$$

The inequalities for the = cocycles $a_n^{(j)}$, $j = 2, \dots, m$, follow similarly. □

4.2. **Lyapunov norm for upper and lower Lyapunov exponents.** Since the Multiplicative Ergodic Theorem does not apply in the infinite dimensional setting, we use a cruder version of Lyapunov norm which takes into account only upper and lower Lyapunov exponents. We fix $\epsilon > 0$ and for any point $x \in \Lambda$ we define the *Lyapunov norm* of $u \in \mathcal{E}_x$ as follows:

$$\|u\|_x = \|u\|_{x,\epsilon} = \sum_{n=0}^{\infty} \|\mathcal{A}_x^n \, u\| \, e^{-(\lambda_+ + \epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}_x^{-n} \, u\| \, e^{(\lambda_- - \epsilon)n}. \tag{4.9}$$

By the definition (4.5) of Λ, both series converge exponentially. Properties of this norm were obtained in [KS2, Proposition 3.1]. They are similar to those of the usual Lyapunov norm discussed in Section 2.1. In particular,

$$\|\mathcal{A}_x^n\|_{x_n \leftarrow x_0} \leq e^{n(\lambda_+ + \epsilon)} \quad \text{and} \quad \|(\mathcal{A}_x^n)^{-1}\|_{x_0 \leftarrow x_n} \leq e^{n(-\lambda_- + \epsilon)} \quad \text{for } x \in \Lambda \tag{4.10}$$

and the ratio to the background norm $\|u\|_x / \|u\|$ is "tempered". Specifically, there exist an f-invariant set $\Lambda' \subset \Lambda$ with $\mu(\Lambda') = 1$ and a measurable function $K_\epsilon(x)$ on Λ' satisfying conditions (2.4) and (2.3). For any $\ell > 1$ we define

$$\Lambda_\ell = \{x \in \Lambda' : \ \ K(x) \leq \ell\}, \tag{4.11}$$

and note that $\mu(\Lambda_\ell) \to 1$ as $\ell \to \infty$.

We use this Lyapunov norm to obtain estimates similar to (4.10) for *any* point $p \in X$ whose trajectory is close to that of a point $x \in \Lambda_\ell$. Since the Lyapunov norm may not exist at points $f^i p$ we will use the Lyapunov norms at the corresponding points $f^i x$ for the estimates.

Lemma 4.3. [KS2, Lemma 4.1] *Let f be an ergodic invertible measure-preserving transformation of a probability space (X, μ), let $\mathcal{A}$ be an α-Hölder cocycle over f with the set $\{\mathcal{A}_x : x \in X\}$ bounded in $GL(V)$, and let λ_+ and λ_- be the upper and lower Lyapunov exponents of $\mathcal{A}$.*

Then for any $\epsilon > 0$ and γ with $\epsilon < \gamma\alpha$ there exists a constant $c = c\,(\mathcal{A}, \alpha\gamma - \epsilon)$ such that for any point x in $\Lambda_{\epsilon,\ell}$ with $f^k x$ in $\Lambda_{\epsilon,\ell}$ and any point $p \in X$ such that the orbit segments $x, fx, \dots, f^k x$ and $p, fp, \dots, f^k p$ satisfy with some $\delta > 0$

$$dist(f^i x, f^i p) \leq \delta e^{-\gamma \min\{i, k-i\}} \quad \text{for every } \ i = 0, \dots, k \tag{4.12}$$

we have for every $i = 0, \dots, k$

$$\|\mathcal{A}_p^i\| \le \ell \, \|\mathcal{A}_p^i\|_{x_i \leftarrow x_0} \le \ell \, e^{c\,\ell\delta^\alpha} e^{i(\lambda_+ + \epsilon)} \quad \text{and} \tag{4.13}$$

$$\|(\mathcal{A}_p^i)^{-1}\| \le \ell e^{\epsilon \min\{i,\, k-i\}} \|(\mathcal{A}_p^i)^{-1}\|_{x_0 \leftarrow x_i} \le \ell e^{\epsilon \min\{i,\, k-i\}} e^{c\,\ell\delta^\alpha} e^{i(-\lambda_- + \epsilon)}. \tag{4.14}$$

5. Proof of Theorem 1.5

We fix a sufficiently small $\epsilon' > 0$ and sufficiently large $\ell' \in \mathbb{N}$ so that $\mu(\mathcal{R}_{\epsilon',\ell'}) > 0.9$ and so that we can apply the Closing Lemma 2.1 and get $\gamma = \gamma(\epsilon', \ell') > 0$. We define $\epsilon_0 = \min\{\alpha\gamma/4,\, \epsilon'\} > 0$. We fix $0 < \epsilon < \epsilon_0$ and consider the sets $\Lambda_{\ell,\epsilon}$ for the cocycle $\mathcal{A}$. We choose ℓ sufficiently large so that $\mu(P) > 0.8$ where

$$P = \Lambda_{\ell,\epsilon} \cap \mathcal{R}_{\ell',\epsilon'}.$$

We apply Corollary 4.2 to the subadditive cocycles

$$a_n^{(1)}(x) = a_n(x) = \log \|\mathcal{A}_x^n\| \quad \text{and} \quad a_n^{(2)}(x) = \tilde{a}_n(x) = \log \|(\mathcal{A}_x^n)^{-1}\|$$

and obtain the set E of full measure for these two cocycles.

We take a compact set $K \subset (P \cap E)$ with $\mu(K) > 0.7$, let ν be the restriction of μ to K, and denote by G the support of ν. Then G is a compact subset of K satisfying $\nu(X \setminus G) = 0$, $\nu(G) = \mu(K) > 0.7$, and $\nu(U) > 0$ for any subset U (relatively) open in G. We fix a countable basis $\{U_j\}$, $U_j \subset G$, for the topology of G and note that $\mu(U_j) = \nu(U_j) > 0$. We denote by G' the subset of full μ-measure in G on which the Birkhoff Ergodic Theorem holds for the indicator functions of each U_j, that is for each $x \in G'$

$$\lim_{n\to\infty} n^{-1} \left|\{i : 0 \le i \le n-1 \text{ and } f^i x \in U_j\}\right| = \mu(U_j) > 0 \quad \text{for all } j \in \mathbb{N}. \tag{5.1}$$

For the remainder of the proof we fix a point $x \in G'$ and choose $\sigma = 4\epsilon/(\alpha\gamma)$. We establish the following lemma, which is a refinement of [GG, Lemma 8] for our setting.

Lemma 5.1. *For each* $x \in G'$, $\beta > 0$ *and* $\sigma > 0$ *there exists an integer* $N = N(x, \sigma, \beta)$ *such that for each* $n \ge N$ *there is an integer* k *satisfying*

$$n(1+\sigma) \le k \le n(1+2\sigma), \qquad f^k x \in G, \quad \text{and} \quad dist(x, f^k x) < \beta.$$

Proof. We fix $x \in G'$ and $\beta > 0$ and consider a set U_j from the countable base that is contained in the open ball $B(x, \beta/2)$. Since $x \in G'$, condition (5.1) holds for x. Denoting the set of the return times i to U_j by T and letting $\phi(n) = |\,T \cap [0, n-1]\,|$ we get that $\phi(n)/n \to t = \mu(U_j) > 0$. For any $\sigma > 0$ we can take $c > 0$ such that

$$(1+c)/(1-c) < (1+2\sigma)/(1+\sigma)$$

and then M such that

$$(1-c)tn < \phi(n) < (1+c)tn \quad \text{for all } n \ge M.$$

Using this and the fact that $(1-c)(1+2\sigma)-(1+c)(1+\sigma)>0$ by the choice of c, we obtain that there exists $N>M$ such that for all $n\ge N$ we have

$$\phi(n(1+2\sigma))-\phi(n(1+\sigma))>[(1-c)(1+2\sigma)-(1+c)(1+\sigma)]\,tn>1,$$

which means there exists k between $n(1+\sigma)$ and $n(1+2\sigma)$ such that f^kx is in $U_j\subset(G\cap B(x,\beta/2))$. $\square$

Since $x\in G'\subset E$, by Corollary 4.2 there exists an integer $L=L(x,\epsilon)$ such that for any $n\in S_x$ and any i with $L\le i\le n$

$$a_n(x)-a_{n-i}(f^ix)\ge(\lambda_+-\epsilon)i \quad\text{and}\quad \tilde a_n(x)-\tilde a_{n-i}(f^ix)\ge(-\lambda_--\epsilon)i. \tag{5.2}$$

We conclude that for any $\beta>0$ there exist arbitrarily large n for which the above property holds and a corresponding $k=k(n)$ satisfying the conclusion of Lemma 5.1. We will later choose $\delta>0$ sufficiently small so that (5.14) below is satisfied, and take $\beta=\beta(\delta,\ell,\gamma)>0$ from the Closing Lemma 2.1, which then gives a periodic point $p=f^kp$ such that

$$\operatorname{dist}(f^ix,f^ip)\le\delta e^{-\gamma\,\min\{i,\,k-i\}} \quad\text{for every } i=0,\dots,k. \tag{5.3}$$

Obtaining upper estimates for $\|\mathcal{A}_p^k\|$ and $\|(\mathcal{A}_p^k)^{-1}\|$. Since x and p satisfy (5.3) we can apply Lemma 4.3 with $i=k$ to get

$$\|\mathcal{A}_p^k\|\le\ell e^{c\,\ell\delta^\alpha}e^{k(\lambda_++\epsilon)} \quad\text{and}\quad \|(\mathcal{A}_p^k)^{-1}\|\le\ell e^{c\,\ell\delta^\alpha}e^{k(-\lambda_-+\epsilon)}.$$

For k sufficiently large compared to ℓ and $c=c(\mathcal{A},\,\alpha\gamma-\epsilon)$, we obtain

$$k^{-1}\log\|\mathcal{A}_p^k\|\le\lambda_++\epsilon+k^{-1}(\log\ell+c\,\ell\delta^\alpha)\le\lambda_++2\epsilon, \quad\text{and} \tag{5.4}$$

$$k^{-1}\log\|(\mathcal{A}_p^k)^{-1}\|\le-\lambda_-+\epsilon+k^{-1}(\log\ell+c\,\ell\delta^\alpha)\le-\lambda_-+2\epsilon. \tag{5.5}$$

Obtaining a lower estimate for $\|\mathcal{A}_p^k\|$. First we bound $\|\mathcal{A}_x^n-\mathcal{A}_p^n\|$.

$$\begin{aligned}
\mathcal{A}_x^n-\mathcal{A}_p^n &= \mathcal{A}_{x_1}^{n-1}\circ(\mathcal{A}_x-\mathcal{A}_p)+(\mathcal{A}_{x_1}^{n-1}-\mathcal{A}_{p_1}^{n-1})\circ\mathcal{A}_p\\
&= \mathcal{A}_{x_1}^{n-1}\circ(\mathcal{A}_x-\mathcal{A}_p)+\big(\mathcal{A}_{x_2}^{n-2}\circ(\mathcal{A}_{x_1}-\mathcal{A}_{p_1})+(\mathcal{A}_{x_2}^{n-2}-\mathcal{A}_{p_2}^{n-2})\circ\mathcal{A}_{p_1}\big)\circ\mathcal{A}_p\\
&= \mathcal{A}_{x_1}^{n-1}\circ(\mathcal{A}_x-\mathcal{A}_p)+\mathcal{A}_{x_2}^{n-2}\circ(\mathcal{A}_{x_1}-\mathcal{A}_{p_1})\circ\mathcal{A}_p+(\mathcal{A}_{x_2}^{n-2}-\mathcal{A}_{y_2}^{n-2})\circ\mathcal{A}_p^2\\
&=\dots=\sum_{i=0}^{n-1}\mathcal{A}_{x_{i+1}}^{n-(i+1)}\circ(\mathcal{A}_{x_i}-\mathcal{A}_{p_i})\circ\mathcal{A}_p^i.
\end{aligned}$$

Hence we can estimate the norm as follows

$$\|\mathcal{A}_x^n-\mathcal{A}_p^n\|\le\sum_{i=0}^{n-1}\|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\|\cdot\|\mathcal{A}_{x_i}-\mathcal{A}_{p_i}\|\cdot\|\mathcal{A}_p^i\|. \tag{5.6}$$

Since n satisfies (5.2) with $a_n(x)=\log\|\mathcal{A}_x^n\|$,

$$\|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\|\le\|\mathcal{A}_x^n\|\,e^{-(i+1)(\lambda_+-\epsilon)} \quad\text{for all } i \text{ with } L\le i\le n-1. \tag{5.7}$$

Since $n < k$, applying Lemma 4.3 we get

$$\|\mathcal{A}_p^i\| \le \ell\, e^{c\,\ell\delta^\alpha} e^{i(\lambda_+ + \epsilon)} \quad \text{for } i = 0, \dots, n. \tag{5.8}$$

Using (5.3) and Hölder continuity of $\mathcal{A}$ we obtain

$$\|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \le M \operatorname{dist}(x_i, p_i)^\alpha \le M(\delta\, e^{-\gamma \min\{i,k-i\}})^\alpha = M\delta^\alpha\, e^{-\alpha\gamma \min\{i,k-i\}}.$$

We claim that the exponent satisfies

$$\alpha\gamma \min\{i, k-i\} \ge 4\epsilon i \quad \text{for } \ i = 0, \dots, n. \tag{5.9}$$

If $i = \min\{i, k-i\}$ this holds since $\epsilon < \alpha\gamma/4$. If $k - i = \min\{i, k-i\}$ then

$$\alpha\gamma \min\{i, k-i\} = \alpha\gamma(k-i) \ge 4\epsilon i \quad \text{is equivalent to} \quad i \le k/(1 + 4\epsilon/(\alpha\gamma)),$$

which holds for $i \le n$ since $n \le k/(1+\sigma)$ and $\sigma = 4\epsilon/(\alpha\gamma)$. Thus we conclude that

$$\|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \le M\delta^\alpha\, e^{-4\epsilon i} \quad \text{for } i = 0, \dots, n. \tag{5.10}$$

Combining (5.7), (5.10), and (5.8) we obtain that for $L \le i \le n-1$

$$\begin{aligned} &\|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\| \cdot \|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \cdot \|\mathcal{A}_p^i\| \le \\ &\le \|\mathcal{A}_x^n\|\, e^{-(i+1)(\lambda_+ - \epsilon)} \cdot M\delta^\alpha\, e^{-4\epsilon i} \cdot \ell\, e^{c\,\ell\delta^\alpha} e^{i(\lambda_+ + \epsilon)} < C_1(\delta)\, \|\mathcal{A}_x^n\|\, e^{-\epsilon i}, \end{aligned} \tag{5.11}$$

where $C_1(\delta) = \delta^\alpha M \ell\, e^{c\,\ell\delta^\alpha - \lambda_+ + \epsilon}$, and we conclude that

$$\begin{aligned} &\sum_{i=L}^{n-1} \|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\| \cdot \|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \cdot \|\mathcal{A}_p^i\| \le \\ &\le\ C_1(\delta)\, \|\mathcal{A}_x^n\| \sum_{i=L}^{n-1} e^{-\epsilon i}\ \le\ C_1(\delta)\, \|\mathcal{A}_x^n\|\, \frac{1}{1 - e^{-\epsilon}}\ =\ C_2(\delta)\, \|\mathcal{A}_x^n\|. \end{aligned} \tag{5.12}$$

Since the set $\{\mathcal{A}_x : x \in X\}$ is bounded in $GL(V)$, there exists $\lambda_* \le \lambda_- \le \lambda_+$ such that $\|(\mathcal{A}_x)^{-1}\| \le e^{-\lambda_*}$ for all $x \in X$. For $i < L$ we estimate

$$\|\mathcal{A}_{x_i}^{n-i}\| \le \|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^i)^{-1}\| \le \|\mathcal{A}_x^n\|\, e^{-\lambda_* i}.$$

Then using (5.8) and (5.10) we obtain

$$\begin{aligned} &\sum_{i=0}^{L-1} \|\mathcal{A}_{x_{i+1}}^{n-(i+1)}\| \cdot \|\mathcal{A}_{x_i} - \mathcal{A}_{p_i}\| \cdot \|\mathcal{A}_p^i\| \le \\ &\le \sum_{i=0}^{L-1} \|\mathcal{A}_x^n\|\, e^{-(i+1)\lambda_*} \cdot M\delta^\alpha e^{-4\epsilon i} \cdot \ell e^{c\,\ell\delta^\alpha} e^{i(\lambda_+ + \epsilon)} \le \\ &\le L \cdot \delta^\alpha M \ell e^{c\,\ell\delta^\alpha}\, e^{-\lambda_* + (\lambda_+ - \lambda_*)L} \cdot \|\mathcal{A}_x^n\| = C_3(\delta)\, \|\mathcal{A}_x^n\|. \end{aligned} \tag{5.13}$$

Combining estimates (5.6), (5.12) and (5.13) we obtain

$$\|\mathcal{A}_x^n - \mathcal{A}_p^n\|\ \le\ \|\mathcal{A}_x^n\|\, (C_2(\delta) + C_3(\delta))\ \le\ \|\mathcal{A}_x^n\|/2$$

since by the choice of $\delta > 0$ we have

$$C_2(\delta) + C_3(\delta) = \delta^\alpha M \ell e^{c\ell\delta^\alpha} \left((1-e^{-\epsilon})^{-1} e^{-\lambda_+ + \epsilon} + L e^{-\lambda_* + (\lambda_+ - \lambda_*)L}\right) < 1/2. \tag{5.14}$$

Hence

$$\|\mathcal{A}_p^n\| \ \geq\ \|\mathcal{A}_p^n\| - \|\mathcal{A}_x^n - \mathcal{A}_p^n\| \ \geq\ \|\mathcal{A}_x^n\|/2 > e^{(\lambda_+ - \epsilon)n}/2,$$

provided that n is sufficiently large for the limit in (4.2). Since $\mathcal{A}_p^n = (\mathcal{A}_{f^n p}^{k-n})^{-1} \circ \mathcal{A}_p^k$,

$$\|\mathcal{A}_p^n\| \leq \|(\mathcal{A}_{f^n p}^{k-n})^{-1}\| \cdot \|\mathcal{A}_p^k\| \leq \|\mathcal{A}_p^k\|\, e^{-\lambda_*(k-n)}.$$

Hence

$$\|\mathcal{A}_p^k\| \geq e^{\lambda_*(k-n)} \|\mathcal{A}_p^n\| > e^{(\lambda_+ - \epsilon)n + \lambda_*(k-n)}/2 > e^{(\lambda_+ - \epsilon)k - (\lambda_+ - \lambda_*)(k-n)}/2, \text{ and so}$$

$$k^{-1} \log \|\mathcal{A}_p^k\| > k^{-1}[(\lambda_+ - \epsilon)k - (\lambda_+ - \lambda_*)(k-n) - \log 2].$$

Since $k - n < 2\sigma n < 2\sigma k$ and $\sigma = 4\epsilon/(\alpha\gamma)$ we obtain

$$k^{-1} \log \|\mathcal{A}_p^k\| > \lambda_+ - \epsilon - k^{-1} \log 2 - (\lambda_+ - \lambda_*)2\sigma > \lambda_+ - 2\epsilon - 8\epsilon(\lambda_+ - \lambda_*)/(\alpha\gamma)$$

if n and hence k are sufficiently large.

Since $0 < \epsilon < \epsilon_0$ is arbitrary and λ_+, $\lambda_*, \alpha, \gamma$ do not depend on ϵ, this inequality above together with (5.4) imply the approximation of $\lambda_+(\mathcal{A}, \mu)$ by $k^{-1} \log \|\mathcal{A}_p^k\|$.

Obtaining a lower estimate for $\|(\mathcal{A}_p^k)^{-1}\|$. We use the equation

$$(\mathcal{A}_x^n)^{-1} - (\mathcal{A}_p^n)^{-1} \ =\ \sum_{i=0}^{n-1} (\mathcal{A}_p^i)^{-1} \circ \left((\mathcal{A}_{x_i})^{-1} - (\mathcal{A}_{p_i})^{-1}\right) \circ (\mathcal{A}_{x_{i+1}}^{n-(i+1)})^{-1}$$

and estimate the norm of the difference similarly to (5.6):

$$\|(\mathcal{A}_x^n)^{-1} - (\mathcal{A}_p^n)^{-1}\| \ =\ \sum_{i=0}^{n-1} \|(\mathcal{A}_p^i)^{-1}\| \cdot \|(\mathcal{A}_{x_i})^{-1} - (\mathcal{A}_{p_i})^{-1}\| \cdot \|(\mathcal{A}_{x_{i+1}}^{n-(i+1)})^{-1}\|. \tag{5.15}$$

Since n satisfies the second part of (5.2) with $\tilde{a}_n(x) = \log \|(\mathcal{A}_x^n)^{-1}\|$,

$$\|(\mathcal{A}_{x_{i+1}}^{n-(i+1)})^{-1}\| \leq \|(\mathcal{A}_x^n)^{-1}\|\, e^{-(i+1)(-\lambda_- - \epsilon)} \quad \text{for all } i \text{ with } L \leq i \leq n.$$

Since $n < k$, using (4.14) of Lemma 4.3 we get

$$\|(\mathcal{A}_p^i)^{-1}\| \leq \ell e^{\epsilon \min\{i, k-i\}}\, e^{c\ell\delta^\alpha} e^{i(-\lambda_- + \epsilon)} \leq \ell\, e^{c\ell\delta^\alpha} e^{i(-\lambda_- + 2\epsilon)} \qquad \text{for } i = 1, \dots, n.$$

Using Hölder continuity and the exponent estimate (5.9) we obtain as in (5.10) that

$$\|(\mathcal{A}_{x_i})^{-1} - (\mathcal{A}_{p_i})^{-1}\| \leq M \operatorname{dist}(x_i, p_i)^\alpha \leq M\delta^\alpha\, e^{-\alpha\gamma \min\{i, k-i\}} \leq M\delta^\alpha\, e^{-4\epsilon i}$$

for $i = 1, \dots, n$. Combining these estimates we obtain for the terms in (5.15) the same estimate as in (5.11)

$$\|(\mathcal{A}_p^i)^{-1}\| \cdot \|(\mathcal{A}_{x_i})^{-1} - (\mathcal{A}_{p_i})^{-1}\| \cdot \|(\mathcal{A}_{x_{i+1}}^{n-(i+1)})^{-1}\| \leq C_1(\delta)\, \|(\mathcal{A}_x^n)^{-1}\|\, e^{-\epsilon i} \tag{5.16}$$

for all i with $L \leq i \leq n-1$. The remainder of the argument is essentially identical to estimating $\|\mathcal{A}_p^k\|$ with $\mathcal{A}$ replaced by $\mathcal{A}^{-1}$.

It is clear from the argument that we can choose arbitrarily large n and hence k.

Simultaneous approximation for several cocycles. Finally, we remark on how to obtain simultaneous approximation for cocycles $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(m)}$ in Remark 1.6. First, we take the set $\Lambda_{\epsilon,\ell}$ to be the intersection of the corresponding sets for all $\mathcal{A}^{(j)}$ and define the set $P = \Lambda_{\epsilon,\ell} \cap \mathcal{R}_{\epsilon',\ell'}$ as in the proof. We also take E to be the full measure set given by Corollary 4.2 for $2m$ subadditive cocycles

$$a^{(j)}(x) = \log \|\mathcal{A}^{(j)}\|, \quad \tilde{a}^{(j)}(x) = \log \|(\mathcal{A}^{(j)})^{-1}\|, \quad j = 1, \dots, m.$$

Then the argument goes through showing that the constructed point $p = f^k p$ gives the approximation of the upper and lower exponents for all cocycles $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(m)}$.

References

[B] L. Backes. *On the periodic approximation of Lyapunov exponents for semi-invertible cocycles.* Preprint arXiv:1612.04159

[BP] L. Barreira and Ya. Pesin. *Nonuniformly hyperbolicity: dynamics of systems with nonzero Lyapunov exponents.* Encyclopedia of Mathematics and Its Applications **115**, Cambridge University Press, 2007.

[GG] G. Grabarnik and M. Guysinsky. *Livšic theorem for Banach rings.* Discrete and Continuous Dynamical Systems, Vol. 37, Issue 8 (2017), 4379-4390.

[GoKa] S. Gouëzel and A. Karlsson. *Subadditive and multiplicative ergodic theorems.* Preprint.

[K] B. Kalinin. *Livšic Theorem for matrix cocycles.* Annals of Math. 173 (2011), no. 2, 1025-1042.

[KS1] B. Kalinin and V. Sadovskaya. *Cocycles with one exponent over partially hyperbolic systems.* Geometriae Dedicata, 167, Issue 1 (2013), 167-188.

[KS2] B. Kalinin and V. Sadovskaya. *Periodic approximation of Lyapunov exponents for Banach cocycles.* To appear in Ergodic Theory and Dynamical Systems.

[KaM] A. Karlsson and G. Margulis. *A multiplicative ergodic theorem and nonpositively curved spaces.* Communications in Mathematical Physics **208** (1999) 107-123.

[Kt] A. Katok. *Lyapunov exponents, entropy and periodic points of diffeomorphisms.* Publ. Math. IHES, 51, 137-173, 1980.

[P1] Ya. Pesin. *Families of invariant manifolds corresponding to nonzero characteristic exponents.* Math. USSR-Izv. 40 (1976) 1261-1305.

[P2] Ya. Pesin. *Characteristic Ljapunov exponents and smooth ergodic theory.* Russ. Math. Surv. 32 (1977) 55-114.

[Po] Mark Pollicott. *Lectures on Ergodic Theory and Pesin Theory on Compact Manifolds.* Cambridge University Press, 1993.

[O] V. Oseledets. *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems.* Trans. Mosc. Math. Soc. 19 (1968), 197-221.

[WS] Z. Wang and W. Sun. *Lyapunov exponents of hyperbolic measures and hyperbolic periodic orbits.* Trans. Amer. Math. Soc., 362 (2010) no. 8, 4267-4282, .

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