Linear cocycles over hyperbolic systems: periodic data and rigidity

Boris Kalinin

Anosov diffeomorphisms

f – a diffeomorphism of a compact Riemannian manifold $\mathcal{M}$.

Definition: f is **Anosov** if there exist
a continuous invariant decomposition $T\mathcal{M} = E^s \oplus E^u$
and constants $K > 0$, $\lambda > 0$ such that for all $n \in \mathbb{N}$,

$$\|Df^n(v)\| \le Ke^{-\lambda n}\|v\| \quad \text{for all } v \in E^s,$$
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Basic examples: Anosov automorphisms of tori.
A – a hyperbolic matrix in $SL(d, \mathbb{Z})$ (no eigenvalue of modulus 1)
$A : \mathbb{R}^d \to \mathbb{R}^d$ projects to an automorphism of $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

Periodic points

Let $f : \mathcal{M} \to \mathcal{M}$ be a transitive Anosov diffeomorphism.

Anosov Closing Lemma. If $\text{dist}(f^n x, x) \le \epsilon$, then there exists $p = f^n p$ such that $\text{dist}(f^i p, f^i x) \le C\epsilon$ for $i = 0, \dots, n$.

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Livšic Theorem. Let $\alpha : \mathcal{M} \to \mathbb{R}$ be a Hölder function. Then

$$\alpha(x) = \varphi(fx) - \varphi(x) \qquad (*)$$

has a Hölder continuous solution φ if and only if whenever $f^n p = p$

$$\textstyle\sum_{i=0}^{n-1} \alpha(f^i p) = 0$$

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Corollary. f preserves a volume iff $\det Df^n(p) = 1$ for $f^n p = p$.

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Fix a volume form ω and let $J_\omega(x)$ be the Jacobian of f w.r.t. ω.

If $\omega' = \frac{1}{c(x)}\omega$ then $J_{\omega'}(x) = c(fx)^{-1} c(x) J_\omega(x)$.

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f preserves $\omega' \Leftrightarrow J_{\omega'}(x) = 1$ for all $x \Leftrightarrow J_\omega(x) = c(fx)\,c(x)^{-1}$

$\Leftrightarrow (*)$ with $\alpha(x) = \log J_\omega$ and $\varphi(x) = \log c(x)$.

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A solution exists if and only if

$$0 = \sum_{i=0}^{n-1} \alpha(f^i p) = \log\det Df^n(p) \quad \text{whenever} p = f^n p.$$

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with Hölder continuous Riemannian metric;

$F : \mathcal{E} \to \mathcal{E}$ a Hölder continuous linear cocycle over f, i.e.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E} \\ P\downarrow & & P\downarrow \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M} \end{array}$$

and $F_x : \mathcal{E}_x \to \mathcal{E}_{fx}$ is a linear isomorphism
which depends Hölder continuously on x.

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Example: Derivative cocycle.

The differential Df is a cocycle on the tangent bundle $\mathcal{E} = T\mathcal{M}$.
If $\mathcal{E}'$ is a Df-invariant sub-bundle, then $Df|_{\mathcal{E}'}$ is also a cocycle.

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$\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, so $\mathcal{E}_x = \mathcal{E}_{fx} = \mathbb{R}^d$, $F_x \in GL(d, \mathbb{R})$,
and F can be viewed as a Hölder function $F : \mathcal{M} \to GL(d, \mathbb{R})$.

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More generally, let $f : \mathcal{M} \to \mathcal{M}$ be a hyperbolic system:
Anosov diffeomorphism, locally maximal hyperbolic set,
or a symbolic system such as subshift of finite type.

Periodic data of a cocycle

$F : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over f.
For a periodic point $p = f^n p$ in $\mathcal{M}$, consider the return map

$$F_p^n = F_{f^{n-1}p} \circ \cdots \circ F_{fp} \circ F_p : \quad \mathcal{E}_p \to \mathcal{E}_p$$

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In particular, what can be said about F based on Lyapunov exponents at periodic points?

The Lyapunov exponents of F at a periodic point $p = f^n p$ are given by eigenvalues of F_n^p:

$$\lambda_i^{(p)} = \frac{1}{n} \log |i^{th} \text{ eigenvalue of } F_p^n|$$

Oseledets' Multiplicative Ergodic Theorem (1965)

Let f be an ergodic measure preserving transformation of a Lebesgue probability measure space (X, μ) and let $F : X \to GL(d, \mathbb{R})$ be a measurable cocycle over f.

If $\log \|F_x\|, \log \|F_x^{-1}\| \in L^1(X, \mu)$ then there exist numbers $\lambda_1 < \cdots < \lambda_l$, an f-invariant set $\mathcal{R}$ of full measure, and an F-invariant decomposition of $\mathbb{R}^d$ for $x \in \mathcal{R}$

$$\mathbb{R}^d_x = E_{\lambda_1}(x) \oplus \cdots \oplus E_{\lambda_l}(x)$$

such that for any nonzero $v \in E_{\lambda_i}(x)$, $\lim\limits_{n \to \pm\infty} \frac{1}{n} \log \|F_x^n v\| = \lambda_i$.

The numbers $\lambda_1, \dots, \lambda_l$ are called the **Lyapunov exponents** of F.

Periodic approximation of Lyapunov exponents

Theorem (B.K. 2008)

Let $f : \mathcal{M} \to \mathcal{M}$ *be a hyperbolic system.*
Let $F : \mathcal{E} \to \mathcal{E}$ *be a Hölder continuous linear cocycle over* f*.*
Let μ *be an ergodic invariant measure for* f*.*

Then the Lyapunov exponents $\lambda_1 \leq \ldots \leq \lambda_d$ *of* F *with respect to* μ *(listed with multiplicities) can be approximated by the Lyapunov exponents of* F *at periodic points.*

More precisely, for any $\epsilon > 0$ *there exists a periodic point* $p \in \mathcal{M}$ *for which the Lyapunov exponents* $\lambda_1^{(p)} \leq \ldots \leq \lambda_d^{(p)}$ *of* F *satisfy* $|\lambda_i - \lambda_i^{(p)}| < \epsilon$ *for* $i = 1, \ldots, d$*.*

Corollary

Let F be a Hölder linear cocycle over a hyperbolic system f.

Suppose that for each periodic point $p = f^n p$ the largest Lyapunov exponent of F at p is at most λ.

Then for every $\epsilon > 0$ there exists a constant C_ϵ such that for all $x \in M$ and $n \in \mathbb{N}$

$$\|F_x^n\| \leq C_\epsilon e^{(\lambda+\epsilon)n}$$

The largest Lyapunov exponents of F with respect to μ:

$$\lambda_+(F, \mu) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\| \qquad \text{for } \mu \text{ a.e. } x \in \mathcal{M}.$$

Conformality

Quasiconformal distortion

$$K_F(x,n) = \|F_x^n\| \cdot \|(F_x^n)^{-1}\| = \frac{\max\{\,\|\,F_x^n\,(v)\,\|\!: \, v\!\in\!\mathcal{E}_x,\ \|v\|\!=\!1\,\}}{\min\{\,\|\,F_x^n\,(v)\,\|\!: \, v\!\in\!\mathcal{E}_x,\ \|v\|\!=\!1\,\}}$$

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Three equivalent necessary conditions: whenever $f^n p = p$,

(1) F_p^n is conformal with respect to an inner product on $\mathcal{E}_p$;

(2) F_p^n is diagonalizable over $\mathbb{C}$ with eigenvalues equal in modulus;

(3) $K_F(p,n) \leq C(p)$ for all n such that $f^n p = p$.

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Theorem (B.K., V. Sadovskaya)

Suppose that the fibers of $\mathcal{E}$ are two-dimensional. If F satisfies (1) or (2) or (3) at each periodic point, then F is ***conformal*** *with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.*

Conformality in higher dimension

The Theorem does not hold in dimension ≥ 3:

There exists F such that at every periodic point
F_p^n is *isometric* with respect to an inner product on $\mathcal{E}_p$,
but F is not conformal with respect to any continuous metric.

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If there exists a constant C_{per} such that

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Isometries

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In general: if there exists a constant C'_{per} such that

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Isometric at periodic points, but not conformal, in dim 3

Example (dim $\mathcal{E}_x \geq 3$)

There exists $F : \mathcal{E} \to \mathcal{E}$ such that whenever $f^n p = p$
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Let $\mathcal{E} = \mathcal{M} \times \mathbb{R}^3$, $\qquad F_x = \begin{bmatrix} \cos\alpha(x) & -\sin\alpha(x) & \epsilon \\ \sin\alpha(x) & \cos\alpha(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

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Let S be a closed f-invariant set in $\mathcal{M}$ without periodic points;

$\alpha : \mathcal{M} \to \mathbb{R}, \quad \alpha(x) = 0$ for $x \in S$ and $0 < \alpha(x) \leq \epsilon$ for $x \notin S$

For $x \in S$, $\quad F_x^n = \begin{bmatrix} 1 & 0 & \mathbf{n}\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$K_F(x, n) \to \infty$ as $n \to \infty \implies$
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At $p = f^n p$, $\quad F_p^n$ is diagonalizable with eigenvalues of modulus 1.

Local rigidity for Anosov diffeomorphisms

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f is called **locally rigid** if conjugacy Df_p^n and $Dg_{h(p)}^n$ implies smoothness of h for every C^1-small perturbation g.

Local rigidity for Anosov automorphisms

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Yes if $Df^n|_{E^s(p)}$ and $Df^n|_{E^u(p)}$ are conformal and
$\dim E^u = \dim E^s = 2$ (B.K, V. Sadovskaya)

Local rigidity for Anosov automorphisms

Theorem (Gogolev, B.K., Sadovskaya)

Let $L : \mathbb{T}^d \to \mathbb{T}^d$ *be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus. Then* L *is* **locally rigid**, *more precisely*

If g *is a* C^1*-small perturbation of* L *with conjugate periodic data, then* g *is* $C^{1+Hölder}$ *conjugate to* L*.*

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Proposition

Toral automorphisms satisfying the assumptions are **generic**: *the proportion of matrices L in $SL(d, \mathbb{Z})$ with $\|L\| \le T$ that do* **not** *satisfy the assumptions can be estimated by $cT^{-\delta}$ for some $\delta > 0$.*

Role of conformality

Let L be an Anosov automorphism of $\mathbb{T}^d$. $L \in SL(d, \mathbb{Z})$.

Let $1 < \rho_1 < \rho_2 < \cdots < \rho_m$ be the distinct moduli of the unstable eigenvalues of L.

The corresponding splitting: $E^{u,L} = E_1^L \oplus E_2^L \oplus \cdots \oplus E_m^L$

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Since L is irreducible, it is diagonalizable over $\mathbb{C}$, and hence L is conformal on each E^L_i.

As the periodic data are conjugate, the cocycle $Dg|_{E^g_i}$ is conformal at the periodic points. Since $\dim E^g_i = \dim E^L_i \leq 2$, the Theorem implies that g is conformal on E^g_i.

Conformality of g allows to establish smoothness of h along E^L_i.

Proposition (V. Sadovskaya)

Let $F : \mathcal{M} \to GL(2,\mathbb{R})$ be an orientation-preserving cocycle such that for each $p = f^n p$, the eigenvalues of F_p^n are equal in modulus.

Then, possibly after passing to a double cover, F is conjugate to

$$k(x)\begin{bmatrix} 1 & \beta(x) \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad k(x)Id \quad \text{or} \quad k(x)\begin{bmatrix} \cos\beta(x) & -\sin\beta(x) \\ \sin\beta(x) & \cos\beta(x) \end{bmatrix}$$

$F : \mathcal{E} \to \mathcal{E}$ a Hölder continuous linear cocycle, $\dim \mathcal{E}_x = d$.

Theorem (B.K, V. Sadovskaya)

Suppose that for each periodic point $p = f^n p$, the eigenvalues of F_p^n are equal in modulus.

Structural Theorem

$F : \mathcal{E} \to \mathcal{E}$ a Hölder continuous linear cocycle, $\dim \mathcal{E}_x = d$.

Theorem (B.K, V. Sadovskaya)

Suppose that for each periodic point $p = f^n p$, the eigenvalues of F_p^n are equal in modulus. Then there exists a flag of Hölder continuous F-invariant sub-bundles

$$\mathcal{E}^1 \subset \ldots \subset \mathcal{E}^{k-1} \subset \mathcal{E}^k = \mathcal{E}$$

and Hölder continuous Riemannian metrics on $\mathcal{E}^1$ and on the factor bundles $\mathcal{E}^{i+1}/\mathcal{E}^i$, $i = 1, \ldots, k-1$, such that

- *$F|_{\mathcal{E}_1}$ is conformal and*
- *the factor-maps induced by F on $\mathcal{E}^{i+1}/\mathcal{E}^i$ are conformal.*

If the flag is trivial then F is conformal on $\mathcal{E}$.

Structural Theorem – Special Case

If there are d continuous vector fields which give bases for all $\mathcal{E}^i$, then F is Hölder cohomologous to a cocycle of the form

$$\begin{bmatrix} A_1(x) & * & \dots & * \\ 0 & A_2(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k(x) \end{bmatrix}$$

$A_i(x) = l_i(x) O_i(x)$ is a scalar multiple of an orthogonal matrix.

Polynomial growth

Quasiconformal distortion of F

$$K_F(x,n) = \|F_x^n\| \cdot \|(F_x^n)^{-1}\| = \frac{\max\{\,\|\,F_x^n\,(v)\,\|:\; v\in\mathcal{E}_x,\ \|v\|=1\,\}}{\min\{\,\|\,F_x^n\,(v)\,\|:\; v\in\mathcal{E}_x,\ \|v\|=1\,\}}$$

Theorem

Suppose that for each periodic point $p = f^n p$,
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Then $\mathbf{K_F(x,n) \le Cn^{2m}}$ *for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$.*

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If the eigenvalues of F_p^n are of modulus 1 whenever $p = f^n p$,

then $\|\mathbf{F_x^n}\| \leq \mathbf{Cn^m}$ *for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$.*

$\mathbf{m} =$ the number of non-trivial sub-bundles in the flag $\leq d-1$.