ON THE CLASSIFICATION OF CARTAN ACTIONS

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Abstract. We study higher rank Cartan actions on compact manifolds preserving an ergodic measure with full support. In particular, we classify actions by $\mathbb{R}^k$ with $k \geq 3$ whose one-parameter groups act transitively as well as nondegenerate totally nonsymplectic $\mathbb{Z}^k$-actions for $k \geq 3$.

1. Introduction

The classification of Anosov systems is a deep and central problem in dynamics. For single diffeomorphisms, a long outstanding conjecture asserts that they are all topologically conjugate to automorphisms of tori, nilmanifolds and finite factors of such. Little progress has been made since Franks, Manning, and Newhouse proved this for Anosov diffeomorphisms on tori and nilmanifolds, and for codimension one Anosov diffeomorphisms on arbitrary compact manifolds [5, 24, 25]. There is no analogue to this conjecture for flows. Geodesic flows of closed manifolds of negative curvature are always Anosov. There are many non-algebraic examples of such manifolds such as the ones constructed by Gromov and Thurston [8]. Moreover, various examples of Anosov flows not even topologically conjugate to geodesic flows have been constructed by Franks and Williams [6] and Handel and Thurston [14]. Recently, Benoist found a new source for Anosov flows via geodesic flows of Hilbert metrics on compact quotients of convex sets in projective space [1]. For single Anosov diffeomorphisms and flows one can easily change the derivative at periodic points. Thus topological conjugacies are rarely smooth. As Farrell and Jones have constructed Anosov diffeomorphisms on exotic tori, one cannot even hope for smooth classification of the underlying manifold structure [4].

The situation is quite different for higher rank Anosov actions, i.e. actions of higher rank abelian groups such that at least one element acts normally hyperbolically to the orbit foliation. The simplest examples of such actions arise from products of Anosov diffeomorphisms or flows. More interestingly, there are $\mathbb{Z}^k$-actions on tori and nilmanifolds by automorphisms, and Anosov $\mathbb{R}^k$-actions on homogeneous spaces G/Λ by left translations, and more generally on bi-quotients. These form the class of *algebraic actions*. Many of these are not products. The irreducible higher rank algebraic actions enjoy very strong rigidity properties such as scarcity of invariant measures and cocycle rigidity (see surveys in [23, 26]).

Intriguingly, the only known examples of higher rank Anosov actions are either algebraic or are reducible, i.e. some finite cover admits an Anosov flow or diffeomorphism as a factor.

* Supported in part by NSF grants DMS-0140513.
** Supported in part by NSF grant DMS-0203735.

By work of Palis and Yoccoz, the centralizer of a generic Anosov diffeomorphism f on a torus consists just of the powers f^n of f [27]. A. Katok and the second author showed that C^1-small perturbations of higher-rank algebraic Anosov actions with semisimple linear parts are *smoothly* conjugate to the original action [21]. This followed earlier work by Katok and Lewis for the special case of a maximal commuting set of toral automorphisms [15]. Katok and Lewis also showed a global rigidity result for suitable higher rank actions on tori [16]. Recently Damjanovic and Katok generalized local rigidity to partially hyperbolic actions on tori using KAM arguments [3]. Rodriguez Hertz classified abelian actions with an Anosov element for the 3-torus under additional conditions on the action on homology. These are all smoothly conjugate to a linear action by automorphisms [10]. All of these results motivate the following conjecture:

Conjecture 1.1. *All irreducible higher rank $\mathbb{Z}^k$ and $\mathbb{R}^k$ Anosov actions on any compact manifold have finite covers* smoothly *conjugate to an algebraic action.*

While this conjecture remains wide open in this generality, we will prove strong classification results for the subclass of Cartan actions in this paper. Cartan actions are Anosov actions such that the maximal non-trivial intersections of stable manifolds of distinct elements are one-dimensional. This paper and part of its approach was motivated by similar results by E. Goetze and the second author for Cartan actions by higher rank semisimple groups and their lattices [7].

Our main technical result is the following theorem. We call a one-parameter subgroup of a Lyapunov hyperplane in $\mathbb{R}^k$ *generic* if it is not contained in any other Lyapunov hyperplane. Call an $\mathbb{R}^k$ Cartan action *totally Cartan* if the set of Anosov elements is dense in $\mathbb{R}^k$.

Theorem 1.2. *Let α be a totally Cartan action of $\mathbb{R}^k$, $k \geq 2$, on a compact smooth manifold M preserving an ergodic probability measure μ with full support. Suppose that every Lyapunov hyperplane contains a generic one-parameter subgroup with a dense orbit. Then there exists a Hölder continuous Riemannian metric g on M such that for any $a \in \mathbb{R}^k$ and any Lyapunov exponent χ*

$$\|a_*(v)\| = e^{\chi(a)}\|v\| \qquad \text{for any } \ v \in E_\chi.$$

We use this theorem to get the following classification of $\mathbb{R}^k$ Cartan actions.

Theorem 1.3. *Let α be a C^∞ totally Cartan action of $\mathbb{R}^k$, $k \geq 3$, on a compact smooth connected manifold M preserving an $\mathbb{R}^k$-ergodic probability measure μ with full support. Suppose that every one-parameter subgroup of $\mathbb{R}^k$ has a dense orbit. Then α is C^∞ conjugate to an almost algebraic action, i.e. the lift of the action to some finite cover of M is C^∞ conjugate to an $\mathbb{R}^k$-action by left translations on a homogeneous space G/Λ for some Lie group G and cocompact lattice Λ.*

The major difficulty in this work is the construction of a Hölder metric on the various Lyapunov foliations which is expanded and contracted precisely according to a linear functional. The proof requires a new idea of construction. This is closely linked to cohomology triviality for cocycles. Indeed, Proposition 3.1 says precisely that the restriction of the derivative cocycle in a Lyapunov direction is Hölder cohomologous to a linear functional. Cohomology triviality has been established for general cocycles for homogeneous actions [19, 18, 26]. Nothing however seems to be known for general actions. Our approach here is specific to the derivative cocycle, and draws on the proof of the Livsic' theorem. Let us comment that in [7], Goetze and the second author used topological super-rigidity techniques to trivialize the derivative cocycle for actions of semi-simple groups. These techniques are not available for abelian Anosov actions. We also note that our results may provide another approach to the main results of [7].

Finally, we will apply a technical variation Theorem 5.1 of our Theorem 1.3 to classify certain $\mathbb{Z}^k$ Cartan actions. Call a Cartan action *totally nonsymplectic* or TNS if no two nonzero Lyapunov exponents are negatively proportional. Further call an Anosov action *non-degenerate* if the intersection of two Lyapunov hyperplanes is never contained in a third Lyapunov hyperplane.

Corollary 1.4. *Let α be a C^∞ nondegenerate TNS Cartan action of $\mathbb{Z}^k$, $k \geq 3$, on a compact smooth manifold M such that each non-trivial element is an Anosov diffeomorphism. Suppose also that one of the diffeomorphisms is transitive. Then a finite cover of α is C^∞ conjugate to a $\mathbb{Z}^k$ action by automorphisms of a torus.*

In this corollary, the requirement that the action is TNS Cartan is equivalent to the requirement that all Lyapunov exponents are simple and there are no proportional Lyapunov exponents.

The second author would like to thank E. Goetze for numerous discussions related to this problem which yielded partial results and suggested part of the approach in this paper.

2. Basic Structures

2.1. **Anosov actions.**
Let us recall the definitions and basic properties of Anosov actions.

Definition 2.1. *Let α be a locally faithful action of $\mathbb{R}^k$ by smooth diffeomorphisms on a compact manifold M. Call an element $a \in \mathbb{R}^k$* Anosov *or* normally hyperbolic *for α if there exist real constants $\lambda > 0$, $C > 0$ and a continuous α-invariant splitting of the tangent bundle*

$$TM = E^u_a \oplus E^0 \oplus E^s_a$$

such that E^0 is the tangent distribution of the $\mathbb{R}^k$-orbits, and for all $p \in M$, for all $v \in E^s_a(p)$ ($v \in E^u_a(p)$ respectively) and $n > 0$ ($n < 0$ respectively) the differential $a_* : TM \to TM$ satisfies

$$\| a^n_*(v) \| \leq Ce^{-\lambda|n|} \| v \| .$$

Hirsch, Pugh and Shub introduced the notion of a diffeomorphism acting normally hyperbolically with respect to an invariant foliation. Our Anosov elements are precisely the elements in $\mathbb{R}^k$ which act normally hyperbolically with respect to the orbit foliation of $\mathbb{R}^k$ [11]. By [11], we can define stable and unstable distributions E^s_a and E^u_a for any Anosov element $a \in \mathbb{R}^k$. These are Hölder distributions and integrate in the usual fashion to stable and unstable foliations which we will denote by $\mathcal{W}_a^s$ and $\mathcal{W}_a^u$. These are Hölder foliations with C^∞-leaves. (cf. [11] for all this).

The set of Anosov elements $\mathcal{A}$ in $\mathbb{R}^k$ is always an open subset of $\mathbb{R}^k$. In fact, by the structural stability theorem for normally hyperbolic maps by Hirsch, Pugh and Shub a map C^1-close to a normally hyperbolic map is again normally hyperbolic for a suitable foliation [11]. For an element in $\mathbb{R}^k$ close to an Anosov element, this suitable foliation is forced to be the orbit foliation of the action.

Definition 2.2. Call α an *Anosov action* if some element $a \in \mathbb{R}^k$ is Anosov. Furthermore call α *totally Anosov* if the set of Anosov elements $\mathcal{A}$ is dense in $\mathbb{R}^k$.

It is not known if all Anosov actions are totally Anosov. We will make the standing assumption that all our actions are totally Anosov and preserve a probability measure with full support unless otherwise specified.

2.2. **Lyapunov theory.** Recall that for any diffeomorphism ϕ of a compact manifold M preserving an ergodic probability measure μ, there are finitely many numbers χ^i and a measurable splitting of the tangent $TM = \bigoplus E^i$ such that the forward and backward Lyapunov exponents of $v \in E^i$ are exactly χ^i. This is the Lyapunov decomposition of TM for ϕ.

Now consider an $\mathbb{R}^k$ action α on a compact manifold M by diffeomorphisms preserving an ergodic probability measure μ. Then we can refine the Lyapunov decompositions of the individual elements $a \in \mathbb{R}^k$ to a joint invariant splitting.

Proposition 2.3. *There are finitely many linear functionals χ on $\mathbb{R}^k$, a set of full measure $\mathcal{P}$ and a measurable splitting of the tangent bundle $TM = \bigoplus E^\chi$ over $\mathcal{P}$, invariant under α, such that for all $a \in \mathbb{R}^k$ and $v \in E^\chi$, the Lyapunov exponent of v is $\chi(a)$, i.e.*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \| a_*^n(v) \| = \chi(a)$$

where $\| .. \|$ is any continuous norm on TM.

We call $\bigoplus E^\chi$ the *Lyapunov splitting* and the nonzero linear functionals χ the *Lyapunov exponents* or *weights* of α. We will call the hyperplanes $\ker\chi$ the *Lyapunov hyperplanes* or *Weyl chamber walls*, and the connected components of $\mathbb{R}^k - \cup_\chi \ker\chi$ the *Weyl chambers* of α.

Define the *coarse Lyapunov space* $E_\chi = \oplus E^\lambda$, where the sum ranges over all positive multiples $\lambda = c\,\chi$ of χ. We will also denote E_χ by E_H where H is the half space of $\mathbb{R}^k$ on which χ is negative. Call such a half space H a *Lyapunov half space*. Note that H is determined by the hyperplane $\ker\chi$ and an orientation of this hyperplane, i.e. a choice of one of the two half spaces $\ker\chi$ bounds. In our case, the orientation is given by which of the two half spaces χ is negative on. Then we obtain a measurable decomposition $TM = \oplus E_H$ where H ranges over all Lyapunov half spaces. Note that for an $\mathbb{R}$-action we just retrieve the stable and unstable distributions.

2.3. **Coarse Lyapunov foliations.** In this section we consider a totally Anosov $\mathbb{R}^k$-action α preserving an ergodic probability measure μ with full support. We show that the coarse Lyapunov splitting given by μ can be extended to a Hölder splitting of TM consisting of distributions tangent to foliations which we will call the coarse Lyapunov foliations. We also show that the Lyapunov hyperplanes, Weyl chambers, and the coarse Lyapunov foliations agree for all invariant measures.

First we observe that for each Anosov element $a \in \mathbb{R}^k$ we have $E^s_a = \bigoplus_{\chi(a)<0} E^\chi$ and $E^u_a = \bigoplus_{\chi(a)>0} E^\chi$ at any point of the set of full measure where the Lyapunov splitting is defined. Note that the following proposition applies to actions more general than totally Anosov.

Proposition 2.4. *Let α be an Anosov $\mathbb{R}^k$-action on a compact smooth manifold M preserving an ergodic probability measure μ. Suppose that every Weyl chamber defined by μ contains an Anosov element. Then for each Lyapunov half space H the distribution*

$$\bigcap_{a\in\mathcal{A}\cap H} E^s_a$$

is Hölder continuous on the support of μ and coincides μ-a.e. with the coarse Lyapunov space E_H. If $supp\,\mu = M$ then E_H extends to a Hölder continuous distribution on M tangent to the Hölder foliation $\mathcal{W}_H := \bigcap_{a\in\mathcal{A}\cap H} \mathcal{W}^s_a$ with C^∞ leaves.

Proof: For a Lyapunov half space H let $L_H = \bigcap_{a\in\mathcal{A}\cap H} E^s_a$. This defines a distribution on all of M, possibly discontinuous and of varying dimension. By ergodicity and invariance of the Lyapunov splitting under α, there is a set $\mathcal{P}$ of full measure where the Lyapunov splitting is defined and the dimensions of the Lyapunov distributions are constant. We will show that $L_H = E_H$ on $\mathcal{P}$, and that L_H is continuous on the support of μ, $supp\,\mu$. Since L_H is the intersection of Hölder foliations, it follows easily that L_H is Hölder. Hence if $supp\,\mu = M$, the distribution L_H is Hölder continuous on M and tangent to the corresponding foliation $\mathcal{W}_H$. Since foliations $\mathcal{W}^s_a$ have C^∞ leaves so does $\mathcal{W}_H$.

Let us first show that $L_H = E_H$ on the set $\mathcal{P}$ of full measure. Let $p \in \mathcal{P}$. First note that $E_H(p) \subset L_H(p)$ since, by definition, E_H is contained in every $E^s_a(p)$ of the intersection. To prove the reverse inclusion, suppose that for some $v \in T_pM$, $v \in L_H(p)$ and $v \notin E_H(p)$. Using the coarse Lyapunov splitting $T_pM = \bigoplus E_{H_i}(p)$ we decompose $v = \sum v_i$. Then $v_i \neq 0$ for some H_i different from H. Since every Weyl chamber contains an Anosov element there exists $a \in \mathcal{A} \cap H \cap (-H_i)$. This implies that $v_i \in E_{H_i}(p) \subset E^u_a(p)$ which contradicts the fact that $v \in L_H \subset E^s_a(p)$.

Now we will show the continuity of L_H on $supp\,\mu$. Since $L_H = E_H$ on $\mathcal{P}$, the dimensions of all distributions L_H are constant on $\mathcal{P}$. It follows easily that these distributions are continuous on $\mathcal{P}$ and form the direct sum $\bigoplus L_H$ of dimension $(\dim M - k)$ over $\mathcal{P}$, where the summation is over all Lyapunov half spaces.

Now consider a point $q \in supp\,\mu$ and a sequence of points $p_n \in \mathcal{P}$ converging to q. For each Lyapunov half space H let R_H be some limit point of $L_H(p_n)$ in the Grassman bundle of subspaces of dimension $\dim L_H$. To prove the continuity it suffices to show that $R_H = L_H(q)$ for each H. The continuity of distributions E^s_a implies that $R_H \subset L_H(q)$ for all H.

To prove the reverse inclusion first notice that the sum $\sum L_H(q)$ is contained in $E^s_b \oplus E^u_b$ for any Anosov element b. In particular $\dim \sum L_H(q) \leq n-k$. Now suppose that for some H the dimension of $L_H(q)$ is greater than that of R_H. Then the sum $\sum L_H(q)$ is no longer direct. Thus for some H the intersection $L_H(q) \cap L$ is nontrivial for $L = \sum L_{H'}(q)$, where the sum is not assumed to be direct and the summation is over all Lyapunov half spaces H' different from H. We can choose two Anosov elements a and b such that H is the only Lyapunov half space that contains both a and b. This can be done as follows. Restrict the action to a generic 2-plane P which intersects all Lyapunov hyperplanes in distinct lines. Then for each of the two half lines of $\partial H \cap P$ there is exactly one Weyl chamber in H whose intersection with P borders the half line. Now we can take one Anosov element from each of these two Weyl chambers.

Let $E_1 = E^s_a(q) \cap L$ and $E_2 = E^u_a(q) \cap L$. Note that by definition any space $L_{H'}(q)$ is contained either in $E^s_a(q)$ or in $E^s_{-a}(q) = E^u_a(q)$. Hence both E_1 and E_2 are sums of subspaces $L_{H'}$ for various H'. It follows that $L = E_1 \oplus E_2$. Also note that $E_1 \subset E^u_b(q)$. Indeed, if some $L_{H'} \subset E_1$ then $L_{H'} \subset E^s_a(q)$ and hence $a \in H'$. If H' is different from H then, by the choice of elements a and b, we must have $b \notin H'$ and thus $L_{H'} \subset E^u_b(q)$.

Let v be a nonzero vector in the nontrivial intersection $L_H(q) \cap L$. We write $v = v_1 + v_2$, with $v_1 \in E_1, v_2 \in E_2$, and iterate v by na. Since $v \in L_H(q) \subset E^s_a(q)$ and $v_1 \in E_1 \subset E^s_a(q)$ we see that $(na)v \to 0$ and $(na)v_1 \to 0$. Since $v_2 \in E_2 \subset E^u_a(q)$ we conclude that $v_2 = 0$ and $v = v_1$. Since $v \in L_H(q) \subset E^s_b(q)$ and $v_1 \in E_1 \subset E^u_b(q)$ we conclude that $v = v_1 = 0$. This shows that $R_H = L_H(q)$ for each H and completes the proof of the proposition. $\diamond$

Lemma 2.5. *The set $\mathcal{A}$ of Anosov elements for α is the union of the Weyl chambers in $\mathbb{R}^k$.*

Proof: Suppose that $a, b \in \mathcal{A}$ belong to the same Weyl chamber $\mathcal{C}$. Since $E^s_a = \bigoplus_{\chi(a)<0} E^\chi$, we get $E^s_a = E^s_b$. By commutativity, the stable distributions of Anosov elements are invariant under α. Let $c = t\,a + s\,b$ where $s, t > 0$ are real numbers. If $v \in E^s_a(p)$ for $p \in M$, then the derivative $c^n_* = (ns\,b)_* \circ (nt\,a)_*$ contracts v exponentially fast as $(nt\,a)_*$ does and $(nt\,a)_*(v) \in E^s_b((nt\,a)(p))$. We conclude that E^s_c is defined and $E^s_c = E^s_a = E^s_b$. Similarly, $E^u_c = E^u_a = E^u_b$ and hence c is Anosov. Thus the intersection of $\mathcal{A}$ with $\mathcal{C}$ is an open and dense convex cone in $\mathcal{C}$. Therefore $\mathcal{C} \subset \mathcal{A}$. Clearly, no element on a Weyl chamber wall can be in $\mathcal{A}$. $\diamond$

The elements of $\mathbb{R}^k$ which belong to the union of the Weyl chambers are called *regular*. All other elements of $\mathbb{R}^k$ are called *singular*. A singular element is called *generic* if belongs to only one Lyapunov hyperplane.

For a singular element $a \in \mathbb{R}^k$ we can define its neutral, stable, and unstable distributions as

$$E_a^0 = T\mathcal{O} \oplus \bigoplus_{\chi(a)=0} E^\chi \qquad E_a^s = \bigoplus_{\chi(a)<0} E^\chi \qquad E_a^u = \bigoplus_{\chi(a)>0} E^\chi.$$

Lemma 2.6. *The distributions E_a^0, E_a^s, E_a^u are Hölder continuous. E_a^s and E_a^u integrate to Hölder continuous foliations $\mathcal{W}_a^s$ and $\mathcal{W}_a^u$ with smooth leaves. E_a^s is uniformly contracted and E_a^u is uniformly expanded by a.*

Proof: The Hölder continuity of the distributions follows immediately from Proposition 2.4. To show the integrability we note that $E_a^s = \bigcap E_b^s$ and $E_a^u = \bigcap E_b^u$, where the intersection is taken over all Anosov elements b close to a. Indeed, if b is close enough to a, the signs of $\chi(b)$ and $\chi(a)$ are the same for any Lyapunov exponent χ with $\chi(a) \neq 0$. This shows that E_a^s is contained in every E_b^s and thus in the intersection. For the reverse inclusion we note that for every nonzero Lyapunov exponent χ with $\chi(a) = 0$ we can choose an Anosov element b close to a such that $\chi(b) > 0$. The uniform contraction and expansion can be obtained as in the proof of Lemma 2.5 since a can be represented as a positive combination of the nearby Anosov elements. $\diamond$

Remark 2.7. *In contrast to the individual distributions E^χ, E_a^0 is not necessarily integrable. Moreover, we do not assume any uniform estimates on the possible expansion or contraction of E_a^0 by a, so a is not necessarily a partially hyperbolic element in the usual sense.*

Now we will show that the structures of Lyapunov hyperplanes and Weyl chambers agree for all invariant measures. Note that this does not entail that the Lyapunov functionals are the same. Indeed, they need not be as the case of products of Anosov flows easily shows.

Proposition 2.1. *Suppose that the Lyapunov splitting and the Lyapunov exponents exist at a point p. Then the Lyapunov hyperplanes and Weyl chambers defined by these exponents coincide with the Lyapunov hyperplanes and Weyl chambers defined by the exponents of the ergodic invariant measure with full support. Moreover, the coarse Lyapunov splitting of T_pM defined by the exponents at p coincides with the Hölder continuous coarse Lyapunov splitting defined in Proposition 2.4.*

Proof: It suffices to show that for any Lyapunov half space H defined by the ergodic invariant measure with full support and for any $v \in E_H(p)$ the Lyapunov exponent $\chi(\cdot, v)$: $\mathbb{R}^k \to \mathbb{R}$ has kernel ∂H and is negative on H. Suppose that this is not the case. Then there exists $b \in H$ such that $\chi(b, v) > 0$. Since the Anosov elements are dense in $\mathbb{R}^k$ we may choose b to be Anosov, i.e. $b \in \mathcal{A} \cap H$. Then by the definition of E_H, $v \in E_H = \bigcap_{a \in \mathcal{A} \cap H} E_a^s \subset E_b^s$. But for $v \in E_b^s$, $\chi(b, v) > 0$ is impossible. $\diamond$

We immediately get that the Lyapunov half spaces on $\mathcal{P}$ are consistent with those at all the periodic points.

Corollary 2.8. *The Lyapunov hyperplanes, Weyl chambers, and coarse Lyapunov splitting for any compact orbit of the action coincide with the Lyapunov hyperplanes, Weyl chambers, and coarse Lyapunov splitting defined by the ergodic invariant measure with full support.*

2.4. **Cartan Actions.** Here we define Cartan actions which are closely related to Hurder's trellised actions [12, 13].

Definition 2.9. Call a (totally) Anosov action of $\mathbb{R}^k$ a *(totally) Cartan* if all coarse Lyapunov foliations are one-dimensional.

Totally Cartan actions satisfy the following properties tantamount to being a trellised action. First, let us call two foliations *pairwise transverse* if their tangent spaces intersect trivially. This is different from the standard notion in differential topology which also requires the sum of the tangent spaces to span the tangent space of the manifold.

Consider a totally Cartan action of $\mathbb{R}^k$ on M. Then the coarse Lyapunov foliations $\{\mathcal{W}_i\}$ form a collection of one dimensional, pairwise transverse foliations such that

1. the tangent distributions have internal direct sum $T\mathcal{W}_1 \oplus \cdots \oplus T\mathcal{W}_r \oplus T\mathcal{O} \cong TM$, where $T\mathcal{O}$ is the distribution tangent to the $\mathbb{R}^k$ orbits,
2. for each $x \in M$ the leaf $\mathcal{W}_i(x)$ of $\mathcal{W}_i$ through x is a C^∞ immersed submanifold of M,
3. the C^∞ immersions $\mathcal{W}_i(x) \to M$ depend uniformly Hölder continuously on the base-point x in the C^∞ topology on immersions, and
4. each $\mathcal{W}_i$ is invariant under every $a \in \mathbb{R}^k$.

3. Proof of Theorem 1.2

It is clearly sufficient to show the existence of such a metric for each coarse Lyapunov distribution. After that, the desired metric on M can be obtained using these metrics and the natural metric on the orbit distribution, by requiring that the coarse Lyapunov distributions and the orbit distribution are pairwise orthogonal. The metric on the orbit distribution can be defined as follows. For $v \in T_x\mathcal{O}$ define $||v|| = ||b||_{\mathbb{R}^k}$, where $b \in \mathbb{R}^k$ is such that $v = \frac{d}{dt}((tb)x)$. Thus the theorem reduces to the following proposition.

Proposition 3.1. *Let χ be a Lyapunov exponent, H be its negative Lyapunov half space, and $E = E_H$ be the corresponding one-dimensional coarse Lyapunov distribution. Under the assumptions of Theorem 1.2 there exists a Hölder continuous Riemannian metric on E for which*

$$||a_*(v)|| = e^{\chi(a)}||v||$$

for any $a \in \mathbb{R}^k$ and $v \in E$.

Proof: Let $\mathcal{W} = \mathcal{W}_H$ be the coarse Lyapunov foliation of the coarse Lyapunov distribution E. Denote by E' and $\mathcal{W}'$ the (possibly trivial) coarse Lyapunov distribution and the coarse Lyapunov foliation corresponding to the Lyapunov half space $-H$.

Notations. In Sections 3 and 4 for any element $b \in \mathbb{R}^k$ we denote by $D^E_x b$ the restriction of its derivative at $x \in M$ to $E(x)$. We fix some background Riemannian metric g_0 and denote the norm of $D^E_x b$ with respect to g_0 by $d^E_x b = ||D_x b(v)||_{bx} \cdot ||v||_x^{-1}$, where $v \in E_x$ and $||.||_x$ is the norm given by g_0 at x.

By the assumption there exist an element $a_0 \in \partial H$ not contained in any other Lyapunov hyperplane and a point x^* such that the orbit $\mathcal{O}^* := \{(ta_0)x^*\}$ is dense. We define a new metric g^* on E over $\mathcal{O}^*$ by taking the background metric g_0 on E_{x^*} and propagating it along this dense orbit by the derivative $D^E_{x^*}(ta_0)$. By the construction, the derivative $D^E_x(ta_0)$ is isometric with respect to g^* for any t and any $x \in \mathcal{O}^*$.

The main part of the proof is to show that the metric g^* is Hölder continuous on $\mathcal{O}^*$ and thus extends to a Hölder continuous Riemannian metric g on the whole distribution E. Clearly, such g is also preserved by $D^E_x(ta_0)$ for any t and any $x \in M$. For any other element $b \in \mathbb{R}^k$ consider the metric $b_* g$. By commutativity, this metric is again preserved by ta_0 for any t. Since E is one-dimensional and $\mathcal{O}^*$ is dense, it is easy to see that, up to constant scaling, there is only one metric invariant under ta_0. Hence $b_* g = c \cdot g$ on M, where c is a constant. Clearly, the logarithm of this constant gives the Lyapunov exponent for b of any $v \in E$ and hence $c = e^{\chi(b)}$. To complete the proof of the proposition we will now show that g^* is Hölder continuous on $\mathcal{O}^*$.

To prove that g^* is Hölder continuous on $\mathcal{O}^*$ we need to show that for any point $x \in \mathcal{O}^*$ which returns close to itself under an element $a = ta_0$ the norm $d^E_x a$ defined above is Hölder close to 1. The specific Hölder exponent depends on the Hölder exponents of certain invariant foliations. Let $\alpha_0 > 0$ be such that all coarse Lyapunov distributions are Hölder continuous with exponent α_0, and let $\alpha = \min\{\alpha_0, \frac{1}{2}\}$. We will show that for any positive $\beta < \alpha/(1+\alpha)$ there exists a positive constant ε_* such that

$$|d^E_x a - 1| < \operatorname{dist}(x, ax)^\beta \qquad \text{for any } x \in \mathcal{O}^* \text{ and } a \text{ with } \operatorname{dist}(x, ax) < \varepsilon_* \tag{1}$$

To show this we will use special closing arguments given in Propositions 4.1 and 4.2. They establish the existence of a nearby point which returns under a to the same leaf of $\mathcal{O} \oplus \mathcal{W}'$.

Fix any positive $\beta < \alpha/(1+\alpha)$. Then $1 - \beta > 1/(1+\alpha)$. Fix a positive γ smaller than $1 - \beta$ but greater than $1/(1+\alpha)$. From this we obtain

$$\alpha\gamma > \alpha/(1+\alpha) > \beta \tag{2}$$

For such β and γ, Proposition 4.1 gives a positive constant ε_0. We choose $\varepsilon_* > 0$ such that $\varepsilon_* < \varepsilon_0$ and $\varepsilon_*^\gamma < \varepsilon_1$, where ε_1 is a positive constant given by Proposition 4.2.

Consider $x_0 \in \mathcal{O}^*$ and $a = ta_0$ with $\varepsilon = \operatorname{dist}(x_0, ax_0) < \varepsilon_*$. It suffices to assume that the return time t is positive and large enough. Suppose that the inequality (1) does not hold for x_0. We will use Propositions 4.1 and 4.2 to obtain an estimate for $d^E_{x_0} a$ which contradicts this assumption if ε_* satisfies the inequality (9). This will imply that with such ε_* the inequality (1) holds for any $x \in \mathcal{O}^*$ and $a = ta_0$ with $\operatorname{dist}(x, ax) < \varepsilon_*$.

If the inequality (1) does not hold for x_0, by taking inverses we may assume without loss of generality that $d^E_{x_0} a < 1 - \operatorname{dist}(x_0, ax_0)^\beta$. Thus we can use Proposition 4.1 to obtain the corresponding point x_1. Since $\operatorname{dist}(x_1, ax_1) < \varepsilon_*^\gamma < \varepsilon_1$ by the choice of ε_*, we can use Proposition 4.2 to obtain the corresponding point x_2 and $\delta \in \mathbb{R}^k$ for which $(a+\delta)x_2 \in \mathcal{W}'(x_2)$.

Denote $b = a + \delta$. Take an Anosov element c such that $\mathcal{W}^s_c = \mathcal{W}^s_a \oplus \mathcal{W}'$. Let $y = \lim(t_n c)x_2$ be an accumulation point of c-orbit of x_2. Since $bx_2 \in \mathcal{W}'(x_2)$ and c contracts $\mathcal{W}'$ we obtain $y = \lim(t_n c)(bx_2)$. Then $by = \lim b((t_n c)x_2) = \lim(t_n c)(bx_2) = y$ and thus y is a fixed point for b.

We will now show that $d^E_y b$ is close to 1. Since $b - a = \delta$ is small, we may assume that b does not belong to any Lyapunov hyperplane different from ∂H. Hence either b is Anosov or b belongs to ∂H. In the latter case, since $by = y$ we immediately obtain $d^E_y b = 1$. Indeed, suppose for example that $d^E_y b = \lambda > 1$. Arbitrarily close to b there are Anosov elements for which E is contained in the stable distribution. For any such element c and for n sufficiently large so that $-nc$ expands E we can estimate $d^E_y(n(b-c)) = d^E_y(-nc) \cdot d^E_y(nb) \geq \lambda^n$, but this is impossible if the element $b - c$ is sufficiently close to $0 \in \mathbb{R}^k$. If, on the other hand, b is Anosov we can conclude that the orbit $\mathbb{R}^k b$ is compact (cf. [29]). Hence the Lyapunov splitting and Lyapunov exponents are defined at all points of this compact orbit. We denote by $\tilde{\chi}$ the Lyapunov exponent of vectors in E on this compact orbit. Note that while $\tilde{\chi}$ may not coincide with χ, by Proposition 2.1 their kernels are the same: $\ker\tilde{\chi} = \partial H$. Since y is fixed by b, $\tilde{\chi}(b) = \log(d^E_y b)$. Since $a \in \ker\tilde{\chi} = \partial H$, we obtain

$$|\tilde{\chi}(b)| = |\tilde{\chi}(a) + \tilde{\chi}(\delta)| = |\tilde{\chi}(\delta)| < C_1||\delta|| < C_2\varepsilon^\gamma \text{ , where } \varepsilon = \operatorname{dist}(x_0, ax_0)$$

Thus we conclude that

$$|d^E_y b - 1| < C_3\varepsilon^\gamma \tag{3}$$

We will now show that $d^E_{x_0} a$ is Hölder close to $d^E_y b$ using the following estimates. First we get from parts (1) and (2) of Proposition 4.1 together with Lemma 4.3 that

$$|d^E_{x_0} a - d^E_{x_1} a| < C_4\varepsilon^{\alpha\gamma} \tag{4}$$

Next combine parts (1) and (2) of Proposition 4.2 together with the analog of Lemma 4.3 for $\mathcal{W}^u_{a_0}$.

$$|d^E_{x_1} a - d^E_{x_2} a| < C_5\varepsilon^{\alpha\gamma} \tag{5}$$

Now apply part (5) of Proposition 4.2 with $b - a = \delta$.

$$|d^E_{x_2} a - d^E_{x_2} b| < C_6\varepsilon^\gamma \tag{6}$$

Finally, we show that

$$|d^E_{x_2} b - d^E_y b| < C_7\varepsilon^\gamma \tag{7}$$

This can be seen as follows. By commutativity $b = (-t_n c) \circ b \circ (t_n c)$, so for any iterate $(t_n c)x_2$ we have

$$d^E_{x_2} b = d^E_{b(t_n c)x_2}(-t_n c) \cdot d^E_{(t_n c)x_2} b \cdot d^E_{x_2}(t_n c)$$

The middle term on the right side tends to $d^E_y b$, while the ratio of the other two terms is Hölder close to 1. The latter follows from a standard argument in the proof of Livsic' theorem since $d^E_x c$ is Hölder continuous and the orbits of x_2 and bx_2 under c are exponentially close.

We conclude that equations (3), (4), (5), (6), (7) imply

$$|d^E_{x_0} a - 1| < C_8 \varepsilon^{\alpha\gamma} \tag{8}$$

However, by equation (2), $\alpha\gamma > \beta$. Thus, possibly after decreasing ε_* further to satisfy

$$C_8 \varepsilon_*^{\alpha\gamma} < \varepsilon_*^{\beta} \tag{9}$$

we obtain that the inequality (8) contradicts the assumption that the inequality (1) does not hold for x_0. Hence we conclude that with this ε_* equation (1) holds for any $x \in \mathcal{O}^*$.

This establishes the desired Hölder estimate for the constructed metric and thus completes the proof of Proposition 3.1 and Theorem 1.2. $\diamond$

4. Closing lemmas

In this section we fix a coarse Lyapunov subspace H and a singular element $a_0 \in \partial H$ which is generic, i.e. is not contained in any other Lyapunov hyperplane. Hence the neutral distribution of a_0 is $E_H \oplus E_{(-H)}$, where $E_{(-H)}$ is trivial if there is no Lyapunov exponent positive on H. We will use notations $E = E_H$, $E' = E_{(-H)}$, $E^s = E^s_{a_0}$, and $E^u = E^u_{a_0}$. We assume that the corresponding distributions $\mathcal{W}$, $\mathcal{W}'$, $\mathcal{W}^s$, and $\mathcal{W}^u$ are Hölder continuous with exponent $\alpha_0 > 0$, and denote $\alpha = \min\{\alpha_0, \frac{1}{2}\}$.

The propositions below can be compared to the Anosov closing lemma for Anosov flows. The main differences are the following. The neutral distribution $E \oplus E'$ of a_0 is not integrable in general, this forces us to consider $\mathcal{W}$ and $\mathcal{W}'$ separately. The holonomies under consideration are only Hölder continuous, this forces us to use a topological fixed point argument rather than the contracting mapping theorem. Proposition 4.2 can be formulated and proved in the context of a single partially hyperbolic diffeomorphism a_0 to become a relatively standard version of the closing lemma for diffeomorphisms normally hyperbolic to an invariant foliation (cf. [11]). We formulate it in the specific form that we need with the Lipschitz estimates absent in [11]. The main novelty is in Proposition 4.1 where we utilize the weak contraction in the neutral foliation $\mathcal{W}$. This is substantially higher rank, since the proof relies on the nonstationary linearization along the leaves of $\mathcal{W}$ used in Lemma 4.3. The existence of this nonstationary linearization is provided by an element in $\mathbb{R}^k$ which uniformly contracts $\mathcal{W}$.

Proposition 4.1. *For any positive $\beta < \alpha/(1+\alpha)$ and positive $\gamma < 1-\beta$ there exists $\varepsilon_0 > 0$ such that for any $x_0 \in M$ and $t > 1$ with $\varepsilon = dist(x_0, (ta_0)x_0) < \varepsilon_0$ and $||D^E_{x_0}(ta_0)|| < 1 - \varepsilon^\beta$ there exists a point $x_1 \in M$ such that*

1. $dist(x_0, x_1) < \varepsilon^\gamma$
2. $x_1 \in (\mathcal{W}^s \oplus \mathcal{W})(x_0)$
3. $dist((ta_0)x_0, (ta_0)x_1) < \varepsilon^\gamma$
4. $(ta_0)x_1 \in (\mathcal{O} \oplus \mathcal{W}^u \oplus \mathcal{W}')(x_1)$
5. $dist(x_1, (ta_0)x_1) < \varepsilon^\gamma$

Proposition 4.2. *There exist positive constants* ε_1 *and* C *such that for any* $x_1 \in M$ *and* $t > 1$ *with* $(ta_0)x_1 \in (\mathcal{O} \oplus \mathcal{W}^u \oplus \mathcal{W}')(x_1)$ *and* $\varepsilon = dist(x_1, (ta_0)x_1) < \varepsilon_1$*, there exist a point* $x_2 \in M$*,* $\delta \in \mathbb{R}^k$*, such that*

(1) $dist(x_1, x_2) < C\varepsilon$
(2) $x_2 \in \mathcal{W}^u(x_1)$
(3) $dist((ta_0)x_1, (ta_0)x_2) < C\varepsilon$
(4) $(ta_0 + \delta)x_2 \in \mathcal{W}'(x_2)$
(5) $||\delta|| < C\varepsilon$

Below we give a proof of Proposition 4.1. Proposition 4.2 can be proved similarly, its proof would avoid the main difficulty caused by the neutral direction $\mathcal{W}$ and yield the Lipschitz estimates.

Proof: Recall that $\beta < \alpha/(1+\alpha)$ implies $1-\beta > 1/(1+\alpha)$. Since it is clearly sufficient to consider only γ which are close to $1-\beta$, we may assume that $\gamma > 1/(1+\alpha)$. From this we obtain that $\gamma > 1-\alpha$ and $\alpha\gamma > \alpha/(1+\alpha) > \beta$. We summarize the inequalities we have:

$$0 < \beta < \frac{\alpha}{1+\alpha} < \alpha\gamma < \alpha \leq \frac{1}{2} \leq 1-\alpha < \gamma < 1-\beta < 1 \tag{10}$$

We introduce the following notations $a = ta_0$, $y_0 = ax_0$, and $F = \mathcal{W}^s \oplus \mathcal{W}$. By assumption $\varepsilon = \mathrm{dist}(x_0, y_0) < \varepsilon_0$. Consider balls $B_1 \subset \mathcal{W}^s(x_0)$ and $B_2 \subset \mathcal{W}(x_0)$ of radius $k_1\varepsilon^\gamma$ centered at x_0. For $w_1 \in B_1$ and $w_2 \in B_2$ we denote by $[w_1, w_2]$ the unique local intersection of $\mathcal{W}(w_1)$ and $\mathcal{W}^s(w_2)$ in $F = \mathcal{W}^s \oplus \mathcal{W}$. We define a "rectangle" $P = \{[w_1, w_2] : w_1 \in B_1, w_2 \in B_2\} \subset F(x_0)$. Since the minimal angle between the leaves of foliations $\mathcal{W}$ and $\mathcal{W}^s$ is bounded away from 0, the constant k_1 can be chosen so small that P is contained in a ball of radius ε^γ centered at x_0. Denote $f = a|_P : P \to F(y_0)$ and let $h : f(P) \to F(x_0)$ be the holonomy map of the foliation $\mathcal{O} \oplus \mathcal{W}^u \oplus \mathcal{W}'$. We will show that for small enough ε_0 we can ensure that $h(f(P)) \subset P$. Since P is homeomorphic to a ball, this implies the existence of a fixed point x_1 for $h \circ f$ which satisfies the conclusions of the proposition.

We fix a Riemannian metric on M and the induced metric on $T_{x_0}M$. We identify a neighborhood of x_0 in M with $T_{x_0}M$ using local coordinates for which the differential at the base point is identity, and the leaf $F(x_0)$ identifies with its tangent space. By abuse of notations we will write F for the flat leaf $F(x_0)$ and F' for the leaf $F(y_0)$. We denote by Pr the orthogonal projection from F' to F.

First we give an estimate of the distance between the leaves F and F'. For a point $x \in F$ let $x' \in F'$ be such that $x = \mathrm{Pr}\,(x')$. Denote by $d(x)$ the distance between $x \in F$ and the corresponding $x' \in F'$. Let $d_0 = d(x_0)$. Since the tangent distribution of the foliation F is Hölder with exponent α, the maximal angle between F and $T_{x'}F'$ is at most $K_2 d(x)^\alpha$. Hence the function d satisfies inequality $|\mathrm{grad}\, d(x)| \leq K_2 d(x)^\alpha$. We conclude that d is bounded by the solution of $y' = K_2 y^\alpha$ with the initial condition $y(0) = d_0$:

$$d(x) \leq \left(d_0^{1-\alpha} + (1-\alpha)K_2 \mathrm{dist}(x, x_0)\right)^{1/(1-\alpha)}.$$

We observe that d_0 is of order $\varepsilon = \operatorname{dist}(x_0, y_0)$. Recall that P is contained in a ball of radius ε^γ centered at x_0, and that $\varepsilon^\gamma < \varepsilon^{1-\alpha}$ by (10). Thus the first term in the sum dominates for small ε and hence the maximum of d on P is bounded by $K_3\varepsilon$ for some constant K_3. We conclude that

$$\operatorname{dist}(P, f(P)) = \max \operatorname{dist}(x, x') \leq K_3\varepsilon. \tag{11}$$

Now we estimate how the holonomy h from F' to F deviates from the orthogonal projection $\Pr$. Since the minimal angle between the leaves of foliations F and $\mathcal{O} \oplus \mathcal{W}^u \oplus \mathcal{W}'$ is bounded away from 0, there exists a constant K_4 such that for any $x' \in f(P)$ we can estimate

$$\Delta(x') := \operatorname{dist}(h(x'), \Pr(x')) = \operatorname{dist}(h(x'), x)) \leq K_4 \operatorname{dist}(x, x') \leq K_4 K_3 \varepsilon \tag{12}$$

Now we will study $f(P)$ and its projection $\Pr(f(P))$ to F. Our goal is to show that $\Pr(f(P)) \subset P$ and that the distance form $\Pr(f(P))$ to the relative boundary ∂P of P in F is greater than the upper bound we just obtained for the distance Δ between the holonomy h and the projection $\Pr$. This will imply that $h(f(P)) \subset P$ and complete the proof.

We first estimate the derivatives of $a = ta_0$ on P. Recall that a_0 contracts $\mathcal{W}^s$. We may assume that the return time t has to be large for the return under a to be ε-close. Hence we may assume that the norm of the derivative of a restricted to E^s is bounded above by $\frac{1}{4}$ on P :

$$||D_x^{E^s} a|| < \frac{1}{4} \qquad \text{for any } x \in P \tag{13}$$

Now we will estimate the derivative $D_x^E a$ in E direction using the following lemma.

Lemma 4.3. *The dependence of the derivative $D_y^E(ta_0)$ on y is Lipschitz continuous along $\mathcal{W}$ and Hölder continuous along $\mathcal{W}^s_{a_0}$ with exponent α and constants independent of y and t.*

Proof: To show the Lipschitz continuity along one-dimensional leaves of $\mathcal{W}$ we use non-stationary linearization of the action. For an element $b \in \mathbb{R}^k$ which contracts $\mathcal{W}$ the following lemma from [16] gives the nonstationary linearization of b along $\mathcal{W}$.

Lemma 4.4. *If a diffeomorphism b of a manifold M contracts an invariant one-dimensional foliation $\mathcal{W}$, then there exists a unique family of C^∞ diffeomorphisms $h_x : \mathcal{W}(x) \to T_x\mathcal{W}$, $x \in M$, such that*

(i) $h_{bx} \circ b = D_x b \circ h_x$,

(ii) $h_x(x) = 0$ *and* $D_x h_x$ *is the identity map,*

(iii) h_x *depends continuously on* x *in* C^∞ *topology.*

Since $a = ta_0$ commutes with b, the same family linearizes a and we obtain

$$a|_{\mathcal{W}(x)}(y) = (h_{ax}^{-1} \circ D_x^E a \circ h_x)(y) : \mathcal{W}(x) \to \mathcal{W}(ax)$$

Since the second derivatives of h_x are uniformly bounded, the first derivatives vary Lipschitz continuously. This implies that $D^E_y a$ varies Lipschitz continuously along $\mathcal{W}(x)$ in a small neighborhood of an arbitrary point x.

Now we show the Hölder continuity along $\mathcal{W}^s$. Since a_0 exponentially contracts $\mathcal{W}^s_{a_0}$, the orbits under $t_0 a$ of any two nearby points y_1 and $y_2 \in \mathcal{W}^s_{a_0}(y_1)$ are exponentially close. The derivative cocycle in the direction of E is a Hölder cocycle Hence the standard argument from the proof of Livsic' theorem shows that $D^E_{y_1}(ta_0)$ is Hölder close to $D^E_{y_2}(ta_0)$ with exponent α and a uniform constant. $\diamond$

By the assumption, $||D^E_{x_0}a|| < 1 - \varepsilon^\beta$. Since P is contained in a ball of radius ε^γ, using Lemma 4.3 and the fact that $\beta < \alpha\gamma$ we obtain

$$||D^E_x a|| < 1 - \frac{\varepsilon^\beta}{2} \qquad \text{for any } x \in P \tag{14}$$

provided that ε_0 is small enough.

Now we study how $f(P)$ projects to F and estimate the distance form $\Pr(f(P))$ to the boundary ∂P. Recall that B_2 is just an interval in $\mathcal{W}(x_0)$ centered at x_0 of length $2k_1\varepsilon^\gamma$. Using estimates similar to (11) we obtain

$$\operatorname{dist}(B_2, \Pr(f(B_2)) \le \operatorname{dist}(B_2, f(B_2)) \le K_3\varepsilon. \tag{15}$$

The "rectangle" P is the union of "horizontal layers" $B_x = B_1 \times \{x\} = \{[w_1, x]_b : w_1 \in B_1\} \subset \mathcal{W}^s(x)$ for $x \in B_2$. Let us fix $x \in B_2$ and the corresponding B_x. Using (13) we obtain that $f(B_x)$ is contained in a ball of radius $\frac{1}{3}k_1\varepsilon^\gamma$ centered at $y = f(x)$ in $\mathcal{W}^s(y)$. By (15), $\operatorname{dist}(\Pr(y), B_2) \le K_3\varepsilon$. Hence it is easy to see that the projection of this ball to P is at least $k_5\,\varepsilon^\gamma$ away from the "vertical part" $\partial B_1 \times B_2$ of the boundary ∂P. Thus we obtain

$$\operatorname{dist}(\Pr(f(B_x), \partial B_1 \times B_2) \ge k_5\,\varepsilon^\gamma. \tag{16}$$

Now we will estimate the distance from $\Pr(f(B_x))$ to $B_{x_1} = B_1 \times \{x_1\} \subset \partial P$, where x_1 is one of the endpoints of the interval B_2. We denote $z = \Pr(y)$, and $z_0 = \Pr(y_0)$. The derivative estimate (14) above implies that $\operatorname{dist}(y_0, y) \le k_1\varepsilon^\gamma(1 - \frac{1}{2}\varepsilon^\beta)$, hence after projecting we have $\operatorname{dist}(z_0, z) \le k_1\varepsilon^\gamma(1 - \frac{1}{2}\varepsilon^\beta)$. Since $\operatorname{dist}(x_0, y_0) \le \varepsilon$ by assumption, we obtain $\operatorname{dist}(x_0, z_0) \le \varepsilon$. Hence $\operatorname{dist}(x_0, z) \le \varepsilon + k_1\varepsilon^\gamma(1 - \frac{1}{2}\varepsilon^\beta)$ and

$$\operatorname{dist}(z, x_1) \ge \operatorname{dist}(x_0, x_1) - \operatorname{dist}(x_0, z) \ge k_1\varepsilon^\gamma - \left(\varepsilon + k_1\varepsilon^\gamma - \frac{1}{2}k_1\varepsilon^\gamma\varepsilon^\beta\right) \ge \frac{1}{2}k_1\varepsilon^\gamma\varepsilon^\beta - \varepsilon \ge \frac{1}{3}k_1\varepsilon^{\gamma+\beta}$$

since $\gamma + \beta < 1$, provided that $\varepsilon < \varepsilon_0$ is small enough. By (15), z is $K_3\varepsilon$ close to the interval $B_2 \subset \mathcal{W}(x_0)$. Since the angles between $\mathcal{W}$ and $\mathcal{W}^s$ are bounded away from zero, it is easy to see that

$$\operatorname{dist}(z, B_{x_1}) \ge k_6\varepsilon^{\gamma+\beta}$$

We will now complete the estimate of the distance between $\Pr(f(B_x))$ and B_{x_1}. Note that $f(B_x)$ and B_{x_1} lie on two nearby leaves of the foliation $\mathcal{W}^s$ and the distance from any point in $\Pr(f(B_x))$ to any point B_{x_1} is at most of order ε^γ. By Hölder continuity of the corresponding distribution we see that the angles between tangent spaces to $f(B_x)$ and B_{x_1} differ no more than $K_7(\varepsilon^\gamma)^\alpha$. Hence, after projecting, the angles between tangent spaces to $\mathrm{Pr}\,(f(B_x))$ and B_{x_1} also differ no more than $K_7(\varepsilon^\gamma)^\alpha$. Thus, the distance from a point on $\mathrm{Pr}\,(f(B_x))$ to B_{x_1}, as a function of this point, cannot change by more than

$$K_7(\varepsilon^\gamma)^\alpha \cdot (2k_1\varepsilon^\gamma) = 2k_1K_7\varepsilon^{\gamma+\alpha\gamma}.$$

Since $\alpha\gamma > \beta$ we obtain

$$\mathrm{dist}(\mathrm{Pr}\,((f(B_x)), B_{x_1}) \geq k_6\varepsilon^{\gamma+\beta} - 2k_1K_7\varepsilon^{\gamma+\alpha\gamma} \geq k_8\varepsilon^{\gamma+\beta}$$

provided that $\varepsilon < \varepsilon_0$ is small enough. Combining this with (16) we conclude that

$$\mathrm{dist}(\mathrm{Pr}\,(f(B_x), \partial P) \geq \max\{k_5\,\varepsilon^\gamma, k_8\varepsilon^{\gamma+\beta}\}.$$

Since $x \in B_2$ was arbitrary we conclude that

$$\mathrm{dist}(\mathrm{Pr}\,(f(P), \partial P) \geq k_9\varepsilon^{\gamma+\beta}.$$

Since $\gamma + \beta < 1$ we see this distance is larger than that the estimate (12) for the deviation of the holonomy from the projection, provided that $\varepsilon < \varepsilon_0$ is small enough. This shows that $h(f(P)) \subset P$ and proves the existence of a fixed point which satisfies the conclusions of Proposition 4.1. $\diamond$

5. Classification of $\mathbb{R}^k$-actions

In this section we will prove the following generalization of Theorem 1.3. We will use this stronger technical version to prove Corollary 1.4 in the next section. Given two Lyapunov hyperplanes H_i and H_j with $H_j \neq \pm H_i$, we denote by E_{ij} the smallest direct sum of coarse Lyapunov distributions which is integrable and contains E_{H_i} and E_{H_j}. We denote the foliation tangent to E_{ij} by $\mathcal{W}_{ij}$

Theorem 5.1. *Let α be a C^∞ totally Cartan action of $\mathbb{R}^k$, $k \geq 3$, on a compact smooth connected manifold M preserving an ergodic probability measure μ with full support. Suppose that every Lyapunov hyperplane contains a generic one-parameter subgroup with a dense orbit. Assume further that for any two Lyapunov half spaces H_i and $H_j \neq \pm H_i$ there is an element $a \in \partial H_i \cap \partial H_j$ and a point $x \in M$ for which the closure of the a-orbit contains the whole leaf $\mathcal{W}_{ij}(x)$. Then α is C^∞ conjugate to an almost algebraic action.*

Proof: First we note that Theorem 1.2 provides us with a Hölder continuous Riemannian metric g on M such that for any $a \in \mathbb{R}^k$ and any Lyapunov exponent χ

$$||a_*(v)|| = e^{\chi(a)}||v|| \qquad \text{for any } v \in E_\chi.$$

The next step is to show that this metric g and the coarse Lyapunov splitting are C^∞. For this we will use Theorem 2.4 from [7]. Our assumptions on the one-dimensionality of the coarse Lyapunov foliations and the existence of the metric g given by Theorem 1.2 guarantee that the assumptions of Theorem 2.4 are satisfied with the exception of the assumptions of invariant volume and ergodicity of one-parameter subgroups. We note however that the proof goes through verbatim under the weaker assumption of preservation of an ergodic invariant measure with full support. Furthermore, ergodicity of one-parameter subgroups is used only to guarantee that for certain one-parameter subgroups in the Lyapunov hyperplanes there are points x whose orbits accumulate on the whole leaves $\mathcal{W}_{ij}(x)$. The last assumption of Theorem 5.1 is precisely what is required in the proof. Thus we obtain that the metric g and the coarse Lyapunov splitting are C^∞. Now we can complete the proof of Theorem 5.1 as follows.

Passing to a finite cover of M if necessary, we may assume for any coarse Lyapunov direction E_χ that there are nowhere vanishing vectorfields tangent to E_χ. Consider all the vectorfields V_χ pointing in the various one-dimensional Lyapunov foliations $\mathcal{W}_\chi$ of length 1 with respect to the metric g as well as the generating fields of the $\mathbb{R}^k$-action. Then all vectorfields V_χ are smooth and are expanded or contracted uniformly by $e^{\chi(a)}$ for $a \in \mathbb{R}^k$. Hence the Lie bracket of two such fields $[V_\chi, V_\xi]$ is expanded by $e^{(\chi+\xi)(a)}$, and hence is a constant multiple of $V_{\chi+\xi}$ if the latter exists. Otherwise $[V_\chi, V_\xi] = 0$. Since the $\mathbb{R}^k$-action normalizes the V_χ, the V_χ together with the generating fields of the $\mathbb{R}^k$-action span a finite dimensional Lie algebra $\mathfrak{g}$. Let G be the corresponding simply connected Lie group with Lie algebra $\mathfrak{g}$. Since the V_χ are globally defined and bounded with respect to an ambient Riemannian metric, G acts on M. By the construction this action is locally simply transitive and thus transitive as M is connected. Hence M is a homogeneous space G/Γ for a lattice Γ in G. Moreover, the $\mathbb{R}^k$-action embeds into the left action of G by construction. $\diamond$

6. Proof of Corollary 1.4

To prove Corollary 1.4 we will apply Theorem 5.1 to the suspension α' of the $\mathbb{Z}^k$ action α. By the assumption $\mathbb{Z}^k$ contains an element a which is a transitive Anosov diffeomorphism. It is well known that such an element has a unique measure μ of maximal entropy. In fact, transitive Anosov diffeomorphisms are topologically mixing [9, Corollary 18.3.5] and thus have the specification property [9, Theorem 18.3.9]. For the latter, [9, Theorem 20.1.3] proves existence and uniqueness of the measure of maximal entropy. Finally, it has full support due to the estimate from below of the measure of a dynamical ball [9, Lemma 20.1.1]. By uniqueness, μ is also α -invariant. Indeed, for any $b \in \mathbb{Z}^k$ by commutativity $b_*\mu$ is also a-invariant and has the same entropy. Hence μ lifts to an α' -invariant measure μ' on the suspension manifold M'. Clearly μ' is ergodic with respect to α' and has full support on M'.

We will now verify that the suspensions of nondegenerate TNS Cartan $\mathbb{Z}^k$ actions satisfy the assumptions of Theorem 5.1. First we notice the easy

Lemma 6.1. *The suspension action of a $\mathbb{Z}^k$ action all of whose non-trivial elements act by Anosov diffeomorphisms is totally Anosov.*

Proof: If $a \in \mathbb{Z}^k$ is an Anosov diffeomorphisms then any non-trivial element on the line $\mathbb{R} \cdot a$ is an Anosov element for the suspension action. Since all non-trivial elements of the original action are assumed to be Anosov it is clear that $\mathbb{R} \cdot \mathbb{Z}^k$ forms a dense set in $\mathbb{R}^k$. $\diamond$

Next we will verify the two hypotheses on transitivity of Theorem 5.1 adapting an argument of [20]. First we check the transitivity assumption that every Lyapunov hyperplane contains a generic one-parameter subgroup with a dense orbit.

Lemma 6.2. *For every Lyapunov hyperplane ∂H almost every element in ∂H is transitive.*

Proof: Fix a Lyapunov half-space H. Pick a generic element $a \in \partial H$ and an Anosov element $c \in (-H)$ so close to a as to preserve the signs of all Lyapunov exponents nonzero on a. In particular we get $\mathcal{W}^s_a = \mathcal{W}^s_c$. We can take element c to be a time-t map of an Anosov element b which fixes the fibers of the suspension and induces an Anosov diffeomorphism on them. Note that this Anosov diffeomorphism is transitive. Indeed, it preserves a finite measure of full support. Thus the nonwandering set is the whole manifold. It is well-known that this implies topological transitivity [9, Corollary 18.3.5]. As above, such an element has a unique measure ν of maximal entropy which is also α -invariant and lifts to an α' -invariant measure ν' with full support on the suspension manifold M'. We will use measure ν' for the rest of the proof.

Birkhoff averages with respect to a of any continuous function are constant on the leaves of $\mathcal{W}^s_a$. Since such averages generate the algebra of a-invariant functions we conclude that the partition ξ_a into ergodic components of a is coarser than the measurable hull $\xi(\mathcal{W}^s_a)$ of the foliation $\mathcal{W}^s_a$ which coincides with the Pinsker algebra $\pi(c)$ ([22], Theorem B). Thus we conclude

$$\xi_a \leq \xi(\mathcal{W}^s_a) = \xi(\mathcal{W}^s_c) = \pi(c).$$

This shows the partition $\xi_{\partial H}$ into ergodic components of ∂H is coarser than $\xi(\mathcal{W}^s_c) = \pi(c)$. Since the Pinsker algebra of b with respect to ν on M is trivial, the Pinsker algebra of c (with respect to ν') is coarser than the partition into the fibers of the suspension. Hence so is $\xi_{\partial H}$, the partition into the ergodic components of ∂H. Hence we can project $\xi_{\partial H}$ along the fibers of the suspension to the factor. The factor is $\mathbb{T}^k$ with the standard $\mathbb{R}^k$ action by translations. Since every element of $\mathbb{Z}^k$ is Anosov, no element in $\mathbb{Z}^k$ can belong to a Lyapunov hyperplane. Thus ∂H contains no element in $\mathbb{Z}^k$ and hence ∂H action on $\mathbb{T}^k$ is uniquely ergodic. This implies that the projection of $\xi_{\partial H}$ to $\mathbb{T}^k$ is trivial, and hence so is $\xi_{\partial H}$ itself. This establishes the ergodicity of ∂H with respect to ν'. It is general that if an abelian group acts ergodically then so does a.e. element in the group. Hence we get transitivity of almost every element. $\diamond$

Now we verify the second transitivity assumption of Theorem 5.1. Consider two (negative) Lyapunov half spaces H_i and $H_j \neq \pm H_i$ and denote by $\mathcal{W}_i$ and $\mathcal{W}_j$ the corresponding coarse Lyapunov foliations. By nondegeneracy, we can choose an element $a \in \partial H_i \cap \partial H_j$ which does not belong to any other Lyapunov hyperplane. Then we can take an Anosov element c in $-(H_i \cap H_j)$ so close to a as to preserve the signs of all Lyapunov exponents nonzero on a. By the choice of a, any Lyapunov exponent zero on a has either ∂H_i or ∂H_j as the kernel. Since the action is TNS, such an exponent must be negative on either H_i or H_j, and hence is positive on c. Thus $E^u_c = E^u_a \oplus E_i \oplus E_j$ and $\mathcal{W}^s_c = \mathcal{W}^s_a$. Therefore, as in the proof of the previous lemma, we get the following inequalities

$$\xi_a \leq \xi(\mathcal{W}^s_a) = \xi(\mathcal{W}^s_c) = \pi(c).$$

Again as in the previous lemma the Pinsker algebra of c (with respect to the measure of the maximal entropy for c) is coarser than the partition into the fibers of the suspension. Thus the ergodic components of a consist of the whole fibers. Since the conditional measures on the fibers have full support, the closure of a-orbit of a typical point contains the whole fiber through that point and, in particular, $\mathcal{W}_{ij}$. $\diamond$

We conclude from Theorem 1.3 that α' is C^∞ conjugate to an algebraic action. To compete the proof of Theorem 1.4 we use Lemma 6.3 below. We note that in our case the infranlimanifold is finitely covered by a torus. Indeed, if the corresponding nilpotent Lie algebra was not abelian then there would be a resonance relation $\chi_1 + \chi_2 = \chi_3$ among the Lyapunov exponents, this can be seen as in the end of the proof of Theorem 5.1. It is easy to see, however, that such a relation is impossible due to the nondegeneracy assumption.

Lemma 6.3. *If the suspension of a $\mathbb{Z}^k$ Anosov action is C^∞-conjugate to an algebraic action then the original $\mathbb{Z}^k$-action is C^∞-conjugate to an action by automorphisms of infranilmanifolds*

Proof: The induced $\mathbb{R}^k$-action contains the original $\mathbb{Z}^k$-action by restricting to a fiber. Since the algebraic $\mathbb{R}^k$-action expands and contracts a smooth Riemannian metric such that

$$\|a_*(v)\| = e^{\chi(a)}\|v\| \qquad \text{for any } \ v \in E_\chi$$

so does the original $\mathbb{Z}^k$-action. Also notice that the non-orbit coarse Lyapunov spaces are all tangent to the fibers of the suspension. Then it follows by the same argument as at the end of the proof of Theorem 1.4 that the $\mathbb{Z}^k$-action is algebraic. It is well-known that the only algebraic $\mathbb{Z}^k$ actions are the ones by automorphisms of infranilmanifolds. We refer to [7, Proposition 3.13] for a proof.

$\diamond$

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