ON ANOSOV DIFFEOMORPHISMS WITH ASYMPTOTICALLY CONFORMAL PERIODIC DATA

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Abstract. We consider transitive Anosov diffeomorphisms for which every periodic orbit has only one positive and one negative Lyapunov exponent. We establish various properties of such systems including strong pinching, $C^{1+\beta}$ smoothness of the Anosov splitting, and C^1 smoothness of measurable invariant conformal structures and distributions. We apply these results to volume preserving diffeomorphisms with two-dimensional stable and unstable distributions and diagonalizable derivatives of the return maps at periodic points. We show that a finite cover of such a diffeomorphism is smoothly conjugate to an Anosov automorphism of $\mathbb{T}^4$. As a corollary we obtain local rigidity for such diffeomorphisms. We also establish a local rigidity result for Anosov diffeomorphisms in dimension three.

1. Introduction

The goal of this paper is to study Anosov diffeomorphisms for which every periodic orbit has only one positive and one negative Lyapunov exponent. Our main motivation comes from the problem of local rigidity for higher-dimensional Anosov systems, i.e. the question of regularity of conjugacy to a small perturbation. If f is an Anosov diffeomorphism and g is sufficiently C^1 close to f, then it is well known that g is also Anosov and topologically conjugate to f. However, the conjugacy is typically only Hölder continuous. A necessary condition for the conjugacy to be C^1 is that Jordan normal forms of the derivatives of the return maps of f and g at the corresponding periodic points are the same. If this condition is also sufficient for any g which is C^1 close to f, then f is called *locally rigid*. The problem of local rigidity has been extensively studied, and Anosov diffeomorphisms with one-dimensional stable and unstable distributions were shown to be locally rigid [11], [12], [15], [18].

In contrast, higher dimensional systems are not always locally rigid. In [12, 13] R. de la Llave constructed examples of Anosov automorphisms of the torus $\mathbb{T}^4$ which are not C^1 conjugate to certain small perturbations with the same periodic data. One of the examples has two (un)stable eigenvalues of different moduli and the other one has a double (un)stable eigenvalue with a nontrivial Jordan block. This suggests that it is natural to consider local rigidity for automorphisms that

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are diagonalizable over $\mathbb{C}$ with all (un)stable eigenvalues equal in modulus. For such an automorphism the expansion of the unstable and contraction of the stable distribution are *conformal* with respect to some metric on the torus. For the case of an Anosov diffeomorphism of a compact manifold, a more natural notion is that of *uniform quasiconformality*. It means that at each point all vectors in the (un)stable subspace are expanded/contracted at essentially the same rate (see Section 3.2).

The study of local rigidity of conformal and uniformly quasiconformal Anosov systems was initiated in [13] and continued in [8] and [14]. It is closely related to the study of *global rigidity* of such systems, or their classification up to a smooth conjugacy. In [8] we established the following global rigidity result:

[8, Theorem 1.1] *Let f be a transitive C^∞ Anosov diffeomorphism of a compact manifold $\mathcal{M}$ which is uniformly quasiconformal on the stable and unstable distributions. Suppose either that both distributions have dimension at least three, or that they have dimension at least two and $\mathcal{M}$ is an infranilmanifold. Then f is C^∞ conjugate to an affine Anosov automorphism of a finite factor of a torus.*

In the case of two-dimesional distributions, the additional assumption that $\mathcal{M}$ is an infranilmanifold can be replaced by preservation of a volume [2].

This theorem implies, in particular, that the conjugacy of such a diffeomorphism f to a perturbation g is smooth if and only if g is also uniformly quasiconformal [8, Corollary 1.1]. Hence to establish local rigidity of f it suffices to show that any C^1 small perturbation with the same periodic data is also uniformly quasiconformal. Establishing uniform quasiconformality is also the first step in [12, 13] where smoothness of the conjugacy is obtained directly.

The most general local rigidity result so far was for uniformly quasiconformal Anosov diffeomorphisms f satisfying the following additional assumption:

$$\text{for any periodic point } p, \;\; df^m|_{E^s(p)} = a^s(p)\cdot \mathrm{Id} \;\text{ and }\; df^m|_{E^u(p)} = a^u(p)\cdot \mathrm{Id},$$

where m is the period of p, E^s and E^u are stable and unstable distributions, and $a^s(p)$, $a^s(p)$ are real numbers [13, 8]. When one considers a perturbation g of f with the same periodic data, the derivatives $dg^m|_{E^s(p)}$ are again multiples of identity. Such a map preserves any conformal structure, i.e. induces the identity map on the space of conformal structures on $E^s(p)$. There is a major difference between this special case and the general one. Indeed, for a uniformly quasiconformal Anosov diffeomorphism f these derivatives are conjugate to multiples of isometries, and are not necessarily multiples of identity. This gives surprisingly little information about $dg^m|_{E^s(p)}$. It is still conjugate to a multiple of isometry, but one has no information on how such conjugacy varies with p. Such a map preserves *some* conformal structure, i.e. the induced map on the space of conformal structures on $E^s(p)$ has a fixed point. However, it may significantly affect other conformal structures. In particular, there is no control over quasiconformal distortion. This makes it difficult to show uniform quasiconformality of g in the general case. Indeed, so far there have been no results related to quasiconformality or local rigidity for systems with such periodic data.

In this paper we introduce some new techniques to study the case of general quasiconformal periodic data. We establish local, as well as global, rigidity for such volume preserving systems with two-dimensional stable and unstable distributions.

We begin with the study of transitive Anosov diffeomorphisms for which every periodic orbit has only one positive and one negative Lyapunov exponent. We note that this assumption does not exclude the possibility of Jordan blocks and hence such systems are not necessarily uniformly quasiconformal. Still we obtain strong results for these systems which form the basis for further analysis. In particular, we establish continuity and C^1 smoothness of measurable invariant conformal structures and distributions. We apply these results to volume preserving diffeomorphisms with $\dim E^u = \dim E^s = 2$ and diagonalizable derivatives of the return maps at periodic points. In addition, we use the Amenable Reduction Theorem. We thank A. Katok for bringing this result to our attention. In our context this theorem implies the existence of a measurable invariant conformal structure or a measurable invariant one-dimensional distribution for E^u (E^s). This allows us to establish uniform quasiconformality and hence local and global rigidity. We also obtain a local rigidity result for Anosov diffeomorphisms in dimension 3.

We formulate our main results in the next section and prove them in Section 4. In Section 3 we introduce the notions used throughout this paper.

2. Statements of Results

First we consider transitive Anosov diffeomorphisms for which every periodic orbit has only one positive and one negative Lyapunov exponent, i.e. all (un)stable eigenvalues of the derivative of the return map have the same modulus. We do not assume that the derivatives of the return maps at periodic points are diagonalizable. This class includes systems which are not uniformly quasiconformal and not locally rigid. However, the following theorem shows that they exhibit a variety of useful properties.

Theorem 2.1. *Let f be a transitive Anosov diffeomorphism of a compact manifold $\mathcal{M}$. Suppose that for each periodic point p there is only one positive Lyapunov exponent $\lambda_+^{(p)}$ and only one negative Lyapunov exponent $\lambda_-^{(p)}$. Then*

(1) *Any ergodic invariant measure for f has only one positive and only one negative Lyapunov exponent. Moreover, for any $\varepsilon > 0$ there exists C_ε such that for all x in $\mathcal{M}$ and n in $\mathbb{Z}$*

$$K^u(x,n) = \frac{\max\{\,\|\,df^n(v)\,\| : v\in E^u(x),\ \|v\|=1\,\}}{\min\{\,\|\,df^n(v)\,\| : v\in E^u(x),\ \|v\|=1\,\}} \le C_\varepsilon e^{\varepsilon|n|} \quad \textit{and}$$

$$K^s(x,n) = \frac{\max\{\,\|\,df^n(v)\,\| : v\in E^s(x),\ \|v\|=1\,\}}{\min\{\,\|\,df^n(v)\,\| : v\in E^s(x),\ \|v\|=1\,\}} \le C_\varepsilon e^{\varepsilon|n|}$$

(2) *The stable and unstable distributions* E^s *and* E^u *are* $C^{1+\beta}$ *for some* $\beta > 0$.

(3) *There exist* $C > 0$, $\beta > 0$, *and* $\delta_0 > 0$ *such that for any* $\delta < \delta_0$, $x, y \in \mathcal{M}$ *and* $n \in \mathbb{N}$ *with* $dist(f^i(x), f^i(y)) \le \delta$ *for* $0 \le i \le n$, *we have*
$$\|(df^n_x)^{-1} \circ df^n_y - Id\| \le C\delta^\beta.$$

(4) *Any* f*-invariant measurable conformal structure on* E^u (E^s) *is* C^1. *Measurability here can be understood with respect to the measure of maximal entropy or with respect to the invariant volume, if it exists.*

(5) *If in addition* f *is volume-preserving, then any measurable* f*-invariant distribution in* E^u (E^s) *defined almost everywhere with respect to the volume is* C^1.

Remark 2.2. *To consider the composition of the derivatives in (3) we identify the tangent spaces at nearby points* x *and* y. *This can be done by fixing a smooth background Riemannian metric and using the parallel transport along the unique shortest geodesic connecting* x *and* y. *Such identification can be adjusted using projections to preserve the Anosov splitting (or an invariant distribution). In this case the identification will be as regular as the Anosov splitting (or the invariant distribution).*

Next we consider the case when $\dim E^u = \dim E^s = 2$. Now we assume that the matrix of the derivative the return map is diagonalizable. We note that the example given by de la Llave in [13] shows that this assumption is necessary in the following theorem and its corollary. The theorem establishes global rigidity for systems with conformal periodic data, and the corollary yields local rigidity.

Theorem 2.3. *Let* $\mathcal{M}$ *be a compact 4-dimensional manifold and let* $f : \mathcal{M} \to \mathcal{M}$ *be a transitive* C^∞ *Anosov diffeomorphism with 2-dimensional stable and unstable distributions. Suppose that for each periodic point* p *the derivative of the return map is diagonalizable over* $\mathbb{C}$ *and its eigenvalues* $\lambda_1^{(p)}, \lambda_2^{(p)}, \lambda_3^{(p)}, \lambda_4^{(p)}$ *satisfy*
$$|\lambda_1^{(p)}| = |\lambda_2^{(p)}|, \quad |\lambda_3^{(p)}| = |\lambda_4^{(p)}|, \quad \text{and} \quad |\lambda_1^{(p)}\lambda_2^{(p)}\lambda_3^{(p)}\lambda_4^{(p)}| = 1.$$
Then f *is uniformly quasiconformal and its finite cover is* C^∞ *conjugate to an Anosov automorphism of* $\mathbb{T}^4$.

Note that the last condition on the eigenvalues is equivalent to the fact that f preserves a smooth volume [10, Theorem 19.2.7].

Corollary 2.4. *Let* $\mathcal{M}$ *and* f *be as in Theorem 2.3 and let* $g : \mathcal{M} \to \mathcal{M}$ *be a* C^∞ *Anosov diffeomorphism conjugate to* f *by a homeomorphism* h. *Suppose that for any point* p *such that* $f^m(p) = p$, *the derivatives* df^m_p *and* $dg^m_{h(p)}$ *have the same Jordan normal form. Then* h *is a* C^∞ *diffeomorphism, i.e.* g *is* C^∞ *conjugate to* f.

This corollary applies to any Anosov automorphism of $\mathbb{T}^4$ which is diagonalizable over $\mathbb{C}$ and whose eigenvalues satisfy $|\lambda_1| = |\lambda_2| < 1 < |\lambda_3| = |\lambda_4|$.

We also obtain local rigidity in dimensional three even though there is no global rigidity result for this case.

Corollary 2.5. *Let f be a C^∞ volume-preserving Anosov diffeomorphism of a 3-dimensional manifold $\mathcal{M}$. Suppose that for each periodic point p the derivative of the return map is diagonalizable over $\mathbb{C}$ and two of its eigenvalues have the same modulus. Let g be a C^∞ diffeomorphism of $\mathcal{M}$ which is C^1 close to f and has the same periodic data. Then g is C^∞ conjugate to f.*

We note that the matrix of an Anosov automorphism of $\mathbb{T}^3$ has either a pair of complex eigenvalues and a real eigenvalue or three real eigenvalues of different moduli. Corollary 2.5 applies, in particular, to the former case. The local rigidity for the latter case with $C^{1+\text{Hölder}}$ smoothness of the conjugacy was recently proved by Gogolev and Guysinsky [3]. Thus Anosov automorphisms of $\mathbb{T}^3$ are locally rigid:

Corollary 2.6. *Let f be an Anosov automorphism of $\mathbb{T}^3$ and g be a C^∞ diffeomorphism of $\mathbb{T}^3$ which is C^1 close to f and has the same periodic data. Then g is $C^{1+Hölder}$ conjugate to f.*

3. PRELIMINARIES

In this section we briefly introduce the main notions used throughout this paper.

3.1. **Anosov diffeomorphisms.** Let f be a diffeomorphism of a compact Riemannian manifold $\mathcal{M}$. The diffeomorphism f is called Anosov if there exist a decomposition of the tangent bundle $T\mathcal{M}$ into two f-invariant continuous subbundles E^s and E^u, and constants $C > 0$, $0 < \lambda < 1$ such that for all $n \in \mathbb{N}$,

$$\|df^n(v)\| \le C\lambda^n \|v\| \text{ for } v \in E^s \quad \text{and} \quad \|df^{-n}(v)\| \le C\lambda^n \|v\| \text{ for } v \in E^u.$$

The distributions E^s and E^u are called stable and unstable. It is well-known that these distributions are tangential to the foliations W^s and W^u respectively (see, for example [10]). The leaves of these foliations are C^∞ injectively immersed Euclidean spaces, but in general the distributions E^s and E^u are only Hölder continuous transversally to the corresponding foliations.

3.2. **Uniformly quasiconformal diffeomorphisms.** Let f be an Anosov diffeomorphism of a compact Riemannian manifold $\mathcal{M}$. We say that f is uniformly quasiconformal on the unstable distribution if the quasiconformal distortion

$$K^u(x,n) = \frac{\max\{\,\|\,df^n(v)\,\| : v \in E^u(x),\ \|v\|=1\,\}}{\min\{\,\|\,df^n(v)\,\| : v \in E^u(x),\ \|v\|=1\,\}}$$

is uniformly bounded for all $n \in \mathbb{Z}$ and $x \in \mathcal{M}$. If $K^u(x,n) = 1$ for all x and n, then f is conformal on E^u. Similarly, one can define the corresponding notions for E^s, or any other continuous invariant distribution. If a diffeomorphism is uniformly quasiconformal (conformal) on both E^u and E^s then it is called uniformly quasiconformal (conformal). An Anosov toral automorphism is uniformly quasiconformal if and only if its matrix is diagonalizable over $\mathbb{C}$ and all its (un)stable eigenvalues are equal in modulus.

Unlike the notion of conformality, the weaker notion of uniform quasiconformality does not depend on the choice of a Riemannian metric on the manifold. However, any transitive uniformly quasiconformal Anosov diffeomorphism is conformal with respect to some continuous Riemannian metric [20, Theorem 1.3].

3.3. **Conformal structures.** A conformal structure on $\mathbb{R}^n$, $n \geq 2$, is a class of proportional inner products. The space $\mathcal{C}^n$ of conformal structures on $\mathbb{R}^n$ identifies with the space of real symmetric positive definite $n \times n$ matrices with determinant 1, which is isomorphic to $SL(n,\mathbb{R})/SO(n,\mathbb{R})$. $GL(n,\mathbb{R})$ acts transitively on $\mathcal{C}^n$ via

$$X[C] = (\det X^T X)^{-1/n}\, X^T C\, X, \quad \text{where } X \in GL(n,\mathbb{R}), \text{ and } C \in \mathcal{C}^n.$$

It is known that $\mathcal{C}^n$ becomes a Riemannian symmetric space of non-positive curvature when equipped with a certain $GL(n,\mathbb{R})$-invariant metric. The distance to the identity in this metric is given by

$$\operatorname{dist}(\mathrm{Id}, C) = \frac{\sqrt{n}}{2}\left((\log \lambda_1)^2 + \cdots + (\log \lambda_n)^2\right)^{1/2},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of C. The distance between two structures C_1 and C_2 can be computed as $\operatorname{dist}(C_1, C_2) = \operatorname{dist}(\mathrm{Id}, X[C_2])$, where $X[C_1] = \mathrm{Id}$. We note that on any compact subset of $\mathcal{C}^n$ this distance is bi-Lipschitz equivalent to the distance induced by the operator norm on matrices.

Now, let f be a diffeomorphism of a compact manifold $\mathcal{M}$ and let $E \subset T\mathcal{M}$ be a subbundle invariant under df with $\dim E \geq 2$. A conformal structure on $E(x) \subset T_x\mathcal{M}$ is a class of proportional inner products on $E(x)$. Using a background Riemannian metric, we can identify an inner product with a symmetric linear operator with determinant 1 as before. For each $x \in \mathcal{M}$, we denote the space of conformal structures on $E(x)$ by $\mathcal{C}(x)$. Thus we obtain a bundle $\mathcal{C}$ over $\mathcal{M}$ whose fiber over x is $\mathcal{C}(x)$. We equip the fibers of $\mathcal{C}$ with the Riemannian metric defined above. A continuous (measurable) section of $\mathcal{C}$ is called a continuous (measurable) conformal structure on E. A measurable conformal structure τ on E is called bounded if the distance between $\tau(x)$ and $\tau_0(x)$ is uniformly bounded on $\mathcal{M}$ for a continuous conformal structure τ_0 on E.

The diffeomorphism f induces a natural pull-back action F on conformal structures as follows. For a conformal structure $\tau(fx) \in \mathcal{C}(fx)$, viewed as the linear operator on $E(fx)$, $F_x(\tau(fx)) \in \mathcal{C}(x)$ is given by

$$F_x(\tau(fx)) = \left(\det\left((df_x)^* \circ df_x\right)\right)^{-1/n} (df_x)^* \circ \tau(fx) \circ df_x,$$

where $(df_x)^* : T_{fx}\mathcal{M} \to T_x\mathcal{M}$ denotes the conjugate operator of df_x. We note that $F_x : \mathcal{C}_{fx} \to \mathcal{C}_x$ is an isometry between the fibers $\mathcal{C}(fx)$ and $\mathcal{C}(x)$.

We say that a conformal structure τ is f-invariant if $F(\tau) = \tau$. For an Anosov diffeomorphism f, a subbundle E can carry an invariant conformal structure only if

$E \subset E^s$ or $E \subset E^u$. Clearly, a diffeomorphism is conformal with respect to a Riemannian metric on E if and only if it preserves the conformal structure associated with this metric. If f preserves a continuous or bounded conformal structure on E then f is uniformly quasiconformal on E. If f is a transitive Anosov diffeomorphism and E is Hölder continuous then the converse is also true: if f is uniformly quasiconformal on E then f preserves a continuous conformal structure on E (see Theorem 1.3 in [20] and Theorem 2.7 in [9]).

4. Proofs

4.1. **Proof of Theorem 2.1.** Parts (1)-(5) of the theorem are proven in Propositions 4.1-4.5 respectively. The propositions are somewhat more detailed than the corresponding statements in the theorem.

We would like to point out that while the statements (4) and (5) of the theorem look similar, their proofs are completely different. The derivative of f induces an isometry between the spaces of conformal structures at x and $f(x)$, but these spaces are not compact. On the other hand, the Grassman manifold of the subspaces at x is compact, but the induced map between the manifolds at x and $f(x)$ is not an isometry. This calls for different approaches. Moreover, the continuity of a measurable invariant conformal structure holds in greater generality. Its proof relies only on the statement (3) of the theorem. It holds, for example, for *any* C^1 small perturbation of a conformal Anosov automorphism, with no assumption on the coincidence of periodic data [8, Lemma 5.1]. In contrast, the proof of continuity of a measurable invariant distribution relies on the statement (1) of the theorem which requires the coincidence of periodic data. In fact, one may expect that a typical small perturbation of a conformal Anosov automorphism has simple Lyapunov exponents and measurable Lyapunov distributions which are not continuous.

Proposition 4.1. *Let f be a transitive Anosov diffeomorphism of a compact manifold $\mathcal{M}$. Suppose that for each periodic point there is only one positive Lyapunov exponent (i.e. all unstable eigenvalues of the derivative of the return map have the same modulus).*

Then any ergodic invariant measure for f has only one positive Lyapunov exponent. Moreover, for any $\varepsilon > 0$ there exists C_ε such that

$$K^u(x,n) \le C_\varepsilon e^{\varepsilon|n|} \quad \text{for all } x \text{ in } \mathcal{M} \text{ and } n \text{ in } \mathbb{Z}.$$

A similar statement holds for the stable distribution.

Proof. The fact that any ergodic invariant measure for f has only one positive Lyapunov exponent follows from the following result of W. Sun and Z. Wang (see also [7]).

[21, Theorem 3.1] *Let $\mathcal{M}$ be a d-dimensional Riemannian manifold. Let $f : \mathcal{M} \to \mathcal{M}$ be a $C^{1+\beta}$ diffeomorphism and let μ be an ergodic hyperbolic measure with Lyapunov exponents $\lambda_1 \leq ... \leq \lambda_d$. Then the Lyapunov exponents of μ can be approximated by the Lyapunov exponents of hyperbolic periodic orbits, more precisely for any $\varepsilon > 0$ there exists a hyperbolic periodic point p with Lyapunov exponents $\lambda_1^{(p)} \leq ... \leq \lambda_d^{(p)}$ such that $|\lambda_i - \lambda_i^{(p)}| < \varepsilon$ for $i = 1, \dots, d$.*

To establish the estimate for the quasiconformal distortion $K^u(x, n)$, we use the following result.

[19, Proposition 3.4] *Let $f : \mathcal{M} \to \mathcal{M}$ be a continuous map of a compact metric space. Let $a_n : \mathcal{M} \to \mathbb{R}$, $n \geq 0$ be a sequence of continuous functions such that*

$$a_{n+k}(x) \leq a_n(f^k(x)) + a_k(x) \quad \text{for every } x \in \mathcal{M}, \quad n, k \geq 0 \tag{4.1}$$

and such that there is a sequence of continuous functions b_n, $n \geq 0$ satisfying

$$a_n(x) \leq a_n(f^k(x)) + a_k(x) + b_k(f^n(x)) \quad \text{for every } x \in \mathcal{M}, \quad n, k \geq 0. \tag{4.2}$$

If $\inf_n \left(\frac{1}{n} \int_{\mathcal{M}} a_n d\mu \right) < 0$ for every ergodic f-invariant measure, then there is $N \geq 0$ such that $a_N(x) < 0$ for every $x \in \mathcal{M}$.

We take $\varepsilon > 0$ and apply this result to $a_n(x) = \log K^u(x, n) - \varepsilon n$. It is easy to see that the quasiconformal distortion is submultiplicative, i.e.

$$K^u(x, n + k) \leq K^u(x, k) \cdot K^u(f^k x, n) \quad \text{for every } x \in \mathcal{M}, \quad n, k \geq 0.$$

Hence the functions a_n satisfy (4.1). It is straightforward to verify that

$$K^u(x, n + k) \geq K^u(n, x) \cdot (K^u(f^n x, k))^{-1}$$

This inequality implies $a_{n+k}(x) \geq a_n(x) - b_k(f^n x)$ where $b_n(x) = \log K^u(x, n) + \varepsilon n$. Taking into account (4.1) we obtain (4.2).

Since a_n satisfy (4.1), the Subadditive Ergodic Theorem implies that for every f-invariant ergodic measure μ

$$\lim_{n\to\infty} \frac{1}{n} a_n(x) = \inf_n \frac{1}{n} \int_{\mathcal{M}} a_n d\mu \quad \text{ for } \mu \text{ a.e. } x \in \mathcal{M}.$$

Since μ has only one positive Lyapunov exponent, it is easy to see that

$$\lim_{n\to\infty} \frac{1}{n} \log K^u(x, n) = 0 \text{ and hence } \lim_{n\to\infty} \frac{1}{n} a_n(x) = -\varepsilon < 0 \text{ for } \mu \text{ a.e. } x \in \mathcal{M}.$$

Thus all assumptions of the proposition above are satisfied and hence for any $\varepsilon > 0$ there exists N_ε such that $a_{N_\varepsilon}(x) < 0$, i.e. $K^u(x, N_\varepsilon) \leq e^{\varepsilon N_\varepsilon}$ for all $x \in \mathcal{M}$. We conclude that $K^u(x, n) \leq C_\varepsilon e^{\varepsilon n}$ for all x in $\mathcal{M}$ and n in $\mathbb{N}$, where $C_\varepsilon = \max K^u(x, n)$ with the maximum taken over all $x \in \mathcal{M}$ and $1 \leq n < N_\varepsilon$. Since $K^u(x, n) = K^u(f^n x, -n)$ we obtain $K^u(x, n) \leq C_\varepsilon e^{\varepsilon |n|}$ for all x in $\mathcal{M}$ and n in $\mathbb{Z}$. $\square$

Proposition 4.2. *Let f be an Anosov diffeomorphism of a compact manifold $\mathcal{M}$. Suppose that for any $\varepsilon > 0$ there exists C_ε such that*

$$K^s(x,n) \le C_\varepsilon e^{\varepsilon n} \quad \text{for all } x \text{ in } \mathcal{M} \text{ and } n \text{ in } \mathbb{N}.$$

Then the dependence of $E^u(x)$ on x is $C^{1+\beta}$ for some $\beta > 0$.
Similarly, if $K^u(x,n) \le C_\varepsilon e^{\varepsilon n}$ for all $x \in \mathcal{M}$, $n \in \mathbb{N}$, then E^s is $C^{1+\beta}$.

Proof. We use the C^r Section Theorem of M. Hirsch, C. Pugh, and M. Shub (see Theorems 3.1, 3.2, 3.5, and Remarks 1 and 2 after Theorem 3.8 in [5]).

[5, C^r Section Theorem] *Let f be a C^r, $r \ge 1$, diffeomorphism of a compact C^r manifold $\mathcal{M}$. Let $\mathcal{B}$ be a C^r finite-dimensional normed vector bundle over $\mathcal{M}$ and let $\mathcal{B}_1$ be the corresponding bundle of unite balls in $\mathcal{B}$. Suppose that $F : \mathcal{B} \to \mathcal{B}$ (or $F : \mathcal{B}_1 \to \mathcal{B}_1$) is a C^r extension of f which contracts fibers, i.e. for any $x \in \mathcal{M}$ and any $v, w \in \mathcal{B}(x)$ (resp. $v, w \in \mathcal{B}_1(x)$)*

$$\|F(v) - F(w)\|_{fx} \le k_x \|v - w\|_x \quad \text{with} \quad \sup_{x \in \mathcal{M}} k_x < 1.$$

Then there exists a unique continuous F-invariant section of $\mathcal{B}$ (resp. $\mathcal{B}_1$). If also

$$\sup_{x \in \mathcal{M}} k_x \alpha_x^r < 1 \quad \text{where } \alpha_x = \|(df_x)^{-1}\|,$$

then the unique invariant section is C^r.

Since the stable and unstable distributions are a priori only Hölder continuous, we take C^2 distributions $\bar{E}^u$ and $\bar{E}^s$ which are close to E^u and E^s respectively. We consider a vector bundle $\mathcal{B}$ whose fiber over x is the set of linear operators from $\bar{E}^u(x)$ to $\bar{E}^s(x)$. We endow the fibers of $\mathcal{B}$ with the standard operator norm. We fix a sufficiently large n and consider the graph transform action F induced by the differential of f^n. More precisely, if $A : \bar{E}^u(x) \to \bar{E}^s(x)$ is in $\mathcal{B}(x)$ and $L \subset T_x\mathcal{M}$ is the graph of this operator then $F(A) \in \mathcal{B}(f^n x)$ is defined to be the operator from $\bar{E}^u(f^n x)$ to $\bar{E}^s(f^n x)$ whose graph is $df_x^n(L)$. We note that $F(A)$ is well-defined as long as $df_x^n(L)$ is transversal to $\bar{E}^s(f^n x)$. Let us denote

$$l_x = \min \{ \|df_x^n(v)\| : \ v \in E^u(x), \ \|v\| = 1 \}$$
$$m_x = \min \{ \|df_x^n(v)\| : \ v \in E^s(x), \ \|v\| = 1 \}$$
$$M_x = \max \{ \|df_x^n(v)\| : \ v \in E^s(x), \ \|v\| = 1 \}.$$

In the case when

$$\bar{E}^u(x) = E^u(x), \quad \bar{E}^s(x) = E^s(x), \quad \bar{E}^u(f^n x) = E^u(f^n x), \quad \bar{E}^s(f^n x) = E^s(f^n x)$$

map F is defined on the whole fiber $\mathcal{B}(x)$ and $F_x : \mathcal{B}(x) \to \mathcal{B}(f^n x)$ is the linear map

$$F_x(A) = df_x^n|_{E^s(x)} \circ A \circ (df_x^n|_{E^u(x)})^{-1}, \quad \text{with} \quad \|F_x\| \le M_x / l_x < 1.$$

In particular, $F_x(\mathcal{B}_1(x)) \subset \mathcal{B}_1(f^n x)$. Clearly, the same inclusion holds if $\bar{E}^u$ and $\bar{E}^s$ are close enough to E^u and E^s respectively. Thus $F : \mathcal{B}_1 \to \mathcal{B}_1$ is a well-defined

C^2 extension of f. In general, $F_x : \mathcal{B}_1(x) \to \mathcal{B}_1(f^n x)$ is an algebraic map which depends continuously on the choice of $\bar{E}^u$ and $\bar{E}^s$ at x and $f^n x$. Thus by choosing $\bar{E}^u$ and $\bar{E}^s$ sufficiently close to E^u and E^s one can make F_x sufficiently C^1-close to the linear map above. Therefore, F_x is again a contraction with $k_x \approx M_x/l_x$.

We will now apply the C^r Section Theorem to show the smoothness of the invariant section for F. By uniqueness, the graph of this invariant section is the distribution E^u. The extension F satisfies

$$k_x \approx M_x/l_x < 1 \quad \text{and} \quad \alpha_x = \|(df^n|_x)^{-1}\| = 1/m_x$$

and thus

$$k_x \alpha_x^{1+\beta} \approx \frac{M_x}{l_x m_x^{1+\beta}} \leq \frac{1}{l_x \cdot m_x^\beta} \cdot \frac{M_x}{m_x} \leq \frac{K^s(x,n)}{l_x \cdot m_x^\beta} \leq \frac{C_\varepsilon e^{\varepsilon n}}{Cl^n \cdot (\inf_x\{m_x\})^\beta}$$

for some $C > 0$ and $l > 1$.

We can take ε so small that $e^\varepsilon/l < 1$ and then take n so large that $C_\varepsilon e^{\varepsilon n}/Cl^n < 1$. Then the right hand side is less than 1 for some $\beta > 0$. Once n and β are chosen, we can take $\bar{E}^u$ and $\bar{E}^s$ close enough to E^u and E^s to guarantee that $\sup_{x\in\mathcal{M}} k_x\alpha_x^{1+\beta} < 1$. Hence, by the C^r Section Theorem, the distribution E^u is $C^{1+\beta}$. □

Proposition 4.3. *Let f be an Anosov diffeomorphism of a compact manifold $\mathcal{M}$ such that for any $\varepsilon > 0$ there exists C_ε such that*

$$K^u(x,n) \leq C_\varepsilon e^{\varepsilon n} \quad \text{and} \quad K^s(x,n) \leq C_\varepsilon e^{\varepsilon n} \tag{4.3}$$

for all x in $\mathcal{M}$ and n in $\mathbb{N}$.

Then there exist $C > 0$, $\beta > 0$, and $\delta_0 > 0$ such that for any $\delta < \delta_0$, $x, y \in \mathcal{M}$ and $n \in \mathbb{N}$ with $dist(f^i(x), f^i(y)) \leq \delta$ for $0 \leq i \leq n$, we have

$$\|(df_x^n)^{-1} \circ df_y^n - Id\| \leq C\delta^\beta. \tag{4.4}$$

To consider the composition of the derivatives we identify the tangent spaces at nearby points preserving the Anosov splitting as in Remark 2.2. Since the Anosov splitting is $C^{1+\beta}$ by Proposition 4.2, this identification is also $C^{1+\beta}$.

Proof. We will use non-stationary linearizations along stable and unstable manifolds given by the following proposition.

[20, Proposition 4.1] *Let f be a diffeomorphism of a compact Riemannian manifold $\mathcal{M}$, and let W be a continuous invariant foliation with C^∞ leaves. Suppose that $\|df|_{TW}\| < 1$, and there exist $C > 0$ and $\varepsilon > 0$ such that for any $x \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\| (df^n|_{T_xW})^{-1} \| \cdot \| df^n|_{T_xW} \|^2 \leq C(1-\varepsilon)^n. \tag{4.5}$$

Then for any $x \in \mathcal{M}$ there exists a C^∞ diffeomorphism $h_x^W : W(x) \to T_xW$ such that

(i) $h_{fx}^W \circ f = df_x \circ h_x^W$,
(ii) $h_x^W(x) = 0$ and $(dh_x^W)_x$ is the identity map,
(iii) h_x^W depends continuously on x in C^∞ topology.

Clearly, condition (4.3) implies (4.5) with $W = W^s$ and its analogue for the unstable distribution. Thus we obtain linearizations $h^s_x : W^s(x) \to E^s(x)$ and $h^u_x : W^u(x) \to E^u(x)$. Then we construct a local linearization $h_x : U_x \to T_x\mathcal{M}$, where U_x is a small open neighborhood of $x \in \mathcal{M}$ as follows:

$$h_x|_{W^u(x)} = h^u_x, \quad h_x|_{W^s(x)} = h^s_x,$$

and for $y \in W^u(x) \cap U_x$ and $z \in W^s(x) \cap U_x$ we set

$$h_x([y,z]) = h^u_x(y) + h^s_x(z),$$

where $[y,z] = W^s_{\text{loc}}(y) \cap W^u_{\text{loc}}(z)$. It is easy to see that h satisfies conditions (i) and (ii). Since the maps h^s_x and h^u_x are C^∞ and the local product structure is $C^{1+\beta}$, the maps h_x are $C^{1+\beta}$ with uniform Hölder constant for all x.

Since the differential df_x is the direct sum of the stable differential $df|_{E^s(x)}$ and the unstable differential $df|_{E^u(x)}$, it suffices to prove the proposition for these restrictions. We will give a proof for the unstable differential, the other case is similar.

If δ_0 is small enough, there exists a unique point $z \in W^u_{\text{loc}}(x) \cap W^s_{\text{loc}}(y)$ with

$$\text{dist}(f^i(x), f^i(z)) < C_1\delta \;\text{ and }\; \text{dist}(f^i(z), f^i(y)) < C_1\delta \quad \text{for } 0 \le i \le n.$$

Thus it is sufficient to prove the proposition for x and y lying on the same stable or on the same unstable manifold. First we consider the case when $y \in W^s(x)$. When considering the unstable differential this case appears more difficult, but there will be little difference in our argument.

Since $y \in W^s(x)$ and $\text{dist}\,(x,y) \le \delta$ we obtain $\text{dist}\,(f^n x, f^n y) \le \delta \cdot C_2 \gamma^n$ for some positive $\gamma < 1$. Using the linearizations h_x and $h_{f^n x}$ we can write in a neighborhood of x

$$f^n = (h_{f^n x})^{-1} \circ df^n_x \circ h_x.$$

Differentiating at y we obtain

$$df^n_y = ((dh_{f^n x})_{f^n y})^{-1} \circ df^n_x \circ (dh_x)_y$$

and restricting to E^u

$$df^n|_{E^u(y)} = \big((dh_{f^n x})|_{E^u(f^n y)}\big)^{-1} \circ df^n|_{E^u(x)} \circ (dh_x)|_{E^u(y)}. \tag{4.6}$$

Since the linearizations are $C^{1+\beta}$ we obtain that

$$(dh_z)_w = \text{Id} + R \quad \text{with} \quad \|R\| \le C_3 \,\text{dist}(z,w)^\beta$$

for all $z \in \mathcal{M}$ and all w close to z. Therefore we can write

$$(dh_{f^n x})|_{E^u(f^n y)} = \text{Id} + R_1 \quad \text{and} \quad (dh_x)|_{E^u(y)} = \text{Id} + R_2, \tag{4.7}$$

where

$$\|R_2\| \le C_3 \,\text{dist}(x,y)^\beta \le C_3\, \delta^\beta \tag{4.8}$$

and

$$\|R_1\| \le C_3 \,\text{dist}(f^n x, f^n y)^\beta \le C_3 (C_2\, \delta \gamma^n)^\beta \le C_4\, \delta^\beta \gamma^{\beta n}$$

We also observe that

$$\big((dh_{f^n x})|_{E^u(f^n y)}\big)^{-1} = (\mathrm{Id} + R_1)^{-1} = \mathrm{Id} + R_3 \quad \text{with} \tag{4.9}$$

$$\|R_3\| \le \frac{\|R_1\|}{1 - \|R_1\|} \le 2C_4\, \delta^\beta \gamma^{\beta n} \tag{4.10}$$

provided that $\|R_1\| < 1/2$. Combining (4.6), (4.7), and (4.9), we can write

$$(df^n|_{E^u(x)})^{-1} \circ df^n|_{E^u(y)} = (df^n|_{E^u(x)})^{-1} \circ (\mathrm{Id} + R_3) \circ df^n|_{E^u(x)} \circ (\mathrm{Id} + R_2)$$

$$= \mathrm{Id} + R_2 + (df^n|_{E^u(x)})^{-1} \circ R_3 \circ df^n|_{E^u(x)} \circ (\mathrm{Id} + R_2)$$

Therefore, we can estimate

$$\|(df^n|_{E^u(x)})^{-1} \circ df^n|_{E^u(y)} - \mathrm{Id}\| \le$$

$$\|R_2\| + \|(df^n|_{E^u(x)})^{-1}\| \cdot \|R_3\| \cdot \|df^n|_{E^u(x)}\| \cdot \|\mathrm{Id} + R_2\|$$

We note that $\|(df^n|_{E^u(x)})^{-1}\| \cdot \|df^n|_{E^u(x)}\| \le K^u(x,n) \le C_\varepsilon e^{\varepsilon n}$. Finally, using estimates (4.8) and (4.10) for $\|R_2\|$ and $\|R_3\|$, we obtain

$$\|(df^n|_{E^u(x)})^{-1} \circ df^n|_{E^u(y)} - \mathrm{Id}\| \le C_3\, \delta^\beta + 2C_4\, \delta^\beta \gamma^{\beta n} \cdot C_\varepsilon e^{\varepsilon n} \cdot (1 + C_3\, \delta^\beta)$$

If ε is chosen sufficiently small so that $|\gamma^\beta e^\varepsilon| < 1$, then the term $\gamma^{\beta n} \cdot C_\varepsilon e^{\varepsilon n}$ is uniformly bounded in n. So we conclude that

$$\|(df^n|_{E^u(x)})^{-1} \circ df^n|_{E^u(y)} - \mathrm{Id}\| \le C_5\, \delta^\beta.$$

To complete the proof of the proposition it remains to consider the case when $y \in W^u(x)$. We use the same notation as in the previous case and indicate the necessary changes. In this case we have $\mathrm{dist}\,(f^n x, f^n y) \le \delta$ and $\mathrm{dist}\,(x,y) \le \delta \cdot C_2 \gamma^n$ for some positive $\gamma < 1$. Therefore,

$$\|R_3\| \le \frac{\|R_1\|}{1 - \|R_1\|} \le 2C_3\, \delta^\beta \;\text{ and }\; \|R_2\| \le C_3\, \mathrm{dist}(x,y)^\beta \le C_6\, \delta^\beta \gamma^{\beta n}. \tag{4.11}$$

Now we can write

$$df^n|_{E^u(y)} \circ (df^n|_{E^u(x)})^{-1} = (\mathrm{Id} + R_3) \circ df^n|_{E^u(x)} \circ (\mathrm{Id} + R_2) \circ (df^n|_{E^u(x)})^{-1}$$

$$= \mathrm{Id} + R_3 + (\mathrm{Id} + R_3) \circ (df^n|_{E^u(x)})^{-1} \circ R_2 \circ df^n|_{E^u(x)}.$$

Finally, using the new estimates (4.11) for $\|R_2\|$ and $\|R_3\|$, we obtain similarly to the previous case that

$$\|df^n|_{E^u(y)} \circ (df^n|_{E^u(x)})^{-1} - \mathrm{Id}\| \le C_7\, \delta^\beta.$$

This estimate clearly implies a similar Hölder estimate for $\|(df^n|_{E^u(x)})^{-1} \circ df^n|_{E^u(y)} - \mathrm{Id}\|$ and concludes the proof of the proposition. ☐

Proposition 4.4.

(i) Let f be a a transitive Anosov diffeomorphism of a compact manifold $\mathcal{M}$ and let E be a Hölder continuous invariant distribution. Suppose that there exist $k > 0$, $\delta_0 > 0$, and $\beta > 0$ such that for any $\delta < \delta_0$, $x, y \in \mathcal{M}$ and $n \in \mathbb{N}$ such that dist$(f^i(x), f^i(y)) < \delta$ for $0 \le i \le n$,

$$\| \left(df^n|_{E(x)}\right)^{-1} \circ \left(df^n|_{E(y)}\right) - Id \, \| \le k \, \delta^\beta.$$

Then any f-invariant measurable conformal structure on E is Hölder continuous.

(ii) If f is as in Theorem 2.1, then any f-invariant measurable conformal structure on E^u (E^s) is C^1.

Remark. As follows from the proof, the measurability of the conformal structure in this proposition can be understood with respect to any ergodic measure μ with full support for which the local stable (or unstable) holonomies are absolutely continuous with respect to the conditional measures on local unstable (or stable) manifolds. Note that invariant volume, if it exists, satisfies these properties, and so does the Bowen-Margulis measure of maximal entropy, which always exists for a transitive Anosov diffeomorphism f. More generally, these properties hold for any equilibrium (Gibbs) measure corresponding to a Hölder continuous potential.

Proof. **(i)** In this proposition we identify spaces $E(x)$ and $E(y)$ at nearby points x and y as in Remark 2.2. Since the distribution E is Hölder continuous, the identification is also Hölder continuous, and hence $df|_E$ is Hölder continuous with respect to the identification. The identification allows us to conveniently compare differentials and conformal structures at different points.

For $x \in \mathcal{M}$, we denote by $\tau(x)$ the conformal structure on $E(x)$. First we estimate the distance between the conformal structures at x and at a nearby point $y \in W^s(x)$. We use the distance described in Section 3.3. Let $x_n = f^n(x)$, $y_n = f^n(y)$, and let F^n_x be the isometry from $\mathcal{C}(f^n x)$ to $\mathcal{C}(x)$ induced by $df^n|_{E(x)}$. Since the conformal structure τ is invariant, $\tau(x) = F^n_x(\tau(x_n))$ and $\tau(y) = F^n_y(\tau(y_n))$. Using this and the fact that F^n is an isometry, we obtain

$$\begin{aligned}
\text{dist}(\tau(x), \tau(y)) &= \text{dist}\left(F^n_x(\tau(x_n)), F^n_y(\tau(y_n))\right) \\
&\le \text{dist}\left(F^n_x(\tau(x_n)), F^n_y(\tau(x_n))\right) + \text{dist}\left(F^n_y(\tau(x_n)), F^n_y(\tau(y_n))\right) \\
&= \text{dist}\left(\tau(x_n), ((F^n_x)^{-1} \circ F^n_y)(\tau(x_n))\right) + \text{dist}\left(\tau(x_n), \tau(y_n)\right).
\end{aligned}$$

Let μ be an invariant measure as in the proposition or the remark after it. Since the conformal structure τ is measurable, by Lusin's theorem we can take a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ on which τ is uniformly continuous and bounded.

First we show that for $x_n \in S$ the term $\text{dist}\left(\tau(x_n), ((F^n_x)^{-1} \circ F^n_y)(\tau(x_n))\right)$ is Hölder in $\text{dist}(x, y)$. For this we observe that the map $(F^n_x)^{-1} \circ F^n_y$ is induced by $(df^n_x)^{-1} \circ df^n_y$, and $\|(df^n_x)^{-1} \circ df^n_y - \text{Id}\, \| \le k \cdot \text{dist}(x, y)^\beta$. Let A be the matrix of

$(df_x^n)^{-1} \circ df_y^n$. Then

$$A = \mathrm{Id} + R, \quad \text{where } \|R\| \le k \cdot \mathrm{dist}(x,y)^\beta.$$

Let C be the matrix corresponding to the conformal structure $\tau(x_n)$. Recall that C is symmetric positive definite with determinant 1. Thus there exists an orthogonal matrix Q such that $Q^T C Q$ is a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_i > 0$ of C. Let X be the product of Q and the diagonal matrix with entries $1/\sqrt{\lambda_i}$. Then X has determinant 1 and $X[C] = X^T C X = \mathrm{Id}$. Now we estimate

$$\begin{aligned}
&\mathrm{dist}\left(\tau(x_n), ((F_x^n)^{-1} \circ F_y^n)(\tau(x_n))\right) = \mathrm{dist}\,(C, A[C]) = \ \mathrm{dist}\,(\mathrm{Id}, X[A[C]]) \\
&= \mathrm{dist}\left(\mathrm{Id}, X^T A^T C A X\right) = \mathrm{dist}\left(\mathrm{Id}, X^T(\mathrm{Id} + R^T) C (\mathrm{Id} + R) X\right) \\
&= \mathrm{dist}\,(\mathrm{Id}, \mathrm{Id} + B), \quad \text{where } B = X^T C R X + X^T R^T C X + X^T R^T C R X.
\end{aligned}$$

We observe that $\|B\| \le 3\|X\|^2 \cdot \|C\| \cdot \|R\|$ and $\|X\|^2 \le \|C^{-1}\|$, as follows from the construction of X. Since the conformal structure τ is bounded on S, so are $\|C^{-1}\|$ and $\|C\|$, and hence

$$\|B\| \le 3\|C^{-1}\| \cdot \|C\| \cdot \|R\| \quad \le 3k_1\|R\| \ \le \ 3k_1 k \cdot \mathrm{dist}(x,y)^\beta = k_2 \cdot \mathrm{dist}(x,y)^\beta$$

In particular, $\|B\|$ is small if $\mathrm{dist}(x,y)$ is sufficiently small. Finally, for $x_n \in S$ we obtain

$$\mathrm{dist}\,(\tau(x_n), ((F_x^n)^{-1} \circ F_y^n)(\tau(x_n))) = \mathrm{dist}\,(\mathrm{Id}, \mathrm{Id} + B) \le k_3\|B\| \le k_4 \cdot \mathrm{dist}(x,y)^\beta$$

where constant k_4 depend on the set S. We conclude that if x_n is in S then

$$\mathrm{dist}(\tau(x), \tau(y)) \le \mathrm{dist}(\tau(x_n), \tau(y_n)) + k_4 \cdot \mathrm{dist}(x,y)^\beta.$$

It follows from the Birkhoff ergodic theorem that the set of points for which the frequency of visiting S equals $\mu(S) > 1/2$ has full measure. We denote this set by G. If both x and y are in G, then there exists a sequence (n_i) such that $x_{n_i} \in S$ and $y_{n_i} \in S$. If in addition x and y lie on the same stable leaf, then $\mathrm{dist}(x_{n_i}, y_{n_i}) \to 0$ and hence $\mathrm{dist}(\tau(x_{n_i}), \tau(y_{n_i})) \to 0$ by continuity of τ on S. Thus, we obtain

$$\mathrm{dist}(\tau(x), \tau(y)) \le k^s \cdot \mathrm{dist}(x,y)^\beta.$$

By a similar argument, $\mathrm{dist}(\tau(x), \tau(z)) \le k^u \cdot \mathrm{dist}(x,z)^\beta$ for any two nearby points $x, z \in G$ lying on the same unstable leaf.

Consider a small open set in $\mathcal{M}$ with a product structure. For μ almost all local stable leaves, the set of points of G on the leaf has full conditional measure. Consider points $x, y \in G$ lying on two such stable leaves. Let $H_{x,y}$ be the unstable holonomy map between $W^s(x)$ and $W^s(y)$. Since the holonomy maps are absolutely continuous with respect to the conditional measures, there exists a point $z \in W^s(x) \cap G$ close to x such that $H_{x,y}(z)$ is also in G. By the above argument,

$$\begin{aligned} \operatorname{dist}(\tau(x),\tau(z)) &\le k^s \cdot \operatorname{dist}(x,z)^\beta, \\ \operatorname{dist}(\tau(z),\tau(H_{x,y}(z))) &\le k^u \cdot \operatorname{dist}(z,H_{x,y}(z))^\beta, \quad \text{and} \\ \operatorname{dist}(\tau(H_{x,y}(z)),\tau(y)) &\le k^s \cdot \operatorname{dist}(H_{x,y}(z),y)^\beta. \end{aligned}$$

Since the points x, y, and z are close, it is clear from the local product structure that

$$\operatorname{dist}(x,z)^\beta + \operatorname{dist}(z,H_{x,y}(z))^\beta + \operatorname{dist}(H_{x,y}(z),y)^\beta \le k_5 \cdot \operatorname{dist}(x,y)^\beta.$$

Hence, we obtain $\operatorname{dist}(\tau(x),\tau(y)) \le k_6 \cdot \operatorname{dist}(x,y)^\beta$ for all x and y in a set of full measure $\tilde{G} \subset G$.

We can assume that $\tilde{G}$ is invariant by considering $\bigcap_{n=-\infty}^{\infty} f^n(\tilde{G})$. Since μ has full support the set $\tilde{G}$ is dense in $\mathcal{M}$. Hence we can extend τ from $\tilde{G}$ and obtain an invariant Hölder continuous conformal structure τ on $\mathcal{M}$. This completes the proof of the first part of the proposition.

(ii) Let τ be a measurable conformal structure on E^u. By (i) it is Hölder continuous, and we can show that it is actually smooth as follows. First we note that by Lemma 3.1 in [20] the conformal structure τ is C^∞ along the leaves of W^u. The lemma shows that the C^∞ linearization $h_x : W^u(x) \to E^u(x)$ as in Proposition 4.3 maps τ on $W^u(x)$ to a constant conformal structure on E^u.

By Proposition 4.2, the distributions E^u and E^s are $C^{1+\beta}$. Theorem 1.4 in [20] implies that τ is preserved by the stable holonomies, which are $C^{1+\beta}$ (in fact they are C^∞ for $\dim E^u \ge 2$). Thus τ is smooth along the leaves of W^s. Now it follows easily, for example from Journe Lemma [6], that τ is at least C^1 on $\mathcal{M}$. $\square$

Proposition 4.5. *Let f be a volume-preserving Anosov diffeomorphism of a compact manifold $\mathcal{M}$. Suppose that for each periodic point p there is only one positive Lyapunov exponent $\lambda_+^{(p)}$ and only one negative Lyapunov exponent $\lambda_-^{(p)}$. Then any measurable measurable f-invariant distribution in E^u (E^s) defined almost everywhere with respect to the volume is C^1.*

Proof. Let E be a k-dimensional measurable invariant distribution in E^u. We consider fiber bundle $\bar{\mathcal{M}}$ over $\mathcal{M}$ whose fiber over x is the Grassman manifold G_x of all k-dimensional subspaces in $E^u(x)$. The differential df_x induces a natural map $F_x : G_x \to G_{fx}$ and we obtain an extension $\bar{f} : \bar{\mathcal{M}} \to \bar{\mathcal{M}}$ of our system $f : \mathcal{M} \to \mathcal{M}$ given by $\bar{f}(x,V) = (f(x), F_x(V))$ where $V \in G_x$. Thus we have the following commutative diagrams where π is the projection.

$$\begin{array}{ccc} \bar{\mathcal{M}} & \xrightarrow{\bar{f}} & \bar{\mathcal{M}} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M} \end{array} \qquad\qquad \begin{array}{ccc} T\bar{\mathcal{M}} & \xrightarrow{d\bar{f}} & T\bar{\mathcal{M}} \\ \downarrow d\pi & & \downarrow d\pi \\ T\mathcal{M} & \xrightarrow{df} & T\mathcal{M} \end{array}$$

We note that since the distribution E^u is $C^{1+\beta}$ by Proposition 4.2, the diffeomorphism $\bar{f}$ is also $C^{1+\beta}$.

Let $\bar{E}^c$ be the distribution in $T\bar{\mathcal{M}}$ tangent to the fibers, i.e. $\bar{E}^c = \ker d\pi$. Clearly, $\bar{E}^c$ is invariant under $d\bar{f}$.

Lemma 4.6. *There exists $C > 0$ such that for any $\bar{x} \in \bar{\mathcal{M}}$, $n \in \mathbb{Z}$, and $\varepsilon > 0$*

$$\|d\bar{f}^n|_{\bar{E}^c(\bar{x})}\| \le C \cdot K^u(x,n) \le C \cdot C_\varepsilon e^{\varepsilon|n|}, \quad \text{where } \; x = \pi(\bar{x}). \tag{4.12}$$

Proof. Let w and v be two unit vectors in $E^u(x)$. We denote $D = df_x^n$. Using the formula $2 < Dw, Dv >= \|Dw\|^2 + \|Dv\|^2 - \|Dw - Dv\|^2$ for the inner product, we obtain

$$(2\,\sin(\angle(Dw,Dv)/2)\,)^2 = 2(1-\cos\angle(Dw,Dv)) =$$

$$2 - \frac{2 < Dw, Dv >}{\|Dw\|\cdot\|Dv\|} = \frac{2\|Dw\|\cdot\|Dv\| - \|Dw\|^2 - \|Dv\|^2 + \|Dw - Dv\|^2}{\|Dw\|\cdot\|Dv\|} =$$

$$\frac{\|Dw - Dv\|^2 - (\|Dw\| - \|Dv\|)^2}{\|Dw\|\cdot\|Dv\|} \le \frac{\|D\|^2\cdot\|w-v\|^2}{\|Dw\|\cdot\|Dv\|} \le K^u(x,n)^2\cdot\|w-v\|^2.$$

Suppose that the angle between the unit vectors w and v is sufficiently small so that it remains small when multiplied by $K^u(x,n)$. Then we obtain that the angle $\angle(Dw,Dv)$ is also small and

$$\angle(Dw,Dv) \le 2K^u(x,n)\cdot\angle(w,v)$$

Therefore, for any subspaces $V, W \in G_x$ we have

$$\operatorname{dist}(F_x^n(V), F_x^n(W)) \le 2K^u(x,n)\cdot\operatorname{dist}(V,W)$$

where the distance between two subspaces is the maximal angle. This means that $\bar{f}^n$ expands the distance between any two nearby points in the fiber G_x at most by a factor of $2K^u(x,n)$. We note that the maximal angle distance is Lipschitz equivalent to any smooth Riemannian distance on the Grassman manifold. It follows that there exists $C > 0$ such that for any $\bar{x} \in \bar{\mathcal{M}}$ and any $n > 0$

$$\|d\bar{f}^n|_{\bar{E}^c(\bar{x})}\| \le C\cdot K^u(\pi(x),n) \le C\cdot C_\varepsilon e^{\varepsilon|n|},$$

where the last inequality is given by Proposition 4.1. The case of $n < 0$ can be considered similarly. $\square$

We define distributions $\bar{E}^{uc} = d\pi^{-1}(E^u)$ and $\bar{E}^{sc} = d\pi^{-1}(E^s)$. It follows from the commutative diagram that $\bar{E}^{uc}$ and $\bar{E}^{sc}$ are invariant under $d\bar{f}$. Note that $\bar{E}^c = \bar{E}^{uc} \cap \bar{E}^{sc}$.

Lemma 4.7. *$\bar{f}$ is a partially hyperbolic diffeomorphism. More precisely, there exists a continuous $d\bar{f}$-invariant splitting*

$$T\bar{\mathcal{M}} = \bar{E}^u \oplus \bar{E}^c \oplus \bar{E}^s, \tag{4.13}$$

where the unstable and stable distributions $\bar{E}^u$ and $\bar{E}^s$ are contained in $\bar{E}^{uc}$ and $\bar{E}^{sc}$ respectively and are expanded/contracted uniformly by $\bar{f}$. The expansion/contraction in $\bar{E}^c$ is uniformly slower than expansion in $\bar{E}^u$ and contraction in $\bar{E}^s$.

Proof. We fix a continuous distribution H such that $\bar{E}^{uc} = \bar{E}^c \oplus H$. We can identify H with E^u via $d\pi$ and introduce the metric on $\bar{E}^{uc}$ so that the sum is orthogonal. We consider fiber bundle $\mathcal{B}$ whose fiber over $x \in \bar{\mathcal{M}}$ is the space of linear operators $L_x : H(x) \to \bar{E}^c(x)$ equipped with the operator norm. The differential $d\bar{f}$ induces via the graph transform the natural extension $F_x : \mathcal{B}(x) \to \mathcal{B}(\bar{f}x)$ of $\bar{f}$. It is easy to see that F_x is an affine map. Lemma 4.6 implies that the possible expansion in $\bar{E}^c$ is slower than the uniform expansion in $H \cong E^u$. Similarly to the proof of Proposition 4.2, this implies that F is a uniform fiber contraction. Hence it has a continuous invariant section whose graph gives a continuous $d\bar{f}$ invariant distribution which we denote $\bar{E}^u$. Since $d\pi$ induces an isomorphism between $\bar{E}^u$ and E^u which conjugates the actions of $d\bar{f}$ and df, we conclude that $d\bar{f}$ uniformly expands $\bar{E}^u$. The distribution $\bar{E}^s$ is obtained similarly. □

The theory of partially hyperbolic diffeomorphisms gives the existence of the unstable and stable foliations $\bar{W}^u$ and $\bar{W}^s$ with C^1 smooth leaves tangential to the distributions $\bar{E}^u$ and $\bar{E}^s$ respectively. The leaf $\bar{W}^u(\bar{x})$ is contained in the corresponding center-unstable leaf $\bar{W}^{uc}(\bar{x}) = \pi^{-1}(W^u(x))$, where $\pi(\bar{x}) = x$, and projects diffeomorphically to $W^u(x)$. Similarly, $\bar{W}^s(\bar{x})$ is contained in $\bar{W}^{sc}(\bar{x}) = \pi^{-1}(W^s(x))$ and projects diffeomorphically to $W^s(x)$.

The measurable f-invariant distribution E gives rise to the measurable $\bar{f}$-invariant section $\phi : \mathcal{M} \to \bar{\mathcal{M}}$, $\phi(x) = (x, E(x))$. We will show that ϕ is C^1, and hence so is E. Let μ be the smooth f-invariant volume on $\mathcal{M}$. We denote by $\bar{\mu}$ the lift of μ to the graph Φ of ϕ, i.e. for a set $X \subset \bar{\mathcal{M}}$, $\bar{\mu}(X) = \mu(\pi(X \cap \Phi))$. Since μ is f-invariant and Φ is $\bar{f}$-invariant, the measure $\bar{\mu}$ is also $\bar{f}$-invariant.

Now we consider Lyapunov exponents of $\bar{\mu}$. Lemma 4.6 implies that the Lyapunov exponent of any vector in $\bar{E}^c$ is zero. The uniform contraction in $\bar{E}^s$ and expansion in $\bar{E}^u$ now imply that the partially hyperbolic splitting (4.13) coincides with the Lyapunov splitting into the unstable, neutral, and stable distributions for $\bar{\mu}$.

We recall that by the Ruelle inequality the entropy of any ergodic measure invariant under a diffeomorphism is not greater than the sum of its positive Lyapunov exponents counted with multiplicities. It is well known that the inequality is, in fact, equality for an invariant volume. Thus

$$
\begin{aligned}
h_{\bar{\mu}}(\bar{f}) \; &\leq \; \text{sum of positive exponents of } \bar{\mu}, \\
h_{\mu}(f) \; &= \; \text{sum of positive exponents of } \mu.
\end{aligned} \tag{4.14}
$$

Lemma 4.6 implies that $\|d\bar{f}^n|_{\bar{E}^{uc}(\bar{x})}\| \leq \|df^n|_{E^u(x)}\|$ and hence the largest Lyapunov exponent of $\bar{\mu}$ is not greater than that of μ. By Proposition 4.1 μ has only one positive exponent, and since $\dim E^u = \dim \bar{E}^u$ we obtain

$$\text{sum of positive exponents of } \bar{\mu} \ \leq \ \text{sum of positive exponents of } \mu. \tag{4.15}$$

Since (f, μ) is a factor of $(\bar{f}, \bar{\mu})$ we have $h_\mu(f) \leq h_{\bar{\mu}}(\bar{f})$. Combining this with (4.14) and (4.15) we conclude that

$$\begin{aligned} h_{\bar{\mu}}(\bar{f}) \ &\leq \text{sum of positive exponents of } \bar{\mu} \\ &\leq \text{sum of positive exponents of } \mu \ = \ h_\mu(f) \ \leq \ h_{\bar{\mu}}(\bar{f}), \end{aligned}$$

and hence all the inequalities above are, actually, equalities. In particular, $h_{\bar{\mu}}(\bar{f})$ equals the sum of the positive exponents of $\bar{\mu}$. This implies [17, Theorem A] that the conditional measures on the unstable manifolds of $\bar{f}$ are absolutely continuous with positive densities. We note that the C^2 smoothness assumption for the diffeomorphism in that theorem is only used to establish that the holonomies of the unstable foliation within a center-unstable set are Lipschitz. Since $\bar{f}$ is $C^{1+\beta}$ partially hyperbolic and dynamically coherent diffeomorphism satisfying Lemma 4.6, these holonomies are in fact C^1 [1, Theorem 0.2]. Similarly, we establish that the conditional measures on the stable manifolds of $\bar{f}$ are absolutely continuous with positive densities. We show below that the local product structure given by foliations W^u and W^s on $\mathcal{M}$ lifts to the local product structure on the graph Φ in $\bar{\mathcal{M}}$ given by foliations $\bar{W}^u$ and $\bar{W}^s$, which easily implies the proposition. We note that the absolute continuity of the conditional measures seems essential for this argument.

For a point x in $\mathcal{M}$ we will denote by $\bar{x}$ the unique point on the graph Φ with $\pi(\bar{x}) = x$. The absolute continuity of the conditional measures implies that for a $\bar{\mu}$-typical point $\bar{x}$ almost every point $\bar{y}$ with respect to a volume on the local stable leaf $\bar{W}^s_{\text{loc}}(\bar{x})$ is also typical with respect to $\bar{\mu}$. The projection of the set of such points $\bar{y}$ has full measure in the local stable leaf $W^s_{\text{loc}}(x)$ with respect to a volume on this leaf, which is equivalent to the conditional measure of μ. This means that for a μ-typical point x the leaf $\bar{W}^s_{\text{loc}}(\bar{x})$ is essentially contained in the graph Φ. More precisely, there is a set X in $\mathcal{M}$ with $\mu(X) = 1$ such that if $x, z \in X$ and $z \in W^s_{\text{loc}}(x)$ then $\bar{z} \in \bar{W}^s_{\text{loc}}(\bar{x})$. We can also assume that X has a similar property for the unstable leaves.

Consider nearby points $x, y \in X$. Absolute continuity of the unstable holonomies for f gives the existence of points $z_1, z_2 \in X$ such that $z_1 \in W^s_{\text{loc}}(x)$, $z_2 \in W^s_{\text{loc}}(y)$, and $z_2 \in W^u_{\text{loc}}(z_1)$. By the property of X the corresponding lifts satisfy $\bar{z}_1 \in \bar{W}^s_{\text{loc}}(\bar{x})$, $\bar{z}_2 \in \bar{W}^s_{\text{loc}}(\bar{y})$, and $\bar{z}_2 \in \bar{W}^u_{\text{loc}}(\bar{z}_1)$, i.e. $\bar{x}$ and $\bar{y}$ can be connected by local stable/unstable manifolds. Since these manifolds are C^1 graphs over the corresponding manifolds in $\mathcal{M}$, we conclude that $\text{dist}(\bar{x}, \bar{y}) \leq C \cdot \text{dist}(x, y)$. This implies that the graph Φ and the section ϕ are Lipschitz continuous (up to a change on a set of measure zero). Moreover, the graph of ϕ restricted to $W^s(x)$ is the manifold $\bar{W}^s(\bar{x})$, and hence the restriction is C^1. Similarly, the restriction of ϕ to $W^u(x)$ is C^1 for any $x \in \mathcal{M}$. It follows, for example from Journe Lemma [6], that ϕ is C^1 on $\mathcal{M}$. $\square$

This completes the proof of Theorem 2.1.

4.2. **Proof of Theorem 2.3.** We will use the following particular case of Zimmer's Amenable Reduction Theorem (see [4, Theorem 1.6 and Corollary 1.8]):

[4] *Let $A : X \to GL(n, \mathbb{R})$ be a measurable cocycle over an ergodic transformation T of a measure space (X, μ). Then A is measurably cohomologous to a cocycle B with values in an amenable subgroup of $GL(n, \mathbb{R})$.*

Corollary 4.8. *Let f be a diffeomorphism of a smooth manifold $\mathcal{M}$ preserving an ergodic measure μ and let E be a 2-dimensional measurable invariant distribution. Then the restriction of the derivative df to E has either a measurable invariant one-dimensional distribution or a measurable invariant conformal structure.*

Proof. We apply the Amenable Reduction Theorem to restriction of the derivative cocycle to E. In the case of $n = 2$ any maximal amenable subgroup of $GL(2, \mathbb{R})$ is conjugate either to the subgroup of (upper) triangular matrices or to the subgroup of scalar multiples of orthogonal matrices. The former subgroup preserves a coordinate line, the latter subgroup preserves the standard conformal structure on $\mathbb{R}^2$. The measurable coordinate change given by the cohomology gives the corresponding measurable invariant structure for $df|_E$. □

We will now show that f is uniformly quasiconformal on E^u and E^s. The corollary above applied to $E = E^u$ yields either a measurable invariant one-dimensional distribution in E^u or a measurable invariant conformal structure on E^u. In the latter case Proposition 4.4 implies that the conformal structure is continuous. If follows that f is uniformly quasiconformal on E^u.

If there is a measurable invariant one-dimensional distribution in E^u then it is continuous by Proposition 4.5. For any periodic point p this gives an invariant line for the derivative $df^n|_{E^u(p)}$ of the return map. This implies that the eigenvalues of $df^n|_{E^u(p)}$ are real. Hence, by the assumption of the theorem, they are either λ, λ or $\lambda, -\lambda$. In the former case we have $df^n|_{E^u(p)} = \lambda \cdot \mathrm{Id}$ which preserves any conformal structure on $E^u(p)$. This implies uniform quasiconformality of f on E^u (see [8, Proposition 1.1], [16]). The case $\lambda, -\lambda$ is orientation reversing and can be excluded by passing to a finite cover of $\mathcal{M}$ and replacing f by its power. Note that uniform quasiconformality of a power of f implies that of f.

We conclude that f is uniformly quasiconformal on E^u. Similarly, f is uniformly quasiconformal on E^s. This implies that a finite cover of f is C^∞ conjugate to an Anosov automorphism of a torus. Indeed, if $\mathcal{M}$ is known to be an infranilmanifold then [8, Theorem 1.1] stated in the introduction applies. Since f is volume preserving the result holds even if $\mathcal{M}$ is arbitrary [2, Corollary 1]. □

4.3. **Proof of Corollary 2.4.** The coincidence of the periodic data implies that Theorem 2.3 applies to g as well. Hence, after passing to a finite cover of $\mathcal{M}$, f and g are C^∞ conjugate to automorphisms of $\mathbb{T}^4$. Thus it suffices to show that any two

Anosov automorphisms A and B of $\mathbb{T}^k$ which are topologically conjugate are also C^∞ conjugate. Let h be a topological conjugacy, i.e. a homeomorphism of $\mathbb{T}^k$ such that $A \circ h = h \circ B$. Let H be the induced action of h on the fundamental group $\mathbb{Z}^k$ of $\mathbb{T}^k$. Then H is an integral matrix with determinant ± 1, and hence it induces an automorphism of $\mathbb{T}^k$. From the induced actions of A, B, and h on the fundamental group $\mathbb{Z}^k$ we see that $A \circ H = H \circ B$. Thus, H gives a smooth conjugacy between A and B. In fact, $H = h$ since the conjugacy to an Anosov automorphism is known to be unique in a given homotopy class [10]. $\square$

4.4. **Proof of Corollary 2.5.** Let h be the topological conjugacy between f and g given by the structural stability. Suppose that f and g have two-dimensional unstable and one-dimensional stable distributions. We apply the approaches used by R. de la Llave in [11, 12] for one-dimensional distributions and in [14] for higher-dimensional conformal case. There the smoothness of h is established separately along the stable and the unstable foliations. This implies that h is smooth on the manifold.

The smoothness of h along the two-dimensional unstable foliation follows as in the proof of Theorem 1.1 in [14] once we establish that f and g preserve C^1 conformal structures on their unstable distributions. As in the proof of Theorem 2.3, we obtain that f is uniformly quasiconformal on E^u, and hence it preserves a bounded measurable conformal structure on E^u. By (4) of Theorem 2.1, this conformal structure is C^1. The same argument gives an invariant C^1 conformal structure for g. We conclude that h is smooth along the unstable foliations. The smoothness along the one-dimensional stable foliation can be established as in [11, 12]. $\square$

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