BOUNDEDNESS, COMPACTNESS, AND INVARIANT NORMS FOR BANACH COCYCLES OVER HYPERBOLIC SYSTEMS

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Abstract. We consider group-valued cocycles over dynamical systems with hyperbolic behavior. The base system is either a hyperbolic diffeomorphism or a mixing subshift of finite type. The cocycle $\mathcal{A}$ takes values in the group of invertible bounded linear operators on a Banach space and is Hölder continuous. We consider the periodic data of $\mathcal{A}$, i.e. the set of its return values along the periodic orbits in the base. We show that if the periodic data of $\mathcal{A}$ is uniformly quasiconformal or bounded or contained in a compact set, then so is the cocycle. Moreover, in the latter case the cocycle is isometric with respect to a Hölder continuous family of norms. We also obtain a general result on existence of a measurable family of norms invariant under a cocycle.

1. Introduction and statement of the results

Group-valued cocycles appear naturally and play an important role in dynamics. In particular, cocycles over hyperbolic systems have been extensively studied starting with the work of A. Livšic [Liv71, Liv72]. One of the main problems in this area is to obtain properties of the cocycle from its values at the periodic orbits in the base, which are abundant for hyperbolic systems. The study encompassed various types of groups, from abelian to compact non-abelian and more general non-abelian, see [NT95, PaP97, Pa99, Sch99, PW01, dLW10, K11, KS10, S15, Gu, KS16] and a survey in [KtN]. Cocycles with values in the group of invertible linear operators on a vector space V are the prime examples in the last class. The case of finite dimensional V has been well studied, with various applications including derivative cocycles of smooth dynamical systems and random matrices. The infinite dimensional case is more difficult and is less developed so far. The simplest examples are given by random and Markov sequences of operators. In our setting they correspond to locally constant cocycles over subshifts of finite type. Similarly to finite dimensional case, the derivative of a smooth infinite dimensional system gives a natural example of an operator valued cocycle. We refer to monograph [LL] for an overview of results in this area and to [GKa, M12] for some of the recent developments.

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In this paper we consider cocycles of invertible bounded operators on a Banach space V over dynamical systems with hyperbolic behavior. The space $L(V)$ of bounded linear operators on V is a Banach space equipped with the operator norm $\|A\| = \sup\{\|Av\| : v \in V, \|v\| \le 1\}$. The open set $GL(V)$ of invertible elements in $L(V)$ is a topological group and a complete metric space with respect to the metric

$$d(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|.$$

Definition 1.1. *Let f be a homeomorphism of a metric space X and let A be a function from X to $(GL(V), d)$. The* Banach cocycle over f generated by A *is the map $\mathcal{A} : X \times \mathbb{Z} \to G$ defined by $\mathcal{A}(x, 0) = Id$ and for $n \in \mathbb{N}$*

$$\mathcal{A}(x, n) = \mathcal{A}^n_x = A(f^{n-1}x) \circ \cdots \circ A(x) \quad \textit{and} \quad \mathcal{A}(x, -n) = \mathcal{A}^{-n}_x = (\mathcal{A}^n_{f^{-n}x})^{-1}.$$

Clearly, $\mathcal{A}$ satisfies the *cocycle equation* $\mathcal{A}^{n+k}_x = \mathcal{A}^n_{f^k x} \circ \mathcal{A}^k_x$.

Cocycles can be considered in any regularity, but Hölder continuity is the most natural in our setting. On the one hand continuity of the cocycle is not sufficient for development of a meaningful theory even for scalar cocycles over hyperbolic systems. On the other hand, symbolic systems have a natural Hölder structure but lack a smooth one. Moreover, even for smooth hyperbolic systems higher regularity is rare for many usual examples of cocycles, such as restrictions of the differential to the stable and unstable subbundles. We say that a cocycle $\mathcal{A}$ is *β-Hölder* if its generator A is Hölder continuous with exponent $0 < \beta \le 1$, i.e. there exists $c > 0$ such that

$$d(A(x), A(y)) \le \; c \operatorname{dist}(x, y)^\beta \quad \text{for all } x, y \in X.$$

For a cocycle $\mathcal{A}$, we consider the periodic data set $\mathcal{A}_P$ and the set of all values $\mathcal{A}_X$,

$$\mathcal{A}_P = \{\mathcal{A}^k_p : \; p = f^k p, \; p \in X, \; k \in \mathbb{N}\} \quad \text{and} \quad \mathcal{A}_X = \{\mathcal{A}^n_x : \; x \in X, \; n \in \mathbb{Z}\}.$$

Our main result is that uniform quasiconformality, uniform boundedness, and pre-compactness of the cocycle can be detected from its periodic data. Moreover, pre-compactness implies that the cocycle is isometric with respect to a Hölder continuous family of norms.

Definition 1.2. *The* quasiconformal distortion *of a cocycle $\mathcal{A}$ is the function*

$$Q_{\mathcal{A}}(x, n) = \|\mathcal{A}^n_x\| \cdot \|(\mathcal{A}^n_x)^{-1}\|, \quad x \in X \; \textit{and} \; n \in \mathbb{Z}.$$

Theorem 1.3. *Let (X, f) be either a transitive Anosov diffeomorphism of a compact connected manifold or a topologically mixing diffeomorphism of a locally maximal hyperbolic set or a mixing subshift of finite type (see Section 2 for definitions). Let $\mathcal{A}$ be a Hölder continuous Banach cocycle over f.*

(i) *If there exists a constant C_{per} such that $Q_{\mathcal{A}}(p, k) \le C_{per}$ whenever $f^k p = p$, then $\mathcal{A}$ is* uniformly quasiconformal, *i.e. there exist a constant C such that*

$$Q_{\mathcal{A}}(x, n) \le C \; \textit{ for all } x \in X \textit{ and } n \in Z.$$

(ii) *If the set $\mathcal{A}_P$ is bounded in $(GL(V), d)$, then so is the set $\mathcal{A}_X$.*

(iii) *If the set $\mathcal{A}_P$ has compact closure in $(GL(V), d)$, then so does the set $\mathcal{A}_X$.*

(iv) *If the set $\mathcal{A}_X$ has compact closure in $(GL(V), d)$ then there exists a Hölder continuous family of norms $\|.\|_x$ on V such that*

$$\mathcal{A}_x : (V, \|.\|_x) \to (V, \|.\|_{fx}) \text{ is an isometry for each } x \in X.$$

We note that the closures in (iii) are not the same in general. For example, if $\mathcal{A}$ is a *coboundary*, i.e. is generated by $A(x) = C(fx) \circ C(x)^{-1}$ for a function $C : X \to GL(V)$, then $\mathcal{A}_P = \{\mathrm{Id}\}$ while $\mathcal{A}_X$ is usually not. The question whether $\mathcal{A}_P = \{\mathrm{Id}\}$ characterizes coboundaries has been studied over several decades and answered positively for various groups in [Liv72, NT95, PW01, K11, Gu].

Remark 1.4. *We can view the cocycle $\mathcal{A}$ as an automorphism of the trivial vector bundle $\mathcal{V} = X \times V$ which covers f in the base and has fiber maps $\mathcal{A}_x : \mathcal{V}_x \to \mathcal{V}_{fx}$. Theorem 1.3 holds in the more general setting where $X \times V$ is replaced by a Hölder continuous vector bundle $\mathcal{V}$ over X with fiber V and the cocycle $\mathcal{A}$ is replaced by an automorphism $\mathcal{F} : \mathcal{V} \to \mathcal{V}$ covering f. This setting is described in detail in Section 2.2 of* [KS13] *and our proofs work without any significant modifications.*

Theorem 1.3 extends results for finite dimensional V in [KS10, dLW10, K11]. The infinite dimensional case is substantially different. The initial step of obtaining fiber-bunching of the cocycle from its periodic data relies on our new approximation results [KS16]. The finite dimensional boundedness result is extended in two directions: boundedness and pre-compactness, as the latter does not follow automatically. Existence of a continuous family of norms requires a new approach. Indeed, on a finite dimensional space the set of Euclidean norms has a structure of a symmetric space of nonpositive curvature which was used in the arguments, but in infinite dimensional case there is no analogous metric structure. We consider a natural distance on the set of norms but the resulting space is not separable so we work with a small subset. The following general result yields a measurable invariant family of norms and then we show its continuity.

Proposition 1.5. *Let f be a homeomorphism of a metric space X and let $\mathcal{A}$ be a continuous Banach cocycle over f. If the set of values $\mathcal{A}_X$ has compact closure in $GL(V)$, then there exists a bounded Borel measurable family of norms $\|.\|_x$ on V such that $\mathcal{A}_x : (V, \|.\|_x) \to (V, \|.\|_{fx})$ is an isometry for each $x \in X$.*

Theorem 1.3 yields cocycles with a "small" set of values $\mathcal{A}_X$, which are relatively well understood. For example, a cocycle satisfying the conclusion (ii) of the theorem has *bounded distortion* in the sense of [Sch99], i.e. there exists a constant c such that

$$d(AB_1, AB_2) \le c\, d(B_1, B_2) \quad \text{and} \quad d(B_1 A, B_2 A) \le c\, d(B_1, B_2)$$

for all $A \in \mathcal{A}_X$ and all $B_1, B_2 \in GL(V)$. Some definitive results on cohomology of such cocycles were obtained by K. Schmidt in [Sch99]. These results can be extended to cocycles satisfying the conclusion of (i) by considering the quotient by the group of scalar operators.

2. Systems in the base

Transitive Anosov diffeomorphisms. Let X be a compact connected manifold. A diffeomorphism f of X is called *Anosov* if there exist a splitting of the tangent bundle TX into a direct sum of two Df-invariant continuous subbundles E^s and E^u, a Riemannian metric on X, and continuous functions ν and $\hat{\nu}$ such that

$$\|Df_x(v^s)\| < \nu(x) < 1 < \hat{\nu}(x) < \|Df_x(v^u)\| \tag{2.1}$$

for any $x \in X$ and unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$. The subbundles E^s and E^u are called stable and unstable. They are tangent to the stable and unstable foliations W^s and W^u respectively (see, for example [KtH]). Using (2.1) we choose a small positive number ρ such that for every $x \in \mathcal{M}$ we have $\|Df_y\| < \nu(x)$ for all y in the ball in $W^s(x)$ centered at x of radius ρ in the intrinsic metric of $W^s(x)$. We refer to this ball as the local stable manifold of x and denote it by $W^s_{loc}(x)$. Local unstable manifolds are defined similarly. It follows that for all $n \in \mathbb{N}$ and $x \in X$,

$$\begin{aligned} &\operatorname{dist}(f^n x, f^n y) < \nu_x^n \cdot \operatorname{dist}(x, y) \quad \text{for all } y \in W^s_{loc}(x), \\ &\operatorname{dist}(f^{-n} x, f^{-n} y) < \hat{\nu}_x^{-n} \cdot \operatorname{dist}(x, y) \quad \text{for all } y \in W^u_{loc}(x), \end{aligned}$$

where $\nu_x^n = \nu(f^{n-1}x) \cdots \nu(x)$ and $\hat{\nu}_x^{-n} = (\hat{\nu}(f^{-n}x))^{-1} \cdots (\hat{\nu}(f^{-1}x))^{-1}$. We also assume that ρ is sufficiently small so that $W^s_{loc}(x) \cap W^u_{loc}(z)$ consists of a single point for any sufficiently close x and z in X. This property is called *local product structure.*

A diffeomorphism is said to be *(topologically) transitive* if there is a point x in X with dense orbit. All known examples of Anosov diffeomorphisms have this property.

Mixing diffeomorphisms of locally maximal hyperbolic sets. (See Section 6.4 in [KtH] for more details.) More generally, let f be a diffeomorphism of a manifold $\mathcal{M}$. A compact f-invariant set $X \subset \mathcal{M}$ is called *hyperbolic* if there exist a continuous Df-invariant splitting $T_X\mathcal{M} = E^s \oplus E^u$, and a Riemannian metric and continuous functions ν, $\hat{\nu}$ on an open set $U \supset X$ such that (2.1) holds for all $x \in X$. Local stable and unstable manifolds are defined similarly for any $x \in X$ and we denote their intersections with X by $W^s_{loc}(x)$ and $W^y_{loc}(y)$. The set X is called *locally maximal* if $X = \bigcap_{n\in\mathbb{Z}} f^{-n}(U)$ for some open set $U \supset X$. This property ensures that $W^s_{loc}(x) \cap W^y_{loc}(y)$ exists in X, so that X has local product structure. The map $f|_X$ is called *topologically mixing* if for any two open non-empty subsets U, V of X there is $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

In the case of $X = \mathcal{M}$ this gives an Anosov diffeomorphism. It is known that mixing holds automatically for transitive Anosov diffeomorphisms of connected manifolds.

Mixing subshifts of finite type. Let M be $k \times k$ matrix with entries from $\{0, 1\}$ such that all entries of M^N are positive for some N. Let

$$X = \{\, x = (x_n)_{n\in\mathbb{Z}} : \ 1 \leq x_n \leq k \ \text{ and } \ M_{x_n, x_{n+1}} = 1 \ \text{ for every } n \in \mathbb{Z} \,\}.$$

The shift map $f : X \to X$ is defined by $(fx)_n = x_{n+1}$. The system (X, f) is called a *mixing subshift of finite type*. We fix $\nu \in (0,1)$ and consider the metric

$$\operatorname{dist}(x,y) = d_\nu(x,y) = \nu^{n(x,y)}, \quad \text{where} \quad n(x,y) = \min\{\, |i| \,:\, x_i \neq y_i \}.$$

The set X with this metric is compact. The metrics d_ν for different values of ν are Hölder equivalent. The following sets play the role of the local stable and unstable manifolds of x

$$W^s_{loc}(x) = \{\, y : \; x_i = y_i, \;\; i \geq 0 \,\}, \quad W^u_{loc}(x) = \{\, y : \; x_i = y_i, \;\; i \leq 0 \,\}.$$

Indeed, for all $x \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} \operatorname{dist}(f^n x, f^n y) &= \nu^n \operatorname{dist}(x,y) \quad \text{for all } y \in W^s_{loc}(x), \\ \operatorname{dist}(f^{-n} x, f^{-n} y) &= \nu^n \operatorname{dist}(x,y) \quad \text{for all } y \in W^u_{loc}(x), \end{aligned}$$

and for any $x, z \in X$ with $\operatorname{dist}(x,z) < 1$ the intersection of $W^s_{loc}(x)$ and $W^u_{loc}(z)$ consists of a single point, $y = (y_n)$ such that $y_n = x_n$ for $n \geq 0$ and $y_n = z_n$ for $n \leq 0$.

3. Proofs of Theorem 1.3 and Proposition 1.5

3.1. **Fiber bunching and closing property.** First we show that the cocycle $\mathcal{A}$ is *fiber bunched*, i.e. $Q(x,n)$ is dominated by the contraction and expansion in the base in the following sense.

Definition 3.1. *A β-Hölder cocycle $\mathcal{A}$ over a hyperbolic diffeomorphism f is* fiber bunched *if there exist numbers $\theta < 1$ and L such that for all $x \in X$ and $n \in \mathbb{N}$,*

$$Q_{\mathcal{A}}(x,n) \cdot (\nu^n_x)^\beta < L\, \theta^n \quad \text{and} \quad Q_{\mathcal{A}}(x,-n) \cdot (\hat{\nu}^{-n}_x)^\beta < L\, \theta^n. \tag{3.1}$$

For a subshift of finite type, $\nu(x) = \nu$ and $\hat{\nu}(x) = 1/\nu$, and so the conditions become

$$Q_{\mathcal{A}}(x,n) \cdot \nu^{\beta|n|} < L\, \theta^{|n|} \quad \text{for all } n \in \mathbb{Z}.$$

Fiber bunching plays an important role in the study of cocycles over hyperbolic systems. In particular, it ensures certain closeness of the cocycle at the points on the same stable/unstable manifold.

Proposition 3.2. [KS13, Proposition 4.2(i)] *If $\mathcal{A}$ is fiber bunched, then there exists $c > 0$ such that for any $x \in X$ and $y \in W^s_{loc}(x)$,*

$$\|(\mathcal{A}^n_y)^{-1} \circ \mathcal{A}^n_x - Id\| \leq c\, dist(x,y)^\beta \quad \text{for every } n \in \mathbb{N},$$

and similarly for any $x \in X$ and $y \in W^u_{loc}(x)$,

$$\|(\mathcal{A}^{-n}_y)^{-1} \circ \mathcal{A}^{-n}_x - Id\| \leq c\, dist(x,y)^\beta \quad \text{for every } n \in \mathbb{N}.$$

This proposition was proven in the finite dimensional case but the argument holds for Banach cocycles without modifications.

The base systems that we are considering satisfy the following *closing property.*

Lemma 3.3. *(Anosov Closing Lemma* [KtH, 6.4.15-17]*) Let (X, f) be a topologically mixing diffeomorphism of a locally maximal hyperbolic set. Then there exist constants D, $\delta_0 > 0$ such that for any $x \in X$ and $k \in \mathbb{N}$ with $dist(x, f^k x) < \delta_0$ there exists a periodic point $p \in X$ with $f^k p = p$ such that the orbit segments $x, fx, ..., f^k x$ and $p, fp, ..., f^k p$ remain close:*

$$dist(f^i x, f^i p) \leq D\, dist(x, f^k x) \quad \text{for every } i = 0, ..., k.$$

For subshifts of finite type this property can be observed directly. Moreover, for the systems we consider there exist $D' > 0$ and $0 < \gamma < 1$ such that for the above trajectories

$$\text{dist}(f^i x, f^i p) \leq D' \text{dist}(x, f^k x)\, \gamma^{\min\{i, k-i\}} \quad \text{for every } i = 0, ..., k. \tag{3.2}$$

Indeed, the local product structure gives existence of a point $y = W^s_{loc}(p) \cap W^u_{loc}(x)$. Then the contraction/expansion along stable/unstable manifolds yields the exponential closeness in (3.2).

We obtain fiber bunching of the cocycle $\mathcal{A}$ from the following proposition. Clearly, the assumption in part (i) of the theorem is weaker then the ones in (i-iv), and so it suffices to deduce fiber bunching from the assumption in (i).

Proposition 3.4. [KS16, Corollary 1.6(ii)] *Let f be a homeomorphism of a compact metric space X satisfying the closing property* (3.2) *and let $\mathcal{A}$ be a Hölder continuous Banach cocycle over f. If for some numbers C and s we have*

$$Q(p, k) \leq Ce^{sk} \quad \text{whenever } p = f^k p,$$

then for each $\epsilon > 0$ there exists a number C'_ϵ such that

$$Q(x, n) \leq C'_\epsilon\, e^{(s+\epsilon)|n|} \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

We apply the proposition with $s = 0$ and take $\epsilon > 0$ such that

$$e^\epsilon \nu^\beta < 1 \;\text{ and }\; e^\epsilon (\hat{\nu}^{-1})^\beta < 1, \;\text{ where }\; \nu = \max_x \nu(x) \;\text{ and }\; \hat{\nu}^{-1} = \max_x \hat{\nu}(x)^{-1}.$$

Then the fiber bunching condition (3.1) are satisfied with

$$\theta = \max\{e^\epsilon \nu^\beta,\; e^\epsilon (\hat{\nu}^{-1})^\beta\} \quad \text{and} \quad L = C'_\epsilon.$$

3.2. **Proof of (i).** Now we show that quasiconformal distortion of $\mathcal{A}$ is bounded along a dense orbit. Since f is transitive, there is a point $z \in X$ such that its orbit

$$O(z) = \{f^n z : \; n \in \mathbb{Z}\} \quad \text{is dense in } X.$$

We take δ_0 sufficiently small to apply Anosov Closing Lemma 3.3 and so that

$$(1 + c\delta_0^\beta)/(1 - c\delta_0^\beta) \leq 2, \quad \text{where } c \text{ is as in Proposition 3.2.}$$

Let $f^{n_1} z$ and $f^{n_2} z$ be two points of $O(z)$ with $\delta := \text{dist}(f^{n_1} z, f^{n_2} z) < \delta_0$. We assume that $n_1 < n_2$ and denote

$$w = f^{k_1} z \quad \text{and} \quad k = n_2 - n_1, \quad \text{so that } \; \delta = \text{dist}(w, f^k w) < \delta_0.$$

Then there exists $p \in X$ with $f^k p = p$ such that $\text{dist}(f^i w, f^i p) \le D\delta$ for $i = 0, ..., k$. Let y be the point of intersection of $W^s_{loc}(p)$ and $W^u_{loc}(w)$. We apply Proposition 3.2 to p and y and to $f^k y$ and $f^k w$ and obtain

$$\begin{aligned} &\|(\mathcal{A}^k_y)^{-1} \circ \mathcal{A}^k_p - \text{Id}\,\| \le c\delta^\beta \quad \text{and} \\ &\|\,\mathcal{A}^k_w \circ (\mathcal{A}^k_y)^{-1} - \text{Id}\| = \|(\mathcal{A}^{-k}_{f^k w})^{-1} \circ \mathcal{A}^{-k}_{f^k y} - \text{Id}\,\| \le c\delta^\beta. \end{aligned} \tag{3.3}$$

Lemma 3.5. *Let $A, B \in GL(V)$. If either $\|A^{-1}B - Id\| \le r$ or $\|AB^{-1} - Id\| \le r$ for some $r < 1$, then*

$$(1-r)/(1+r) \le Q(A)/Q(B) \le (1+r)/(1-r),$$

where $Q(A) = \|A\| \cdot \|A^{-1}\|$ and $Q(B) = \|B\| \cdot \|B^{-1}\|$

Proof. Clearly, $Q(A) = Q(A^{-1})$ and $Q(A_1 A_2) \le Q(A_1)Q(A_2)$.

Suppose that $\|A^{-1}B - \text{Id}\,\| \le r$. We denote $\Delta = A^{-1}B - \text{Id}$. Since for any unit vector v, $1 - r \le \|(\text{Id} + \Delta)v\| \le 1 + r$, we have $Q(\text{Id} + \Delta) \le (1+r)/(1-r)$.

Since $B = A\,(\text{Id} + \Delta)$ we obtain

$$Q(B) \le Q(A) \cdot Q(\text{Id} + \Delta) \le Q(A) \cdot (1+r)/(1-r).$$

Also, $A^{-1} = (\text{Id} + \Delta)B^{-1}$ and hence

$$Q(A) = Q(A^{-1}) \le Q(\text{Id} + \Delta) \cdot Q(B^{-1}) \le (1+r)/(1-r) \cdot Q(B),$$

and the estimate for $Q(A)/Q(B)$ follows. The case of $\|AB^{-1} - \text{Id}\,\| \le r$ is similar. □

It follows from the Lemma 3.5 and the choice of δ_0 that

$$Q(y,k)/Q(p,k) \le (1 + c\delta^\beta)/(1 - c\delta^\beta) \le 2 \quad \text{and} \quad Q(w,k)/Q(y,k) \le 2,$$

and hence

$$Q(w,k) = Q(f^{n_1} z,\, n_2 - n_1) \le 4Q(p,k) \le 4C_{per}.$$

We take $m \in \mathbb{N}$ such that the set $\{f^j z;\ |j| \le m\}$ is δ_0-dense in X. Let

$$Q_m = \max\{\, Q(z,j) : \ |j| \le m\}.$$

Then for any $n > m$ there exists j, $|j| \le m$, such that $\text{dist}(f^n z, f^j z) \le \delta_0$ and hence

$$Q(z,n) \le Q(z,j) \cdot Q(f^j z,\, n - j) \le Q_m \cdot 4C_{per}.$$

The case of $n < -m$ is similar. Thus $Q(z,n)$ is uniformly bounded in $n \in \mathbb{Z}$, and hence $Q(f^\ell z, n)$ is uniformly bounded in $\ell, n \in \mathbb{Z}$ since

$$Q(f^\ell z, n) \le Q(f^\ell z, -\ell) \cdot Q(z, n + l) = Q(z, \ell) \cdot Q(z, n + l).$$

Since $O(z)$ is dense in X and $Q(x,n)$ is continuous on X for each n, this implies that $Q(x,n)$ is uniformly bounded in $x \in X$ and $n \in \mathbb{Z}$.

3.3. **Proof of (ii).** Since the set $\mathcal{A}_P$ is bounded, there is a constant C'_{per} such that

$$\max\{\|\mathcal{A}_p^k\|,\ \|(\mathcal{A}_p^k)^{-1}\|\} \le C'_{per} \quad \text{whenever } f^k p = p.$$

We show that there exists a constant C' such that

$$\max\{\|\mathcal{A}_x^n\|,\ \|(\mathcal{A}_x^n)^{-1}\|\} \le C' \quad \text{for all } x \text{ and } n.$$

Let z, n_1, n_2, $w = f^{n_1}z$, $k = n_2 - n_1$, y and p be as in (i). Since

$$(\mathcal{A}_y^k)^{-1} = \big(\mathrm{Id} + ((\mathcal{A}_y^k)^{-1} \circ \mathcal{A}_p^k - \mathrm{Id})\big) \circ (\mathcal{A}_p^k)^{-1},$$

the first inequality in (3.3) implies

$$\|(\mathcal{A}_y^k)^{-1}\| \le (1 + c\delta^\beta) \cdot \|(\mathcal{A}_p^k)^{-1}\| \le (1 + c\delta_0^\beta) C'_{per} \le 2C'_{per}$$

by the choice of δ_0. Interchanging p and y we obtain $\|(\mathcal{A}_p^k)^{-1} \circ \mathcal{A}_y^k - \mathrm{Id}\,\| \le c\delta^\beta$ and it follows that $\|\mathcal{A}_y^k\| \le 2C'_{per}$. Similarly, the second inequality in (3.3) yields

$$\|\mathcal{A}_w^k\| \le (1 + c\delta^\beta) \cdot \|\mathcal{A}_y^k\| \le 2\|\mathcal{A}_y^k\| \quad \text{and} \quad \|(\mathcal{A}_w^k)^{-1}\| \le 2\|(\mathcal{A}_y^k)^{-1}\|,$$

and we conclude that $\|\mathcal{A}_w^k\| \le 4C'_{per}$ and $\|(\mathcal{A}_w^k)^{-1}\| \le 4C'_{per}$. It follows similarly to (i) that $\max\{\|\mathcal{A}_x^n\|,\ \|(\mathcal{A}_x^n)^{-1}\|\}$ is uniformly bounded in $x \in X$ and in $n \in \mathbb{Z}$.

3.4. **Proof of (iii).** Now we show that if the set $\mathcal{A}_P$ has compact closure, then so does $\mathcal{A}_X$. It suffices to prove that $\mathcal{A}_X$ is totally bounded, i.e. for any $\epsilon > 0$ it has a finite ϵ-net. Since $\mathcal{A}_X$ is bounded by (iii), we can choose a constant M such that

$$\|A\|,\ \|A^{-1}\| \le M \ \text{ for all } A \in \mathcal{A}_X.$$

We fix $\delta_0 > 0$ sufficiently small to apply Anosov Closing Lemma 3.3 and so that $4M^2 c\,\delta_0^\beta < \epsilon/2$ and take ϵ' such that $4M^2 c\,\delta_0^\beta + M\epsilon' < \epsilon$. We fix a finite ϵ'-net $P_{\epsilon'} = \{P_1, ..., P_\ell\}$ in $\mathcal{A}_P$. As in (i), we take a point z with dense orbit and choose $m \in \mathbb{N}$ such that the set $\{f^j z :\ |j| \le m\}$ is δ_0-dense in X. We will show that the set

$$\tilde{P}_\epsilon = \{\, P_i \circ \mathcal{A}_z^j \ :\ i = 0, 1, ..., \ell,\ |j| \le m\}, \quad \text{where } P_0 = \mathrm{Id},$$

is a finite ϵ-net for $\{\mathcal{A}_z^n : n \in \mathbb{Z}\}$. Clearly, $\mathcal{A}_z^n \in \tilde{P}_\epsilon$ for $|n| \le m$.

Suppose $n > m$, the argument for $n < -m$ is similar. Then there exists j with $|j| \le m$ such that $\delta = \mathrm{dist}(f^j z, f^n z) \le \delta_0$ and hence for $x = f^j z$ and $k = n - j$ there is $p = f^k p$ such that such that $\mathrm{dist}(f^i x, f^i p) \le D\delta$ for $i = 0, ..., k$. Then it follows from the first inequality in (3.3) that for $y = W^s_{loc}(p) \cap W^u_{loc}(x)$,

$$\|\mathcal{A}_p^k - \mathcal{A}_y^k\| = \|\mathcal{A}_y^k \circ ((\mathcal{A}_y^k)^{-1} \circ \mathcal{A}_p^k - \mathrm{Id})\,\| \le \|\mathcal{A}_y^k\| \cdot \|(\mathcal{A}_y^k)^{-1} \circ \mathcal{A}_p^k - \mathrm{Id})\,\| \le Mc\delta^\beta,$$

similarly,

$$\|(\mathcal{A}_p^k)^{-1} - (\mathcal{A}_y^k)^{-1}\| = \|((\mathcal{A}_y^k)^{-1} \circ \mathcal{A}_p^k - \mathrm{Id}) \circ (\mathcal{A}_p^k)^{-1}\| \le Mc\delta^\beta,$$

and so

$$d(\mathcal{A}_p^k,\ \mathcal{A}_y^k) = \|\mathcal{A}_p^k - \mathcal{A}_y^k\| + \|(\mathcal{A}_p^k)^{-1} - (\mathcal{A}_y^k)^{-1}\| \le 2Mc\delta^\beta.$$

It follows similarly from the second inequality in (3.3) that $d(\mathcal{A}_y^k,\, \mathcal{A}_x^k) \le 2Mc\delta^\beta$. Thus

$$d(\mathcal{A}_p^k,\, \mathcal{A}_{f^j z}^k) = d(\mathcal{A}_p^k,\, \mathcal{A}_x^k) \le 4Mc\delta^\beta =: c'\delta^\beta.$$

Also, there exists an element P_i of the ϵ'-net $P_{\epsilon'}$ such that $d(\mathcal{A}_p^k, P_i) < \epsilon'$. So we have $d(\mathcal{A}_{f^j z}^k, P_i) < c'\delta^\beta + \epsilon'$. Then, as $\mathcal{A}_z^n = \mathcal{A}_{f^j z}^k \circ \mathcal{A}_z^j$, by Lemma 3.6 below we have

$$d(\mathcal{A}_z^n,\, P_i \circ \mathcal{A}_z^j) = d(\mathcal{A}_{f^j z}^k \circ \mathcal{A}_z^j,\, P_i \circ \mathcal{A}_z^j) \le M(c'\delta^\beta + \epsilon') \le \epsilon.$$

Lemma 3.6. *If for each of $A, \tilde{A}, B, \tilde{B} \in GL(V)$ the norm and the norm of the inverse are at most M, then $d(A \circ B, \tilde{A} \circ \tilde{B}) = M(d(A, \tilde{A}) + d(B, \tilde{B}))$.*

Proof. Adding and subtracting $\tilde{A} \circ B$ we obtain

$$\|A \circ B - \tilde{A} \circ \tilde{B}\| \le \|A - \tilde{A}\| \cdot \|B\| + \|\tilde{A}\| \cdot \|B - \tilde{B}\| \le M(\|A - \tilde{A}\| + \|B - \tilde{B}\|).$$

Similarly $\|A^{-1} \circ B^{-1} - \tilde{A}^{-1} \circ \tilde{B}^{-1}\| \le M(\|A^{-1} - \tilde{A}^{-1}\| + \|B^{-1} - \tilde{B}^{-1}\|)$. $\square$

We conclude that the set $\tilde{P}_\epsilon$ is a finite ϵ-net for the set $\{\mathcal{A}_z^n : n \in \mathbb{Z}\}$ and hence this set is totally bounded. It follows that so is the set $\{\mathcal{A}_{f^i z}^n : i, n \in \mathbb{Z}\}$. Indeed, if $\{\tilde{P}_1, ..., \tilde{P}_N\}$ is an ϵ-net for $\{\mathcal{A}_z^n : n \in \mathbb{Z}\}$, then

$$\{\, \tilde{P}_i \circ (\tilde{P}_j)^{-1} :\ 1 \le i, j \le N \,\}$$

is a $2M\epsilon$-net for $\{\mathcal{A}_{f^i z}^n : i, n \in \mathbb{Z}\}$. This follows from Lemma 3.6 since

$$d(A, B) = d(A^{-1}, B^{-1}) \quad \text{and} \quad \mathcal{A}_{f^i z}^k = \mathcal{A}_z^{k+i} \circ (\mathcal{A}_z^i)^{-1}.$$

Since the orbit of z is dense in X, the set $\{\mathcal{A}_{f^i z}^n : i, n \in \mathbb{Z}\}$ is dense in the set $\mathcal{A}_X$ and hence this set is also totally bounded and its closure is compact.

3.5. **Proof of Proposition 1.5.** We denote by N the space of all norms φ on V which are equivalent to the fixed background norm $\varphi_0 = \|.\|$, and by N_K the subset of the norms equivalent to $\|.\|$ with a constant $K > 0$, i.e.

$$N_K = \{\, \varphi :\ K^{-1}\|v\| \le \varphi(v) \le K\|v\| \quad \text{for all } v \in V\}. \tag{3.4}$$

We consider the following metric on N. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the *closed* unit balls in V with respect to norms φ_1 and φ_2 in N. We define

$$\text{dist}(\varphi_1, \varphi_2) = \log \min \{\, t \ge 1 :\ \mathcal{B}_1 \subseteq t\mathcal{B}_2 \ \text{and}\ \mathcal{B}_2 \subseteq t\mathcal{B}_1\}, \tag{3.5}$$

where $t\mathcal{B} = \{tv : v \in \mathcal{B}\}$. It is easy to check that the minimum is attained and that this is a distance on N. Moreover,

$$\text{dist}(\varphi_1, \varphi_2) = \log \min \{t \ge 1 :\ t^{-1}\varphi_2(v) \le \varphi_1(v) \le t\varphi_2(v) \ \text{for each } v \in V\,\}, \tag{3.6}$$

and hence $\text{diam} N_K = 2\log K$ as N_K is the closed ball of radius $\log K$ centered at φ_0. We note that the space N_K with this metric is not separable in general.

Lemma 3.7. *For each $K > 0$, the distance on N_K given by* (3.5) *is equivalent to*

$$\mathit{dist}'(\varphi_1, \varphi_2) = \sup\{\, |\varphi_1(v) - \varphi_2(v)| : \ \|v\| \le 1\,\},$$

and hence the metric space (N_K, dist) is complete.

Proof. Let $a = \operatorname{dist}(\varphi_1, \varphi_2)$. Then by (3.6) for any v with $\|v\| \le 1$ we have

$$\varphi_2(v) \le e^a \varphi_1(v) \quad \text{and hence} \quad \varphi_2(v) - \varphi_1(v) \le (e^a - 1)\varphi_1(v) \le (e^a - 1)K,$$

and similarly, $\varphi_1(v) - \varphi_2(v) \le (e^a - 1)K$. Using the mean value theorem and the fact that $a \le \operatorname{diam} N_K = 2\log K$ we obtain

$$\operatorname{dist}'(\varphi_1, \varphi_2) \le K(e^a - 1) \le K e^{2\log K} a = K^3 \operatorname{dist}(\varphi_1, \varphi_2).$$

Let $b = \operatorname{dist}'(\varphi_1, \varphi_2)$. Then $|\varphi_1(v) - \varphi_2(v)| \le b$ for all v with $\|v\| \le 1$, and hence

$$|\varphi_1(v) - \varphi_2(v)| \le Kb \quad \text{for all } v \text{ with } \|v\| \le K.$$

Suppose that $v \in \mathcal{B}_1$. Then

$$\varphi_1(v) \le 1 \ \Rightarrow\ \|v\| \le K \ \Rightarrow\ \varphi_2(v) \le \varphi_1(v) + Kb \le 1 + Kb$$

and hence $\mathcal{B}_1 \subseteq (1 + Kb)\mathcal{B}_2$. Similarly, $\mathcal{B}_2 \subseteq (1 + Kb)\mathcal{B}_1$ and so

$$\operatorname{dist}(\varphi_1, \varphi_2) \le \log(1 + Kb) \le Kb = K \operatorname{dist}'(\varphi_1, \varphi_2). \tag{3.7}$$

So the two metrics on N_K are equivalent. It is easy to see that $(N_K, \operatorname{dist}')$ is complete as a closed subset of the complete space of bounded continuous functions on the unit ball, and hence $(N_K, \operatorname{dist})$ is also complete. □

Now we construct a Borel measurable family of norms $\varphi_x = \|.\|_x$ in N_K such that $\mathcal{A}_x : (V, \|.\|_x) \to (V, \|.\|_{fx})$ is an isometry for each $x \in X$. For a norm $\varphi \in N$ and an operator $A \in GL(V)$ we denote the pull-back of φ by $A^*\varphi(v) = \varphi(Av)$. The convenience of the metric (3.5) is that, as $A(\mathcal{B}_1) \subseteq tA(\mathcal{B}_2)$ if and only if $\mathcal{B}_1 \subseteq t\mathcal{B}_2$, the pull-back action of $GL(V)$ on N is isometric, i.e.

$$\operatorname{dist}(A^*\varphi_1, A^*\varphi_2) = \operatorname{dist}(\varphi_1, \varphi_2) \quad \text{for any } A \in GL(V) \text{ and } \varphi_1, \varphi_2 \in N.$$

Lemma 3.8. *For any $\varphi \in N$ and $A, \tilde{A} \in GL(V)$ such that $\varphi, A^*\varphi, \tilde{A}^*\varphi \in N_K$, we have $\mathit{dist}(A^*\varphi, \tilde{A}^*\varphi) \le K^4\|A - \tilde{A}\|$.*

Proof. We denote $\|A\|_\varphi = \sup\{\, \varphi(Av) : \varphi(v) \le 1\}$. It follows from (3.4) that

$$\|A\|_\varphi \le \sup\{K\|Av\| : \ \|v\| \le K\} \le K^2\|A\|.$$

For any v with $\|v\| \le 1$ we have

$$|\varphi(Av) - \varphi(\tilde{A}v)| \le \varphi(Av - \tilde{A}v) \le \|A - \tilde{A}\|_\varphi\, \varphi(v) \le K^2\|A - \tilde{A}\| \cdot K\|v\| = K^3\|A - \tilde{A}\|.$$

It follows that

$$\operatorname{dist}'(A^*\varphi,\ \tilde{A}^*\varphi) = \sup\{\, |\,\varphi(Av) - \varphi(\tilde{A}v)\,| : \ \|v\| \le 1\,\} \le K^3\|A - \tilde{A}\|,$$

and using (3.7) we conclude that

$$\operatorname{dist}(A^*\varphi,\ \tilde{A}^*\varphi) \le K \operatorname{dist}'(A^*\varphi,\ \tilde{A}^*\varphi) \le K^4\|A - \tilde{A}\|.$$

□

We denote $\bar{\mathcal{A}} = \mathrm{Cl}(\mathcal{A}_X)$. We fix K such that $K > \|A\|, \|A^{-1}\|$ for all $A \in \bar{\mathcal{A}}$ and consider the corresponding space of norms N_K. Then $A^*\varphi_0 \in N_K$ for each $A \in \bar{\mathcal{A}}$ as $K^{-1}\|v\| \le \|(A^*\varphi_0)(v)\| = \|Av\| \le K\|v\|$. Lemma 3.8 implies that the function $A \mapsto A^*\varphi_0$ is continuous in A, and since the set $\bar{\mathcal{A}}$ is compact in $GL(V)$, its image under this function

$$N_{\mathcal{A}} = \mathrm{Cl}\,\{\, (\mathcal{A}_x^n)^*\varphi_0 : \ x \in X, \ n \in \mathbb{Z} \,\} \subseteq N_K \ \text{ is compact.}$$

We also note that $A^*\varphi \in N_{K^2}$ for any $A \in \bar{\mathcal{A}}$ and $\varphi \in N_K$.

Since all norms in $N_{\mathcal{A}}$ are equivalent, the intersection of the unit balls $\mathcal{B}_1, ..., \mathcal{B}_n$ for finitely many of these norms $\varphi_1, ..., \varphi_n \in N_{\mathcal{A}}$ is the unit ball of an equivalent norm $\hat{\varphi} = \max\{\varphi_1, ..., \varphi_n\}$ in N_K. We consider the set $\hat{N}_{\mathcal{A}}$ of all such norms $\hat{\varphi}$,

$$\hat{N}_{\mathcal{A}} = \{\, \hat{\varphi} = \max\{\varphi_1, ..., \varphi_n\} : \ n \in \mathbb{N}, \ \varphi_1, ..., \varphi_n \in N_{\mathcal{A}} \,\} \subseteq N_K.$$

Lemma 3.9. *The set $\bar{N}_{\mathcal{A}} = Cl(\hat{N}_{\mathcal{A}})$ is a compact subset of N_K.*

Proof. It suffices to show that $\hat{N}_{\mathcal{A}}$ is totally bounded. Let $P = \{\varphi_1, ..., \varphi_N\}$ be an ϵ-net in $N_{\mathcal{A}}$ and let $\hat{P}$ be the set of all possible maxima of subsets of P. Then $\hat{P}$ is an ϵ-net in $\hat{N}_{\mathcal{A}}$ since

$$\mathrm{dist}(\max\{\varphi_1, ..., \varphi_n\}, \max\{\tilde{\varphi}_1, ..., \tilde{\varphi}_n\}) \le \max\{\mathrm{dist}(\varphi_1, \tilde{\varphi}_1), ..., \mathrm{dist}(\varphi_n, \tilde{\varphi}_n)\}. \tag{3.8}$$

Indeed, if the right hand side equals $\log t$ then for the corresponding unit balls we have $\mathcal{B}_1 \subset t\tilde{\mathcal{B}}_1$, ..., $\mathcal{B}_n \subset t\tilde{\mathcal{B}}_n$ and it follows that

$$\mathcal{B}_1 \cap ... \cap \mathcal{B}_n \subseteq t\tilde{\mathcal{B}}_1 \cap ... \cap t\tilde{\mathcal{B}}_n = t(\tilde{\mathcal{B}}_1 \cap ... \cap \tilde{\mathcal{B}}_n).$$

Similarly, $\tilde{\mathcal{B}}_1 \cap ... \cap \tilde{\mathcal{B}}_n \subseteq t(\mathcal{B}_1 \cap ... \cap \mathcal{B}_n)$, and so the left hand side of (3.8) is at most $\log t$. □

For each $x \in X$ we consider the pullbacks of the background norm by $\mathcal{A}_x^n$ and let

$$\varphi_x^m = \max\{\, (\mathcal{A}_x^n)^*\varphi_0 : \ |n| \le m \,\} \quad \text{and} \quad \varphi_x = \sup\{\, (\mathcal{A}_x^n)^*\varphi_0 : \ n \in \mathbb{Z} \,\}.$$

We note that φ_x is the norm whose unit ball is the intersection of the unit balls of φ_x^m, $m \in \mathbb{N}$, or equivalently the unit balls of $(\mathcal{A}_x^n)^*\varphi_0$, $n \in \mathbb{Z}$. We claim that

$$\varphi_x = \lim_{m\to\infty} \varphi_x^m \ \text{ in } (\bar{N}_{\mathcal{A}}, \mathrm{dist}) \ \text{ for each } x \in X.$$

Indeed, for each $v \in V$ the sequence $\varphi_x^m(v)$ increases and converges to $\varphi_x(v)$. Since the sequence φ_x^m lies in the compact set $\bar{N}_{\mathcal{A}}$, any subsequence has a subsequence converging in $(\bar{N}_{\mathcal{A}}, \mathrm{dist})$, whose limit must be φ_x. This implies, by contradiction, that φ_x^m converges to φ_x in $(\bar{N}_{\mathcal{A}}, \mathrm{dist})$. Note that compactness of $\bar{N}_{\mathcal{A}}$ is crucial here.

Since $(\mathcal{A}_x^n)^*\varphi_0$ depends continuously on x for each n, the inequality (3.8) yields that φ_x^m is a continuous function on X for each m. We conclude that the pointwise limit φ_x is a Borel measurable function from X to $N_{\mathcal{A}} \subseteq N_K$. By the construction, $\varphi_x = (\mathcal{A}_x^n)^*\varphi_{f^n x}$ for all $x \in X$ and $n \in \mathbb{Z}$. In other words, φ_x is an invariant section of the bundle $\mathcal{N} = X \times \bar{N}_{\mathcal{A}}$ over X.

3.6. **Proof of (iv).** We keep the notations of the previous section. By Proposition 1.5, there exists a Borel measurable family φ_x of norms in N_K invariant under the cocycle. Let μ be the Bowen-Margulis measure of maximal entropy for (X, f). We show that the family φ_x coincides μ almost everywhere with a Hölder continuous invariant family of norms. First we consider $x \in X$ and $z \in W^s_{loc}(x)$. We denote $x_n = f^n x$ and $z_n = f^n z$. Since the family of norms φ_x is invariant and the action of $GL(V)$ on norms is isometric, we have

$$\begin{aligned}\operatorname{dist}(\varphi_x, \varphi_z) &= \operatorname{dist}\left((\mathcal{A}^n_x)^* \varphi_{x_n},\, (\mathcal{A}^n_z)^* \varphi_{z_n}\right)\\ &\le \operatorname{dist}\left((\mathcal{A}^n_x)^* \varphi_{x_n},\, (\mathcal{A}^n_x)^* \varphi_{z_n}\right) + \operatorname{dist}\left((\mathcal{A}^n_x)^* \varphi_{z_n},\, (\mathcal{A}^n_y)^* \varphi_{z_n}\right)\\ &\le \operatorname{dist}(\varphi_{x_n},\, \varphi_{z_n}) + \operatorname{dist}\left(\varphi_{z_n},\, (\mathcal{A}^n_z \circ (\mathcal{A}^n_x)^{-1})^* \varphi_{z_n}\right).\end{aligned}$$

By Proposition 3.2 for all $n \in \mathbb{N}$ we have $\|(\mathcal{A}^n_x)^{-1} \circ \mathcal{A}^n_z - \mathrm{Id}\,\| \le c \operatorname{dist}(x, y)^\beta$ and hence

$$\|\mathcal{A}^n_z \circ (\mathcal{A}^n_x)^{-1} - \mathrm{Id}\| \le \|\mathcal{A}^n_z\| \cdot \|(\mathcal{A}^n_x)^{-1} \circ \mathcal{A}^n_z - \mathrm{Id}\,\| \cdot \|(\mathcal{A}^n_z)^{-1}\| \le K^2 c \operatorname{dist}(x, z)^\beta$$

since $\|\mathcal{A}^n_x\|$, $\|(\mathcal{A}^n_x)^{-1}\| \le K$ for all x and n. Then by Lemma 3.8 we have

$$\operatorname{dist}\left(\varphi_{z_n},\, (\mathcal{A}^n_z \circ (\mathcal{A}^n_x)^{-1})^* \varphi_{z_n}\right) \le K^8 \|\mathcal{A}^n_z \circ (\mathcal{A}^n_x)^{-1} - \mathrm{Id}\,\| \le K^{10} c \operatorname{dist}(x, z)^\beta$$

as $\varphi_{z_n} \in N_K$ and hence $(\mathcal{A}^n_z \circ (\mathcal{A}^n_x)^{-1})^* \varphi_{z_n} \in N_{K^2}$.

Since the the space $(\bar{N}_{\mathcal{A}}, \operatorname{dist})$ is compact and hence separable, we can apply Lusin's theorem to the function $\varphi : x \mapsto \varphi_x$ from X to $\bar{N}_{\mathcal{A}}$. So there exists a compact set $S \subset X$ with $\mu(S) > 1/2$ on which φ is uniformly continuous. Let Y be the set of points in X for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff ergodic theorem $\mu(Y) = 1$. If both x and z are in Y, then there exists a sequence $\{n_i\}$ such that $x_{n_i} \in S$ and $z_{n_i} \in S$. Since $z \in W^s_{loc}(x)$,

$$\operatorname{dist}(x_{n_i}, z_{n_i}) \to 0 \quad \text{and hence} \quad \operatorname{dist}(\varphi_{x_{n_i}}, \varphi_{z_{n_i}}) \to 0$$

by uniform continuity of φ on S. Thus we conclude that for $x, z \in Y$ with $z \in W^s_{loc}(x)$

$$\operatorname{dist}(\varphi_x, \varphi_z) \le K^{10} c \operatorname{dist}(x, z)^\beta =: c_1 \operatorname{dist}(x, z)^\beta.$$

Similarly, for $x, y \in Y$ with $y \in W^u_{loc}(x)$ we have $\operatorname{dist}(\varphi_x, \varphi_y) \le c_1 \operatorname{dist}(x, y)^\beta$.

We consider a small open set in X with product structure, which for the shift case is just a cylinder with a fixed 0-coordinate. For almost every local stable leaf, the set of points of Y on the leaf has full conditional measure of μ. We consider $x, y \in Y$ which lie on two such local stable leaves and denote by $H_{x,y}$ the holonomy map along unstable leaves from $W^s_{loc}(x)$ to $W^s_{loc}(y)$:

$$\text{for } z \in W^s_{loc}(x), \quad H_{x,y}(z) = W^u_{loc}(z) \cap W^s_{loc}(y) \,\in W^s_{loc}(y).$$

It is known that the holonomy maps are absolutely continuous with respect to the conditional measures of μ, which implies that there exists a point $z \in W^s_{loc}(x) \cap Y$ close to x such that $z' = H_{x,y}(z)$ is also in Y. By the argument above we have

$$\operatorname{dist}(\varphi_x, \varphi_z) \le c_1 \operatorname{dist}(x, z)^\beta,\ \operatorname{dist}(\varphi_z, \varphi_{z'}) \le c_1 \operatorname{dist}(z, z')^\beta,\ \operatorname{dist}(\varphi_{z'}, \varphi_y) \le c_1 \operatorname{dist}(z', y)^\beta.$$

Since the points x, y, and z are close, by the local product structure we have

$$\operatorname{dist}(x,z)^\beta + \operatorname{dist}(z,z')^\beta + \operatorname{dist}(z',y)^\beta \le c_2 \operatorname{dist}(x,y)^\beta.$$

Hence, we obtain $\operatorname{dist}(\varphi_x, \varphi_y) \le c_3 \operatorname{dist}(x,y)^\beta$ for all x and y in a set of full measure $\tilde{Y} \subset Y$. We can assume that $\tilde{Y}$ is invariant by taking $\bigcap_{n=-\infty}^{\infty} f^n(\tilde{Y})$. Since μ has full support, the set $\tilde{Y}$ is dense in X, and hence we can extend φ from $\tilde{Y}$ and obtain an invariant Hölder continuous family of norms $\|.\|_x$ on X. $\square$

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