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REPRESENTATIONS AND PROPERTIES OF WEIGHT FUNCTIONS IN THE TAUBERIAN THEOREMS

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Abstract. We continue the studies of weight functions in the Tauberian theorems for random fields. We obtain the rate of convergence of function series in the representation of a weight function. We prove a recurrence relation for weight functions in the spaces of different dimensions.

1. Introduction

The Abelian and Tauberian theorems are often used when studying asymptotic properties of random fields. Most of those theorems contain a relationship between the asymptotic behavior of the spectral and covariance functions at infinity and zero, respectively.

The variance of the integrals of random fields is studied in the paper [1]. The exact form of the weight function $g_{n,r,a}(|t|)$ corresponding to the spectral function

$$\Phi_a(\lambda) = \Phi(a+\lambda) - \Phi(a)$$

is evaluated in [1]. A new method for the evaluation of weight functions with the help of recurrence relations is proposed in the paper [2].

The current paper continues studies of weight functions in the Tauberian theorems for random fields. We consider the rate of convergence of function series in the representation of the function $g_{n,r,a}(|t|)$.

A recurrence relation for weight functions is found for the spaces of different dimensions. We exhibit some numerical examples by using *Mathematica 5.0*. We also use *Maple 9.5* to check all numerical results and compare them with the tables [9] whenever is possible.

2. Some definitions and notation

Let $\mathbf{R}^n$ be the Euclidean space of dimension $n \geq 2$ and let $\xi(t)$, $t \in \mathbf{R}^n$, be a real valued measurable mean square continuous random field being homogeneous and isotropic in the wide sense (see [4]). Assume that its mean value is zero and denote the covariance function by $\mathbb{B}_n(r) = \mathbb{B}_n(\|t\|) = \mathsf{E}\,\xi(0)\xi(t)$, $t \in \mathbf{R}^n$.

It is known that there exists a bounded nondecresing function $\Phi(x)$, $x \geq 0$, called the spectral function of the field $\xi(t)$, such that

$$\mathbb{B}_n(r) = 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)\int_0^\infty \frac{J_{(n-2)/2}(rx)}{(rx)^{(n-2)/2}}\,d\Phi(x)$$

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where $J_\nu(z)$ is the Bessel function of the first kind and of order $\nu > -\frac{1}{2}$ (see, for example, [3, 4]).

Some asymptotic properties of the variance of integrals of random fields

$$b_n(r) = \mathbf{D}\left[\int_{v(r)} \xi(t)\,dt\right] = (2\pi)^n r^{2n} \int_0^\infty \frac{J^2_{n/2}(rx)}{(rx)^n}\,d\Phi(x)$$

are studied in the papers [3, 4] in terms of both covariance and spectral functions where $v(r) = \{t \in \mathbf{R}^n \colon |t| \le r\}$ is a ball in $\mathbf{R}^n$.

The asymptotic behavior of the function $\Phi_a(\lambda) := \Phi(a+\lambda) - \Phi(a)$ as $\lambda \to +0$ is considered in the paper [1]. The Tauberian theorem in [1] is stated in terms of the functionals

$$\tilde{b}^a(r) := (2\pi)^n \int_0^\infty \frac{J^2_{n/2}(rx)}{(rx)^n}\,d\Phi_a(x).$$

It is also shown in [1] that there exists a real valued function $g_{n,r,a}(|t|)$ such that

$$\tilde{b}^a(r) = D\left[\int_{\mathbb{R}^n} g_{n,r,a}(|t|)\xi(t)\,dt\right].$$

Moreover

$$g_{n,r,a}(|t|) = \frac{1}{|t|^{n/2-1}} \int_0^\infty \underbrace{(\lambda+a)^{n/2} J_{n/2-1}(|t|(\lambda+a)) \frac{J_{n/2}(r\lambda)}{(r\lambda)^{n/2}}}_{G(\lambda)}\,d\lambda, \qquad |t| \ne r. \tag{1}$$

In what follows we use the same symbol C for different constants whose exact values may vary in the course of proofs.

3. Discussion of the problem

The rate of convergence of series in the representation of a weight function $f_{n,r,a}(|t|)$ in the Tauberian theorem is obtained in the paper [5]. Some numerical results for the approximation of weight functions by partial sums of the representation are also given in [5]. We are interested in obtaining similar and new results of this type for the function $g_{r,a}(|t|)$.

Using the asymptotic behavior of the Bessel function (see [7])

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right), \qquad z \to \infty, \tag{2}$$

we derive the asymptotic behavior of $G(\lambda)$ defined in (1), namely

$$\begin{aligned} G(\lambda) &\sim \frac{C}{\lambda} \cos\left(|t|(\lambda+a) - \frac{\pi n}{4} + \frac{\pi}{4}\right) \cos\left(r\lambda - \frac{n\pi}{4} - \frac{\pi}{4}\right) \\ &\sim \frac{C}{\lambda}\left(\cos\left((r+|t|)\lambda + |t|a - \frac{\pi n}{2}\right) - \sin\left((|t|-r)\lambda + |t|a\right)\right), \qquad \lambda \to \infty. \end{aligned}$$

Thus the integral in representation (1) converges conditionally.

The classical approach uses the Poisson formula ([7, §3.3])

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^1 (1-x^2)^{\nu-1/2} \cos(zx)\,dx \tag{3}$$

and is based on the following representation of the functional $g_{r,a}(|t|)$:

$$\begin{aligned} g_{n,r,a}(|t|) &= \frac{1}{2^{n/2-1}\Gamma(\frac{n-1}{2})\sqrt{\pi} r^{n/2}} \\ &\quad \times \int_0^\infty \frac{(\lambda+a)^{n-1}}{\lambda^{n/2}} J_{n/2}(r\lambda) \int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} \cos(|t|(\lambda+a)x)\,dx\,d\lambda \end{aligned}$$

and on the change of variables formula

$$
\begin{aligned}
g_{n,r,a}(|t|) = {} & \frac{1}{2^{n/2-1}\Gamma(\frac{n-1}{2})\sqrt{\pi}r^{n/2}} \\
& \times \Bigg(\int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}} \cos(|t|ax) \int_0^\infty \frac{(\lambda+a)^{n-1}}{\lambda^{n/2}} J_{n/2}(r\lambda)\cos(|t|\lambda x)\,d\lambda\,dx \\
& - \int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}} \sin(|t|ax) \int_0^\infty \frac{(\lambda+a)^{n-1}}{\lambda^{n/2}} J_{n/2}(r\lambda)\sin(|t|\lambda x)\,d\lambda\,dx \Bigg).
\end{aligned}
$$

The rest of the proof is to evaluate the above integrals.

One cannot use such a change of variables in the above integral unfortunately (the same is true in the case of the functional $f_{r,a}(|t|)$ considered in [5]), since the integrals with respect to λ are divergent for $n \in \mathbb{N}$ (this follows from the asymptotic formula (2).

Another approach is used in the paper [1]. This approach is based on a representation of the function $g_{r,a}(|t|)$ in the form of the sum of the following function series

$$
\begin{aligned}
g_{n,r,a}(|t|) = {} & \frac{a^{n/2}\Gamma\left(\frac{n-2}{2}\right)}{2^{n/2}|t|^{n/2}} \\
& \times \sum_{m=0}^{\infty} (-1)^m \Big(\frac{n}{2}+m-1\Big) C_{n+m-3}^{m} J_{\frac{n-2}{2}+m}(|t|a) \\
& \times \sum_{k=0}^{n-1} C_{n-1}^{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \\
& \times \begin{cases} \dfrac{{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right)}{|t|^k\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m+n-k-1}{2}\right)}, & |t| > r, \\[2ex] \dfrac{|t|^{m+1}{}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}; \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)}, & |t| < r, \end{cases}
\end{aligned} \tag{4}
$$

where the corresponding term vanishes in the cases of $|t| < r$ and $(m+k-n-1)/2 \in \mathbf{N} \cup \{0\}$, since $\Gamma((n-m-k+1)/2) = \infty$.

We use the following notation in what follows

$$
\begin{aligned}
d_m(n,r,a,t) := {} & (-1)^m \left(\frac{n}{2}+m-1\right) C_{n+m-3}^{m} J_{\frac{n-2}{2}+m}(|t|a) \\
& \times \sum_{k=0}^{n-1} C_{n-1}^{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \\
& \times \frac{|t|^{m+1}{}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}; \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
s_m(n,r,a,|t|) := {} & (-1)^m \left(\frac{n}{2}+m-1\right) C_{n+m-3}^{m} J_{\frac{n-2}{2}+m}(|t|a) \\
& \times \sum_{k=0}^{n-1} C_{n-1}^{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \\
& \times \frac{{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right)}{|t|^k\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m+n-k-1}{2}\right)}.
\end{aligned} \tag{6}
$$

4. Some asymptotic properties of the hypergeometric function

We need some properties of the hypergeometric function

$$ {}_2F_1(a,b,c;z) $$

to evaluate the sum in (4) (see [8]).

First we consider the asymptotic behavior of

$$ {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) $$

as $m \to \infty$.

Using the Stirling formula,

$$ \Gamma(k+1) \sim \sqrt{2\pi}k^{k+1/2}e^{-k}e^{\theta_k/k} \quad \text{and} \quad k! \sim \sqrt{2\pi}k^{k+1/2}e^{-k}e^{\theta_k/k} $$

where $\theta_k \in \left(0; \frac{1}{12}\right)$.

The Watson result (see [8, §2.3.2]) is often helpful when deriving the asymptotic results. Following Watson's method with

$$ a = \frac{k+1}{2}, \qquad c = \frac{n}{2}+1, \qquad b = \frac{k-n+3}{2}, \qquad \lambda = \frac{m}{2}, \qquad \frac{2}{1-z} = \left(\frac{|t|}{r}\right)^2 $$

we obtain

$$ \begin{aligned} &{}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) \\ &\sim \frac{\sqrt{2\pi}\Gamma(n/2+m)2^{k+2-n/2}}{\Gamma\left(\frac{k-n+1+m}{2}\right)\Gamma\left(n+\frac{m-k-1}{2}\right)m^{1/2}}\left(\frac{r}{|t|}\right)^{m+k+1} O\left(C^m\right) \\ &\sim O\left(\frac{m^{n/2+m-1/2}C^m}{m^{1/2}m^{(m+k-n)/2}m^{(m-k+2n-2)/2}}\right) = O(C^m) \end{aligned} $$

as $m \to \infty$. This formula holds in the domain $z \in \mathbf{C}\backslash(-\infty, 1)$ and, unfortunately, cannot be used explicitly in our case. Nevertheless the asymptotic behavior of order $O(C^m)$ can be proved.

Lemma 4.1. *For $|t| < r$,*

$$ {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) = O\left(C^m\right), \qquad m \to \infty. $$

Proof. Using the definition of the hypergeometric function (see [5, 3.]),

$$ \begin{aligned} {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) &= \sum_{l=0}^{\infty} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{m+k+1-n}{2}\right)_l}{\left(\frac{n}{2}+m\right)_l l!} z^l \\ = \sum_{l=0}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{m+k+1-n}{2}+l\right)z^l}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{n}{2}+m+l\right)\Gamma\left(\frac{m+k+1-n}{2}\right)\Gamma\left(l+1\right)} \\ = 1 + \frac{\Gamma\left(\frac{n}{2}+m\right)}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{m+k+1-n}{2}\right)} \sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)\Gamma\left(\frac{m+k+1-n}{2}+l\right)z^l}{\Gamma\left(\frac{n}{2}+m+l\right)\Gamma\left(l+1\right)}. \end{aligned} $$

According to Stirling's formula, we get

$$ \frac{\Gamma\left(\frac{n}{2}+m\right)}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{m+k+1-n}{2}\right)} \sim \frac{2^{m+k-n/2}m^{n-k-1/2}}{\sqrt{2\pi}} \tag{7} $$

as $m \to \infty$. Using Stirling's formula again we rewrite and estimate the series:

$$\sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)\Gamma\left(\frac{m+k+1-n}{2}+l\right)z^l}{\Gamma\left(\frac{n}{2}+m+l\right)\Gamma(l+1)}$$

$$\leq C\sum_{l=1}^{\infty} \frac{\left(\frac{m+k-1}{2}+l\right)^{1/2}\left(\frac{m+k-n-1}{2}+l\right)^{(m+k-n)/2} b^{(m+k-n-1)/2} z^l}{l^{1/2}\left(\frac{n}{2}+m+l-1\right)^{(m+n-k)/2} a^{(n+m-k-1)/2}} \tag{8}$$

$$\leq C\sum_{l=1}^{\infty} \frac{C_1^m z^l}{l^{1/2} m^{n-k-1/2}}.$$

After some algebra in (7) and (8) we complete the proof of the lemma. □

Theorem 4.1. *For $|t| < r$,*

$$\sum_{m=N}^{\infty} d_m(n,r,a,t) = O\left(\sum_{m=N}^{\infty} \frac{C^m}{m^{m-n/2+2}}\right), \qquad N \to \infty, \tag{9}$$

$$\sum_{m=N}^{\infty} d_m(n,r,a,t) = O\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right), \qquad N \to \infty, \tag{10}$$

where $\varepsilon > 0$ is arbitrary.

Proof. Consider the asymptotic behavior of the factors in representation (4) for the function $g_{n,r,a}(|t|)$:

$$C_{n+m-3}^m \sim \frac{m^{n-3}}{(n-3)!}. \tag{11}$$

Applying formula [5, (22)] we obtain

$$J_{n/2+m-1}(|t|a) = O\left(\frac{C^m}{m^{n/2+m-1/2}}\right), \qquad m \to \infty. \tag{12}$$

Consider

$$d_k^*(m,n,r,a,t)$$

$$:= \sum_{k=0}^{n-1} C_{n-1}^k \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \frac{|t|^{m+1}{}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}; \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)}.$$

We use Stirling's formula and follow the lines of the above proof:

$$d_k^*(m,n,r,a,t) = \sum_{k=0}^{n-1} O\left(\frac{C^m}{m^{n-k-1/2}}\right) = O\left(\frac{C^m}{m^{1/2}}\right), \qquad m \to \infty. \tag{13}$$

The asymptotic behavior of (10) follows from

$$\sum_{m=N}^{\infty} \frac{C^m}{m^{(m-n/2+2)}} \leq \sum_{m=N}^{\infty} \frac{C^m}{m^{(m-\varepsilon m)}} \leq \sum_{m=N}^{\infty} \left(\frac{C}{N^{(1-\varepsilon)}}\right)^m = \frac{C^N N^{1-\varepsilon}}{N^{N(1-\varepsilon)}(N^{1-\varepsilon}-C)}$$

$$= O\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right), \qquad N \to \infty,$$

for all $\varepsilon > 0$. □

Now we study the asymptotic behavior of

$${}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right) \quad \text{as } m \to \infty.$$

We apply the result of [8, §2.3.2] once more. Similarly to the case considered above we get

$$ {}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right) \sim O\left(\frac{C^m}{m^{(n+1)/2}}\right), \qquad m \to \infty. $$

This result holds in the domain $z \in \mathbf{C} \setminus (-\infty, 1)$ and cannot be used explicitly in our case. Nevertheless we prove that the asymptotic behavior is of order $O(C^m/m^{(n+1)/2})$.

Lemma 4.2. *For $|t| > r$,*

$$ {}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}, \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right) = O\left(\frac{C^m}{m^{n/2+1/2}}\right), \quad m \to \infty. $$

Proof. First we consider the case where $(m+n-k-3)$ is an even number. Let $m+n-k-3 = 2p$. Then

$$ \begin{aligned} \left|{}_2F_1\left(\frac{m+k+1}{2}, -p, \frac{n}{2}+1; z\right)\right| &= \left|\sum_{l=0}^{p} \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)(-1)^l p!\,\Gamma\left(\frac{n}{2}+1\right) z^l}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{n}{2}+1+l\right)(p-l)!\,l!}\right| \\ &\le \frac{\Gamma(\frac{n}{2}+1)}{\Gamma\left(\frac{m+k+1}{2}\right)} \sum_{l=0}^{p} C_p^l z^l \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)}{\Gamma\left(\frac{n}{2}+1+l\right)} \\ &\le \frac{\Gamma\left(\frac{m+k+1}{2}+p\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1+p\right)\Gamma\left(\frac{m+k+1}{2}\right)}(1+z)^p, \end{aligned} $$

since

$$ \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)}{\Gamma\left(\frac{n}{2}+1+l\right)} \le \frac{\Gamma\left(\frac{m+k+1}{2}+p\right)}{\Gamma\left(\frac{n}{2}+1+p\right)} $$

for all $l = 0, \ldots, p$. By Stirling's formula

$$ \frac{\Gamma\left(\frac{m+k+1}{2}+p\right)}{\Gamma\left(\frac{n}{2}+1+p\right)} = \frac{\Gamma\left(m-2+\frac{n}{2}+1\right)}{\Gamma\left(n+\frac{m-k-3}{2}+1\right)} \sim C^m \frac{m^{m+n/2-3/2}}{m^{m/2+n-k/2-1}} = C^m m^{(m-n+k-1)/2}, $$

$$ {}_2F_1\left(\frac{m+k+1}{2}+l, -p, \frac{n}{2}+1; z\right) = O\left(\frac{C^m}{m^{(n+1)/2}}\right). $$

Now let $(m+n-k-3)$ be an odd number. Then

$$ \begin{aligned} &{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}, \frac{n}{2}+1; z\right) \\ &= \sum_{l=0}^{\left[\frac{m+n-k-3}{2}\right]+1} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!} z^l \\ &\quad + \sum_{\left[\frac{m+n-k-3}{2}\right]+2}^{\infty} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!} z^l. \end{aligned} \tag{14} $$

We have

$$\begin{aligned}
&\left|\left(\frac{m+k+1}{2}\right)_l\left(\frac{3+k-m-n}{2}\right)_l\right| \\
&\quad=\left|\left(\frac{m+k+1}{2}\right)\dots\left(\frac{m+k+1}{2}+l-1\right)\right. \\
&\qquad\qquad\left.\times\left(\frac{m+n-k-3}{2}\right)\dots\left(\frac{m+n-k-3}{2}-l+1\right)\right| \\
&\quad\le\left|\left(\frac{m+k+1}{2}+\frac{1}{2}\right)\dots\left(\frac{m+k+1}{2}+l-1+\frac{1}{2}\right)\right. \\
&\qquad\qquad\left.\times\left(\frac{m+n-k-3}{2}+\frac{1}{2}\right)\dots\left(\frac{m+n-k-3}{2}-l+1+\frac{1}{2}\right)\right| \\
&\quad=\left|\left(\frac{m+k}{2}+1\right)_l\left(\frac{2+k-m-n}{2}\right)_l\right|.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\sum_{l=0}^{\left[\frac{m+n-k-3}{2}\right]+1}\frac{\left(\frac{m+k+1}{2}\right)_l\left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!}z^l\right| &\le\sum_{l=0}^{\frac{m+n-k-2}{2}}\frac{\left(\frac{m+k+2}{2}\right)_l\left(\frac{2+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!}z^l \\
&=O\left(\frac{C^m}{m^{(n+1)/2}}\right),\qquad m\to\infty.
\end{aligned}\tag{15}$$

Now we estimate the second term in (14):

$$\begin{aligned}
&\sum_{l=\left[\frac{m+n-k-3}{2}\right]+2}^{\infty}\frac{\left(\frac{m+k+1}{2}\right)_l\left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!}z^l \\
&\quad=\sum_{l=\left[\frac{m+n-k-3}{2}\right]+2}^{\infty}\frac{\Gamma\left(\frac{m+k+1}{2}+l\right)\Gamma\left(\frac{n}{2}+1\right)\left(-\frac{m+n-k-3}{2}\right)\left(-\frac{m+n-k-3}{2}+1\right)\dots}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{n}{2}+1+l\right)\Gamma(l+1)} \\
&\qquad\qquad\times\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\dots\left(-\frac{m+n-k-3}{2}+l-1\right)z^l \\
&\quad=\left|\begin{array}{l} s=l-\left[\frac{m+n-k-3}{2}\right]-1 \\ l=s+\frac{m+n-k-2}{2}\end{array}\right| \\
&\quad=\frac{(-1)^{\frac{m+n-k-2}{2}}\Gamma\left(\frac{m+n-k-1}{2}\right)\Gamma\left(\frac{n}{2}+1\right)z^{\frac{m+n-k-2}{2}}}{\pi\Gamma\left(\frac{m+k+1}{2}\right)} \\
&\qquad\times\sum_{s=1}^{\infty}\frac{\Gamma\left(s+m+\frac{n-1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(s+n+\frac{m-k}{2}\right)\Gamma\left(s+\frac{m+n-k}{2}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{s=1}^{\infty}\frac{\Gamma\left(s+m+\frac{n-1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(s+n+\frac{m-k}{2}\right)\Gamma\left(s+\frac{m+n-k}{2}\right)} &\le C^m\sum_{s=1}^{\infty}\frac{b^{\frac{\frac{m-n+k-1}{2}}{s+\frac{m+n-k-2}{2}}\left(s+m+\frac{n}{2}-1\right)}z^s}{\left(s+\frac{m+n-k-2}{2}\right)^{n-k}} \\
&\le\frac{C^m}{\left(\frac{m+n-k-2}{2}\right)^{n-k}}\sum_{s=1}^{\infty}z^s.
\end{aligned}\tag{16}$$

Representation (16) and Stirling's formula imply that

$$\left| \sum_{l=\left[\frac{m+n-k-3}{2}\right]+2}^{\infty} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!} z^l \right| \\ = O\left(\frac{C^m \Gamma\left(\frac{m+n-k-1}{2}\right)}{\left(\frac{m+n-k-2}{2}\right)^{n-k} \Gamma\left(\frac{m+k+1}{2}\right)} \right) = O\left(\frac{C^m}{m^{(n+1)/2}} \right), \qquad m \to \infty. \tag{17}$$

Substituting (15) and (17) to (14) we complete the proof of the lemma. □

Theorem 4.2. *If $|t| > r$, then*

$$\sum_{m=N}^{\infty} s_m(n,r,a,t) = O\left(\sum_{m=N}^{\infty} \frac{C^m}{m^{m-n/2+2}} \right), \qquad N \to \infty, \tag{18}$$

$$\sum_{m=N}^{\infty} s_m(n,r,a,t) = O\left(\frac{C^N}{N^{N(1-\varepsilon)}} \right), \qquad N \to \infty, \tag{19}$$

for all $\varepsilon > 0$.

Proof. First we consider the asymptotic behavior of the expression for $s_m(n,r,a,t)$ in equality (6). Put

$$s_m^{\star}(n,r,a,t) = \sum_{k=0}^{n-1} C_{n-1}^k \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \frac{{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right)}{|t|^k \Gamma\left(\frac{n}{2}+1\right) \Gamma\left(\frac{m+n-k-1}{2}\right)}.$$

Using Stirling's formula and Lemma 4.2 we obtain

$$s_m^{\star}(n,r,a,t) = \sum_{k=0}^{n-1} O\left(\frac{C^m}{m^{n-k-1/2}} \right) = O\left(\frac{C^m}{m^{1/2}} \right). \tag{20}$$

Applying bounds (11), (12), and (20) we prove relation (18).

Equality (19) is proved similarly to the proof of Theorem 4.1. □

5. Recurrence relations for the weight functions $g_{n,r,a}(t)$

Recalling the well-known identities for the derivatives of the Bessel functions (see §3.2 in [7]), we get

$$\frac{d}{dz}\left(z^{\nu} J_{\nu}(z)\right) = z^{\nu} J_{\nu-1}(z), \qquad \frac{d}{dz}\left(\frac{J_{\nu}(z)}{z^{\nu}}\right) = -\frac{J_{\nu+1}(z)}{z^{\nu}}.$$

Integrating by parts the integral in representation (9) (see [1]) for the weight function $g_{n,r,a}(t)$ and considering the asymptotic behavior of the Bessel function at the origin and infinity we get

$$\begin{aligned}
g_{n,r,a}(|t|) &= \frac{1}{|t|^{n/2-1}} \int_0^\infty (\lambda+a)^{n/2} J_{n/2-1}(|t|(\lambda+a)) \frac{J_{n/2}(r\lambda)}{(r\lambda)^{n/2}}\, d\lambda \\
&= \frac{J_{n/2}(r\lambda)}{|t|^n (r\lambda)^{n/2}} \left(|t|(\lambda+a)\right)^{n/2} J_{n/2}(|t|(\lambda+a))\Big|_0^\infty \\
&\quad + \frac{r}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2}}\, d\lambda \\
&= -\left(\frac{a}{2|t|}\right)^{n/2} \cdot \frac{J_{n/2}(|t|a)}{\Gamma\left(\frac{n}{2}+1\right)} \\
&\quad + \frac{r}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2}}\, d\lambda.
\end{aligned}$$

Further

$$\begin{aligned}
g_{n,r,a}(|t|) &= -\left(\frac{a}{2|t|}\right)^{n/2} \cdot \frac{J_{n/2}(|t|a)}{\Gamma(\frac{n}{2}+1)} \\
&\quad + \frac{r^2}{|t|^{n+1}} \int_0^\infty (|t|(\lambda+a))^{n/2+1} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}}\, d\lambda \\
&\quad - \frac{ar^2}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}}\, d\lambda \\
&= -h_{n,a}(|t|) + r^2 g_{n+2,r,a}(|t|) - r^2 G_n(r,a,|t|)
\end{aligned}$$

where

$$h_{n,a}(|t|) = \left(\frac{a}{2|t|}\right)^{n/2} \cdot \frac{J_{\frac{n}{2}}(|t|a)}{\Gamma(\frac{n}{2}+1)}, \tag{21}$$

$$G_n(r,a,|t|) = \frac{a}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}}\, d\lambda. \tag{22}$$

The recurrence relation implies the following result.

Lemma 5.1. *For an arbitrary $n \geq 1$,*

$$g_{n+2,r,a}(|t|) = \frac{g_{n,r,a}(|t|) + h_{n,a}(|t|)}{r^2} + G_n(r,a,|t|)$$

where $h_{n,a}(|t|)$ and $G_n(r,a,|t|)$ are defined in (21) *and* (22)*, respectively.*

Corollary 5.1. *Let $n > m \geq 1$. Then*

(i) *if n and m are even numbers,*

$$g_{n,r,a}(|t|) = \frac{g_{m,r,a}(|t|)}{r^{n-m}} + \sum_{k=m/2+1}^{n/2} \frac{G_{2k-2}(r,a,|t|)}{r^{n-2k}} + \sum_{k=m/2+1}^{n/2} \frac{h_{2k-2,a}(|t|)}{r^{n-2k+2}};$$

(ii) *if n and m are odd numbers,*

$$g_{n,r,a}(|t|) = \frac{g_{m,r,a}(|t|)}{r^{n-m}} + \sum_{k=(m+1)/2}^{(n-1)/2} \frac{G_{2k-1}(r,a,|t|)}{r^{n-2k-1}} + \sum_{k=(m+1)/2}^{(n-1)/2} \frac{h_{2k-1,a}(|t|)}{r^{n-2k+1}}.$$

6. Evaluation of $g_{1,r,a}(|t|)$

Lemma 6.1. *The function $g_{1,r,a}(|t|)$ is given by*

$$g_{1,r,a}(|t|) = \frac{\log\left(\frac{|t|-r}{|t|+r}\right)\sin(a|t|)}{\pi r} \tag{23}$$

for $|t| > r$ and by

$$g_{1,r,a}(|t|) = \frac{\pi\cos(a|t|) + \log\left(\frac{r-|t|}{r+|t|}\right)\sin(a|t|)}{\pi r} \tag{24}$$

for $|t| < r$.

Proof. Applying [7, §3.4] we get

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z), \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z).$$

Hence

$$g_{1,r,a}(|t|) = \int_0^\infty \frac{J_{\frac{1}{2}}(r\lambda)}{\sqrt{r\lambda}}\sqrt{|t|(\lambda+a)}J_{-\frac{1}{2}}(|t|(\lambda+a))\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\cos(|t|(a+\lambda))\sin(r\lambda)}{r\lambda}\,d\lambda.$$

Now we use *Mathematica 5.0* and [9] to obtain equalities (23) and (24). □

7. Evaluation of $g_{2,r,a}(|t|)$

Lemma 7.1. *If $|t| \neq r$, then*

$$\begin{aligned} g_{2,r,a}(|t|) = {} & \frac{2}{\pi r^2}\int_0^{\min(1,r/|t|)} \frac{a\left(r^2-|t|^2x^2\right)\cos(a|t|x) - |t|x\sin(a|t|x)}{\sqrt{1-x^2}\sqrt{r^2-|t|^2x^2}}\,dx \\ & + \frac{J_0(a|t|) - a|t|J_1(a|t|)}{r^2} \\ & + \frac{2}{\pi r^2}\int_{\min(1,r/|t|)}^1 \frac{a\left(|t|^2x^2-r^2\right)\sin(a|t|x) - |t|x\cos(a|t|x)}{\sqrt{1-x^2}\sqrt{|t|^2x^2-r^2}}\,dx. \end{aligned} \tag{25}$$

Proof. We use the Poisson integral (3) to evaluate $g_{2,r,a}(|t|)$:

$$\begin{aligned} g_{2,r,a}(|t|) &= \int_0^\infty \frac{\lambda+a}{r\lambda}J_0(|t|(\lambda+a))J_1(r\lambda)\,d\lambda \qquad (26)\\ &= \frac{1}{r\pi}\int_0^\infty \frac{\lambda+a}{\lambda}J_1(r\lambda)\int_{-1}^1 \frac{\cos(|t|(\lambda+a)x)}{\sqrt{1-x^2}}\,dx\,d\lambda \\ &= \frac{1}{r\pi}\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\int_0^\infty \frac{\lambda+a}{\lambda}J_1(r\lambda)\cos\left(|t|(\lambda+a)x\right)\,d\lambda\,dx. \qquad (27) \end{aligned}$$

Now we justify the change of order of integration. The expression (27) can be rewritten as follows

$$\begin{aligned} & \int_0^\infty\int_{-1}^1 Q_{|t|,r,a}(x,\lambda)\,dx\,d\lambda \\ & = \int_0^C\int_{-1}^1 Q_{|t|,r,a}(x,\lambda)\,dx\,d\lambda + \int_C^\infty\int_{-1}^1 Q_{|t|,r,a}(x,\lambda)\,dx\,d\lambda \end{aligned} \tag{28}$$

where

$$Q_{|t|,r,a}(x,\lambda) := \frac{\lambda+a}{\lambda\sqrt{1-x^2}}J_1(r\lambda)\cos(|t|(\lambda+a)x).$$

The domain of integration for the integral (28) is a compact set and the integral converges absolutely. Thus the change of order is justified for this integral.

Now we show that

$$\int_C^\infty \int_{-1}^1 Q_{|t|,r,a}(x,\lambda)\,dx\,d\lambda \to 0, \qquad \int_{-1}^1 \int_C^\infty Q_{|t|,r,a}(x,\lambda)\,d\lambda\,dx \to 0 \tag{29}$$

as $C \to +\infty$.

The first part of (29) follows from representation (26):

$$\int_C^\infty \int_{-1}^1 Q_{|t|,r,a}(x,\lambda)\,dx\,d\lambda = \pi \int_C^\infty \frac{\lambda+a}{\lambda} J_0(|t|(\lambda+a))J_1(r\lambda)\,d\lambda$$

and from the known asymptotic behavior of the Bessel function (see [5, (2)]). We introduce the notation

$$\int_C^\infty \frac{\lambda+a}{\lambda} J_1(r\lambda)\cos(|t|(\lambda+a)x)\,d\lambda = \cos(|t|ax) \underbrace{\int_C^\infty \frac{\lambda+a}{\lambda} J_1(r\lambda)\cos(|t|\lambda x)\,d\lambda}_{I_1}$$
$$- \sin(|t|ax) \underbrace{\int_C^\infty \frac{\lambda+a}{\lambda} J_1(r\lambda)\sin(|t|\lambda x)\,d\lambda}_{I_2}.$$

to prove the second part of (29).

It is known that

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2}$$

where $H_\nu^{(k)}(z)$, $k = 1, 2$, are the Bessel functions of the third kind (see [7, (1), §3.61]).

Now we use the asymptotic representations of the Bessel functions (3) and (4) for $\nu = 1$ and $p = 1$ (see [7, §7.2]). Then

$$H_1^{(k)}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i(z-3\pi/4)} \left(1 + O\left(\frac{1}{z}\right)\right)$$

for $k = 1$ and 2, respectively.

Consider the following bound

$$\begin{aligned}
|I_1| &\le \left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} e^{ir\lambda} \cos(|t|\lambda x)\,d\lambda \right| + \left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} e^{ir\lambda} \cos(|t|\lambda x) O\left(\frac{1}{\lambda}\right) d\lambda \right| \\
&\quad + \left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} e^{-ir\lambda} \cos(|t|\lambda x)\,d\lambda \right| + \left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} e^{-ir\lambda} \cos(|t|\lambda x) O\left(\frac{1}{\lambda}\right) d\lambda \right| \\
&\le \left| \int_C^\infty \frac{\sqrt{2}(\lambda+a)}{\lambda\sqrt{\pi r\lambda}} \cos(r\lambda)\cos(|t|\lambda x)\,d\lambda \right| + \left| \int_C^\infty \frac{\sqrt{2}(\lambda+a)}{\lambda\sqrt{\pi r\lambda}} \sin(r\lambda)\cos(|t|\lambda x)\,d\lambda \right| \\
&\quad + \int_C^\infty \frac{C_1\,d\lambda}{2\lambda^{3/2}} \\
&\le \underbrace{\left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} \cos(\lambda(r+|t|x))\,d\lambda \right|}_{\mathcal{I}_1} + \underbrace{\left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} \cos(\lambda(r-|t|x))\,d\lambda \right|}_{\mathcal{I}_2} \\
&\quad + \underbrace{\left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} \sin(\lambda(r+|t|x))\,d\lambda \right|}_{\mathcal{I}_3} + \underbrace{\left| \int_C^\infty \frac{\lambda+a}{\lambda\sqrt{2\pi r\lambda}} \sin(\lambda(r-|t|x))\,d\lambda \right|}_{\mathcal{I}_4} + \frac{C_1}{\sqrt{C}}.
\end{aligned}$$

Then

$$\left|\int_{-1}^{1} \frac{\cos(|t|ax) I_1}{\sqrt{1-x^2}}\,dx\right| \leq \int_{-1}^{1} \left(|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3| + |\mathcal{I}_4| + \frac{C_1}{\sqrt{C}}\right) \frac{dx}{\sqrt{1-x^2}}.$$

Now

$$\int_{-1}^{1} \frac{|\mathcal{I}_1|}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \left|\int_{C|r+|t|x|}^{\infty} \frac{z + a(r+|t|x)}{\sqrt{2\pi r z|r+|t|x|}} \cdot \frac{\cos(z)}{z}\,dz\right| \frac{dx}{\sqrt{1-x^2}}.$$

We have

$$\frac{1}{\sqrt{1-x^2}} \left|\int_{C|r+|t|x|}^{\infty} \frac{z + a(r+|t|x)}{\sqrt{2\pi r z|r+|t|x|}} \cdot \frac{\cos(z)}{z}\,dz\right| \to 0 \quad \text{as } C \to +\infty$$

and

$$\begin{aligned}&\frac{1}{\sqrt{1-x^2}} \left|\int_{C|r+|t|x|}^{\infty} \frac{z + a(r+|t|x)}{\sqrt{2\pi r z|r+|t|x|}} \cdot \frac{\cos(z)}{z}\,dz\right| \\ &\qquad \leq \frac{1}{\sqrt{1-x^2}\sqrt{2\pi r z|r+|t|x|}} \cdot \sup_{C\geq 0} \left|\int_{C}^{\infty} \frac{z + a|r+|t|x|}{z} \cdot \frac{\cos(z)}{\sqrt{z}}\,dz\right| \in L_1([-1,1])\end{aligned}$$

for all $|t| \neq r$ and $x \in (-1,1)$. By the Lebesgue dominated convergence theorem

$$\int_{-1}^{1} \frac{|\mathcal{I}_1|}{\sqrt{1-x^2}}\,dx \to 0 \quad \text{as } C \to \infty.$$

The proof of

$$\int_{-1}^{1} \frac{|\mathcal{I}_k|}{\sqrt{1-x^2}}\,dx \to 0 \quad \text{as } C \to \infty$$

for $k = 2, 3, 4$ is similar. Thus the second relation in (29) holds. Running *Mathematica 5.0* we get expression (25) for the function $g_{2,r,a}(|t|)$. □

8. The main representation

Theorem 8.1. *If n is an odd number, then*

$$g_{n,r,a}(|t|) = \frac{g_{1,r,a}(|t|)}{r^{n-1}} + \sum_{k=1}^{\frac{n-1}{2}} \frac{G_{2k-1}(r,a,|t|)}{r^{n-2k-1}} + \sum_{k=1}^{\frac{n-1}{2}} \frac{h_{2k-1,a}(|t|)}{r^{n-2k+1}}$$

where $g_{1,r,a}(|t|)$ is defined in Lemma 6.1.

If n is an even number, then

$$g_{n,r,a}(|t|) = \frac{g_{2,r,a}(|t|)}{r^{n-2}} + \sum_{k=2}^{n/2} \frac{G_{2k-2}(r,a,|t|)}{r^{n-2k}} + \sum_{k=2}^{n/2} \frac{h_{2k-2,a}(|t|)}{r^{n-2k+2}}$$

where $g_{2,r,a}(|t|)$ is defined in Lemma 7.1.

The functions $h_{k,a}(|t|)$ and $G_k(r,a,|t|)$ are given by equalities (21) *and* (22)*, respectively.*

Remark. In contrast to representation (1) of Theorem 8.1, the function $g_{n,r,a}(|t|)$ is expressed in terms of absolutely convergent integrals.

9. Properties of weight functions

We study some properties of weight functions $g_{n,r,a}(|t|)$.

Theorem 9.1. *If $n \geq 1$, then*

$$\lim_{|t|\to+\infty} g_{n,r,a}(|t|) = 0.$$

Proof. Representation (23) obtained in Lemma 6.1 implies that

$$|g_{1,r,a}(|t|)| \leq \frac{1}{\pi r}\left|\log\left(\frac{|t|-r}{|t|+r}\right)\right| \to 0, \qquad |t| \to +\infty,$$

for $|t| > r$.

The function $g_{2,r,a}(|t|)$ defined by equality (26) can be rewritten as a sum of two integrals, namely

$$\begin{aligned} g_{2,r,a}(|t|) = &\int_0^1 \frac{\lambda+a}{r\lambda} J_0(|t|(\lambda+a))J_1(r\lambda)\,d\lambda \\ &+ \int_1^\infty \frac{\lambda+a}{r\lambda} J_0(|t|(\lambda+a))J_1(r\lambda)\,d\lambda. \end{aligned} \tag{30}$$

Consider the first integral in equality (30). The representation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta$$

(see §3.1 [7]) implies that

$$\left|\frac{J_\nu(z)}{z^\nu}\right| \leq \frac{\sqrt{\pi}}{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}. \tag{31}$$

Thus

$$\left|\frac{J_1(r\lambda)}{r\lambda}\right| \leq 1, \qquad |J_0(|t|(\lambda+a))| \leq 1$$

by inequality (31) for all $\lambda \in [0,1]$, $r > 0$, and $|t| \geq 0$.

Now we use the asymptotic expansion (2) and obtain

$$J_0(|t|(\lambda+a)) \to 0, \qquad |t| \to \infty,$$

for all $\lambda \in [0,1]$. By the Lebesgue dominated convergence theorem

$$\left|\int_0^1 \frac{\lambda+a}{r\lambda} J_0(|t|(\lambda+a))J_1(r\lambda)\,d\lambda\right| \to 0, \qquad |t| \to \infty.$$

We consider the second integral in representation (30). Using the bound obtained in Theorem 4.1 of the paper [6] we show that

$$\left|J_\nu(x) - \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right| \leq \frac{d_\nu}{x^{3/2}} \tag{32}$$

where $\nu \geq -1/2$, $x > 0$, and d_ν are some constants.

Applying inequality (32) to the second integral on the right hand of equality (30) we get

$$\begin{aligned}
&\left|\int_1^\infty \frac{\lambda+a}{r\lambda} J_0(|t|(\lambda+a))J_1(r\lambda)\,d\lambda\right| \\
&\quad= \Bigg|\int_1^\infty \frac{\lambda+a}{r\lambda}\left(\sqrt{\frac{2}{\pi|t|(\lambda+a)}}\cos(|t|(\lambda+a)-\pi/4)+\varepsilon_0(|t|,a,\lambda)\right) \\
&\qquad\qquad\qquad \times\left(\sqrt{\frac{2}{\pi r\lambda}}\cos(r\lambda-3\pi/4)+\varepsilon_1(r,\lambda)\right)d\lambda\Bigg| \\
&\quad\le \underbrace{\left|\frac{2}{\pi r\sqrt{r|t|}}\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\cos(|t|(\lambda+a)-\pi/4)\cos(r\lambda-3\pi/4)\,d\lambda\right|}_{\mathcal{J}_1} \\
&\qquad+\underbrace{\left|\frac{\sqrt{2}}{r\sqrt{\pi|t|}}\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda}\cos(|t|(\lambda+a)-\pi/4)\varepsilon_1(r,\lambda)\,d\lambda\right|}_{\mathcal{J}_2} \\
&\qquad+\underbrace{\left|\frac{\sqrt{2}}{\sqrt{\pi}r^{3/2}}\int_1^\infty \frac{\lambda+a}{\lambda^{3/2}}\cos(r\lambda-3\pi/4)\varepsilon_0(|t|,a,\lambda)\,d\lambda\right|}_{\mathcal{J}_3} \\
&\qquad+\underbrace{\left|\frac{1}{r}\int_1^\infty \frac{\lambda+a}{\lambda}\varepsilon_0(|t|,a,\lambda)\varepsilon_1(r,\lambda)\,d\lambda\right|}_{\mathcal{J}_4}
\end{aligned}$$

where

$$|\varepsilon_0(|t|,a,\lambda)| \le \frac{d_0}{(|t|(\lambda+a))^{3/2}}, \qquad |\varepsilon_1(r,\lambda)| \le \frac{d_1}{(r\lambda)^{3/2}}.$$

Now we study the asymptotic behavior of the integrals $\mathcal{J}_i$, $i=1,\dots,4$, as $|t|\to\infty$. It is clear that

$$\begin{aligned}
|\mathcal{J}_1| &= \Bigg|\frac{1}{\pi r\sqrt{r|t|}}\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\cos(|t|(\lambda+a)+r\lambda-\pi)\,d\lambda \\
&\qquad\qquad + \frac{1}{\pi r\sqrt{r|t|}}\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\cos(|t|(\lambda+a)-r\lambda+\pi/2)\,d\lambda\Bigg| \\
&\le \underbrace{\left|\frac{1}{\pi r\sqrt{r|t|}}\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\cos(|t|(\lambda+a)+r\lambda)\,d\lambda\right|}_{\mathcal{K}_1} \\
&\quad+\underbrace{\left|\frac{1}{\pi r\sqrt{r|t|}}\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\sin(|t|(\lambda+a)-r\lambda)\,d\lambda\right|}_{\mathcal{K}_2}.
\end{aligned}$$

The integrand in the integral $\mathcal{K}_1$ changes its sign and assumes zero values at the points

$$\lambda = \frac{\pi/2+\pi k-|t|a}{|t|+r}, \qquad k\in\mathbf{N}\cup 0. \tag{33}$$

Since the lower limit of the integral $\mathcal{K}_1$ equals 1, we derive from (33) that $k \ge s$ where

$$s := \frac{r + |t|(1+a) - \pi/2}{\pi}.$$

Then

$$\begin{aligned}
&\left|\int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a)+r\lambda)\,d\lambda\right| \\
&\qquad = \left|\int_1^{\frac{3\pi/2+\pi[s]-|t|a}{|t|+r}} \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a)+r\lambda)\,d\lambda\right. \\
&\qquad\qquad \left. + \sum_{k\ge[s]+1} \int_{\frac{\pi/2+\pi k-|t|a}{|t|+r}}^{\frac{\pi/2+\pi(k+1)-|t|a}{|t|+r}} \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a)+r\lambda)\,d\lambda\right|.
\end{aligned} \tag{34}$$

The series in representation (34) is sign alternating and the absolute value of every term does not exceed the absolute value of the preceding term, since $\sqrt{\lambda+a}/(\lambda\sqrt{\lambda})$ is a decreasing function with respect to $\lambda > 1$. By Leibnitz's theorem

$$\begin{aligned}
&\left|\sum_{k\ge[s]+1} \int_{\frac{\pi/2+\pi k-|t|a}{|t|+r}}^{\frac{\pi/2+\pi(k+1)-|t|a}{|t|+r}} \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a)+r\lambda)\,d\lambda\right| \\
&\qquad \le \left|\int_{\frac{3\pi/2+\pi[s]-|t|a}{|t|+r}}^{\frac{5\pi/2+\pi[s]-|t|a}{|t|+r}} \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a)+r\lambda)\,d\lambda\right|.
\end{aligned} \tag{35}$$

We derive from (34) and (35) that

$$\begin{aligned}
|\mathcal{K}_1| &\le \frac{1}{\pi r\sqrt{r|t|}} \int_1^{\frac{5\pi/2+\pi[s]-|t|a}{|t|+r}} \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\,d\lambda \\
&\le \frac{1}{\pi r\sqrt{r|t|}} \int_1^{1+\frac{2\pi}{|t|+r}} \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}}\,d\lambda \to 0, \qquad |t|\to\infty.
\end{aligned}$$

A similar reasoning shows that $|\mathcal{K}_2| \to 0$ as $|t|\to\infty$. Therefore $|\mathcal{J}_1| \to 0$ as $|t|\to\infty$.

Estimating the second integral $\mathcal{J}_2$ we prove that

$$|\mathcal{J}_2| \le \frac{\sqrt{2}d_1}{r^2\sqrt{\pi r|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda^{5/2}}\,d\lambda \to 0, \qquad |t|\to\infty,$$

since the integral $\int_1^\infty \sqrt{\lambda+a}/\lambda^{5/2}\,d\lambda$ converges. Similarly we show that

$$|\mathcal{J}_i| \to 0, \qquad |t|\to\infty,$$

$i = 3, 4$. Thus $|g_{2,r,a}(|t|)| \to 0$ as $|t|\to\infty$.

It is clear that

$$\lim_{|t|\to+\infty} |h_{n,a}(|t|)| = \frac{a^n}{2^{n/2}\Gamma(\frac{n}{2}+1)} \lim_{|t|\to+\infty} \frac{|J_{n/2}(|t|a)|}{(|t|a)^{n/2}} = 0.$$

Now we show that

$$|G_n(r,a,|t|)| = \frac{a}{|t|^n}\left|\int_0^\infty \underbrace{(|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}}}_{S(n,r,a,|t|)}\,d\lambda\right| \to 0 \tag{36}$$

as $|t| \to +\infty$. To check relation (36) we use the Lebesgue dominated convergence theorem again. Accounting the asymptotic behavior of the Bessel function at infinity and at zero we see that

$$\frac{a}{|t|^n}\,|S(n,r,a,|t|)| \le \frac{C}{|t|^{n/2}}\frac{|J_{n/2+1}(r\lambda)|}{(r\lambda)^{n/2+1}}(\lambda+a)^{n/2} \in L_1([0,+\infty], \tag{37}$$

$$\frac{a}{|t|^{n/2}}\left|\frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}}(\lambda+a)^{n/2}J_{n/2}(|t|(\lambda+a))\right| \to 0$$

as $|t| \to +\infty$. Therefore relation (36) holds and the proof of Theorem 9.1 is complete. $\square$

Theorem 9.2. *If $n \ge 1$, then*

$$\lim_{r\to\infty} g_{n,r,a}(|t|) = 0.$$

Proof. Representation (24) of Lemma 6.1 implies that

$$|g_{1,r,a}(|t|)| = \frac{1}{\pi r}\left|\pi\cos(a|t|) + \log\left(\frac{r-|t|}{r+|t|}\right)\sin(a|t|)\right| \to 0$$

as $r \to \infty$.

By (3) and [9] we know that

$$J_0(a|t|) = \frac{2}{\pi}\int_0^1 \frac{\cos(a|t|x)}{\sqrt{1-x^2}}\,dx \; ia|t|J_1(a|t|) = \frac{2}{\pi}\int_0^1 \frac{a|t|x}{\sqrt{1-x^2}}\sin(a|t|x)\,dx,$$

$$\begin{aligned} |g_{2,r,a}(|t|)| = \frac{2}{\pi r^2}\Bigg|\int_0^1 \Big(&\underbrace{(1+a\sqrt{r^2-|t|^2x^2})\cos(a|t|x)}_{R_1(|t|,r,a,x)} \\ &- \underbrace{|t|x(a+\frac{1}{\sqrt{r^2-|t|^2x^2}})\sin(a|t|x)}_{R_2(|t|,r,a,x)}\Big)\frac{dx}{\sqrt{1-x^2}}\Bigg| \to 0 \end{aligned} \tag{38}$$

as $r \to \infty$. Indeed

$$\begin{aligned} &\frac{1}{r^2}\left|\frac{R_1(|t|,r,a,x)-R_2(|t|,r,a,x)}{\sqrt{1-x^2}}\right| \\ &\qquad \le \frac{1}{\sqrt{1-x^2}}\left|\frac{1+a\sqrt{r^2-|t|^2x^2}}{r^2}\right|\cdot\left|1+\frac{|t|x}{\sqrt{r^2-|t|^2x^2}}\right| \in L_1([0,1]) \end{aligned}$$

and

$$\frac{1}{r^2}\left|\frac{R_1(|t|,r,a,x)-R_2(|t|,r,a,x)}{\sqrt{1-x^2}}\right| \to 0 \quad \text{as } r\to\infty.$$

This proves relation (38) by the Lebesgue dominated convergence theorem. Similarly to the proof of (36) (but for the case of $r \to \infty$) we apply the Lebesgue dominated convergence theorem and obtain

$$\frac{a}{|t|^n}\,|S(n,r,a,|t|)| \le \frac{C}{|t|^{n/2}}(\lambda+a)^{n/2}\left|J_{n/2}(|t|(\lambda+a)\right| \in L_1([0,C_1],$$

$$\frac{a}{|t|^n}\,|S(n,r,a,|t|)| \le \frac{C}{|t|^{n/2}}\frac{(\lambda+a)^{n/2}}{\lambda^{n/2+1}}\left|J_{n/2}(|t|(\lambda+a)\right| \in L_1([C_1,+\infty]),$$

whence

$$|S(n,r,a,|t|)| \to 0 \quad \text{as } r \to +\infty.$$

Hence

$$|G_n(r,a,|t|)| \to \infty \quad \text{as } r\to\infty.$$

Every term in the representation of $g_{n,r,a}(|t|)$ considered in Theorem 8.1 approaches zero as $r \to \infty$. The theorem is proved. $\square$

Example 1. Let $n = 3$, $a = 10$, $|t| = 1$ and $|t| = 4$. Then the result of Theorem 9.2 is exhibited in Figures 1 and 2.

figure=\inputpath/plot1.eps,width=5.5cm	figure=\inputpath/plot2.eps,width=5.5cm

FIGURE 1. $g_{3,r,1.2}(1)$ FIGURE 2. $g_{3,r,1.2}(5)$

Theorem 9.3. *The weight function $g_{n,r,a}(|t|)$, $n \geq 1$, is discontinuous if $|t| = r$. This is a discontinuity of the first kind if $ar/\pi \in \mathbf{N}$, otherwise the discontinuity is of the second kind.*

Proof. The continuity of $G_n(r, a, |t|)$ as a function of $|t|$ follows from representation (22) and bound (37). The functions $g_{1,r,a}(|t|)$ and $g_{2,r,a}(|t|)$ are continuous for $|t| \neq r$. Consider the behavior of these functions for $|t| = r$. Lemmas 6.1 and 7.1 imply that

$$\begin{aligned}
\lim_{|t|\to r+} g_{1,r,a}(|t|) &= \frac{1}{\pi r} \lim_{|t|\to r+} \sin(a|t|) \log\left(\frac{|t|-r}{|t|+r}\right) \\
&= \begin{cases} 0, & \text{if } \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k+1),\ k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k+1, 2k+2),\ k \in \mathbf{Z}_+, \end{cases} \\
\lim_{|t|\to r-} g_{1,r,a}(|t|) &= \frac{1}{\pi r} \lim_{|t|\to r-} \left(\pi \cos(a|t|) + \log\left(\frac{r-|t|}{r+|t|}\right) \sin(a|t|)\right) \\
&= \begin{cases} \frac{(-1)^{ar/\pi}}{r}, & \text{if } \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k+1),\ k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k+1, 2k+2),\ k \in \mathbf{Z}_+. \end{cases}
\end{aligned}$$

Running *Mathematica 5.0* and using [9] we obtain

$$\begin{aligned}
\lim_{|t|\to r-} g_{2,r,a}(|t|) &= \frac{J_0(ar) - arJ_1(ar)}{r^2} \\
&\quad + \frac{2}{\pi r^2} \lim_{|t|\to r-} \int_0^1 \frac{a(r^2 - |t|^2x^2)\cos(a|t|x) - tx\sin(a|t|x)}{\sqrt{1-x^2}\sqrt{r^2-|t|^2x^2}}\, dx \\
&= \frac{J_0(ar) - arJ_1(ar)}{r^2} + \frac{2}{\pi r^2} \sin(ar) \\
&\quad + \begin{cases} \frac{(-1)^{ar/\pi}\mathrm{Si}(2ar\pi)}{\pi r^2}, & \text{if } \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k+1),\ k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k+1, 2k+2),\ k \in \mathbf{Z}_+, \end{cases}
\end{aligned}$$

where $\mathrm{Si}(\cdot)$ is the integral sinus, see [2].

Further

$$\lim_{|t|\to r+} g_{2,r,a}(|t|) = \frac{J_0(ar) - arJ_1(ar)}{r^2}$$

$$+ \frac{2}{\pi r^2} \lim_{|t|\to r+} \int_0^{r/|t|} \frac{a\left(r^2 - |t|^2x^2\right)\cos(a|t|x) - |t|x\sin(a|t|x)}{\sqrt{1-x^2}\sqrt{|t|^2 - r^2x^2}}\,dx$$

$$+ \frac{2}{\pi r^2} \lim_{|t|\to r+} \int_{r/|t|}^{1} \frac{a\left(|t|^2x^2 - r^2\right)\sin(a|t|x) - |t|x\cos(a|t|x)}{\sqrt{1-x^2}\sqrt{|t|^2x^2 - r^2}}\,dx$$

$$= \frac{J_0(ar) - arJ_1(ar)}{r^2} + \frac{2}{\pi r^2}\sin(ar) + \frac{\cos(ar)}{r^2}$$

$$+ \begin{cases} \frac{(-1)^{ar/\pi}\mathrm{Si}(2ar\pi)}{\pi r^2}, & \text{if } \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k+1),\ k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k+1, 2k+2),\ k \in \mathbf{Z}_+. \end{cases}$$

The above calculations allow one to determine the kind of the discontinuity of the function $g_n(r, a, |t|)$ for $|t| = r$. This completes the proof of the theorem. □

Example 2. Let $n = 3$, $r = 1$, $a = 2\pi$ and $a = 5$. The result of Theorem 9.3 is exhibited in Figures 3 and 4.

figure=\inputpath/plot3.eps,width=5.5cm figure=\inputpath/plot4.eps,width=5.5cm

FIGURE 3. $g_{3,1,2\pi}(|t|)$

FIGURE 4. $g_{3,1,5}(|t|)$

10. CONCLUDING REMARKS

We obtained the rate of convergence of function series in the representation of weight functions in Tauberian theorems. We constructed the recurrence relations for weight functions. We studied some asymptotic behavior of weight functions.

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There are 0 remarks of the translator

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[1]Provide english data for [9]

[2]Compare English data for [9] with [8]