

When using the Gaussian kernel, we can estimate the copula-based kernel independence measure  $I^2(X)$  as

$$\begin{aligned}\mathcal{M}_u^2[\mathcal{F}, \mathbf{Z}_{1:m}, P_U] &= \frac{1}{m(m-1)} \sum_{i \neq j} k(\mathbf{Z}_i, \mathbf{Z}_j) \\ &\quad - \frac{2}{m} \sum_{i=1}^m \prod_{j=1}^d \int_0^1 \exp\left(\frac{-(\mathbf{Z}_i^j - u)^2}{2\sigma^2}\right) du \\ &\quad + \left( \int_0^1 \int_0^1 \exp\left(\frac{-(u - u')^2}{2\sigma^2}\right) du du' \right)^d\end{aligned}\quad (1)$$

Let us write the integral terms as a function of the Gaussian error function, and using  $\gamma = \frac{1}{2\sigma^2}$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

We have that

$$\int_0^1 \exp\left(\frac{-(\mathbf{Z}_{ij} - u)^2}{2\sigma^2}\right) du = 2\sigma^2 \int_{\frac{-\mathbf{Z}_{ij}}{2\sigma^2}}^{\frac{1-\mathbf{Z}_{ij}}{2\sigma^2}} \exp(-t^2) dt \quad (2)$$

$$= \frac{\sqrt{\pi}}{2\gamma} [\text{erf}(\gamma(1 - \mathbf{Z}_{ij})) + \text{erf}(\gamma \mathbf{Z}_{ij})] \quad (3)$$

and since

$$\int_0^z \text{erf}(x) dx = z \text{erf}(z) + \frac{\exp(-z^2)}{\sqrt{\pi}} \quad (4)$$

we have that

$$\int_0^1 \int_0^1 \exp\left(\frac{-(u - u')^2}{2\sigma^2}\right) du du' = 2\sigma^2 \int_0^1 \int_{\frac{-u'}{2\sigma^2}}^{\frac{1-u'}{2\sigma^2}} \exp(-z^2) dz du' \quad (5)$$

$$= \frac{\sqrt{\pi}}{2\gamma} \int_0^1 [\text{erf}(\gamma(1 - u')) + \text{erf}(\gamma u')] du' \quad (6)$$

$$= \frac{\sqrt{\pi}}{\gamma} \left[ \text{erf}(\gamma) - \frac{\exp(-\gamma^2)}{\gamma\sqrt{\pi}} - \frac{1}{\gamma\sqrt{\pi}} \right] \quad (7)$$

This finally yields

$$\begin{aligned}\mathcal{M}_u^2[\mathcal{F}, \mathbf{Z}_{1:m}, P_U] &= \frac{1}{m(m-1)} \sum_{i \neq j} k(\mathbf{Z}_i, \mathbf{Z}_j) \\ &\quad - \frac{2}{m} \sum_{i=1}^m \prod_{j=1}^d \left[ \frac{\sqrt{\pi}}{2\gamma} [\text{erf}(\gamma(1 - \mathbf{Z}_{ij})) + \text{erf}(\gamma \mathbf{Z}_{ij})] \right] \\ &\quad + \left[ \frac{\sqrt{\pi}}{\gamma} \left[ \text{erf}(\gamma) - \frac{\exp(-\gamma^2)}{\gamma\sqrt{\pi}} - \frac{1}{\gamma\sqrt{\pi}} \right] \right]^d\end{aligned}\quad (8)$$