Трубачеев Б. А.

$X_1 \dots X_n$ – выборка из $Exp(\lambda) \sim \lambda e^{-\lambda x}$

$\hat{\lambda}_{MLE}$ – ?, $\hat{\lambda}_{MAP}$?, $P(x|D)$ – ?

L = Likehood = $P(X_1|\lambda) \cdot \ldots \cdot P(X_n|\lambda) = \prod \lambda \cdot e^{-\lambda x_i} = \lambda^n \cdot \prod e^{-\lambda x_i}$

$\ln L = n \ln \lambda - \sum_{i=1}^{n} \lambda x_i = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$

$(\ln L)'_\lambda = \frac{n}{\lambda} - \sum x_i = 0$

$\hat{\lambda}_{MLE} = \frac{n}{\sum x_i}$

$\hat{\lambda}_{MAP} = \underset{\lambda}{\arg\max}\,(P(\lambda|D)) = \underset{\lambda}{\arg\max}\,(P(\lambda) P(D|\lambda)) =$

$= \underset{\lambda}{\arg\max}\left(P(\lambda) \cdot \lambda^n \cdot \prod e^{-\lambda \sum x_i}\right)$

Сопряжённое к $\lambda^n \cdot \prod e^{-\lambda \sum x_i}$ распределение – $\Gamma(\alpha, \beta)$

$\Gamma(\alpha, \beta) \propto \lambda^{\alpha-1} \cdot e^{-\beta\lambda}$

$\hat{\lambda}_{MAP} = \underset{\lambda}{\arg\max}\,(\lambda^{\alpha-1} \cdot e^{-\beta\lambda} \cdot \lambda^n \cdot e^{-\lambda \sum_{i=1}^{n} x_i}) =$

$= \underset{\lambda}{\arg\max}\,(\lambda^{\alpha-1+n} \cdot e^{-\lambda(\beta + \sum x_i)})$

$\Gamma(\alpha + n, \beta + \sum_{i=1}^{n} x_i)$

Для поиска максимума возьмём логарифм:

$(\alpha - 1 + n) \cdot \ln \lambda - \lambda(\beta + \sum_{i=1}^{n} x_i) \to \underset{\lambda}{\max}$

$\frac{\alpha - 1 + n}{\lambda} - \beta - \sum_{i=1}^{n} x_i = 0$

$\lambda_{MAP} = \frac{\alpha - 1 + n}{\beta + \sum x_i}$

$\hat{\lambda}_{MAP} = \frac{\alpha - 1 + n}{\beta + \sum x_i}$

$$P(x|D) = \int_\lambda P((x,\lambda)|D)\cdot d\lambda = \int_\lambda P(x|\lambda, D)\cdot P(\lambda|D)\cdot d\lambda =$$

$$= \int_\lambda P(x|\lambda)\cdot P(\lambda|D)\cdot d\lambda = \int_\lambda \lambda e^{-\lambda x}\cdot P(\lambda|D)\cdot d\lambda =$$

$$P(\lambda|D) \propto \lambda^{\alpha-1}\cdot e^{-\beta\lambda}\cdot \lambda^{n}\cdot e^{-\lambda\sum x_i} \sim \Gamma(\alpha+n, \beta+\sum x_i)$$

$$= \int_\lambda \lambda e^{-\lambda x}\cdot \lambda^{\alpha-1+n}\cdot e^{-\lambda(\beta+\sum_{i=1}^{n} x_i)}\cdot \frac{(\beta+\sum x_i)^{\alpha+n}}{\Gamma(\alpha+n)}\, d\lambda =$$

$$= \frac{(\beta+\sum x_i)^{\alpha+n}}{\Gamma(\alpha+n)}\cdot \int_\lambda \lambda^{\alpha+n}\cdot e^{-\lambda(\beta+\sum x_i+x)}\cdot d\lambda =$$

$$= \frac{\Gamma(\alpha+n+1)}{(\beta+\sum x_i+x)^{\alpha+n+1}}\cdot \frac{(\beta+\sum x_i)^{\alpha+n}}{\Gamma(\alpha+n)} = \frac{(\alpha+n)\cdot(\beta+\sum x_i)^{\alpha+n}}{(\beta+\sum x_i+x)^{\alpha+n+1}} \sim$$

$\sim$ Paretto II $(\alpha+n, \beta+\sum x_i)$

Ответ: $\hat{\lambda}_{MLE} = \frac{n}{\sum x_i}$, $\hat{\lambda}_{MAP} = \frac{\alpha-1+n}{\beta+\sum x_i}$

$P(x|D) \sim$ Paretto II $(\alpha+n, \beta+\sum x_i)$