MATH 146B: Ordinary and Partial Differential Equations

Final Bonus

An example of non-analytic smooth function

Problem 1. Consider the function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

(a) (2 points) Show that the function f(x) is continuous at x = 0.

Hint: compare the one-sided limits $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

(b) (3 points) Show that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$ and compute the value of f'(0).

Remark. Similarly, it can be shown that $\lim_{x\to 0^-} f^{(k)}(x) = \lim_{x\to 0^+} f^{(k)}(x) = 0$ for any positive integer k>0. Consequently, $f^{(k)}(0)=0$ for all positive integers k>0.

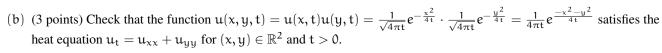
(c) (2 points) Use your findings from (a) and (b) along with the remark to construct the Taylor series expansion for f(x) centered at 0. Explain why this series does not converge to the actual value of f(x) for all x in any interval about 0.

Separation of variables

Problem 2. Consider the partial differential equation $tu_{xx} + 5u_t = 0$.

(a) (3 points) Look for a solution in the form u(x,t) = X(x)T(t) and transform the equation into a system of two ODEs depending on separation constant λ .

(b)	(4 points) Solve the equations that you obtained in (a) and give the family of solutions (depending on $\lambda > 0$) of the initial equation produced this way.
	Heat equation on $\mathbb R$ and $\mathbb R^2$
P	Problem 3.
(a)	(3 points) Check that the function $u(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ satisfies the heat equation $u_t=u_{xx}$ for $x\in\mathbb{R}$ and $t>0$.



Hint: use the product rule and the result from part (a).

Remark. Note that you may have recognized that u(x,t) is the density function for the normal (Gaussian) distribution from probability theory. This connection is further emphasized by the fact that Gaussian distributions are closely related to Brownian motion, a stochastic process that models the random movement of particles suspended in a fluid. In particular, the density function u(x,t) describes the probability distribution of the position of a particle undergoing Brownian motion at time t, starting from an initial position of x=0. This connection highlights the deep interplay between probability theory and the study of diffusion processes, such as heat conduction, which are fundamental in various scientific disciplines.