Summer 2020

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MATH 4581: Statistics and Stochastic Processes

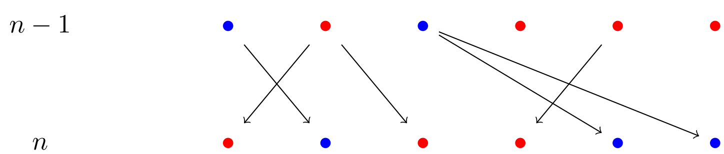
Take-Home Exam

Wright-Fisher model

Consider N haploid individuals, each carrying 1 copy of a specific genetic locus (a location of interest in the genome). Each individual (= gene) can be of two types (= alleles), A or a, which correspond to two different pieces of genetic information at the same locus.

Suppose that at each time unit each individual randomly and uniformly chooses another individual (possibly itself) from the population and adopts its type ("parallel updating"). This is called resampling, and is a form of random reproduction. Suppose that all individuals update independently from each other and independently of how they updated at previous times. We are interested in the evolution of the following quantity: X_n = number of A's at time n.

Example 1. Let $N = 6$ and denote an individual of type A by $\color{red}{\bullet}$, while an individual of type a by $\color{blue}{\bullet}$. We would like to compute the probability that there will be 3 individuals of type A at time n given that there are 4 individuals of type A at time $(n-1)$, i.e. $P(X_n = 3 \mid X_{n-1} = 4)$. First, there are $\binom{6}{3}$ possible choices of type A individuals in the n^{th} time frame. Next, the probability of choosing an individual of type A for the update is $\frac{4}{6} = \frac{2}{3}$ and $\frac{1}{3}$ of type a, hence, $P(X_n = 3 \mid X_{n-1} = 4) = \binom{6}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3$.



Problem 1 We consider the case $N = 3$.

(a) [5 **pts**] Let A_i be the state with i individuals of type A (here $i = 0, 1, 2, 3$). Show that the process $\{X_0, X_1, X_2, \ldots,\}$ is a Markov chain[1] and list the absorbing states.

Solution:

Absorbing states: A_0 and A_3. The Markov property $P(X_n = k_n \mid X_{n-1} = k_{n-1}, \ldots, X_0 = k_0) = P(X_n = k_n \mid X_{n-1} = k_{n-1})$ by definition.

(b) [5 **pts**] Compute the transition probabilities.

Solution:

The transition probability from state k to state s is given by $\binom{3}{k}\left(\frac{k}{3}\right)^s\left(\frac{3-k}{3}\right)^{3-s}$ (see the argument in Example 1).

from/to	A_0	A_1	A_2	A_3
A_0	1	0	0	0
A_1	8/27	4/9	2/9	1/27
A_2	1/27	2/9	4/9	8/27
A_3	0	0	0	1

Definition 2. A discrete-time stochastic process $\{Y_0, Y_1, Y_2, \ldots\}$ is called a *martingale* if $\mathbb{E}(Y_n \mid Y_{n-1}, \ldots, Y_0) = Y_{n-1}$ for any time $n \geq 0$. In other words, the conditional expected value of the next observation, given all the past observations, is equal to the most recent observation.

[1] Check the Markov property

$(c)^*$ [10 **pts**] Check that $\{X_0, X_1, X_2, \ldots\}$ is a martingale. [2]

Solution:

Markov property implies that $\mathbb{E}(X_n \mid X_{n-1}, \ldots, X_0) = \mathbb{E}(X_n \mid X_{n-1})$. Let $X_{n-1} = k$, then $\mathbb{E}(X_n \mid X_{n-1} = k) = \sum_{s=0}^{3} P(X_n = s \mid X_{n-1} = k)s = \sum_{s=0}^{3} \binom{3}{s} \left(\frac{k}{3}\right)^s \left(\frac{3-k}{3}\right)^{3-s} s = 3 \cdot \frac{k}{3} \cdot \left(\left(\frac{3-k}{3}\right)^2 + 2\left(\frac{k}{3}\right)\left(\frac{3-k}{3}\right) + \left(\frac{k}{3}\right)^2\right) = \left(\frac{k}{3} + \frac{3-k}{3}\right)^2 = k.$

(d) [5 **pts**] Assume we start in state A_j. What is the probability that population A takes over?

Solution:

The transition matrix is $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{9} & \frac{1}{27} \\ \frac{1}{27} & \frac{2}{9} & \frac{4}{9} & \frac{8}{27} \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Canonical form of P is $\begin{pmatrix} \frac{4}{9} & \frac{2}{9} & \frac{8}{27} & \frac{1}{27} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{27} & \frac{8}{27} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$Q = \begin{pmatrix} \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{4}{9} \end{pmatrix}$ and $R = \begin{pmatrix} \frac{8}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{8}{27} \end{pmatrix}$.

The fundamental matrix is $N = (I - Q)^{-1} = \begin{pmatrix} \frac{15}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{15}{7} \end{pmatrix}$.

$NR = \begin{pmatrix} \frac{126}{189} & \frac{63}{189} \\ \frac{63}{189} & \frac{126}{189} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, hence,

j	p_j
0	0
1	1/3
2	2/3
3	1

(e) [5 **pts**] What is the average number of updates until the population is stabilized?

Solution:

Absorbing states are stable, for transient states the answer is given by the sum of the elements in the corresponding row of matrix N.

j	# of updates
0	0
1	3
2	3
3	0

Optional stopping theorem

Definition 3. Given a stochastic process $\{X_0, X_1, \ldots, \}$, a non-negative integer-valued random variable τ is called a *stopping time* if for every integer $k \geq 0$, the event $\tau \leq k$ depends only on the events $X_0, X_1, \ldots, X_k$.

Example 4. Consider a gambler playing roulette with a typical house edge, starting with \$100 and betting \$1 on red in each game:

[2]**Hint:** Markov property implies that $\mathbb{E}(X_n \mid X_{n-1}, \ldots, X_0) = \mathbb{E}(X_n \mid X_{n-1})$. Set $X_{n-1} = k$ and verify that $\mathbb{E}(X_n \mid X_{n-1}) = k$ as well.

(1) Playing exactly five games corresponds to $\tau = 5$, and is a stopping time.

(2) Playing until he either runs out of money or has played 500 games is a stopping time.

(3) Playing until he is the maximum amount ahead he will ever be is not a stopping time, as it requires information about the future as well as the present and past.

(4) Playing until he doubles his money (borrowing if necessary) is not a stopping time, as there is a positive probability that he will never double his money.

The **optional stopping theorem** (or Doob's optional sampling theorem) says that, under certain conditions, the expected value of a martingale at a stopping time is equal to its initial expected value. Since martingales can be used to model the wealth of a gambler participating in a fair game, the optional stopping theorem says that, on average, nothing can be gained by stopping play based on the information obtainable so far (i.e., without looking into the future).

Theorem 5 (optional stopping theorem)**.** *Suppose that the stochastic process $\{X_0, X_1, \dots,\}$ is a martingale and τ is a stopping time such that there exists a constant $c > 0$ with $P(\tau \leq c) = 1$. Then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.*

Problem 2 Suppose that Ann plays the following game. At each turn the dealer throws an unbiased coin, and if the outcome is heads Ann wins \$10, while if it is tails she loses \$10. Assume that each coin toss is independent and Ann starts with \$100. What is the probability that she wins \$50 before running out of money?

$(a)^*$ [15 **pts**] Let τ be the turn when Ann either has \$150 or runs out of money. Show that τ is a stopping time and satisfies the assumption of optional stopping theorem.

Solution:

We will establish that $P(\tau > c) = 0$. Notice that for a path $\mathcal{X} = \{p_0, p_1, \dots,\}$ one has $\tau(\mathcal{X}) > c$ iff none of $\{p_0, p_1, \dots, p_c\}$ are located in the region shaded in green on the picture below. It suffices to associate an infinite collection of paths $\mathcal{C}_{\mathcal{X}} := \{\mathcal{X}_1, \mathcal{X}_2, \dots,\}$ to every $\mathcal{X}$ with $\tau(\mathcal{X}) > c$, s.t.

(a) for every $\mathcal{X}_i$, $\tau(\mathcal{X}_i) \leq c$.

(b) $\mathcal{X}_i \neq \mathcal{X}_j$ for any $i \neq j$,

(c) $\mathcal{X}_i \neq \mathcal{Y}_j$ for any i, j, where $\mathcal{Y}_j$ is some path from the collection associated to $\mathcal{Y}$.

It remains to find a collection as above. One such collection consists of $\mathcal{X}_i = \{100, 110, 120, \dots, 150(i+1), 150(i+1) + p_1, 150(i+1)p_2, \dots\}$.

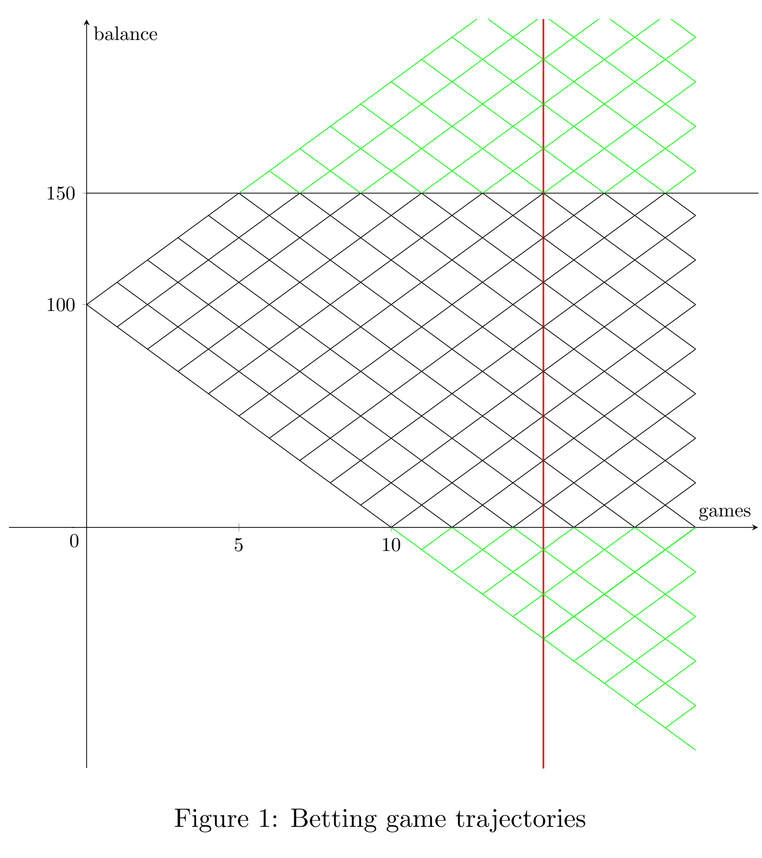


Figure 1: Betting game trajectories

(*b*) [10 **pts**] Apply the optional stopping theorem to answer the question.

Solution:

The optional stopping theorem asserts that $\mathbb{E}[X_\tau] = \mathbb{E}[X_0] = 100$. Thus we get an equation $150p + 0(1-p) = 100$, where p is the probability that Ann wins \$50. This allows to conclude that $p = \frac{2}{3}$.

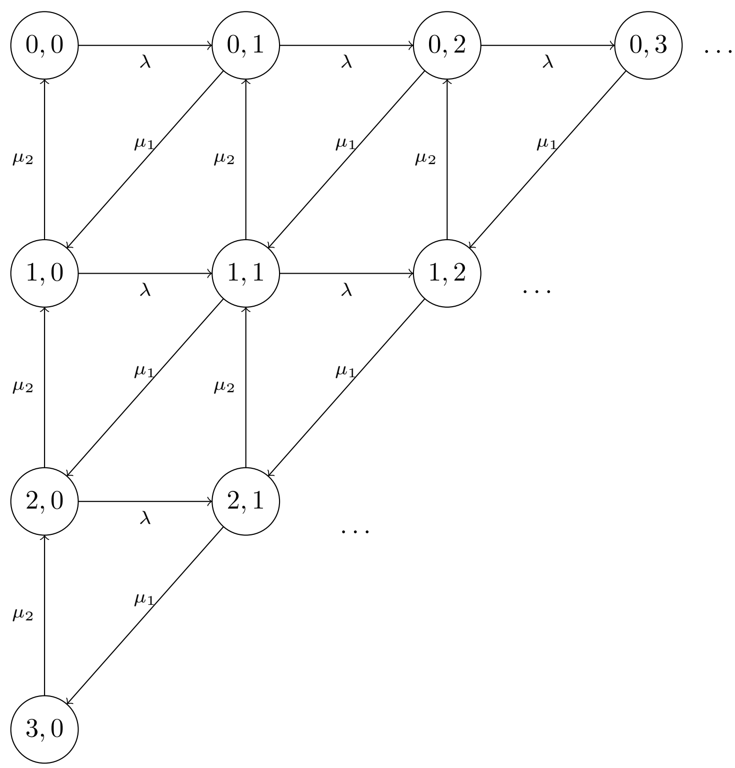
Problem 3[10 **pts**] Consider the Wright-Fisher model and let τ be the time when either population A or a takes over. Show that τ is a stopping time, but does not satisfy the assumption of optional stopping theorem.

Solution:
Notice that $X_\tau = X_{\tau+1} = \ldots$, so X_τ determines the remaining states and is independent of them.
For any $c > 0$ there are only finitely many sequences $\mathcal{X} = \{X_0, X_1, \ldots, \}$ with $\tau(\mathcal{X}) \le c$, since $X_c = X_{c+1} = \ldots$ are either all equal to 0 or 1 by definition of τ. Let us denote this number by k. Since the number of all possible sequences $\mathcal{Y} = \{X_0, X_1, \ldots, \}$ is infinite, we have $P(\tau \le c) = \frac{k}{\infty} = 0$. Hence, τ does not satisfy the assumption of optional stopping theorem.

Queues

Theorem 6. *(Burke) Consider an $M/M/1$, $M/M/c$ or $M/M/\infty$ queue with arrival rate λ. In the steady state, the departures from the system form a Poisson process with rate λ, independently of μ (so long as $\mu > \lambda$). Furthermore, at time t the number of customers in the queue is independent of the departure process prior to time t.*

Problem 4 Consider a two-stage tandem network composed of two nodes with service rates μ_1 and μ_2, respectively. This means that a customer joins the end of the wait line at the first counter upon arrival and moves to the end of the wait line for the second counter after being served at the first. The external arrival rate is λ ($\lambda < \mu_i, i = 1, 2$) and the arrival process is Poisson. Assume that the service times at each node are exponentially distributed and mutually independent, and independent of the arrival processes, the total number of visitors is not bounded.



(a) [5 **pts**] Give an example of a real life occurence of such a queueing system.

(b) [15 **pts**] Show that the steady-state probabilities are given by $p_{nm} = (1-\rho_1)\rho_1^n(1-\rho_2)\rho_2^m$, where $\rho_1 = \frac{\lambda}{\mu_1}$ and $\rho_2 = \frac{\lambda}{\mu_2}$. [3]

(c)* [15 **pts**] Give a generalization of (b) to the case of n counters.

Black-Scholes equation

Problem 5 Recall that the solution of Black Scholes equation corresponding to the European Call option is $C_E(S,t) = S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2)$, where $d_1 = \frac{\ell n(S/E)+(r+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ell n(S/E)+(r-\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $\Phi(d) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{d} e^{-\frac{x^2}{2}}dx$.

(a) [5 **pts**] Find the solution of Black Scholes equation corresponding to the European Put option.[4]

Solution:

$P_E(S,t) = -S + C_E(S,t) + Ee^{-r(T-t)} = -S + S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2) + Ee^{-r(T-t)} = S(\Phi(d_1)-1) + Ee^{-r(T-t)}(1-\Phi(d_2))$.

(b) [10 **pts**] Analyze the solution.

(1) What is the value of the put if the market is wild; it is modeled by $\sigma \to \infty$?

Solution:

Here $d_1 \to +\infty$, while $d_2 \to -\infty$ and $\Phi(d_1) \to 1$, with $\Phi(d_2) \to 0$ giving $P_E(S,t) \to Ee^{-r(T-t)}$.

(2) What is the value of the put if the market is stable; it is modeled by $\sigma \to 0$?

Solution:

Case 1 : $\ell n(S/E) + r(T-t) > 0$. We have $d_1, d_2 \to +\infty$ and $\Phi(d_1), \Phi(d_2) \to 1$, giving $P_E(S,t) \to 0$.
Case 2 : $\ell n(S/E) + r(T-t) < 0$. We have $d_1, d_2 \to -\infty$ and $\Phi(d_1), \Phi(d_2) \to 0$, giving $P_E(S,t) \to -S + Ee^{-r(T-t)}$.
Case 3 : $\ell n(S/E) + r(T-t) = 0$. We have $d_1, d_2 \to 0$ and $\Phi(d_1), \Phi(d_2) \to 1/2$, giving $P_E(S,t) \to -S/2 + Ee^{-r(T-t)}/2$.

(3) What is the value of the put if $S \to \infty$?

Solution:

We have $\ell n(S/E) \to +\infty$, hence, $d_1, d_2 \to +\infty$ and $\Phi(d_1), \Phi(d_2) \to 1$, giving $P_E(S,t) \to -S + S + 0 = 0$.

(4) What is the value of the put if $S \to 0$?

Solution:

In this case $\ell n(S/E) \to -\infty$, hence, $d_1, d_2 \to -\infty$ and $\Phi(d_1), \Phi(d_2) \to 0$, giving $P_E(S,t) \to Ee^{-r(T-t)}$.

(5) What is the value of the put if $t \to T^-$?

Solution:

Case 1 : $S > E$. We have $\ell n(S/E) > 0$, hence, $d_1, d_2 \to +\infty$ and $\Phi(d_1), \Phi(d_2) \to 1$, giving $P_E(S,t) \to -S + S + 0 = 0$.
Case 2 : $S < E$. We have $\ell n(S/E) < 0$, hence, $d_1, d_2 \to -\infty$ and $\Phi(d_1), \Phi(d_2) \to 0$, giving $P_E(S,t) \to -S + E$.
Case 3 : $S = E$. We have $\ell n(S/E) = 0$, hence, $d_1, d_2 \to 0$ and $\Phi(d_1), \Phi(d_2) \to 1/2$, giving $P_E(S,t) \to -S + S/2 + E/2 = 0$.

We conclude that $P_E(S,T) = max(0, E-S)$.

Feynman's checkers

Definition 7. A *checker path* is a finite sequence of integer points in the plane such that the vector from each point (except the last one) to the next one equals either $(1,1)$ or $(-1,1)$. A turn is a point of the path (not the first and not the last one) such that the vectors from the point to the next and to the previous ones are orthogonal. To a path s terminating at (x,t) we associate a vector $\vec{a}(s)$ with length $|\vec{a}(s)| := 2^{\frac{1-t}{2}}$ and angle formed with the x-axis equal to $\frac{\pi}{2}(1 - turns(s))$. Next we define the vector $\vec{a}(x,t)$ as $\vec{a}(x,t) = \sum\limits_{s}\vec{a}(s)$, where the sum over all checker paths s from $(0,0)$ to (x,t) with the first step to $(1,1)$.

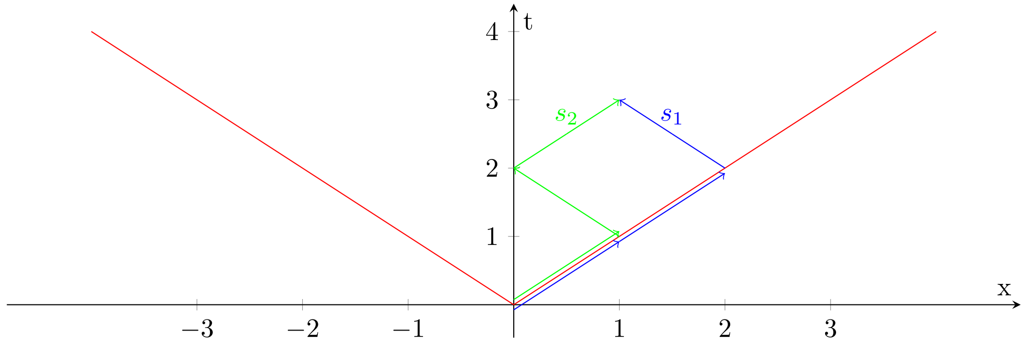
Denote

$P(x,t) := |\vec{a}(x,t)|^2$,
$a_1(x,t) := x -$ coordinate of $a(x,t)$,
$a_2(x,t) := t -$ coordinate of $a(x,t)$.

[3]**Hint:** Using that the queue to the first counter is an $M/M/1$ system with rates λ and μ_1, compute the steady state probability $\widehat{p}_n$. Use Burke's theorem to find the departure rate for this system. Notice that the arrival rate for the second system (which is also $M/M/1$) is equal to the departure rate for the first. Now find $\widetilde{p}_m$, the steady-state probability for the second queue. Show that Burke's theorem allows to conclude $p_{nm} = \widehat{p}_n\widetilde{p}_m$ and get the desired result.

[4]**Hint:** use the put–call parity equation $P_E(S,t) + S = C_E(S,t) + Ee^{-r(T-t)}$.

Example 8. Consider the point $(x,t)=(1,3)$. Then we have two paths s_1, s_2 emerging at $(0,0)$ going to $(1,1)$ on the first step and terminating at $(1,3)$. We find $turns(s_1)=1$, $turns(s_2)=2$ and $t=3$ (so $2^{\frac{1-t}{2}}=\frac{1}{2}$) implies the angle of $\vec{a}(s_1)$ with the $x-$axis is 0 and the angle of $\vec{a}(s_2)$ with the $x-$axis is $-\frac{\pi}{2}$, while $|\vec{a}(s_1)|=|\vec{a}(s_2)|=\frac{1}{2}$. In other words, $\vec{a}(s_1)=(\frac{1}{2},0)$ and $\vec{a}(s_2)=(0,-\frac{1}{2})$. Hence, $\vec{a}(x,t)=\vec{a}(s_1)+\vec{a}(s_2)=(\frac{1}{2},-\frac{1}{2})$, $a_1(x,t)=\frac{1}{2}, a_2(x,t)=-\frac{1}{2}$ and $P(x,t)=\frac{1}{2}$.



Remark 9. Here the x- and t-coordinates are interpreted as the electron position and time, respectively. We work in the natural system of units, where the speed of light, the Plank and the Boltzmann constants are equal to 1. Thus the lines $x=\pm t$ represent motion with the speed of light. Any checker path lies above both lines, i.e in the light cone, which means agreement with relativity: the speed of electron cannot exceed the speed of light.

The square of the length of vector $a(x,t)$ is called the probability to find an electron (with spin $=\frac{1}{2}$) in the square (x,t), if it was emitted from $(0,0)$.

Problem 6

(*a*) [2 **pts**] Find $P(1,1)$ and $P(0,2)$.

Solution:

$P(1,1)=1$ and $P(0,2)=1/2$.

(*b*) [3 **pts**] Find $P(n,n)$ and $P(-n,n+2)$ for $n\geq 1$.

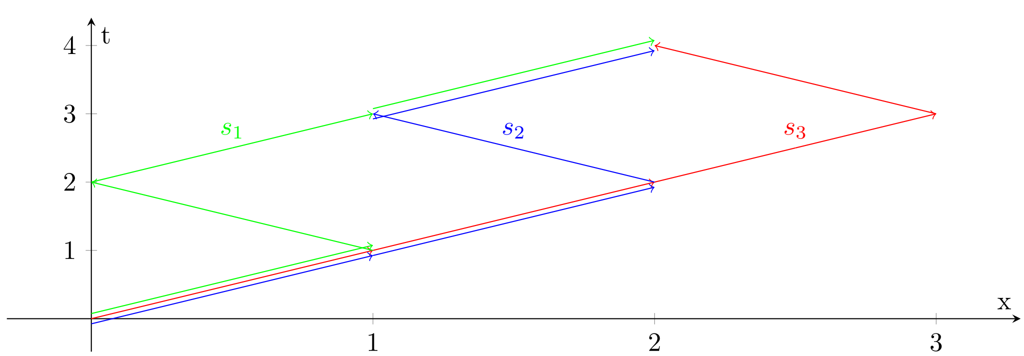
Solution:

Since in both cases there is a single path, we have $P(n,n)=\frac{1}{2^{n-1}}$ and $P(-n,n+2)=\frac{1}{2^{n+1}}$.

(*c*) [5 **pts**] Find $P(2,4)$.

Solution:

We have three paths s_1,s_2,s_3 emerging at $(0,0)$ going to $(1,1)$ on the first step and terminating at $(2,4)$. We find $turns(s_1)=turns(s_3)=1$, $turns(s_2)=2$ and $t=4$ (so $2^{\frac{1-t}{2}}=\frac{1}{2\sqrt{2}}$) implies the angle of $\vec{a}(s_1)$ and $\vec{a}(s_3)$ with the $x-$axis is 0, while the angle of $\vec{a}(s_2)$ with the $x-$axis is $-\frac{\pi}{2}$. Now $|\vec{a}(s_1)|=|\vec{a}(s_2)|=|\vec{a}(s_3)|=\frac{1}{2\sqrt{2}}$. In other words, $\vec{a}(s_1)=\vec{a}(s_3)=\frac{1}{2\sqrt{2}}(1,0)$ and $\vec{a}(s_2)=\frac{1}{2\sqrt{2}}(0,-1)$. Hence, $\vec{a}(x,t)=\vec{a}(s_1)+\vec{a}(s_2)+\vec{a}(s_3)=\frac{1}{2\sqrt{2}}(2,-1)$ and $P(x,t)=\frac{5}{8}$.



Problem 7[10 **pts**] Let m,n be integers, s. t. $n\geq m$ and $m+n\geq 2$. How many different checker paths are there from $(0,0)$ to (m,n)?

Solution:

After the first step, which is always an addition of $(1,1)$, let k denote the number of times the vector $(1,1)$ appears in our path. Then the vector $(1,1)$ must appear $n-k-1$ times. As the x- coordinate of the terminal point is m, we get the equality $k-(n-k-1)=m-1$, which allows to find $k=\frac{m+n-2}{2}$, so the answer is the binomial coefficient $\binom{n-1}{\frac{m+n-2}{2}}$.

Problem 8

$(a)^*$ [10 **pts**] Show that $a_1(x,t)=\frac{1}{\sqrt{2}}(a_1(x+1,t-1)+a_2(x+1,t-1))$ and $a_2(x,t)=\frac{1}{\sqrt{2}}(-a_1(x-1,t-1)+a_2(x-1,t-1))$.

Solution:

We start by observing that if a path s ends with the $(1,1)$ vector, then the number $turns(s)$ is even, thus the vector $\vec{a}(s)$ lies on the $t-$axis. Similarly, if a path s ends with the $(-1,1)$ vector, then the number $turns(s)$ is odd and the vector $\vec{a}(s)$ lies on the $x-$axis. It follows that only the paths containing the point $(x+1,t-1)$ contribute to $a_1(x,t)$. Let s be such a path, then it contains one of the two points $(x+2,t-2)$ or $(x,t-2)$. In the former case it has the last two segments both $(-1,1)$ and contributes to $a_1(x+1,t-1)$, while in the latter it ends with segment $(1,1)$ followed by $(-1,1)$ and contributes to $a_2(x+1,t-1)$ prior to 'turning to negative $x-$ coordinate of $a(x,t)$' on the last step. The first equality in the assertion follows and the second can be verified analogously.

(b) [5 **pts**] Let $t\geq 0$. Using the result in (a) show that $\sum_{x\in\mathbb{Z}} P(x,t)=1$.

Solution:

We argue by induction, the base case $t=1$ follows from (a). Now $\sum_{x\in\mathbb{Z}} P(x,t)=\sum_{x\in\mathbb{Z}}(a_1(x,t)^2+a_2(x,t)^2)=\sum_{x\in\mathbb{Z}}\frac{1}{2}(a_1(x+1,t-1)+a_2(x+1,t-1))^2+\frac{1}{2}(-a_1(x-1,t-1)+a_2(x-1,t-1))^2=\frac{1}{2}\sum_{x\in\mathbb{Z}}(a_1(x+1,t-1)^2+a_2(x+1,t-1)^2+a_1(x-1,t-1)^2+a_2(x-1,t-1)^2)+\sum_{x\in\mathbb{Z}}(a_1(x+1,t-1)a_2(x+1,t-1)-a_1(x-1,t-1)a_2(x-1,t-1))=\frac{1}{2}(\sum_{x'\in\mathbb{Z}}(a_1(x',t-1)^2+a_2(x',t-1)^2)+\sum_{x''\in\mathbb{Z}}(a_1(x'',t-1)^2+a_2(x'',t-1)^2))+\sum_{x'\in\mathbb{Z}}a_1(x',t-1)a_2(x',t-1)-\sum_{x''\in\mathbb{Z}}a_1(x'',t-1)a_2(x'',t-1)=\frac{1}{2}\cdot 2+0=1$.

We have used that $\sum_{x'\in\mathbb{Z}}(a_1(x',t-1)^2+a_2(x',t-1)^2)=1$ by inductive assumption for $n-1$.