MATH 149A: Probability and Mathematical Statistics

Homework 2

Problem 1. (6 points) Suppose we have a random experiment with sample space S and two events A and B with probabilities $P(A) = 0.4$ and $P(B) = 0.3$. Determine whether the following statements are true or false, and justify your answers.

(a) (2 points) For any two events A and B, we have $P(A \cap B) = P(A) \cdot P(B)$.

(b) (2 points) If events A and B are mutually exclusive, then $P(A \cap B) = 0$.

(c) (2 points) If events A and B are independent, then $P(A \mid B) = P(A)$.

Problem 2. (9 points) Let X be a discrete random variable with the following probability mass function:

$$P(X = k) = \begin{cases} 0.2 & \text{if } k = 1 \\ 0.3 & \text{if } k = 2 \\ 0.5 & \text{if } k = 3. \end{cases}$$

Calculate the following probabilities.

(a) (3 points) $P(X = 2)$

(b) (3 points) $P(X < 3)$

(c) (3 points) $P(X \geq 2)$

Problem 3. (6 points) Let X be a continuous random variable with the following probability density function:

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the following probabilities.

(a) (2 points) $P(0.2 < X < 0.6)$

(b) (2 points) $P(X > 0.8)$

(c) (2 points) $P(X = 0.35)$

Problem 4. (12 points) Let X be a continuous random variable with the following cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{4} & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2. \end{cases}$$

Calculate the following probabilities.

(a) (3 points) $P(X < 1)$

(b) (3 points) $P(1 < X < 2)$

(c) (3 points) $P(X > 3)$

(d) (3 points) Find the probability density function $f_X(x)$.

Problem 5. (6 points) Let X be a continuous random variable with PDF $f_X(x) = 0.02x$ for $0 \le x \le 10$. Define a new random variable $Y = X^3 - 2$.

(a) (3 points) Find the cumulative distribution function $F_Y(y)$ of the random variable Y.

(b) (3 points) Use your result from part (a) to compute the probability that Y falls in the interval $[2, 5]$, i.e., compute $P(2 \le Y \le 5)$.

Problem 6. (6 points) Suppose X has a uniform distribution on the interval $[0, 1]$. Let $Y = 2X - 1$. Find the cumulative distribution function (CDF) $F_Y(y)$ of the random variable Y.

Problem 7. (8 points) Suppose we have a random variable X with the following PMF:

$$P(X = x) = \begin{cases} \frac{1}{6} & \text{for } x = \frac{\pi}{6} \\ \frac{1}{3} & \text{for } x = \frac{\pi}{3} \\ \frac{1}{4} & \text{for } x = \frac{\pi}{2} \\ \frac{1}{6} & \text{for } x = \frac{2\pi}{3} \\ \frac{1}{12} & \text{for } x = \frac{5\pi}{6}. \end{cases}$$

(a) (3 points) Determine the range of $Y = \sin(X)$.

(b) (3 points) Find the probability mass function of Y.

(c) (2 points) Calculate $P(0 < Y \leq 1)$.