Marked Length Spectrum Rigidity of Surfaces

by

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Abstract

This dissertation was written for the completion of the "Project" stream of the MSc program in Mathematics and Statistics at Queen's. The goal of this research project was to learn and redo the proof of a Theorem in a recently published paper, "Quantitative Marked Length Spectrum Rigidity" by Karen Butt, as well as learn the surrounding mathematical material. Roughly speaking, marked length spectrum rigidity is a conjecture claiming that the *lengths of certain classes of closed loops* (we call this the marked length spectrum) are enough to *determine the geometry of a space* (that would be the rigidity); it was shown to be true for smooth surfaces of negative curvature independently by Jean-Pierre Otal and Christopher B. Croke in 1990. The main result in this document is a slightly stronger version of Butt's Theorem 1.1, a sort of "approximate" marked length spectrum rigidity, again on smooth surfaces of negative curvature. In lay terms, the idea here is that if the marked length spectrum is known to good approximation, then the geometry is known to good approximation. The proof of the version in this document is essentially an extension of the argument made by Butt; the main idea is the same. It is necessary to use a stronger version of the classical Otal-Croke Theorem on marked length spectrum rigidity, extended to surfaces whose "negative curvature" cannot be defined in the classical sense. Although such a theorem appears implicitly in Butt's paper, we provide a concrete statement. Much of the work in this document lies in building up background information and smaller results necessary for proving this stronger version of the Otal-Croke Theorem on classical marked length spectrum rigidity.

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Contents

Chapter 1

Introduction

Let (M, g) be a Riemannian manifold of negative curvature. Fix a base point x_0 for the fundamental group $\pi_1(M) := \pi_1(M, x_0)$, and denote by $\langle \pi_1(M) \rangle$ its conjugacy classes, which can be identified with the set of free homotopy classes of loops $[\mathbb{S}^1, M]$. The **marked length spectrum** (with respect to g) is the function $\mathcal{L}_g \colon \langle \pi_1(M) \rangle \cong [\mathbb{S}^1, M] \to \mathbb{R}_{\geq 0}$ taking the length of the unique geodesic representative of each free homotopy class:

$$\mathcal{L}_g([\gamma]) = l_g(\gamma_g)$$

where l_g is the length-functional according to g. The negative curvature guarantees that the marked length spectrum is well-defined, as each γ_g exists and is unique with respect to the metric g (see [do Carmo, 1992]).

There is a notion of **marked length spectrum rigidity**, the basic statement of which is that the marked length spectrum specifies the isometry type. This was first conjectured by [Burns and Katok, 1985], then proved in the case of closed surfaces of negative curvature by [Otal, 1990], and shortly after proved by [Croke, 1990] in slightly more generality.[1] One can further wonder to what extent rigidity holds when the marked length spectra are only *approximately* equal. If the marked length spectra are close, are the Riemannian metrics close? Karen Butt provides some answers to this question in [Butt, 2022].

The main goal of this report is to provide a slightly generalized version of [Butt, 2022, Theorem 1.1], which is in the setting of closed surfaces. The basic arguments are the same, following closely ideas from the original proof of Otal, the lecture notes [Wilkinson, 2014], and of course the paper of Butt. Along the way, we also extract a version of the classical marked length spectrum rigidity generalized to locally CAT$(-k)$ spaces, in which the usual notion of (Riemannian) curvature is not necessarily well-defined. In fact, much of the work in this document lies in the

[1] Croke proves that conjugacy of geodesic flows is equivalent to specifying the isometry type for closed surfaces of non-positive curvature. In the strictly negative case, this is equivalent to specification of the marked length spectrum, as stated in [Croke, 1990].

exposition and proof of this latter theorem for locally CAT$(-k)$ spaces. Moreover, the proof of the main theorem is short once the necessary intermediate results are assumed. We thus state the main theorem and its proof here in the introduction.

1.1 Main Theorem

Again, let (M, g) be a Riemannian manifold of negative curvature. Since M is a manifold, its fundamental group $\pi_1(M)$ is countable. We may thus fix an enumeration on $\pi_1(M)$, which passes to an enumeration on $\langle \pi_1(M) \rangle$:

$$\langle \pi_1(M) \rangle = \{1\} \cup \{[\gamma^n] \mid n \in \mathbb{N}\}.$$

Here we may consider $\gamma^0 = 1$ to be the trivial element of $\pi_1(M)$, and that the enumeration is such that $[\gamma^n] \neq [\gamma^{n'}]$ if and only if $n \neq n'$. For $N \in \mathbb{N}$, we denote by $\mathcal{L}_{g,N}$ the first N elements of the marked length spectrum of g:

$$\mathcal{L}_{g,N} = \{l_g(\gamma_g^1), ..., l_g(\gamma_g^N)\}.$$

Here γ_g^n denotes the unique geodesic representative of the n-th element of $\langle \pi_1(M) \rangle$, subject to the enumeration specified above. Given an alternative Riemannian metric h, we further denote:

$$\frac{\mathcal{L}_{g,N}}{\mathcal{L}_{h,N}} := \left\{ \frac{l_g(\gamma_g^1)}{l_h(\gamma_h^1)}, ..., \frac{l_g(\gamma_g^N)}{l_h(\gamma_h^N)} \right\}.$$

Now, let $k, K, v_0, D_0 > 0$ and call $\mathcal{C}(2, k, K, v_0, D_0)$ the space of Riemannian surfaces of bounded geometry; these are (compact) Riemannian manifolds of 2 dimensions, with sectional curvature in the interval $[-K, -k]$, volume bounded below by v_0, and diameter bounded above by D_0.

Theorem 1.1. *Let $k, K, v_0, D_0 > 0$. Fix $L > 1$. Then $\exists\ \varepsilon = \varepsilon(L, k, K, v_0, D_0) > 0$, $\exists N \in \mathbb{N}$ such that for any $(M, g), (M, h) \in \mathcal{C}(2, k, K, v_0, D_0)$ satisfying the property*

$$1 - \varepsilon \leq \frac{\mathcal{L}_{g,N}}{\mathcal{L}_{h,N}} \leq 1 + \varepsilon \tag{$*$}$$

there exists a corresponding L-Bilipschitz map $f\colon (M, g) \to (M, h)$[2]*.*

This theorem is our slight generalization of [Butt, 2022, Theorem 1.1]. It is a stronger statement since the amount of information required from the marked length spectrum is minimized. Restated with more intuitive terminology: given a desired L-bound on the closeness of the metrics, there exists an ε-bound that only the first N elements of the marked length spectrum are required to satisfy in order for an

[2] We mean that f is L-Bilipshitz with respect to the distance functions d_g and d_h (see Appendix C).

L-Lipschitz map $f\colon (M, d_g) \to (M, d_h)$ to exist. It should be clear that [Butt, 2022, Theorem 1.1] is the special case of the above obtained by choosing $N = \infty$.

The proof of Theorem 1.1 follows the strategy of [Butt, 2022], relying on two main facts that can be stated briefly, but which will require the remaining chapters to prove. The first regards the use of the Gromov-Greene-Wu[3] compactness theorem ([Greene and Wu, 1988]). This compactness-theorem states that for any sequence $\{(M, g_k)\}_k \subset \mathcal{C}(2, k, K, v_0, D_0)$, there exists a subsequence $(M, g_{n_k}) \to (M, g)$, converging in the Lipschitz-distance to a C^∞-manifold M with $C^{1,\alpha}$ Riemannian metric g ($\alpha \in (0, 1)$ arbitrary). Further, the space (M, g) is CAT$(-k)$ since it is a suitable limit of CAT$(-k)$ space, by [Bridson and Haefliger, 1999, Theorem II.3.9].[4] As in the case of negative curvature, every free homotopy class in a locally CAT$(-k)$ space has a unique geodesic representative. Proof of the existence of a geodesic representative follows in the usual way employing the Arzelà-Ascoli theorem for compact metric spaces (see [Bridson and Haefliger, 1999, Proposition I.3.16]). Uniqueness follows since a closed geodesic in M lifts to a geodesic segment in the universal cover $\widetilde{M}$, which is (globally) CAT$(-k)$; geodesic segments between points in globally CAT$(-k)$ spaces are always unique by [Bridson and Haefliger, 1999, Proposition 1.4]. Hence the marked length spectrum is well-defined on any locally CAT$(-k)$ space, and in particular on the limiting space (M, g).

Proposition 1.2. *Let $(M, g_N) \to (M, g)$ $(M, h_N) \to (M, h)$ be the Greene-Wu limits of some (sub)sequences in $\mathcal{C}(2, k, K, v_0, D_0)$. If $(M, g_N), (M, h_N)$ satisfy condition $(*)$ for each N, with $\varepsilon = O(1/N)$, then (M, g) and (M, h) have equal marked length spectra.*

The second fact used in the proof below is a generalization of Otal's marked length spectrum rigidity to Riemannian surfaces lacking C^2 regularity in the metric, but which satisfy other conditions. In particular, the classical theorem seen in [Otal, 1990] assumes negative sectional curvature, which is only well-defined when the metric is class C^2 or higher. A more general notion of negatively curved space is a CAT$(-k)$ space (see [Bridson and Haefliger, 1999, Chapter II.1]). Further pairing this with some sufficiently strong regularity conditions, we have the following theorem:

Theorem 1.3. *Let M be a closed smooth surface, let $k > 0$, and let g, h be C^1 Riemannian metrics having $C^{0,1}$ geodesic flow[5], and which make M a locally CAT$(-k)$ space.[6] If g, h have equal marked length spectra, then they are isometric.*

[3]This name should be appropriate since the version by Greene & Wu is a refinement of the original theorem by Gromov.

[4]In particular, following the terminology in the reference, (M, g) is a 4-point limit of $(M, g_N)_N$, hence the Theorem II.3.9 applies. This is an immediate consequence of the uniform convergence on compact sets of the distance functions $d_{g_N} \to d_g$.

[5]We mean the map $\phi_t\colon M \times \mathbb{R} \to M$.

[6]Locally CAT$(-k)$ means each point has a neighborhood which is CAT$(-k)$. The universal cover of a locally CAT$(-k)$ space, with the lifted metric, is (globally) CAT$(-k)$; this is the

This fact appears somewhat implicitly in [Butt, 2022], and its proof is essentially the same as that for the classical theorem of Otal, with some further technicalities arising from the lack of C^2 regularity in the Riemannian metric. In particular, since there is no curvature, many of the useful properties of the Liouville measure and current in negative curvature need to be generalized to the CAT($-k$) setting. There are also some minor differences between the argument seen in [Butt, 2022] and the proof presented in Chapter 4.1. In [Butt, 2022], many of the necessary results leading up to this version of rigidity are proved using the convergence properties of the Greene-Wu limits $(M, g_N) \to (M, g)$. In our case, Theorem 1.3 is treated as independent from Proposition 1.2; the proof makes no assumption on the existence of Gromov-Wu limits, relying only on the CAT($-k$) structure and the Lipschitz continuity of the geodesic flow (as well as the other regularity assumptions).

We now present a proof of Theorem 1.1 assuming that Proposition 1.2 and Theorem 1.3 are given.

Proof of Theorem 1.1. Suppose, for the sake of contradiction, that the theorem is false. That is, $\forall \varepsilon > 0$ and $\forall N \in \mathbb{N}$ (independent of each other), there exists a pair of surfaces $(M, g_{\varepsilon,N}), (M, h_{\varepsilon,N}) \in \mathcal{C}(2, k, K, v_0, D_0)$ which satisfy the property $(*)$, but which do not possess an L-Lipschitz map between them. We may thus collect sequences $\{(M, g_{\varepsilon,N})\}_{N\in\mathbb{N}}$,$\{(M, h_{\varepsilon,N})\}_{N\in\mathbb{N}}$ with $\varepsilon = 1/N$. By the Gromov-Greene-Wu Compactness Theorem ([Greene and Wu, 1988]), we may extract subsequences, say $\{g_N\}_{N\in\mathbb{N}}$, $\{h_N\}_{N\in\mathbb{N}}$, converging in the Lipschitz-distance:

$$\begin{cases} (M, g_N) \to (M, g) \\ (M, h_N) \to (M, h) \end{cases} \qquad (\text{as } N \to \infty).$$

In particular, the limiting metrics g, h are of $C^{1,\alpha}$ regularity, with convergence $g_N \to g$, $h_N \to h$ in the $C^{1,\alpha}$-norm. Moreover, these spaces are CAT($-k$), since they are suitable limits of such spaces (again see [Bridson and Haefliger, 1999, Theorem II.3.9]). Further, [Pugh, 1987, Theorem 1] shows that the geodesic flows of g and h are Lipschitz (i.e. $C^{0,1}$). By Propositions 1.2 and 1.3, $(M, g), (M, h)$ have equal marked length spectra and are hence isometric, under some map $f : (M, g) \to (M, h)$.

Recalling that M is compact, and that the distance functions d_{g_N}, d_{h_N} converge uniformly on compact sets to d_g, d_h, respectively, given any $A > 1$ there exists an integer N large enough such that $\forall p, q \in M$:

$$\begin{cases} A^{-1} d_g(p, q) \le d_{g_N}(p, q) \le A d_g(p, q) \\ A^{-1} d_h(p, q) \le d_{h_N}(p, q) \le A d_h(p, q). \end{cases}$$

In particular, we may let $A = \sqrt{L}$. Applying the above twice and using the isometry

Cartan-Hadamard theorem for CAT(0) spaces ([Bridson and Haefliger, 1999, Theorem 4.1]). Globally CAT($-k$) spaces are always contractible ([Bridson and Haefliger, 1999, Corollary 1.5]).

$f: (M, g) \to (M, h)$, we have that $\forall p, q \in M$, $\exists N \in \mathbb{N}$ such that:

$$d_{g_N}(p, q) \leq \sqrt{L} d_g(p, q) = \sqrt{L} d_h(f(p), f(q)) \leq L d_{h_N}(f(p), f(q))$$

and

$$d_{h_N}(f(p), f(q)) \leq \sqrt{L} d_h(f(p), f(q)) = \sqrt{L} d_h(p, q) \leq L d_{g_N}(p, q).$$

Thus there exists an N such that $f: (M, g_N) \to (M, h_N)$ is L-Bilipschitz. This is a contradiction. $\square$

Outline of the report: The rest of the document is devoted to proving the above two propositions used in the proof of Theorem 1.1. Chapter 2 uses results from the refinements of the Gromov compactness theorem by [Greene and Wu, 1988] and [Pugh, 1987], to prove Proposition 1.2. In Chapter 3 we construct the Liouville measure and current, and discuss their properties, adapted to the reduced regularity of the surfaces introduced by the Gromov-Greene-Wu compactness theorem. In turn, sections 4.1 and 4.2 together prove Theorem 1.3, a version of Otal's classical theorem on the marked length spectrum rigidity of surfaces generalized to locally CAT$(-k)$ spaces. Notes on background material in Riemannian geometry, Symplectic Geometry, and $C^{k,\alpha}$-regularity are relegated to the appendices; we make reference to these when we consider it appropriate.

Chapter 2

Marked Length Spectra of Gromov-Greene-Wu Limits

First recall the statement we want to prove:

Proposition 1.2. *Let $(M, g_N) \to (M, g)$ $(M, h_N) \to (M, h)$ be the Greene-Wu limits of some (sub)sequences in $\mathcal{C}(2, k, K, v_0, D_0)$. If $(M, g_N), (M, h_N)$ satisfy condition $(*)$ for each N, with $\varepsilon = O(1/N)$, then (M, g) and (M, h) have equal marked length spectra.*

There are several intermediate lemmas required to prove Proposition 1.2. The first is a result of [Pugh, 1987], claiming that the limiting metrics g, h have Lipschitz geodesic flows, and the convergence of the corresponding exponential maps is uniform on compact sets. This is equivalently stated in [Butt, 2022, Lemma 4.3] as:

Lemma 2.1. *Let ϕ_n, ϕ denote the geodesic flows on $(M, g_n), (M, g)$, respectively. Fix $T > 0$, and $K \subset TM$ compact. Then $\phi_n^t(v) \to \phi^t(v)$ uniformly for $(t, v) \in [0, T] \times K$.*

i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall (t, v) \in [0, T] \times K$:

$$\hat{d}_g(\phi_n^t v, \phi^t v) < \varepsilon.$$

Here $\hat{d}_g$ denotes the distance function on TM given by the Sasaki metric $\hat{g}$ induced by g (see Appendix A).

It is important to note that, since $d_{g_n} \to d_g$ uniformly by [Pugh, 1987], we also have $\hat{d}_{g_n} \to \hat{d}_g$ uniformly.[1] Hence the above is equivalent to the statement: $\forall \varepsilon > 0$,

[1] This is because, by the horizontal-vertical splitting $T(TM) \cong TM \oplus TM$ (see Appendix A), the length functional of the Sasaki metric is entirely specified by the metric on the base. Specifically, for a smooth segment $[a, b] \ni t \mapsto \gamma(t) = (x(t), v(t)) \in TM$, then length given by $\hat{g}$ is by definition:

$$l_{\hat{g}}(\gamma) = \int_a^b \sqrt{\left\| \frac{dx}{dt} \right\|^2 + \left\| \frac{Dv}{dt} \right\|^2} dt.$$

Convergence of $d_{g_n} \to d_g$ implies that $\| \cdot \|_{g_n} \to \| \cdot \|_g$, yielding the desired result for lengths, and hence distances.

$\exists n \in \mathbb{N}$ large enough that

$$\hat{d}_{g_n}(\phi_n^t v, \phi^t v) < \varepsilon.$$

The Anosov Closing Lemma will also be useful, which we now recall without proof. For ease of use, we restate the version for Anosov flows in the case of the geodesic flow (see [Butt, 2022, Lemma 4.7] and [Fisher and Hasselblatt, 2019, Theorem 5.3.11, Remark 5.3.13]). Note that the geodesic flow is always Anosov on T^1M for a negatively curved Riemannian manifold M (see [Katok and Hasselblatt, 1995, Theorem 17.6.2]).

Lemma 2.2 (Anosov Closing Lemma for ϕ_n)**.** *Let $\phi_n^t \colon T^1M \to T^1M$ be the geodesic flow corresponding to g_n. There exist constants $C, \varepsilon_0 > 0$ and $T_0 > 0$ such that $\forall \varepsilon < \varepsilon_0$: if $v \in T^1M$ and $\hat{d}_{g_n}(\phi_n^T v, v) < \varepsilon$ for some $T \geq T_0$, then there exists $u \in T^1M$ and T' such that $|T - T'| < \varepsilon$, $\phi_n^{T'}(u) = u$ and $\hat{d}_{g_n}(\phi_n^t u, \phi_n^t v) < C\varepsilon$ for all $t \in [0, \min(T, T')]$.*

This means that if $v \in T^1M$ is a tangent vector and its flow returns ε-close to itself at some large enough time T, then there exists a periodic orbit $t \mapsto \phi^t(u)$ of period T' (ε-close to T) which $C\varepsilon$-shadows $t \mapsto \phi_n^t(v)$ on the interval $[0, \min(T, T')]$.

These two previous lemma will imply one key fact regarding the "closeness" between different geodesic representatives of the free homotopy classes of M, given by g_n, g.

Lemma 2.3. *Let $\langle\gamma\rangle$ be a free homotopy class, and let γ_{g_n}, γ_g be the geodesic representatives (of unit speed) uniquely given by g_n, g respectively. We may write $\phi_n^t v_n = (\gamma_{g_n}(t), \dot{\gamma}_{g_n}(t))$, $\phi^t v = (\gamma_g(t), \dot{\gamma}_g(t))$ for some $v_n, v \in T^1M$. Then there exists a sequence of $v_n \in\subset T^1M$ such that for all $0 \leq t \leq l_g(\gamma_g)$, we have $\phi_n^t v_n \to \phi^t v$ in T^1M as $n \to \infty$.*[2]

Proof. Let $C, \varepsilon_0 > 0$ be the constants prescribed by the Anosov Closing Lemma applied to the geodesic flow, as stated above. Let $0 < \varepsilon < \varepsilon_0$.

Let $T = l_g(\gamma_g)$. Write $w_n = v/\|v\|_{g_n}$ as the tangent vector v renormalized with respect to g_n, and write $T_n := \|v\|_{g_n} T$. It is then clear that $\phi^{T_n}(w_n) = w_n$. Now, fix a compact neighborhood $K \subset TM$ of both v and w_n. We may also take $m \in \mathbb{N}$ large enough such that $mT, mT_n \geq T_0$. By Lemma 2.1, $\exists n \in \mathbb{N}$ large enough such that $\forall (t, u) \in [0, \max(mT, mT_n)] \times K$:

$$\hat{d}_{g_n}(\phi_n^t u, \phi^t u) < \varepsilon. \tag{2.1}$$

In particular, since $\phi^{mT_n} w_n = w_n$, we have that:

$$\hat{d}_{g_n}(\phi_n^{mT_n} w_n, w_n) < \varepsilon.$$

[2]Note here that T^1M is taken with respect to g_n, which is different for each n. The vectors in the sequence v_n do not all live in the same T^1M.

Applying the Anosov Closing Lemma (which applies since w_n is unit with respect to g_n), there exists some $u_n \in T^1M$, $T'_n \in \mathbb{R}$ such that $|mT_n - T'_n| < \varepsilon$, $\phi_n^{T'_n} u_n = u_n$ and $t \mapsto \phi_n^t u_n$ $C\varepsilon$-shadows $t \mapsto \phi_n^t w_n$; i.e. $\forall t \in [0, \min(mT_n, T'_n)]$:

$$\hat{d}_{g_n}(\phi_n^t u_n, \phi_n^t w_n) < C\varepsilon. \tag{2.2}$$

In particular this holds for all $t \in [0, T]$ (take m large enough). Moreover, since $g_n \to g$, we have that $w_n \to v$, and hence $\forall t \in [0, T]$ (or on any interval):

$$\hat{d}_{g_n}(\phi_n^t w_n, \phi_n^t v) < \varepsilon \tag{2.3}$$

again provided that n is sufficiently large. Hence, choosing $n \in \mathbb{N}$ sufficiently large, we have that that for all $t \in [0, T]$:

$$\begin{aligned} \hat{d}_{g_n}(\phi_n^t u_n, \phi^t v) &\leq \hat{d}_{g_n}(\phi_n^t u_n, \phi_n^t w_n) + \hat{d}_{g_n}(\phi_n^t w_n, \phi_n^t v) + \hat{d}_{g_n}(\phi_n^t v, \phi^t v) \\ &< C\varepsilon + \varepsilon + \varepsilon = (2 + C)\varepsilon \end{aligned}$$

where we have used the triangle inequality in the first line, and the second line follows from (2.1), (2.2) and (2.3) above. Since $\phi_n^t u_n$, $\phi_n^t v_n$ are both unit-speed geodesic representatives (under g_n) of the free homotopy class $\langle \gamma \rangle$, which must be unique in negative curvature, we have that the images of their orbits are identical. Hence, one can choose $v_n = u_n$. This completes the proof. □

We present just one more lemma in this chapter, a direct consequence of the above:

Lemma 2.4. *In the context of Lemma 2.3, we have that $l_{g_n}(\gamma_{g_n}) \to l_g(\gamma_g)$ as $n \to \infty$.*

Proof. This result follows by Lemma 2.3 and the fact that the distance functions $d_{g_n} \to d_g$, $d_{h_n} \to d_h$ converge uniformly on compact sets ([Pugh, 1987]).

Consider the points $\gamma_{g_n}(0)$ and $\gamma_{g_n}(T)$, where $T = l_g(\gamma_g)$. Using the triangle inequality, since $\gamma_g(0) = \gamma_g(T)$:

$$\begin{aligned} d_{g_n}(\gamma_{g_n}(0), \gamma_{g_n}(T)) &\leq d_{g_n}(\gamma_{g_n}(0), \gamma_g(0)) + d_{g_n}(\gamma_g(T), \gamma_{g_n}(T)) \\ &\leq \hat{d}_{g_n}(\phi_n^0 v_n, \phi^0 v) + \hat{d}_{g_n}(\phi^T v, \phi_n^T v_n). \end{aligned}$$

Applying Lemma 2.3, $\phi_n^t v_n \to \phi^t v$ as $n \to \infty$, the above implies:

$$d_{g_n}(\gamma_{g_n}(0), \gamma_{g_n}(T)) \longrightarrow 0 \text{ as } n \to \infty.$$

Now suppose, without loss of generality, that $l_{g_n}(\gamma_{g_n}) \geq l_g(\gamma_g)$. Since $\phi_{g_n}^t v_n = (\gamma_{g_n}(t), \dot{\gamma}_{g_n}(t))$ is a geodesic, we may write:

$$l_{g_n}(\gamma_{g_n}) = \underbrace{\int_0^T \|\phi_n^t v_n\|_{g_n} dt}_{=T} + \int_T^{l_{g_n}(\gamma_{g_n})} \|\phi_n^t v_n\|_{g_n} dt.$$

By the previous fact, $\gamma_{g_n}(T) \to \gamma_{g_n}(0)$ as $n \to \infty$, hence there exists n large enough such that $\gamma_{g_n}(0), \gamma_{g_n}(T)$ lie in some *totally normal neighborhood.* That is, for n large enough, γ_{g_n} is the unique geodesic joining $\gamma_{g_n}(0), \gamma_{g_n}(T)$. Hence, $\forall \varepsilon > 0$, $\exists n$ large enough such that:

$$\int_T^{l_{g_n}(\gamma_{g_n})} \|\phi_n^t v_n\|_{g_n} dt = d_{g_n}(\gamma_{g_n}(0), \gamma_{g_n}(T)) < \varepsilon.$$

In particular, we have that $l_{g_n}(\gamma_{g_n}) \to T = l_g(\gamma_g)$. □

This will ultimately prove Proposition 1.2.

Proof of Proposition 1.2. Let $k \in \mathbb{N}$ be arbitrary. We wish to show that $l_g(\gamma_g^k) = l_h(\gamma_h^k)$. To do so, let $\varepsilon > 0$.

Since $l_{g_n}(\gamma_{g_n}^k) \to l_g(\gamma_g^k)$ and $l_{h_n}(\gamma_{h_n}^k) \to l_h(\gamma_h^k)$ as $n \to \infty$, by Lemma 2.4, we have that $\exists N_0 \in \mathbb{N}$ large enough such that $\forall n \geq N_0$:

$$\begin{cases} \left|l_{g_n}(\gamma_{g_n}^k) - l_g(\gamma_g^k)\right| < \varepsilon/3 \\ \left|l_{h_n}(\gamma_{h_n}^k) - l_h(\gamma_h^k)\right| < \varepsilon/3. \end{cases}$$

Next, choose $N_1 > N_0$ such that $\varepsilon/3 > 1/N_1$. By assumption, condition $(*)$ holds for N_1. That is, for all $n \in \{1, \ldots, N_1\}$:

$$\left|l_{g_n}(\gamma_{g_n}^k) - l_{h_n}(\gamma_{h_n}^k)\right| < \frac{1}{N_1} < \varepsilon/3.$$

Finally, applying the triangle inequality, we have that $\forall n \in \{N_0, \ldots, N_1\}$:

$$\begin{aligned} \left|l_g(\gamma_g^k) - l_h(\gamma_h^k)\right| &= \left|l_g(\gamma_g^k) - l_{g_n}(\gamma_{g_n}^k) + l_{g_n}(\gamma_{g_n}^k) - l_{h_n}(\gamma_{h_n}^k) + l_{h_n}(\gamma_{h_n}^k) - l_h(\gamma_h^k)\right| \\ &\leq \left|l_g(\gamma_g^k) - l_{g_n}(\gamma_{g_n}^k)\right| + \left|l_{g_n}(\gamma_{g_n}^k) - l_{h_n}(\gamma_{h_n}^k)\right| + \left|l_{h_n}(\gamma_{h_n}^k) - l_h(\gamma_h^k)\right| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $l_g(\gamma_g^k) = l_h(\gamma_h^k)$. Since k is arbitrary, it follows that g and h have the same marked length spectrum. □

Chapter 3

The Liouville Measure and Current

In this section we construct the Liouville Measure and Current on a Riemannian surface (M, g) having C^1 metric and $C^{0,1}$ Anosov geodesic flow. We begin with a discussion of how an (almost) symplectic structure is defined on TM; the lack of C^2 regularity means that several properties of the geodesic flow have to be reinterpreted. Immediately following are the definitions of the Liouville Measure and Current and some of their properties. Along the way, we construct the space of geodesics, and a couple equivalent characterizations of the Liouville current. We conclude with particular coordinate descriptions that will be useful in Chapter 4. Throughout this chapter, we employ the summation convention $\sum_i a^i b_i =: a^i b_i$.

3.1 (Almost) Symplectic Structure on the Tangent Bundle

Any smooth manifold admits a canonical symplectic structure on its cotangent bundle. When the manifold possesses a Riemannian structure, non-degeneracy of the Riemannian metric gives an isomorphism of the tangent and cotangent bundles. If the Riemannian metric is C^2 smooth (or more), this isomorphism pulls-back the symplectic structure on T^*M to a symplectic structure on TM. If the metric is only C^1, TM obtains a structure which cannot be called symplectic, but for which some of the useful properties carry over; in particular we will see that TM obtains a continuous non-degenerate 2-form. Until §3.1.1, the discussion remains in the C^2 setting; the C^1 case will comprise the rest of this chapter, and the document.

Let $(x, \xi) \in T^*M$ be a point on the cotangent bundle. Denote $\zeta \colon T^*M \to M$ the natural projection $\zeta(x, \xi) = x$, which is smooth and has tangent map $d\zeta_{(x,\xi)} \colon T_{(x,\xi)}(T^*M) \to T_xM$. We then define the **tautological 1-form** θ on the cotangent bundle by:

$$\theta_{(x,\xi)} = \xi \circ d\zeta_{(x,\xi)}$$

If $(x^1, ..., x^n)$ are local coordinates on M, we let $(x^1, ..., x^n; \xi_1, ..., \xi_n)$ be the respective local coordinates on T^*M; here $\xi = \xi_i dx^i$. The 1-form θ acts on $\eta = a^i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial \xi_i}$ by:

$$\theta_{(x,\xi)}(\eta) = \xi \circ d\zeta_{(x,\xi)} \left(a^i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial \xi_i} \right) = a^i \xi_i.$$

Hence, the tautological 1-form has local expression:

$$\theta_{(x,\xi)} = \xi_i dx^i.$$

This is a slight abuse of notation, since dx^i is technically a 1-form on M; what is meant is $\zeta^*(dx^i)$ where $\zeta^* \colon \Omega^*(M) \to \Omega^*(T^*M)$ is the pullback map of ζ. Taking the exterior derivative of θ gives a symplectic form $d\theta$ for which *any* coordinates on M yield canonical coordinates on T^*M[1]:

$$d\theta = d\xi_i \wedge dx^i.$$

We thus call $d\theta$ the **canonical symplectic form** on T^*M. Our goal is to endow the tangent bundle TM with a symplectic structure by pulling-back $d\theta$ using the Riemannian structure on M. This symplectic form will be "non-canonical" since, as is shown further below, the coordinates on TM induced by the coordinates on M are not in general "canonical" (see Appendix B for the definition).

The Riemannian metric on (M, g) is by definition positive-definite. In particular, $\det(g)$ is never zero, and hence g induces at every $x \in M$ an isomorphism $T_xM \cong T_x^*M$. We thus have a (fiber-wise) isomorphism of the tangent and cotangent bundles:

$$\begin{aligned} G \colon TM &\to T^*M \\ (x, v) &\mapsto (x, \iota_v g_x) \end{aligned}$$

where $\iota_v g_x$ is the interior product, or tensor contraction, of $g_x \in T_x^*M \otimes T_x^*M$ with $v \in T_xM$. In coordinates, we may express this as:

$$G(x, v^i \partial_i) = (x, g_{jk} dx^j \otimes dx^k(v^i \partial_i)) = (x, g_{ij} v^i dx^j)$$

where $g_{ij} = g_{ji}$ are the matrix components of g in these coordinates. It should be clear that since g is C^2 smooth, so is G (it is in fact a C^2 diffeomorphism, since the matrix g_{ij} is always invertible), and hence defines a pullback $G^* \colon \Omega^*(T^*M) \to \Omega^*(TM)$. This allows us to define a 1-form on TM:

$$\alpha = G^*\theta \in \Omega^1(TM)$$

and a symplectic 2-form on TM:

$$\omega := d\alpha = d(G^*\theta) = G^* d\theta.$$

[1]By "canonical coordinates" we mean nothing other than the presented equation. See also Appendix B for the definition.

Indeed, ω is closed since $d^2\theta = 0$, and the pullback G^* commutes with the exterior derivative.[2] Similarly, ω is non-degenerate because $d\theta$ is non-degenerate.

One should also note that the restriction of α to T^1M makes it a contact 1-form, endowing T^1M with a contact structure (see Appendix B). Since it will be useful to use this contact structure, and to avoid overloaded notation, we will be precise about when we regard α as a form on TM or its restriction to T^1M.

In Coordinates

In the corresponding local coordinates $(x^1, \dots, x^n; v^1, \dots, v^n)$ on TM, the isomorphism $G\colon TM \to T^*M$ has local representation:

$$\tilde{G}(x^1, \dots, x^n; v^1, \dots, v^n) = (x^1, \dots, x^n; g_{1k}v^k, \dots, g_{nk}v^k).$$

Note that this map is not linear, since the g_{ij} depend on x. The Jacobian of this transformation has block-matrix form:

$$D\tilde{G} = \begin{bmatrix} I & 0 \\ \frac{\partial g_{ik}}{\partial x^j}v^k & g_{ij} \end{bmatrix}$$

Meanwhile, the 1-form θ and symplectic 2-form $d\theta$ on T^*M have local (matrix) representations, at arbitrary $(x, \xi) \in T^*M$:

$$\begin{aligned} \theta_{(x,\xi)} &= \begin{bmatrix} \xi_1 & \cdots & \xi_n & 0 & \cdots & 0 \end{bmatrix} \\ d\theta_{(x,\xi)} &= \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \end{aligned}$$

Since the pullback of either of these forms under G is

$$\begin{aligned} \alpha_{(x,v)} &= \theta_{G(x,v)}\left(T_{(x,v)}G(\cdot)\right) \\ \omega_{(x,v)} &= d\theta_{G(x,v)}\left(T_{(x,v)}G(\cdot), T_{(x,v)}G(\cdot)\right) \end{aligned}$$

a straightforward computation then shows that:

$$\alpha = \begin{bmatrix} g_{1i}v^i & \cdots & g_{ni}v^i & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \frac{\partial g_{ik}}{\partial x^j}v^k & g_{ij} \end{bmatrix} = g_{ij}v^i dx^j$$

and

$$\omega = \begin{bmatrix} I & 0 \\ \frac{\partial g_{ik}}{\partial x^j}v^k & g_{ij} \end{bmatrix}^T \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \frac{\partial g_{ik}}{\partial x^j}v^k & g_{ij} \end{bmatrix} = g_{ij}dv^i \wedge dx^j + \frac{\partial g_{jk}}{\partial x^i}v^k dx^i \wedge dx^j.$$

We thus see that in the coordinates (x^i, v^i), ω does not assume a canonical expression.

[2] This only holds when g_{ij} is at least class C^2. Otherwise, $d\omega = d(G^*d\theta)$ is not well defined.

However, given arbitrary local coordinates as above, one can define new *canonical coordinates* by $(x^i, u_i) = (x^i, g_{ij}v^j)$. Hence, for any functional $H\colon TM \to \mathbb{R}$, the equations defining its Hamiltonian flow are (see Appendix B):

$$\begin{cases} \dot{x}^k = \dfrac{\partial H}{\partial u_k} = \dfrac{\partial H}{\partial x^i}\dfrac{\partial x^i}{\partial u_k} + \dfrac{\partial H}{\partial v^i}\dfrac{\partial v^i}{\partial u_k} \\ \dot{u}_k = -\dfrac{\partial H}{\partial x^k} \end{cases}$$

Applying the facts that $\partial x^i/\partial u_k = 0$ and $\partial v^i/\partial u_k = g^{ik}$, as well as $\dot{u}_k = \frac{\partial g_{ik}}{\partial x^j}v^i\dot{x}^j$, we obtain a system of ODEs describing the (Hamiltonian) flow of (the Hamiltonian vector field) X_H in coordinates (x^k, v^k):

$$\begin{cases} \dot{x}^k = g^{ki}\dfrac{\partial H}{\partial v^i} \\ \dot{v}^k = -g^{ki}\left[\dfrac{\partial H}{\partial x^i} + \dfrac{\partial g_{ki}}{\partial x^k}v^i\dot{x}^j\right] \end{cases} \tag{3.1}$$

Note that these are exactly the components of X_H in the specified coordinates:

$$X_H = g^{ki}\frac{\partial H}{\partial v^i}\frac{\partial}{\partial x^k} - g^{ki}\left[\frac{\partial H}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^k}v^i\dot{x}^j\right]\frac{\partial}{\partial v^k}.$$

Geodesic Flow

One of the advantages of introducing the above symplectic structure on TM is that the geodesic flow can be interpreted as a Hamiltonian flow. Indeed, consider the "Kinetic Energy" functional

$$\begin{aligned} &E\colon TM \to \mathbb{R} \\ &E(x,v) = \frac{1}{2}\|v\|_x^2 = \frac{1}{2}g_{ij}v^iv^j \end{aligned}$$

and denote by X_E its Hamiltonian vector field. We may apply the above equations (3.1) to this case, and with a little manipulation of indices, obtain from $dE = \iota_{X_E}\omega$ the geodesic equations:

$$\begin{cases} \dot{x}^k = v^k \\ \dot{v}^k = -g^{km}\left[\dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial x^m}v^iv^j + \dfrac{\partial g_{mi}}{\partial x^j}v^iv^j\right] = -\Gamma^k_{ij}v^iv^j \end{cases} \tag{3.2}$$

Here we have used the symmetry of the term $\frac{\partial g_{mi}}{\partial x^j}v^iv^j = \frac{1}{2}\left[\frac{\partial g_{mi}}{\partial x^j}v^iv^j + \frac{\partial g_{mj}}{\partial x^i}v^jv^i\right]$. That is, the geodesic flow is precisely the Hamiltonian flow of the "Kinetic Energy" Hamiltonian. We call X_E the geodesic field, and denote the geodesic flow by $\phi^t\colon TM \to TM$.

The geodesic field X_E may be restricted to T^1M, and the geodesic flow thus restricted to the unit tangent bundle $\phi^t \colon T^1M \to T^1M$. As with the 1-form α, we will be precise about when these objects are defined on either TM or T^1M.

Now, regarding α as the contact 1-form on T^1M, the Hamiltonian vector field X_E associated to the Energy (also restricted to T^1M) is exactly the Reeb vector field associated to α. Indeed, one can verify that:

$$d\alpha(X_E, \cdot) = 0, \qquad \alpha(X_E) = 1.$$

Note here that these identities hold *only* when considering α and X_E as their restrictions to T^1M[3]. In general on TM we have that $d\alpha(X_E, \cdot) = \omega(X_E, \cdot) = dE \neq 0$, and $\alpha(X_E)_{(x,v)} = \|v\|$.

The properties of Hamiltonian vector fields on symplectic manifolds and the Reeb field on a contact manifold (see Appendix B) thus pass to the geodesic field X_E on the tangent bundle TM and on T^1M. In particular, if we denote by ϕ^t the geodesic flow (of X_E), then:

1. $E \circ \phi^t = E$ on TM. The geodesic flow $\phi^t \colon TM \to TM$ preserves lengths.
2. $(\phi^t)^*\omega = \omega$ on TM. The geodesic flow $\phi^t \colon TM \to TM$ preserves the symplectic structure on TM.
3. $(\phi^t)^*\alpha = \alpha$ on T^1M. The geodesic flow $\phi^t \colon T^1M \to T^1M$ preserves the contact 1-form on T^1M.

3.1.1 The C^1 Case

We now consider the case where the Riemannian metric is only C^1. The main issue that arises from the lack of C^2 regularity is that the expression $d\omega = d^2\alpha = 0$ above no longer makes sense.

Since the metric is C^1, the isomorphism $G : TM \to T^*M$ is also C^1. Thus the pullback G^* is still defined, and hence so are the 1-form $\alpha = G^*\theta$ and 2-form $\omega = G^*d\theta = dG^*\theta$. However, we cannot say what $d\omega$ is, since this involves second order partial derivatives of g_{ij}, hence it cannot be said that ω is closed. In particular, ω is not a symplectic form.

Some structure still remains. The restriction of the 1-form α to T^1M still defines a contact structure, but which is only C^1-regular. The 2-form $\omega = d\alpha$ is non-degenerate and continuous (C^0). In particular, the following properties follow just as in the previous section:

[3]The first identity can be checked by choosing geodesic normal coordinates at any point in $(x, v) \in T^1M$. Then at (x, v), $X_E = v^k\partial_k$ and $g_{ij} = \delta_{ij}$, so $d\alpha(X_E, \cdot) = \sum_i v^i dv^i$. Hence, using the horizontal-vertical splitting (see appendix A), one can show that if $\xi \in T_{(x,v)}(T^1M)$, then $d\alpha(X_E, \xi) = \langle v, \kappa_{(x,v)}\xi\rangle$ where κ is the connector map. Since $\kappa_{(x,v)}\xi$ is tangent to the circle T^1_xM at v, we have that $\kappa_{(x,v)}\xi \perp v$, hence from the above $d\alpha(X_E, \xi) = 0$.

1. The geodesic field X_E can be defined on T^1M by $\iota_{X_E}\omega = dE$. Moreover, restricted to T^1M, it is the (unique) Reeb vector field associated to α: $\alpha(X_E) \equiv 1$ and $\iota_{X_E} d\alpha \equiv 0$. It is however, only of C^0 regularity.

2. The geodesic flow ϕ^t is well-defined (by the geodesic equations $d\phi^t/dt = X_E(\phi^t)$). It is C^1 with respect to t.

3. $E \circ \phi^t = E$. The geodesic flow $\phi^t \colon TM \to TM$ preserves the energy.

4. $\mathcal{L}_{X_E}\alpha = 0$. The contact form α has zero Lie derivative along the geodesic field X_E. Here we use Cartan's formula as the definition of the Lie-derivative of forms (see Appendix B).

5. The top exterior power of the almost-symplectic ω is a volume form ω^n (again, see Appendix B).

Due to the lack of differentiability in the geodesic flow ϕ^t, we cannot say that $(\phi^t)^*\alpha = \alpha$ or $(\phi^t)^*\omega = \omega$, as was possible in the C^2 case. If the flow is Lipschitz continuous, however, then it is differentiable almost everywhere (by Rademacher's Theorem, which extends to manifolds via coordinate charts). Integrals of the form $\int_{\phi^t c}$ are thus well-defined when c is a smooth-chain, since the image of $\phi^t c$ is a *rectifiable set* (see [Morgan, 2016, §3]). There also exists a version of Stokes' theorem for *non-smooth chains*, see [Harrison, 1993, Theorem 3]. This version of Stokes' theorem applies in particular to sets which are locally graphs of Lipschitz functions and C^1 forms. We present the following slightly weaker proposition on the preservation of α and ω under a Lipschitz geodesic flow.

Proposition 3.1. *If the geodesic flow is Lipschitz continuous, then $\omega = d\alpha$ is an integral invariant*[4] *of the geodesic flow on TM, and α is an integral invariant of the geodesic flow restricted to T^1M. That is, for any smooth 2-chain c in TM:*

$$\int_c \omega = \int_{\phi^t c} \omega$$

and for any smooth 1-chain γ in T^1M:

$$\int_\gamma \alpha = \int_{\phi^t \gamma} \alpha.$$

This is essentially the content of *Liouville's theorem* from classical mechanics.[5] Indeed, in the case that g_{ij} were C^2, the above proposition would be equivalent to saying that α and ω are invariant under the pullback of ϕ^t:

$$(\phi^t)^*\alpha = \alpha$$
$$(\phi^t)^*\omega = \omega$$

[4]This is a terminology adopted from [Arnold, 1989].
[5]See for example [Arnold, 1989]

However, in the context of §4.1 and §4.2, the metric is C^1 and the geodesic flow is $C^{0,1}$, hence only Proposition 3.1 holds. It turns out this is enough for our purposes.

We prove Proposition 3.1 employing an argument seen in [Arnold, 1989, §38, p.205].

Proof. It suffices to prove the assertion for any smooth 1- and 2- simplices; the result extends to chains by linearity. For any k-simplex σ, we let $J_\tau\sigma : \Delta^k \times [0,\tau] \ni (x,t) \mapsto \phi^t(\sigma(x))$ be the *track of σ swept out by ϕ^t* for all time $0 \leq t \leq \tau$. It can be shown that the boundary of $J_\tau\sigma$ consists of the the two "end caps" and the "sides":

$$\partial(J_\tau\sigma) = \phi^\tau\sigma - \sigma - J_\tau(\partial\sigma)$$

It should be noted that, in the nomenclature of [Harrison, 1993], $J_\tau\sigma$ is a $(k+1,k+1)$-set. Stokes' theorem ([Harrison, 1993, Theorem 3]) applies on such sets, as long as the integrated form is at least C^1.

(Proof for ω). Let $\sigma\colon \Delta^2 \to TM$ be any smooth 2-simplex. Since ω is only C^0, we cannot apply Stokes' theorem as per [Harrison, 1993, Theorem 3]. However, it is true that ω is the pullback of a smooth form under a C^1 function: $\omega = G^*(d\theta)$. Hence, applying Stokes' theorem to $d\theta$:

$$\int_{\partial(J_\tau\sigma)} \omega = \int_{\partial(J_\tau\sigma)} G^* d\theta = \int_{G(\partial(J_\tau\sigma))} d\theta = \int_{\partial G(J_\tau\sigma)} d\theta = \int_{\partial^2 G(J_\tau\sigma)} \theta = 0.$$

where we have used the fact that $G\colon TM \to T^*M$ is a chain map (i.e. $\partial G = G\partial$), since it is continuous (in fact C^1-diffeo.). Hence:

$$0 = \int_{\partial(J_\tau\sigma)} \omega = \int_{\phi^\tau\sigma} \omega - \int_\sigma \omega - \int_{J_\tau(\partial\sigma)} \omega.$$

We claim that the last integral, along the "sides" $J_\tau\partial\sigma$, is equal to zero (this would conclude the proof for ω).

As $\partial\sigma : [0,1] \to TM$ is a 1-simplex, we may denote $\xi(t,s) = \frac{\partial}{\partial s}\phi^t\sigma(s)$ and $\eta(t,s) = \frac{\partial}{\partial t}\phi^t\sigma(s)$ elements of $T_{\phi^t\sigma(s)}(TM)$. Since ϕ^t is Lipschitz, $J_\tau(\partial\sigma)$ is rectifiable, hence the usual definition of the integral gives:

$$
\begin{aligned}
\int_{J_\tau(\partial\sigma)} \omega &= \int_0^1 \int_0^\tau \omega(\eta(t,s), \xi(t,s)) dt ds \\
&= \int_0^1 \int_0^\tau dE(\xi(t,s)) dt ds \\
&= \int_0^\tau \left(\int_0^1 dE(\xi(t,s)) ds \right) dt && \text{(Fubini' Theorem)} \\
&= \int_0^\tau \left(\int_{\phi^t(\partial\sigma)} dE \right) dt && \text{(Definition of Integral)} \\
&= \int_0^\tau \left(\int_{\partial\phi^t(\partial\sigma)} E \right) dt && \text{(Stokes' Theorem; } E \text{ is } C^2 \subset C^1) \\
&= \int_0^\tau \left(\int_{\phi^t(\partial^2\sigma)} E \right) dt && (\phi^t \text{ continuous (chain map): } \partial\phi^t = \phi^t\partial) \\
&= 0 && (\text{since } \partial^2\sigma = 0)
\end{aligned}
$$

where we have used (in the second line) the fact that $\eta(t,s) = X_E(\phi^t\sigma(s))$, since it is a tangent vector to a trajectory of the flow, and that $dE = \iota_{X_E}\omega = \omega(X_E, \cdot)$.

(Proof for α). Let $\gamma : [0,1] \to T^1M$ be any 1-simplex in the unit tangent bundle. Then by Stokes' theorem (α is C^1), for any τ:

$$
\int_{J_\tau\gamma} d\alpha = \int_{\partial(J_\tau\gamma)} \alpha = \int_{\phi^\tau\gamma} \alpha - \int_\gamma \alpha - \int_{J_\tau(\partial\gamma)} \alpha.
$$

By the definition of the integral and the fact that X_E is the Reeb field associated to α, the left side is equal to zero:

$$
\begin{aligned}
\int_{J_\tau\gamma} d\alpha &= \int_0^1 \int_0^\tau d\alpha \left(\frac{\partial\phi^t\gamma(s)}{\partial t}, \frac{\partial\phi^t\gamma(s)}{\partial s} \right) dt ds \\
&= \int_0^1 \int_0^\tau d\alpha \left(X_E(\phi^t\gamma(s)), \frac{\partial\phi^t\gamma(s)}{\partial s} \right) dt ds \\
&= \int_0^\tau \left(\int_{\phi^t\gamma} d\alpha(X_E, \cdot) \right) dt = 0
\end{aligned}
$$

The last term on the right side is also equal to zero:

$$
\begin{aligned}
\int_{J_\tau(\partial\gamma)} \alpha &= \int_{J_\tau\gamma(1)} \alpha - \int_{J_\tau\gamma(0)} \alpha \\
&= \int_0^\tau \alpha \left(\frac{\partial\phi^t\gamma(1)}{\partial t} \right) dt - \int_0^\tau \alpha \left(\frac{\partial\phi^t\gamma(0)}{\partial t} \right) dt \\
&= \int_0^\tau \alpha\left(X_E\right)|_{\phi^t\gamma(1)} dt - \int_0^\tau \alpha\left(X_E\right)|_{\phi^t\gamma(1)} dt \\
&= \tau - \tau = 0
\end{aligned}
$$

where the last line used the fact that $\alpha(X_E) \equiv 1$. Hence, by the first equality above:

$$0 = \int_{\phi^\tau \gamma} \alpha - \int_\gamma \alpha.$$

□

3.2 Liouville Measure

We continue in the setting of a Riemannian surface (M, g), where M is a smooth manifold of dimension 2, and g is a C^1 Riemannian metric with $C^{0,1}$ (Lipschitz) geodesic flow. The almost symplectic structure on TM and contact structure on T^1M are given as above. Note that the geodesic flow need not be Anosov to define the Liouville measure.

Definition 3.2. *We define the **Liouville Measure** on T^1M to be the volume form*

$$d\lambda \coloneqq \alpha \wedge d\alpha = \alpha \wedge \omega.$$

In the more general case where $\dim M = n$, the Liouville Measure is defined as $d\lambda = \alpha \wedge (d\alpha)^{n-1}$.

The above is really an abuse of notation, as $d\lambda$ is not an exact form. However, it is suggestive of the fact that $d\lambda$ defines a measure on the unit tangent bundle (hence the name). This notation also appears in the notes of [Wilkinson, 2014], which we have followed closely in this document.

Proposition 3.3 (Properties of Liouville Measure)**.** *We present several properties of the Liouville Measure:*

1. *The Liouville Measure has the local product structure of Riemannian volume/area on M and arclength on the fibers of T^1M:*

   $$\alpha \wedge d\alpha = \sqrt{\det(g)} dx^1 \wedge dx^2 \wedge d\theta.$$

   Here x^1, x^2 are coordinates on M and θ is the angular coordinate on the fibers of T^1M.

2. *The unit tangent bundle has finite Liouville volume/measure:*

   $$\lambda(T^1M) = \int_{T^1M} |\alpha \wedge d\alpha| < \infty.$$

3. *The Liouville Measure is an integral invariant of the geodesic flow. That is, for all 3-chains A in T^1M:*

   $$\int_A d\lambda = \int_{\phi^t A} d\lambda \quad \forall t.$$

Proof. Property (1): Since $\dim(M) = 2$, we have that $dx^i \wedge dx^j \wedge dx^k = 0$ always. Hence calculating in coordinates, one obtains:

$$\alpha \wedge d\alpha = g_{ij} g_{kl} v^i dx^j \wedge dx^k \wedge dv^l = (g_{11}g_{22} - g_{12}g_{21}) dx^1 \wedge dx^2 \wedge \left[v^2 dv^1 - v^1 dv^2\right].$$

Now if (r, θ) define the polar coordinates on the fibers of TM given by the metric g, then it becomes possible to compute:

$$i^* d\theta = \sqrt{\det(g)}\, i^* \left[v^2 dv^1 - v^1 dv^2\right]$$

where $i \colon T^1M \hookrightarrow TM$ is the inclusion. Hence, denoting $d\theta$ for $i^* d\theta$:

$$\begin{aligned} d\lambda = \alpha \wedge d\alpha &= \left[\sqrt{g_{11}g_{22} - g_{12}g_{21}}\, dx^1 \wedge dx^2\right] \wedge \left[\sqrt{g_{11}g_{22} - g_{12}g_{21}}(v^2 dv^1 - v^1 dv^2)\right] \\ &= \sqrt{\det(g)}\, dx^1 \wedge dx^2 \wedge d\theta \end{aligned}$$

which is the product structure claimed.

Property (2): Follows from (1) and the fact that T^1M is compact. Indeed, M is compact and the fibers $T^1_x M \cong \mathbb{S}^1$ are compact, hence one can choose a finite cover $\{K_i\}_{i \in I}$ of M, where each K_i is compact and small enough such that the local product structure of $d\lambda$ implies:

$$\int_{T^1 K_i} |d\lambda| = \left[\int_{K_i} \sqrt{\det(g)}\, dx^1 dx^2\right] \cdot \left[\int_{\mathbb{S}^1} v^2 dv^1 - v^1 dv^2\right] < \infty$$

hence

$$\lambda(T^1M) = \sum_i \lambda(T^1 K_i) < \infty.$$

Property (3): This follows from the discussion in the preceding section (i.e. Proposition 3.1). Consider any surface transverse to the geodesic flow, say $S \subset T^1M$. Flowing by time τ generates a "box" in T^1M:

$$A = \{\phi^t(v) \mid v \in S, t \in [0, \tau]\} \cong S \times [0, \tau].$$

By Fubini's theorem, we may first integrate α along the flow lines; since $\alpha(X_E) \equiv 1$, this gives:

$$\int_A \alpha \wedge d\alpha = \tau \cdot \int_S d\alpha.$$

It follows that for all $t \in \mathbb{R}$:

$$\int_{\phi^t A} \alpha \wedge d\alpha = \tau \cdot \int_{\phi^t S} d\alpha = \tau \cdot \int_S d\alpha = \int_A \alpha \wedge d\alpha$$

where in the first and second equality, respectively, we have used that α and $d\alpha$ are integral invariants of the geodesic flow. Since every volume in T^1M can be approximated by sufficiently many (sufficiently small) subsets of the form of A, the claim follows. $\square$

Remark 3.4. *Let $\widetilde{M}$ denote the universal cover of M. The Riemannian metric on M lifts to a Riemannian metric on $\widetilde{M}$, and the corresponding geodesic flow on $T\widetilde{M}$ (resp. almost-symplectic structure on $T\widetilde{M}$, contact structure on $T^1\widetilde{M}$, Liouville measure on $T^1\widetilde{M}$) is exactly the lift of the geodesic flow on TM (resp. almost-symplectic structure on TM, contact structure on T^1M, Liouville measure on T^1M). We do not make a notational distinction between the lifted and the non-lifted versions of these objects; we still denote by g the metric, $\hat{g}$ the Sasaki metric, ϕ^t the flow, α the contact form, λ the Liouville measure. The properties in Proposition 3.1 and Proposition 3.3 still hold in the lifted setting, except that $\lambda(T^1\widetilde{M}) = \infty$.*

3.3 Liouville Current

It is now necessary to additionally assume that the geodesic flow is Anosov. The Liouville current is, roughly speaking, the projection of the Liouville measure from $T^1\widetilde{M}$ onto the space of geodesics $\mathcal{G} = T^1\widetilde{M}/v \sim \phi^t(v)$. It is important to make the distinction here between M and its universal cover $\widetilde{M}$.[6] In this section, we will construct the space of geodesics, and give a precise definition of the Liouville current. We then give several of its equivalent interpretations/characterizations.

Space of Geodesics

The space of geodesics is the set of oriented images of geodesics. Given the geodesic flow on the unit tangent bundle $\phi^t \colon T^1\widetilde{M} \to T^1\widetilde{M}$, one can define an $\mathbb{R}$-action on $T^1\widetilde{M}$ as the translation/flow of vectors by time $t \in \mathbb{R}$:

$$\phi \colon \mathbb{R} \to Lip(T^1\widetilde{M})$$
$$\phi \colon t \mapsto \big(\phi^t \colon v \mapsto \phi^t(v)\big)$$

One should recall here that the geodesic flow ϕ is assumed to be Lipschitz; since it is invertible with Lipschitz inverse ($\phi^t \circ \phi^{-t} = \phi^{-t} \circ \phi^t = Id_{T^1\widetilde{M}}$), it is a Lipeomorphism. The **space of geodesics** is then defined as the quotient of $T^1\widetilde{M}$ under this $\mathbb{R}$-action:

$$\mathcal{G} = T^1\widetilde{M}/\mathbb{R}.$$

We endow $\mathcal{G}$ with the quotient topology, and denote the associated quotient map by $\pi_\phi \colon T^1\widetilde{M} \to T^1\widetilde{M}/\mathbb{R}$. This is a continuous and open map, since it is the quotient map under the action of a topological group [Lee, 2013, Lemma 21.1]. Equivalence

[6] Taking the quotient of T^1M under the geodesic flow yields a space with an undesirable topology. Indeed, every dense orbit of ϕ^t in T^1M would yield a point in the quotient for which any neighborhood contains the whole space. There are however, no dense orbits in $T^1\widetilde{M}$.

classes in $\mathcal{G}$ are thus collections of vectors in $T^1\widetilde{M}$ which lie on the same trajectory of the geodesic flow:

$$\mathcal{G} = \{[v] \mid v \in T^1\widetilde{M}\}$$
$$[v] := \pi_\phi(v) = \{w \in T^1\widetilde{M} \mid \exists t \in \mathbb{R} : \; w = \phi^t(v)\}$$

Every equivalence class $[v] \in \mathcal{G}$ can be identified with a (unique) equivalence class of *oriented* geodesics:

$$[v] \longleftrightarrow [\gamma_v \colon t \mapsto \phi^t(v)]$$

where

$$\gamma_v \sim \gamma_w \iff \exists s \colon \gamma_v(t) = \gamma_w(t+s) \iff \exists s : v = \phi^s(w).$$

Hence the space of geodesics is equivalently:

$$\mathcal{G} \approx \{[\gamma] \mid \gamma(t) = \phi^t(v), \; v \in T^1\widetilde{M}\}.$$

We take it from here that $[v]$ and $[\gamma_v]$ are understood to represent the same geodesic in $\mathcal{G}$.

It is possible to foliate the space of geodesics by collecting (equivalences classes of) forward and backward asymptotic geodesics. For any $[\gamma] = [\gamma_v] \in \mathcal{G}$, we define the leaves of the **stable/unstable foliations** of $\mathcal{G}$:

$$\mathcal{F}^+([\gamma_v]) = \{[\gamma_w] \in \mathcal{G} \mid \lim_{t\to+\infty} d(\gamma_v(t), \gamma_w(t+t_0)) = 0 \text{ for some } t_0\}$$
$$\mathcal{F}^-([\gamma_v]) = \{[\gamma_w] \in \mathcal{G} \mid \lim_{t\to-\infty} d(\gamma_v(t), \gamma_w(t+t_0)) = 0 \text{ for some } t_0\}.$$

This allows us to define another important space, the **boundary at infinity**:

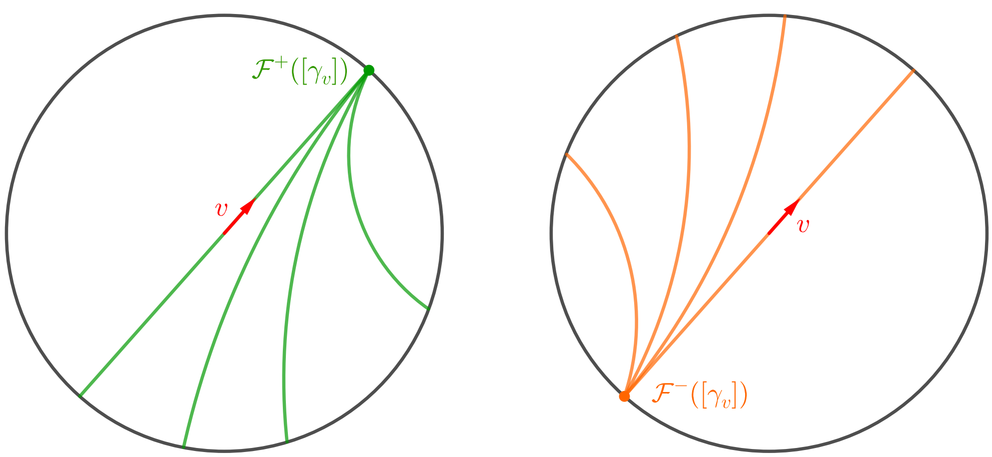

Figure 3.1: Geodesics in the leaves $\mathcal{F}^+([\gamma_v])$ (left) and $\mathcal{F}^+([\gamma_v])$ (right) for some $v \in T^1\widetilde{M}$. For the sketch, we imagine M to be the Poincaré disk, for example.

$$\partial\widetilde{M} = \mathcal{G}/\mathcal{F}^{+} \cong \mathcal{G}/\mathcal{F}^{-}.$$

Again, we endow $\partial\widetilde{M}$ with the quotient topology, and denote by $\mathcal{F}^{+}\colon \mathcal{G} \to \mathcal{G}/\mathcal{F}^{+}$ the associated quotient map. This is also a continuous and open map, as it is the quotient under a foliation on a smooth manifold. The identification $\mathcal{G}/\mathcal{F}^{+} \cong \mathcal{G}/\mathcal{F}^{-}$ comes from the identification of the foliations $\mathcal{F}^{+} \cong \mathcal{F}^{-}$, which is induced by the flip map $v \mapsto -v$ on $T^1\widetilde{M}$:

$$\mathcal{F}^{+}\left([\gamma_v]\right) \mapsto \mathcal{F}^{-}\left([\gamma_{-v}]\right).$$

The boundary at infinity can also be regarded as the image under the map

$$\begin{aligned} F\colon T^1\widetilde{M} &\to \partial\widetilde{M} \\ v &\mapsto \mathcal{F}^{+}([v]) \end{aligned}$$

It is clear that $F = \mathcal{F}^{+} \circ \pi_\phi$ is an open quotient map, since it is the composition of two consecutive open quotient maps.

The foliations $\mathcal{F}^{\pm}$ over $\mathcal{G}$ can also be interpreted as quotients under the geodesic flow of the **weak stable/unstable horocyclic foliations** of $T^1\widetilde{M}$, defined by:

$$\mathcal{W}^{\pm}(v) = \mathcal{F}^{\pm}([v]) \subset T^1\widetilde{M}. \tag{3.3}$$

With the understanding of these foliations of $T^1\widetilde{M}$, we may interpret F as the (open) quotient map which sends $v \in T^1\widetilde{M}$ to the leaf $\mathcal{W}^{+}(v)$. Moreover, we have that

$$\partial\widetilde{M} \cong T^1\widetilde{M}/\mathcal{W}^{+}$$

with the same quotient topology as above.

Lemma 3.5. *The boundary at infinity $\partial\widetilde{M}$ is homeomorphic to the circle $\mathbb{S}^1$. This can be realized through the identification $\partial\widetilde{M} \cong T^1_p\widetilde{M} \cong \mathbb{S}^1$, which is valid for any fixed $p \in \widetilde{M}$.*

Proof. Fix $p \in \widetilde{M}$. We claim that the restriction of F to $T^1_p\widetilde{M}$ is a homeomorphism, which we denote:

$$\begin{aligned} F_p\colon T^1_p\widetilde{M} &\to \partial\widetilde{M} \\ v &\mapsto \mathcal{W}^{+}(v) = \mathcal{F}^{+}([v]) \end{aligned}$$

Note that we may write $F_p = F \circ i$ where $i\colon T^1_p\widetilde{M} \to T^1\widetilde{M}$ is the injection defined by $i(v) = (p, v)$.

The map F_p is injective since if $v_1, v_2 \in T^1_p\widetilde{M}$ and $v_1 \neq v_2$, then the geodesics $\gamma_{v_1}, \gamma_{v_2}$ diverge, hence $F_p(v_1) = \mathcal{W}^{+}(v_1) \neq \mathcal{W}^{+}(v_2) = F_p(v_2)$. It is also surjective, by the following construction. Let $\xi = \mathcal{W}^{+}(v) \in \partial\widetilde{M}$ be represented by some $v \in T^1_q\widetilde{M}$, where $q \in \widetilde{M}$ is some point. Denote by $\gamma_v(t)$ the geodesic in $\widetilde{M}$ such that $\dot{\gamma}_v(0) = v$. For all t, there exists a unique geodesic γ_{w_t} with $p = \gamma_{w_t}(0)$ and $\gamma_v(t) = \gamma_{w_t}(\tau)$ for some $\tau \in \mathbb{R}$. Taking $w = \lim_{t\to\infty} w_t \in T^1_p\widetilde{M}$ we have that

$$F_p(w) = \mathcal{W}^+(w) = \mathcal{W}^+(v) = \xi.$$

Hence F_p is a bijection of $T^1_p\widetilde{M}$ and $\partial\widetilde{M}$.

Further, $F_p = F \circ i$ is continuous and an open map. Continuity should be obvious since F and i are both continuous. To show that F_p is open, let $V \subset T^1_p\widetilde{M}$ be open; we may assume without loss of generality that V is topologically an open interval (an arc in $\mathbb{S}^1$). Since no two vectors in V (or in $T^1_p\widetilde{M}$) will flow to the same point at infinity, V is transverse to the foliation $\mathcal{W}^+$. Hence, since V is open in $T^1_p\widetilde{M}$ and transversal to $\mathcal{W}^+$, its saturation by leaves of $\mathcal{W}^+$ is open in $T^1\widetilde{M}$. Indeed, the collection of leaves:

$$W := \{\mathcal{W}^+(v) \mid v \in V\} \subset \partial\widetilde{M}$$

is an open subset of $\partial\widetilde{M}$ and hence the saturation of V by these leaves:

$$V' := \bigcup_{v\in V} \mathcal{W}^+(v) = \{w \mid \exists v \in V\colon\ \mathcal{W}^+(w) = \mathcal{W}^+(v)\} =: F^{-1}(W)$$

is open in $T^1\widetilde{M}$, by definition of the quotient topology. It is thus clear that $F_p(V) = F(V')$, which is open since F is an open map. $\square$

Remark 3.6. *There is an equivalent notion of the boundary at infinity, called the* ***Gromov Boundary*** *(see [Bridson and Haefliger, 1999, §III.3]). Rather than taking equivalence classes of geodesics (equivalence given by being asymptotic), this definition takes equivalence classes of quasi-geodesic rays (equivalence given by being bounded distance away from each other). We use the Gromov boundary later in Chapter 4, in order to prove Lemma 4.1.*

Every oriented geodesic class $[\gamma] \in \mathcal{G}$ has a unique backwards asymptote $\xi \in \mathcal{G}/\mathcal{F}^-$ and forward asymptote $\eta \in \mathcal{G}/\mathcal{F}^+$; in fact, $\xi = \mathcal{F}^-([\gamma])$ and $\eta = \mathcal{F}^+([\gamma])$. Moreover, since the geodesic flow is Anosov, the assignment $[\gamma] \mapsto (\xi, \eta)$ is injective; i.e. $\mathcal{F}^+([\gamma]) \cap \mathcal{F}^-([\gamma]) = \{[\gamma]\}$. Hence, one can show that:

$$\mathcal{G} \approx \partial^2\widetilde{M} := \partial\widetilde{M} \times \partial\widetilde{M} \backslash \Delta$$

where $\Delta = \{(\xi, \xi) \,\big|\, \xi \in \partial\widetilde{M}\}$ is the diagonal. The exclusion of the diagonal is necessary since no geodesic can have the same point on the boundary at infinity as both its forwards and backwards asymptote.

Definition 3.7. *Denote by* $\Pi\colon T^1M \to \mathcal{G} \approx \partial^2\widetilde{M}$ *the projection from the unit tangent bundle onto the space of geodesics. Let* $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \subset \partial^2\widetilde{M}$ *be a "rectangle" and* *let $\Sigma \subset T^1\widetilde{M}$ be any surface that projects homeomorphically (off the boundary) onto the rectangle. The **Liouville current** is defined as:*

$$m([\xi_1, \xi_2] \times [\eta_1, \eta_2]) = \left| \int_\Sigma d\alpha \right|.$$

Lemma 3.8. *The Liouville current is a well-defined measure on $\mathcal{G}$.*

Proof. This is enough to define a (Borel) measure, since the rectangles in $\mathcal{G}$ generate its Borel σ-algebra.

Moreover, the definition is well-defined since it is independent of choice of Σ. Indeed, if $\Sigma_1, \Sigma_2 \subset T^1\widetilde{M}$ are surfaces that both project under the geodesic flow to $[\xi_1, \xi_2] \times [\eta_1, \eta_2]$, then by definition they are connected by trajectories of the geodesic flow. For each $v \in \Sigma_1$, there exists unique $w \in \Sigma_2$ and unique $l = l(v) \in \mathbb{R}$ such that $w = \phi^{l(v)}(v) = w$. Defining a subset $A \subset T^1M$ by:

$$A = \{\phi^t(v) \mid v \in \Sigma_1, t \in [0, l(v)]\} \cong \{(v, t) \mid v \in \Sigma_1, t \in [0, l(v)]\}$$

we have that

$$\partial A = \Sigma_2 - \Sigma_1 - S$$

where $S = \{\phi^t(v) \mid v \in \partial\Sigma_1, t \in [0, l(v)]\}$ represents the sides of A. Since S is generated by the flow of the geodesic field X_E, and $d\alpha(X_E, \cdot) \equiv 0$ on $T^1\widetilde{M}$, we have

$$\int_{\partial A} d\alpha = \int_{\Sigma_1} d\alpha - \int_{\Sigma_2} d\alpha - \underbrace{\int_S d\alpha}_{=0} = \int_{\Sigma_1} d\alpha - \int_{\Sigma_2} d\alpha.$$

On the left-hand side, we must have zero, as $d\alpha$ integrates to zero over all 2-cycles. Indeed, by Stokes' theorem:

$$\int_{\partial A} d\alpha = \int_{\partial^2 A} \alpha = 0$$

since $\partial^2 = 0$. The claim follows. □

We shall see later in Proposition 3.11 that the Liouville current has a simple expression in a particular coordinate system.

Dynamical Picture

In light of the above definition of the Liouville current, we may choose *any* surface $\Sigma \subset T^1M$ that projects onto the rectangle $[\xi_1, \xi_2] \times [\eta_1, \eta_2]$ (and homeomorphically off the boundary). In fact, by Stokes' theorem, it suffices to choose any curve to serve as the boundary $\partial\Sigma$, as long as it does in fact bound the surface Σ. Here we outline a particularly useful choice of path in T^1M, which happens to bound a surface Σ of the desired type, and gives a dynamical interpretation of the Liouville current (see [Wilkinson, 2014]). One may wish to reference the notes in [Wilkinson, 2014] for the definitions of the strong and weak horocylic foliations ($\mathcal{H}^\pm$ and $\mathcal{W}^\pm$) of $T^1\widetilde{M}$, which generalize to the CAT$(-k)$ setting (see for this purpose [Bridson and Haefliger, 1999, §II.8; Horofunctions and Busemann Functions]).

Construction 3.9. *Choose any representative $v \in T^1\widetilde{M}$ for the geodesic given by $(\xi_1, \eta_1) \in \partial^2\widetilde{M}$. Then:*

1. *Travel along the horocycle $\mathcal{H}^-(v)$ until you reach a new v' on the geodesic (ξ_2, η_1).*
2. *Travel along $\mathcal{H}^+(v')$ until you reach a new v'' on the geodesic (ξ_2, η_2).*
3. *Travel along $\mathcal{H}^-(v'')$ until you reach a new v''' on the geodesic (ξ_1, η_2).*
4. *Travel along $\mathcal{H}^+(v''')$ until you reach a new w on the original geodesic (ξ_1, η_1).*

Since $[\gamma_v] = [\gamma_w] = (\xi_1, \eta_1)$, there must exist a unique $s \in \mathbb{R}$ such that $w = \phi^s(v)$. Then the Liouville current is exactly:

$$m([\xi_1, \xi_2] \times [\eta_1, \eta_2]) = |s|.$$

A sketch description of Construction 3.9 is provided in Figure 3.2. It is also possible to perform the steps backwards; start with a positive horocycle $\mathcal{H}^+(v)$, and continue in the obvious way.

Proof of the Construction's Validity. This algorithm traces out a path in T^1M which projects by Π onto the boundary of $[\xi_1, \xi_2] \times [\eta_1, \eta_2]$. The path also bounds a surface which projects onto all of $[\xi_1, \xi_2] \times [\eta_1, \eta_2]$. Indeed the preimage in $T^1\widetilde{M}$ is $\Pi^{-1}([\xi_1, \xi_2] \times [\eta_1, \eta_2]) \cong [\xi_1, \xi_2] \times [\eta_1, \eta_2] \times \mathbb{R}$, which is topologically a filled in (infinitely long) rectangle. Because the first simplicial homology group of $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \times \mathbb{R}$ is trivial (i.e. every closed 1-simplex is the boundary of a 2-simplex), the path traced out by the algorithm bounds a surface on the interior of $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \times \mathbb{R}$.

Recall that $\alpha(X_E) \equiv 1$, where X_E is the geodesic field, the vector field on $T^1\widetilde{M}$ whose integral curves are trajectories of the geodesic flow. It can be shown that $\alpha(b)|_v = \langle v, T_v\pi(b)\rangle$ for all $b \in T(T^1\widetilde{M})$ (this is an immediate corollary of Lemma 3.12 in the following section). It can further be shown that $T(\mathcal{H}^\pm(v)) \perp H(T^1\widetilde{M})$. It follows that $\alpha(b) = 0$ for all $b \in T(\mathcal{H}^\pm(v))$.

Applying the definition of the Liouville current and Stokes' theorem:

$$m([\xi_1, \xi_2] \times [\eta_1, \eta_2]) = \left| \int_0^s \alpha(X_E(\phi^t(v)))dt \right| = |s|$$

where s is the unique real number such that $w = \phi^s(v)$. In other words, the Liouville current is the distance between v and $w = w(v)$ given by this algorithm. $\square$

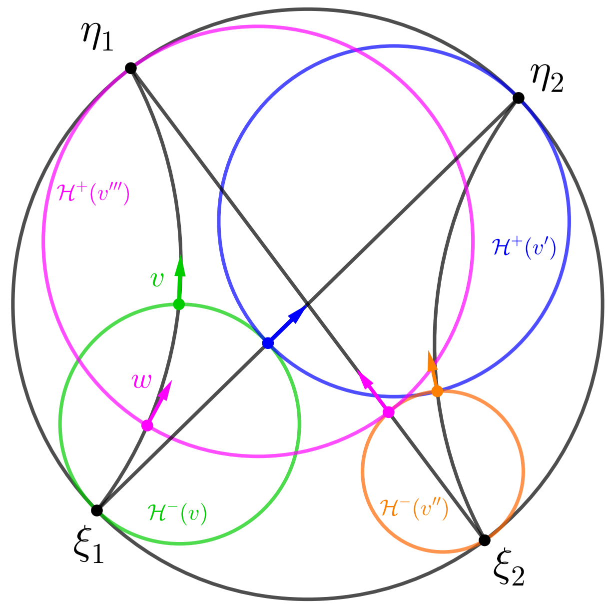

Figure 3.2: A sketch of Construction 3.9 (in the Poincaré Disk, say).

Definition in terms of Cross-Ratio

The definition of the Liouville current as given by [Otal, 1990] uses the **cross-ratio**. This is a function $[\cdot,\cdot,\cdot,\cdot]\colon (\partial\widetilde{M})^4 \to \mathbb{R}_{\geq 0}$ that can be defined in the following way. Let $A, B, C, D \in \partial\widetilde{M}$ be points on the boundary, and choose $[v_1] = (A, C), [v_2] = (B, D), [v_3] = (A, D), [v_4] = (B, C)$. There must exist unique $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that $\phi^{t_1}(v_1) \in \mathcal{H}^+(v_4)$, $\phi^{t_2}(v_2) = \mathcal{H}^+(v_3)$, $\phi^{-t_3}(v_3) = \mathcal{H}^-(v_1)$, $\phi^{-t_4}(v_4) = \mathcal{H}^-(v_2)$, (see Figure 3.3 below). We then define the cross ratio of A, B, C, D as:

$$[A, B, C, D] = |t_1 + t_2 - t_3 - t_4|.$$

One should notice that the t_i are precisely the lengths of geodesic segments connecting the four horospheres in the figure.

We claim that the cross-ratio is well-defined. Indeed, suppose without loss of generality that we change the choice of representative for the first geodesic, to say $[v_1'] = (A, C) \in \mathcal{G}$. Then there exists a unique number τ such that $v_1' = \phi^\tau(v_1)$. Hence $t_1' = t_1 - \tau$ and $t_3' = t_3 - \tau$ are the unique numbers such that $\phi^{t_1'}(v_1') \in \mathcal{H}^+(v_4)$ and $\phi^{-t_3'}(v_3') \in \mathcal{H}^-(v_1)$. It follows that

$$|t_1' + t_2 - t_3' - t_4| = |t_1 - \tau + t_2 - t_3 + \tau - t_4| = |t_1 + t_2 - t_3 - t_4| = [A, B, C, D].$$

This proof works equally well when any other representative tangent vector is changed.

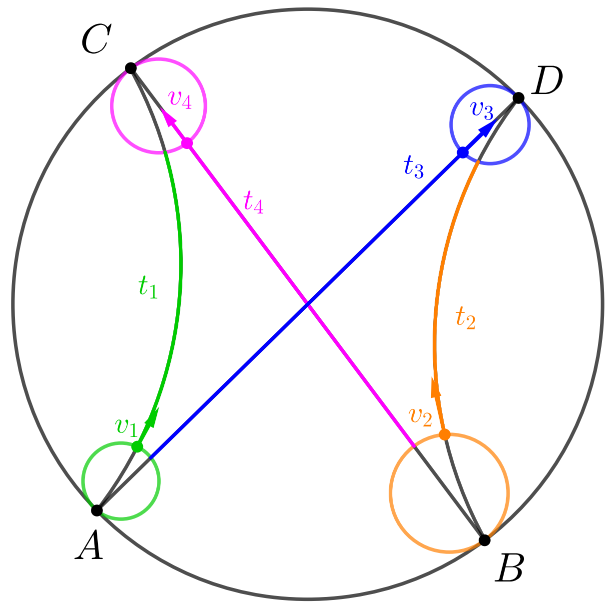

Figure 3.3: A sketch depiction of the lengths used to define the cross-ratio.

Lemma 3.10. *The Liouville current is equivalently expressed as:*

$$m([\xi_1, \xi_2] \times [\eta_1, \eta_2]) = [\xi_1, \xi_2, \eta_1, \eta_2].$$

Proof. Take v, v', v'', v''', w as in the dynamical description (see Construction 3.9). Following the construction of the dynamical description, as well as the above definition for cross-ratio, it is clear that

$$\begin{aligned} v &= v_1 \\ v' &= v_3 \\ v'' &= v_2 \\ v''' &= v_4 \end{aligned}$$

and hence

$$\begin{aligned} t_1 = d(v, w) &= m([\xi_1, \xi_2] \times [\eta_1, \eta_2]) \\ t_2 &= 0 \\ t_3 &= 0 \\ t_4 &= 0. \end{aligned}$$

It follows that

$$[\xi_1, \xi_2, \eta_1, \eta_2] = d(v, w) = m([\xi_1, \xi_2] \times [\eta_1, \eta_2]).$$

$\square$

3.4 Coordinate Descriptions

Both $T^1\widetilde{M}$ and $\mathcal{G}$ can be endowed with a particular coordinate system in which the Liouville measure and current have simple expressions. We describe this coordinate system here.

Fix $w \in T^1\widetilde{M}$ and consider the set

$$U_w = \{v \mid \gamma_v \text{ intersects } \gamma_w \text{ transversally}\} \subset T^1\widetilde{M}.$$

Then for any $v \in U_w$, let $x, y \in \mathbb{R}$ be the unique numbers such that

$$\pi(\phi^{-y}(v)) = \pi(\phi^{-x}(w))$$

where $\pi\colon T^1\widetilde{M} \to \widetilde{M}$ is the natural projection, and let $\theta \in (0, \pi)$ be defined by:

$$\cos\theta = \langle \phi^{-x}(w), \phi^{-y}(v)\rangle.$$

That is, x and y are the distances from the base points of w and v to the point of intersection between γ_w and γ_v, and θ is the angle of intersection; see Figure 3.4 for example. This defines a coordinate system on $T^1\widetilde{M}$:

$$\begin{aligned} B\colon (0,\pi) \times \mathbb{R} \times \mathbb{R} &\to U_w \\ (\theta, x, y) &\mapsto B(\theta, x, y) = v. \end{aligned}$$

These coordinates project to coordinates on $\mathcal{G}$. Consider the set

$$\mathcal{G}_w = \{[\gamma_v] \in \mathcal{G} \mid \gamma_v \text{ intersects } \gamma_w \text{ transversally}\}.$$

Every geodesic $[\gamma_v] \in \mathcal{G}_w$ can be identified with a vector v whose base point lies somewhere on the image of γ_w. Coordinates can thus be defined using B above and fixing $y = 0$:

$$\begin{aligned} b\colon (0,\pi) \times \mathbb{R} &\to \mathcal{G}_w \\ (\theta, x) &\mapsto b(\theta, x) = [\gamma_{B(\theta,x,0)}]. \end{aligned}$$

Proposition 3.11. *In the coordinates described above, the Liouville measure and current take the form:*

$$\begin{aligned} d\lambda &= \sin\theta d\theta dx dy \\ dm &= \sin\theta d\theta dx. \end{aligned}$$

Before proving this proposition, we provide a coordinate free characterization of the 1-form α. Let $V \in \mathfrak{X}(T\widetilde{M})$ be the vector field defined by

$$\begin{aligned} V\colon T\widetilde{M} &\to T(T\widetilde{M}) \\ (p, v) &\to v \oplus \vec{0} \end{aligned}$$

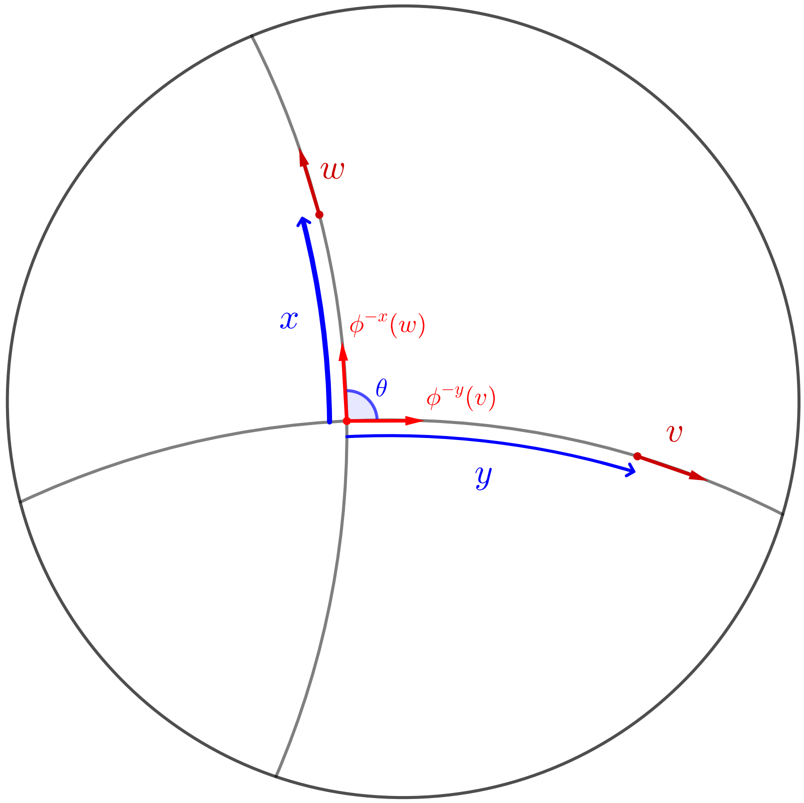

Figure 3.4: A sketch description of the local coordinates for some $v \in U_w \subset T^1\widetilde{M}$.

where we use the horizontal-vertical splitting of the tangent bundle $T_{(p,v)}(T\widetilde{M}) \cong T_p\widetilde{M} \oplus T_p\widetilde{M}$ provided by the Levi-Civita connection (see Appendix A). That is, we may write V as:

$$V = V_{hor} + V_{vert}$$

where $V_{hor}(v) = v$ is the horizontal component and $V_{vert}(v) = 0$ is the vertical component. We claim the following.

Lemma 3.12 (A coordinate free characterization of α)**.** *Let $\hat{g}$ denote the Sasaki metric on $T\widetilde{M}$, and let V be the vector field as defined above. Then as a map from vector fields to (differentiable) functions, $\alpha \in \Omega^1(T\widetilde{M})$ is exactly:*

$$\alpha \colon \mathfrak{X}^1(T\widetilde{M}) \to C^1(T\widetilde{M})$$
$$\alpha = \iota_V \hat{g} = \hat{g}(V, \cdot).$$

Moreover, it follows that the 2-form $d\alpha$ becomes:

$$\begin{aligned}
d\alpha &: \mathfrak{X}(T\widetilde{M}) \times \mathfrak{X}(T\widetilde{M}) \to C^1(T\widetilde{M}) \\
d\alpha(X,Y) &= g(\nabla_{X_{hor}} V_{hor}, Y_{hor}) - g(\nabla_{Y_{hor}} V_{hor}, X_{hor}) \\
&= g(\kappa(X), T\pi(Y)) - g(\kappa(Y), T\pi(X)) \ .
\end{aligned}$$

Proof. Recalling the fiberwise definition, we have that if $X \in \mathfrak{X}(T\widetilde{M})$ is a vector field, then at any $(p,v) \in T\widetilde{M}$:

$$\begin{aligned}
\alpha(X)|_{(p,v)} &= \alpha_{(p,v)}(X_{(p,v)}) \\
&= g(v, T_{(p,v)}\pi(X_{(p,v)})) \\
&= g\left(T_{(p,v)}\pi(V_{(p,v)}), T_{(p,v)}\pi(X_{(p,v)})\right) + 0 \\
&= g\left(T_{(p,v)}\pi(V_{(p,v)}), T_{(p,v)}\pi(X_{(p,v)})\right) + g\left(0, \kappa_v X_{(p,v)}\right) \\
&= g\left(T_{(p,v)}\pi(V_{(p,v)}), T_{(p,v)}\pi(X_{(p,v)})\right) + g\left(\kappa_v V_{(p,v)}, \kappa_v X_{(p,v)}\right) \\
&= \hat{g}(V_{(p,v)}, X_{(p,v)}) \\
&= \iota_V \hat{g}(X) \ |_{(p,v)}
\end{aligned}$$

where $\kappa \colon T(T\widetilde{M}) \to T\widetilde{M}$ denotes the connector-map defined by the Levi-Civita connection (see Appendix A).

It follows that for the 2-form $d\alpha$, using the coordinate-free definition of the exterior derivative, for any X, Y vector fields:

$$\begin{aligned}
d\alpha(X,Y) &= X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \\
&= X(\hat{g}(V,Y)) - Y(\hat{g}(V,X)) - g(V_{hor}, [X,Y]_H) \\
&= X(g(V_{hor}, Y_{hor})) - Y(g(V_{hor}, X_{hor})) - g(V_{hor}, [X_{hor}, Y_{hor}]) \qquad (V_{hor} \perp X_{vert}, Y_{vert};\ V_{vert} = \\
&= X_{hor}(g(V_{hor}, Y_{hor})) - Y_{hor}(g(V_{hor}, X_{hor})) - g(V_{hor}, [X_{hor}, Y_{hor}]) \\
&= g(\nabla_{X_{hor}} V_{hor}, Y_{hor}) - g(\nabla_{Y_{hor}} V_{hor}, X_{hor}) \\
&\quad + [g(V_{hor}, \nabla_{X_{hor}} Y_{hor} - \nabla_{Y_{hor}} X_{hor}) - g(V_{hor}, [X_{hor}, Y_{hor}])] \\
&= g(\nabla_{X_{hor}} V_{hor}, Y_{hor}) - g(\nabla_{Y_{hor}} V_{hor}, X_{hor})
\end{aligned}$$

where in the last line we have used the fact that the Levi-Civita connection is torsion free: $\nabla_{X_{hor}} Y_{hor} - \nabla_{Y_{hor}} X_{hor} = [X_{hor}, Y_{hor}]$. One should also note that here $X_{hor}, Y_{hor}, V_{hor}$ are regarded as vector fields on $\widetilde{M}$, since they live in the horizontal subspace of the $T(T\widetilde{M})$. This is in fact necessary, since the Levi-Civita connection is defined only for the C^1 metric g on $\widetilde{M}$, and not for the C^0 Sasaki metric on $T\widetilde{M}$.

It should also be clear that $\nabla_{X_{hor}} V_{hor} = \kappa(X_{hor})$. Indeed, at any point $(p,v) \in T\widetilde{M}$, if $X(p,v)$ is given by the tangent to some curve in $T\widetilde{M}$, say:

$$\beta(t) = (p(t), v(t))$$

such that $\beta(0) = (p(0), v(0)) = (p, v) =: V_{hor}(p, v)$, then

$$X_{hor}(p, v) = \frac{dp}{dt}(0).$$

Hence, in $T(T\widetilde{M}) \cong T\widetilde{M} \oplus T\widetilde{M}$:

$$\nabla_{X_{hor}} V_{hor}\Big|_{(p,v)} = \nabla_{\frac{dp}{dt}} v\Big|_{t=0} \oplus \vec{0} = \kappa_v(X_{hor}(p, v)) \oplus \vec{0}\ .$$

Regarding this as a vector field on $\widetilde{M}$, taking values in $T\widetilde{M}$:

$$\nabla_{X_{hor}} V_{hor}\Big|_{p} = \kappa_v(X_{hor})\Big|_{p}\ .$$

□

The above Lemma provides a useful way of computing the components of $d\alpha$ in the special coordinate system.

Proof of Proposition 3.11. Since dm is the "projection" of $d\alpha$ from $T^1\widetilde{M}$ onto $\mathcal{G}$, we can first show that $d\alpha = \sin\theta d\theta \wedge dx$ in the coordinates $B\colon (0, \pi) \times \mathbb{R} \times \mathbb{R} \to T^1\widetilde{M}$.

Fix $w \in T^1\widetilde{M}$ to define the local chart (U_w, B), and let $v \in U_w$. Consider the coordinate curves going through v, defined at $(\theta, x, y) = B^{-1}(v)$ by:

$$\begin{cases} \beta_\theta(t) = B(\theta + t, x, y) = (p_\theta(t), v_\theta(t)) \\ \beta_x(t) = B(\theta, x + t, y) = (p_x(t), v_x(t)) \\ \beta_y(t) = B(\theta, x, y + t) = (p_y(t), v_y(t)) \end{cases}$$

where we have denoted $t \mapsto p_i(t) \in \widetilde{M}$ the curve in the base and $t \mapsto v_i(t) \in T^1_{p_i(t)}\widetilde{M}$ the curve in the fibers, for each $i = \theta, x, y$. These curves satisfy:

$$\begin{cases} \beta_\theta(0) = v; \quad \beta'_\theta(0) = \dfrac{\partial}{\partial\theta} \\ \beta_x(0) = v; \quad \beta'_x(0) = \dfrac{\partial}{\partial x} \\ \beta_y(0) = v; \quad \beta'_y(0) = \dfrac{\partial}{\partial y} \end{cases}$$

where $\partial/\partial\theta$, $\partial/\partial x$ and $\partial/\partial y$ are understood to be the coordinate vector fields *evaluated* at $v \in T^1\widetilde{M}$. We can split these vectors into their horizontal and vertical components:

- The curve $t \mapsto \beta_\theta(t)$ traces a path in a fixed fiber $T^1_p\widetilde{M}$, hence $\beta'_\theta(0) = \partial/\partial\theta$ must live in the *vertical component* of $T_{(p,v)}(T\widetilde{M})$. Equivalently, the horizontal component is zero:

$$T_{(p,v)}\pi\left(\frac{\partial}{\partial\theta}\right) = T_{(p,v)}\pi\left(\beta'_\theta(0)\right) = 0 \ .$$

  Since $v \in T^1\widetilde{M}$, we may regard v as living in the horizontal component of $T_{(p,v)}(T\widetilde{M})$. Hence, as the definition of the Sasaki metric $\hat{g}$ on $T^1\widetilde{M}$ requires that the vertical and horizontal subbundles are perpendicular:

$$g\left(\kappa_v(\partial/\partial\theta), v\right) = \hat{g}(\partial/\partial\theta, v) = 0 \ .$$

  In other words,

$$\kappa_v\left(\frac{\partial}{\partial\theta}\right) = \frac{\pi}{2} \cdot v$$

  which is the vector obtained by rotating v by $\pi/2$ in $T_{(p,v)}(T\widetilde{M})$.

- The curve $t \mapsto \beta_x(t)$ has a slightly more complicated description. It describes a unit tangent vector $\beta_x(t) \in T^1\widetilde{M}$ such that $\phi^{-(x+t)}(w)$ and $\phi^{-y}(\beta_x(t))$ have the same base-point and make an angle θ for all t, where θ, x, y are fixed as above.

  We denote $\beta_x(t) = (p(t), v(t))$ where $p(t)$ is the base point, and $v(t) \in T^1_{p(t)}\widetilde{M}$. In a similar fashion, we also write $\phi^{-y}(\beta_x(t)) = (q(t), u(t))$. It should thus be clear that:

$$p(t) = \exp_{q(t)}(y \cdot u(t))$$

  where $\exp_{q(t)} \colon T_{q(t)}\widetilde{M} \to \widetilde{M}$ is the exponential map, and $y \cdot u(t) \in T_{q(t)}\widetilde{M}$ is the scaling of $u(t)$ by y. (One may wish to look at Figure 3.5 for visual aid).

  Recall that the tangent map may equivalently be defined on the tangent bundle, $\exp \colon T\widetilde{M} \to \widetilde{M}$, by:

$$\exp(q, u) = \exp_q(u).$$

  We may thus denote:

$$p(t) = \exp(y \cdot \phi^{-y}(\beta_x(t))) := \exp(q(t), y \cdot u(t)) \ .$$

  Recalling the splitting of $T(TM)$ into its horizontal and vertical subbundles, $T(T\widetilde{M}) \equiv H(T\widetilde{M}) \oplus V(T\widetilde{M}) \cong T\widetilde{M} \oplus T\widetilde{M}$, the tangent map of exp may be written as $T\exp \colon T\widetilde{M} \oplus T\widetilde{M} \to T\widetilde{M}$. Hence, applying the chain rule to $dp(t)/dt$:

$$\frac{dp}{dt}(t) = T\exp\left(\frac{dq}{dt}(t) + y \cdot \frac{Du}{dt}(t)\right) = T_{y \cdot u(t)}\exp_{q(t)}\left(\frac{dq}{dt}(t)\right)$$

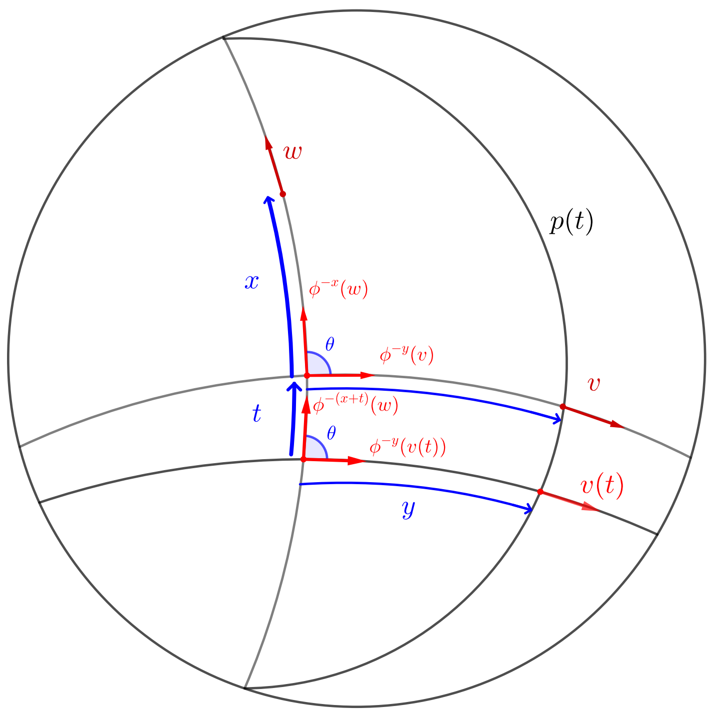

Figure 3.5: Description of the curve $t \to \beta_x(t) = (p(t), v(t))$. Note that $t \to p(t)$ must asymptote to the same boundary points as γ_w, since each $p(t)$ is always a fixed distance y away from γ_w.

since $Du(t)/dt = 0$, as $u(t)$ is parallel transported along $q(t)$ (it always makes an angle θ with the geodesic γ_w, of which $q(t)$ is just a reparametrization).

Moreover, as $u(t)$ is a unit-tangent vector to the geodesic $\gamma_{v(t)}$ at the point $q(t)$, we have:

$$T_{y\cdot u(t)} \exp_{q(t)}(u(t)) = \phi^y(u(t)) = \phi^y \phi^{-y}(v(t)) = v(t) \ .$$

By Gauss's Lemma (see Appendix A), it follows that, for all $t \in \mathbb{R}$:

$$\begin{aligned}
\left\langle \frac{dp}{dt}(t), v(t) \right\rangle &:= \left\langle T_{y\cdot u(t)} \exp_{q(t)} \left(\frac{dq}{dt}(t) \right), T_{y\cdot u(t)} \exp_{q(t)}(u(t)) \right\rangle_{p(t)} \\
&= \left\langle \frac{dq}{dt}(t), u(t) \right\rangle_{q(t)} \\
&= \cos\theta \ .
\end{aligned}$$

To be clear, the second line is due to Gauss' Lemma, while the third/last is the assumption made at the beginning (dq/dt is tangent to the geodesic γ_w, which makes an angle θ with $\phi^{-y}(v(t)) = (q(t), u(t))$). In other words, for all $t \in \mathbb{R}$:

$$\frac{dp}{dt}(t) = \theta \cdot v(t) \ .$$

Since this holds for all $t \in \mathbb{R}$, we have that $v(t)$ is a parallel vector field along the curve $p(t)$, hence:

$$\nabla_{\frac{dp}{dt}} v(t) = 0$$

It follows that (taking $t = 0$ above):

$$T_{(p,v)}\pi \left(\frac{\partial}{\partial x} \right) = \frac{dp}{dt}(0) = \theta \cdot v$$

$$\kappa_v \left(\frac{\partial}{\partial x} \right) = \nabla_{\frac{dp}{dt}} v(t) \Big|_{t=0} = 0.$$

- The curve $t \mapsto \beta_y(t)$ simply translates the vector $v = B(\theta, x, y)$ along the geodesic γ_v; in particular:

  $$\beta_y(t) = \phi^t(v)$$

  Hence

  $$\beta_y'(0) = X_E(v)$$

  where X_E is the geodesic field. Since $(X_E)_{hor}(v) = v$ (recall §3.1; Geodesic Flow), it follows that:

  $$T_{(p,v)}\pi \left(\frac{\partial}{\partial y} \right) = T_{(p,v)}\pi(\beta_y'(0)) = v.$$

Moreover, $\beta_y(t)$ is a geodesic in $T^1\widetilde{M}$, hence:

$$\kappa_v\left(\frac{\partial}{\partial y}\right) = \nabla_v \phi^t(v)|_{t=0} = 0.$$

We may thus compute the components of $d\alpha$ using the formula in Lemma 3.12:

$$\begin{aligned}
d\alpha\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x}\right) &= \langle \pi/2\cdot v, \theta\cdot v\rangle - \langle 0,0\rangle = \cos(\pi/2-\theta) = \sin\theta \\
d\alpha\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y}\right) &= \langle \pi/2\cdot v, v\rangle - \langle 0,0\rangle = 0 \\
d\alpha\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= \langle 0, v\rangle - \langle \theta\cdot v, 0\rangle = 0\ .
\end{aligned}$$

Since $d\alpha$ is skew-symmetric, this gives all components. Hence:

$$d\alpha = \sin\theta d\theta \wedge dx\ .$$

Now recall that the Liouville Measure has local product structure of Riemannian-volume on the base times arc-length on the fibers. In these local coordinates:

$$d\lambda = \sqrt{g_{xx}g_{yy} - g_{xy}g_{yx}} dx\wedge dy\wedge d\theta\ .$$

As we have seen above, ∂_x and ∂_y have only a horizontal component, hence:

$$\begin{aligned}
g_{xy} = g_{yx} &= \langle \partial_x, \partial_y\rangle = \langle \theta\cdot v, v\rangle = \cos\theta \\
g_{xx} &= \langle \theta\cdot v, \theta\cdot v\rangle = \|\theta\cdot v\|^2 = 1 \\
g_{yy} &= \langle v, v\rangle = \|v\|^2 = 1\ .
\end{aligned}$$

Hence $\det g = 1 - \cos^2\theta$. It follows that:

$$d\lambda = \sin\theta d\theta\wedge dx\wedge dy.$$

□

Chapter 4

Otal's Theorem Generalized

Recall the Proposition that we want to prove:

Theorem 1.3. *Let M be a closed smooth surface, let $k > 0$, and let g, h be C^1 Riemannian metrics having $C^{0,1}$ geodesic flow*[1]*, and which make M a locally CAT$(-k)$ space.*[2] *If g, h have equal marked length spectra, then they are isometric.*

For the remainder of this chapter, these surfaces will be denoted $(M, g), (M, h)$, and their universal covers (with the respective lifted Riemannian metrics) will be denoted $(\widetilde{M}, g), (\widetilde{M}, h)$. We recall that the marked length spectra $\mathcal{L}_g, \mathcal{L}_h$ are functions from the set of free homotopy classes of loops in M to $\mathbb{R}_{\geq 0}$, defined on each homotopy class by taking the length of the unique geodesic representative. We will use the notation $\mathcal{L}_g([\gamma]) = l_g(\gamma_g)$, where $[\gamma]$ is a free homotopy class of loops, $\gamma_g \in [\gamma]$ is the unique geodesic representative, and l_g is the length functional given by g. Whenever the fundamental group $\pi_1(M) = \pi_1(M, x_0)$ is mentioned, we take it as implicitly understood that there is a choice of base point $x_0 \in M$, fixed for the entirety of this chapter. We note also that the geodesic flow on locally CAT$(-k)$ spaces is Anosov, hence all of Chapter 3 from §3.1.1 onwards applies.

The proof of Theorem 1.3 comes in two parts, following closely arguments seen in [Otal, 1990], [Butt, 2022] and the lecture notes [Wilkinson, 2014]. The first step is to introduce a correspondence between the geodesics in $(\widetilde{M}, g), (\widetilde{M}, h)$ which, due to the equality of the marked-length-spectra, preserves the Liouville current[3]. The second part of the proof asserts that, since the correspondence of geodesics preserves the Liouville current, it must send triply intersecting geodesics in $(\widetilde{M}, g)$ to triply

[1]We mean the map $\phi_t \colon M \times \mathbb{R} \to M$.

[2]Locally CAT$(-k)$ means each point has a neighborhood which is CAT$(-k)$. The universal cover of a locally CAT$(-k)$ space, with the lifted metric, is (globally) CAT$(-k)$; this is the Cartan-Hadamard theorem for CAT(0) spaces ([Bridson and Haefliger, 1999, Theorem 4.1]). Globally CAT$(-k)$ spaces are always contractible ([Bridson and Haefliger, 1999, Corollary 1.5]).

[3]In contrast, [Wilkinson, 2014] upgrades the correspondence to not just an orbit equivalence of geodesic flows, but a conjugacy of geodesic flow.

intersecting geodesics in $(\widetilde{M}, h)$; this induces an isometry $\tilde{f}: (\widetilde{M}, g) \to (\widetilde{M}, h)$, which is $\pi_1(M)$-equivariant and hence projects to an isometry $f: (M, g) \to (M, h)$.

4.1 Volume Preserving Correspondence of Geodesics

We will now establish a correspondence between the geodesics of $(\widetilde{M}, g)$ and $(\widetilde{M}, h)$, analogous to [Butt, 2022, Construction 2.1]. Beginning with the fact that the surface M in our setting is compact, we will show that the spaces $(\widetilde{M}, g)$ and $(\widetilde{M}, h)$ are *quasi-isometric*, and hence that the *Gromov boundaries* $(\partial\widetilde{M}, g)$, $(\partial\widetilde{M}, h)$ are the same. The "identity" on the double boundary $\mathrm{Id}: (\partial^2\widetilde{M}, g) \to (\partial^2\widetilde{M}, h)$ is thus well-defined, sending boundary points to themselves $(\xi, \eta) \mapsto (\xi, \eta)$. Since $\widetilde{M}$ equipped with either metric is a CAT$(-k)$ space, every geodesic is specified by a unique pair of points on the respective boundary (indeed $\mathcal{G} \cong \partial^2\widetilde{M}$ as discussed in the previous chapter), and hence the correspondence of geodesics is exactly the identity above. Note that even if two geodesics with respect to either metric happen to be identified by the same points on the boundary, they are not necessarily the same curves in $\widetilde{M}$. We thus employ the notation $(\xi, \eta)_g \in (\partial^2\widetilde{M}, g) \cong \mathcal{G}_g$ to specify a geodesic with respect to g, and likewise $(\xi, \eta)_h \in (\partial^2\widetilde{M}, h) \cong \mathcal{G}_h$ for h.

Lemma 4.1. *The CAT$(-k)$ spaces $(\widetilde{M}, g)$,$(\widetilde{M}, h)$ have a $\pi_1(M)$-equivariant homeomorphism between their spaces of geodesics*

$$\begin{aligned} \Phi: (\partial^2\widetilde{M}, g) &\to (\partial^2\widetilde{M}, h) \\ (\xi, \eta)_g &\mapsto (\xi, \eta)_h \ . \end{aligned}$$

Proof. We first show that the identity on the universal cover is a quasi-isometry with respect to the distance functions of either metric. Since M is compact, the identity $\mathrm{Id}: M \to M$ is uniformly bounded with respect to both of the two different distance functions, with some bounding constant $c > 0$. That is, for all $x, y \in M$:

$$\begin{aligned} d_h(x, y) &\leq c \cdot d_g(x, y) \\ d_g(x, y) &\leq c \cdot d_h(x, y) \ . \end{aligned}$$

Since the identity on M lifts to the identity on the universal cover $\widetilde{M}$, the above is true for all $x, y \in \widetilde{M}$ as well, and hence:

$$\frac{1}{c} \cdot d_g(x, y) \leq d_h(x, y) \leq c \cdot d_g(x, y) \ .$$

In particular, $\mathrm{Id}: (\widetilde{M}, g) \to (\widetilde{M}, h)$ is a quasi-isometry.

We now show that the Gromov boundaries with respect to either metric are the same. By definition, the Gromov boundary $(\partial\widetilde{M}, g)$ is the collection of equivalence classes of quasi-geodesic rays $[\gamma\colon \mathbb{R}_{\geq 0} \to \widetilde{M}]$, where equivalence is given by being *asymptotic*:

$$\gamma \sim \gamma' \iff \exists K \geq 0\colon d_g(\gamma(t), \gamma'(t)) \leq K \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Note that by defintion, a quasi-geodesic in M is a quasi-isometric embedding of $\mathbb{R}_{\geq 0}$ into M. By the quasi-isometry established above, it follows that γ is a quasi-geodesic with respect to g if and only if it is a quasi-geodesic with respect to h. Moreover, we find that two quasi-geodesic rays γ,γ' are asymptotic with respect to g if and only if they are asymptotic with respect to h. Indeed:

$$d_h(\gamma(t), \gamma'(t)) \leq c \cdot d_g(\gamma(t), \gamma'(t)) \leq c \cdot K \quad \forall t \in \mathbb{R}_{\geq 0}\ .$$

The converse case follows using the other side of the inequality given by the quasi-isometry. Hence, as sets, the equivalence classes of asymptotic quasi-geodesic rays with respect to g are exactly the same as the equivalence classes of asymptotic quasi-geodesic rays with respect to h; i.e:

$$(\partial\widetilde{M}, g) = (\partial\widetilde{M}, h)\ .$$

And hence

$$(\partial^2\widetilde{M}, g) = (\partial^2\widetilde{M}, h).$$

Since the spaces $(\widetilde{M}, g), (\widetilde{M}, h)$ are CAT$(-k)$, and hence geodesics are uniquely specified by pairs of points on the boundary, the claim follows. Moreover, the map Φ is necessarily $\pi_1(M)$-equivariant and a homeomorphism, since it is given by the identity on the Gromov boundary. $\square$

The above correspondence of geodesics preserves the Liouville current. To arrive at this result, we will first address a couple of useful lemmas. We also introduce the following useful objects. Define the space of geodesic segments $\hat{\mathcal{G}} = \{I\colon [a,b] \to \widetilde{M} \mid I \text{ is a geodesic segment}\}$. Define a function:

$$\begin{cases} G\colon \hat{\mathcal{G}} \to \mathcal{G} \\ I \mapsto G(I) = \{\gamma \in \mathcal{G} \mid \gamma \text{ intersects } I \text{ transversally}\}\ . \end{cases}$$

Lemma 4.2 (Compare [Otal, 1990, Proposition 3]). *Let $[\gamma\colon \mathbb{S}^1 \to M]$ be a free homotopy class of loops, with γ its unique geodesic representative. Let $I\colon [0,T] \to \widetilde{M}$ be a geodesic segment that projects (bijectively) to γ in the base M; in particular $T = l(\gamma)$ and the image of I is contained in some fundamental domain for the action of $\pi_1(M)$.[4] If $G(I) \subset \mathcal{G} \cong \partial^2\widetilde{M}$ denotes the set of geodesics which intersect I, then:*

$$\mathcal{L}([\gamma]) = \frac{1}{2} m(G(I))$$

where m denotes the Liouville current taken as a measure on $\mathcal{G}$.

[4] We call such a segment I a *fundamental domain* of γ.

Proof. Here $T = l(\gamma) = \mathcal{L}([\gamma])$ is the period of γ, which is also the value of the marked length spectrum evaluated at the free homotopy class $[\gamma]$. Now, in the coordinates $b\colon (\theta, x) \mapsto b(\theta, x) \in \mathcal{G}$ outlined in §3.4, the set $G(I)$ can be parametrized as:

$$b^{-1}\left(G(I)\right) = (0, \pi) \times [0, T] \ .$$

Hence:

$$m(G(I)) = \int_0^T \int_0^\pi \sin\theta d\theta dx = 2T = 2\mathcal{L}([\gamma]) \ .$$

□

Note that the above is independent of the choice of I; the map $I \mapsto m(G(I))$ is invariant under deck-transformations. That is, for any $\gamma' \in \pi_1(M)$:

$$m(G(I)) = m(G(\gamma' \cdot I)).$$

Lemma 4.3 (Compare [Otal, 1990, Theorem 2]). *The Liouville current is specified by its value on sets of the form $G(I)$. That is: if $m(G(I))$ is known for every free homotopy class of loops $[\gamma]$ (with unique geodesic representative γ and fundamental domain I), then the value of the Liouville current on every measurable subset of $\mathcal{G}$ is known.*

Proof. We use the characterization of the Liouville current as the cross-ratio (see Lemma 3.10). Since the Borel σ-algebra of $\mathcal{G}$ is generated by "rectangles" $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \subset \partial^2\widetilde{M} \cong \mathcal{G}$, it is enough to show that the Liouville current of any such rectangle can be measured. Moreover, since the geodesic flow $\phi^t\colon T^1M \to T^1M$ is transitive (for the proof of this fact on compact locally CAT$(-k)$ spaces, see [Constantine et al., 2019, Lemma 2.6]), there exists a geodesic γ which is dense in T^1M; it follows that the set of forward and backwards asymptotes of all lifts of γ

$$\{(\mathcal{F}^-([\tilde{\gamma}]), \mathcal{F}^+([\tilde{\gamma}])) \mid \tilde{\gamma} \text{ is a lift of } \gamma\} \subset \partial^2\widetilde{M} \cong \mathcal{G}$$

is dense in $\partial^2\widetilde{M} \cong \mathcal{G}$ (since γ comes arbitrarily close to any $v \in T^1M$). It thus suffices to consider only "rectangles" $[\xi_1, \xi_2] \times [\eta_1, \eta_2]$ where $(\xi_1, \eta_1), (\xi_2, \eta_2)$ are some lifts of γ. Let us denote the geodesics in $\widetilde{M}$ by $(\xi_i, \eta_j) = \tilde{\gamma}_{ij}$, and the geodesics they descend to in M by γ_{ij}. Note that while $\tilde{\gamma}_{11}$ and $\tilde{\gamma}_{22}$ are lifts of the dense orbit γ (i.e. $\gamma_{11} = \gamma_{22} = \gamma$), we do not a priori have that $\tilde{\gamma}_{12}$ and $\tilde{\gamma}_{21}$ are also lifts of γ. However, it is true that the orbits they descend to, γ_{12} and γ_{21}, are dense in M; indeed, each of the lifts $\tilde{\gamma}_{12}$ and $\tilde{\gamma}_{21}$ converges in the past to a lift of γ, and in the future to another lift of γ, and hence are dense in both the future and the past.

The Liouville measure of such "rectangles" is given by the cross-ratio:

$$m([\xi_1,\xi_2]\times[\eta_1,\eta_2]) = [\xi_1,\xi_2;\eta_1,\eta_2] := l(I_{11}) + l(I_{22}) - l(I_{12}) - l(I_{21})$$

where the I_{ij} are the segments of $\tilde{\gamma}_{ij} = (\xi_i, \eta_j)$, respectively, as prescribed by the definition of the cross-ratio (refer to relevant section of §3 and Figure 3.3). Our goal is to shadow each of the geodesic segments I_{ij} with the fundamental domains I'_{ij} of some periodic geodesics, in order to approximate the cross-ratio $[\xi_1,\xi_2;\eta_1,\eta_2]$ arbitrarily close by sets of the form $m(G(I'_{ij}))$, as claimed.

Note that since each of the (ξ_i,η_j) descends to an orbit γ_{ij} dense in the future and the past, then γ_{ij} is ε-pseudo-periodic in both the future and the past[5]. Moreover, the density of γ_{ij} implies that the pseudo-period can be taken as large as desired. Hence the segments I_{ij} of the lifts $\tilde{\gamma}_{ij}$ can also be taken as long as desired, and such that they descend to (segments of γ_{ij} which are) ε-pseudo-periodic orbits in M.

Claim: Let $\varepsilon > 0$. Let $\tilde{\gamma}$ be a lift of a dense orbit γ. Let $T > 0$ be large enough to satisfy the hypothesis of the Anosov Closing Lemma. If $I\colon [0,T] \to \widetilde{M}$ is any segment of $\tilde{\gamma}$ descending to an ε-pseudo-periodic segment of γ in M, then there exists a lift $\tilde{\gamma}'$ of a periodic geodesic γ' such that:

$$|m(G(I)) - m(G(I'))| < \varepsilon$$

where I' is any fundamental domain of γ'.

Proof of Claim: The Anosov closing lemma implies that there exists a periodic geodesic γ' with period T_0 (ε-close to T) which shadows γ:

$$\forall t \in [0, \min(T,T_0)]\colon\ d_{\hat{g}}(\dot{\gamma}(t), \dot{\gamma}'(t)) < \varepsilon.$$

Hence there exists a fundamental domain I' of γ' that shadows the segment I:

$$\forall t \in [0, \min(T,T_0)]\colon\ d_{\hat{g}}(\dot{I}(t), \dot{I}'(t)) < \varepsilon.$$

It follows that the lengths are ε-close, which by the same reasoning as in Lemma 4.2 means:

$$\big|m(G(I)) - m(G(I'))\big| = \big|l(I) - l(I')\big| < \varepsilon.$$

$\square$

As the claim holds for each γ_{ij} (having fundamental domains I_{ij}, resp.), one has that there exist periodic geodesics γ'_{ij} (having fundamental domains I'_{ij}, resp.) such that:

$$\Big|m([\xi_1,\xi_2]\times[\eta_1,\eta_2]) - [m(G(I'_{11}) + m(G(I'_{22}) - m(G(I'_{12}) - m(G(I'_{21}))]\Big| \leq 4\varepsilon$$

[5] ε-pseudo-periodic means that at some (possibly large) time T, the orbit returns ε-close to its starting point. We call T the pseudo-period.

where we have used the triangle inequality. We thus have that the Liouville current of the box $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \subset \mathcal{G}$ is approximated arbitrarily close by a weighted sum of the Liouville currents of sets of the form $G(I'_{ij})$, where each I'_{ij} is a fundamental domain for some periodic geodesic. This concludes the proof. $\square$

We are now prepared to prove that the correspondence of geodesics preserves the Liouville current. We denote $\mathcal{G}_g \cong (\partial^2 \widetilde{M}, g)$ and $\mathcal{G}_h \cong (\partial^2 \widetilde{M}, h)$, the spaces of geodesics with respect to g, h.

Proposition 4.4. *If g, h have equal marked length spectra, then the correspondence of geodesics Φ preserves the Liouville current (as a measure); i.e. for every measurable subset $F \subset \mathcal{G}_g$:*

$$m_h(\Phi(F)) = m_g(F).$$

Proof. The Borel σ-algebra of $\mathcal{G}_g \cong (\partial^2 \widetilde{M}, g)$ is generated by rectangles of the form $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \subset \partial^2 \widetilde{M}$. By the Lemma 4.3, the Liouville current of such sets is approximated arbitrarily close by the Liouville current of sets of the form $G(I)$, where I is some geodesic segment in a fundamental domain for the action of $\pi_1(M)$. Thus, it suffices to show that the above equality holds for every set of the form $G(I)$.

Now by assumption, the metrics g and h have the same marked length spectrum. That is, for any free homotopy class of loops $[\gamma]$:

$$\mathcal{L}_g([\gamma]) = \mathcal{L}_h([\gamma]) \ .$$

Meaning the geodesic representatives with respect to each metric, γ_g and γ_h, have the same length:

$$l_g(\gamma_g) = l_h(\gamma_h).$$

It follows by Lemma 4.2, if I_g, I_h are geodesic segments projecting (bijectively) onto γ_g, γ_h, respectively, then:

$$m_g(G(I_g)) = m_h(G(I_h)).$$

Recall that the free homotopy class $[\gamma]$ corresponds to a conjugacy class of $\pi_1(M)$. In particular, there exists some $\gamma \in \pi_1(M)$ free homotopic to both γ_g and γ_h. Hence, there exist unique lifts $\tilde{\gamma}_g, \tilde{\gamma}_h$ of γ_g, γ_h respectively, such that $\gamma \cdot \tilde{\gamma}_g = \tilde{\gamma}_g$ and $\gamma \cdot \tilde{\gamma}_h = \tilde{\gamma}_h$. It follows that $\Phi(\tilde{\gamma}_g) = \tilde{\gamma}_g$ and $\Phi(\tilde{\gamma}_h) = \tilde{\gamma}_h$, since Φ is $\pi_1(M)$-equivariant. By the definition of the $\pi_1(M)$-action on $\partial \widetilde{M}$, γ fixes a unique pair of points $(\xi_0, \eta_0) \in \partial^2 \widetilde{M}$. Hence, $\tilde{\gamma}_g$ and $\tilde{\gamma}_h$ must have the same boundary points at infinity:

$$\Phi(\tilde{\gamma}_g) = \tilde{\gamma}_h \ .$$

Claim: We may choose geodesic segments I_g, I_h of $\tilde{\gamma}_g, \tilde{\gamma}_h = \Phi(\tilde{\gamma}_g)$, respectively, such that:

$$G(I_h) = \Phi(G(I_g)).$$

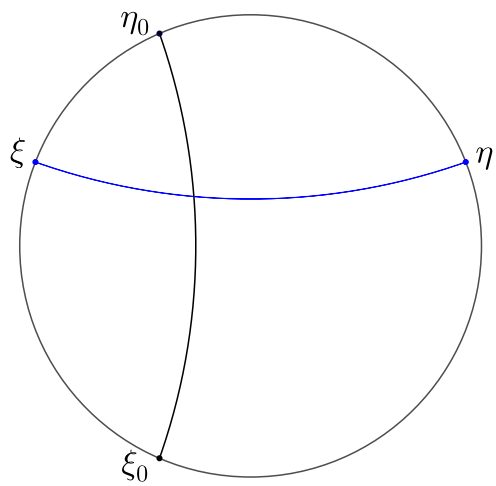

Figure 4.1: Every path between $\xi, \eta \in \mathbb{S}^1 \subset \mathbb{R}^2$ must intersect every path between $\xi_0, \eta_0 \in \mathbb{S}^1$ when positioned as in the figure, as long as the paths remain in the region in $\mathbb{R}^2$ bounded by $\mathbb{S}^1$.

Proof of Claim. Following the above, denote by $(\xi_0, \eta_0) \in \partial^2\widetilde{M}$ the common forwards and backwards asymptotes of the geodesics $\tilde{\gamma}_g, \tilde{\gamma}_h$. Further denote $(\xi_0, \eta_0)_g = \tilde{\gamma}_g$, $(\xi_0, \eta_0)_h = \tilde{\gamma}_h$. Since $\partial^2\widetilde{M} \cong \mathbb{S}^1 \times \mathbb{S}^1 \backslash \Delta$, where $\mathbb{S}^1 \equiv [0,1]/0 \sim 1$ and $\Delta = \{(\xi, \xi) \mid \xi \in \mathbb{S}^1\}$ is the diagonal, it is immediate that:

$$(\xi, \eta)_g \text{ intersects } (\xi_0, \eta_0)_g \iff \xi < \xi_0 < \eta < \eta_0 \iff (\xi, \eta)_h \text{ intersects } (\xi_0, \eta_0)_h \quad .$$

In the middle statement, by $a < b$ for $a, b \in \mathbb{S}^1 = [0,1]/0 \sim 1$ we mean that for a choice of representatives $a, b \in [0, 1)$, we have $a < b$. It should be obvious that the condition above is equivalent to obtaining an intersection between the geodesics (ξ, η) and (ξ_0, η_0); they are positioned on the circle $\mathbb{S}^1$ in such a way that *any* two paths connecting ξ to η and ξ_0 to η_0 must intersect. One may wish to look at the Figure 4.1 for an intuitive description of this fact, which is a topological property of paths between points on the circle[6].

Now, fix a fundamental domain of $\tilde{\gamma}_g$, and let $I_g \colon [0, T] \to \widetilde{M}$ be the geodesic segment along this domain. Let $x = I_g(0)$; then $\gamma \cdot x = I_g(T)$. Let α_g be a geodesic intersecting $\tilde{\gamma}_g$ at x; then $\gamma \cdot \alpha_g$ is a geodesic intersecting $\tilde{\gamma}_g$ at $\gamma \cdot x$. Further, denote by y the point of intersection between $\tilde{\gamma}_h$ and $\Phi(\alpha_g)$, and by z the point of intersection

[6]We acknowledge that the Figure 4.1 does not constitute a proof of this fact, and merely suggests that it should be obvious.

between $\tilde{\gamma}_h$ and $\Phi(\gamma \cdot \alpha_g)$. Then:

$$\begin{aligned} z &:= \Phi(\gamma \cdot \alpha_g) \cap \tilde{\gamma}_h \\ &= \gamma \cdot \Phi(\alpha_g) \cap \gamma \cdot \Phi(\tilde{\gamma}_g) \\ &= \gamma \cdot [\Phi(\alpha_g) \cap \Phi(\tilde{\gamma}_g)] \\ &=: \gamma \cdot y. \end{aligned}$$

The subset $\Phi(G(I_g)) \subset \mathcal{G}$ is exactly the set of geodesics intersecting $\tilde{\gamma}_h = \Phi(\tilde{\gamma}_g)$ between the points y and z. Since $z = \gamma \cdot y$, this segment of $\tilde{\gamma}_g$ is thus a fundamental domain for the action of $\pi_1(M)$. Hence, we may denote this segment by:

$$I_h = \Phi(G(I_g))$$

and the claim holds. □

The claim implies that

$$m_g(G(I_g)) = m_h(\Phi(G(I_g)))$$

for all geodesic segments I_g, which concludes the proof. □

4.2 Constructing the Isometry

As $\dim(M) = 2$, the homeomorphism $\Phi \colon \mathcal{G}_g \to \mathcal{G}_h$ sends a pair of intersecting geodesics to a pair of intersecting geodesics. In this section, it is our goal to show that this defines a $\pi_1(M)$-equivariant isometry of the universal covers $\tilde{\varphi} \colon (\widetilde{M}, g) \to (\widetilde{M}, h)$, defined by sending the point of intersection p between some geodesics γ_g^1, γ_g^2 to the point of intersection p' between the corresponding $\gamma_h^1 = \Phi(\gamma_g^1), \gamma_h^2 = \Phi(\gamma_g^2)$. This map would descend to an isometry of the bases $\varphi \colon (M, g) \to (M, h)$.

Such a map is only well-defined if triply intersecting geodesics are sent to triply intersecting geodesics. That is, if $\gamma^1, \gamma^2, \gamma^3$ all intersect at a common point p, then $\Phi(\gamma^1), \Phi(\gamma^2), \Phi(\gamma^3)$ also intersect at a common point p'. *A priori* however, three intersecting geodesics will be sent by Φ to three geodesics intersecting in a (possibly degenerate) geodesic triangle. We will show that the resulting triangle is always degenerate.

Let $v \in T^1M$ (the unit tangent bundle with respect to g). Let $\theta \in [0, \pi]$. Denote by γ_v and $\gamma_{\theta \cdot v}$ geodesics in $\mathcal{G}_g$ that pass through v and $\theta \cdot v$, respectively. Define $f(v, \theta)$ to be the angle between the geodesics $\Phi(\gamma_v), \Phi(\gamma_{\theta \cdot v})$. Since $\Phi : \mathcal{G}_g \to \mathcal{G}_h$ is a homeomorphism, this defines a continuous function $f \colon T^1M \times [0, \pi] \to [0, \pi]$ which fixes the endpoints: $f(v, 0) = 0$ and $f(v, \pi) = \pi$.

To avoid overloading notation, from this point forward we will take it as implicitly understood when an object is defined with respect to g or h, such as T^1M, the Liouville measure λ, and the space of geodesics $\mathcal{G}$, unless explicitly stated.

Definition 4.5. *Define the* ***average-new-angle function*** $F\colon [0,\pi] \to [0,\pi]$ *by:*

$$F(\theta) = \frac{1}{\lambda(T^1M)} \int_{T^1M} f(v,\theta) d\lambda(v).$$

We prove that F is the identity, using the following facts:

Lemma 4.6. *The function F is continuous, fixes endpoints and has the properties:*

(i) ***Symmetry about*** $\pi - \theta$***:*** $F(\pi-\theta) = \pi - F(\theta)$*. In particular,* $F(\pi/2) = \pi/2$.

(ii) ***Super additivity:*** $\forall \theta_1, \theta_2 \in [0,\pi]$ *with* $\theta_1 + \theta_2 \in [0,\pi]$*:*

$$F(\theta_1 + \theta_2) \geq F(\theta_1) + F(\theta_2).$$

Proof. F is continuous because f is continous. F also clearly fixes the endpoints; indeed $f(v,0) = 0$ and $f(v,\pi) = \pi$ for all $v \in T^1M$ necessarily, hence $F(0) = 0$ and $F(\pi) = \pi$.

Since the function f is continuous, we must have that for all $\theta \in [0,\pi]$:

$$\pi = f(v,\theta) + f(\theta \cdot v, \pi - \theta).$$

Recall that the Liouville measure has the local product structure of Riemannian volume on the base and arc length on the fibers of T^1M, which we denote $dv \wedge dA$, with dv the arc length form and dA the surface area form. We may thus integrate first with respect to the *arc in the fibers* (not to be confused with the angle above). That is, for an arbitrary fixed base point $p \in M$, integrate over all $v \in T^1_pM \cong \mathbb{S}^1$:

$$\pi \cdot 2\pi = \int_{T^1_pM} f(v,\theta)dv + \int_{T^1_pM} f(\theta\cdot v, \pi-\theta)dv = \int_{T^1_pM} f(v,\theta)dv + \int_{T^1_pM} f(v,\pi-\theta)dv$$

where in the second equality we use the fact that the arc length dv is invariant under rotation by a fixed angle θ. Next integrate over the surface M:

$$\pi \cdot 2\pi \cdot A(M) = \int_M \int_{T^1_pM} f(v,\theta)dvdA + \int_M \int_{T^1_pM} f(v,\pi-\theta)dvdA.$$

Since, by the local product structure of the Liouville measure, $\lambda(T^1M) = 2\pi \cdot A(M)$, dividing yields:

$$\pi = F(\theta) + F(\pi - \theta)$$

which is the symmetry about $\pi - \theta$.

To prove the super-additivity, recall that $\widetilde{M}$ is a $\mathrm{CAT}(-k)$ space when endowed with either metric g or h. One characterization of $\mathrm{CAT}(-k)$ spaces is that the interior angles of a geodesic triangle are all less than or equal to the corresponding interior

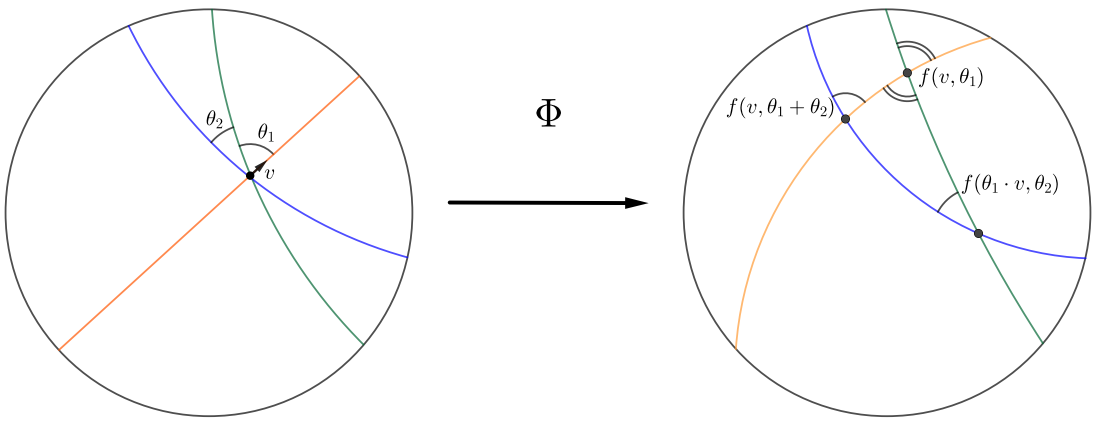

Figure 4.2: A sketch description of the mapping of three intersecting geodesics.

angles of a comparison triangle in the hyperbolic plane of constant sectional curvature $-k$ (see [Bridson and Haefliger, 1999, Proposition 1.7]). In particular, the Gauss-Bonnet theorem applied to the comparison triangle implies that the sum of interior angles is less than or equal to π, with equality if and only if the triangle is degenerate (is a point). Hence, the sum of interior angles of a geodesic triangle in a CAT$(-k)$ space is less than or equal to π, with equality if and only if the triangle is degenerate.

Now let $v \in T^1M$, let $\theta_1, \theta_2 \in [0,\pi]$ such that $\theta_1+\theta_2 \leq \pi$, and consider the geodesics $\gamma_v, \gamma_{\theta_1\cdot v}, \gamma_{(\theta_2+\theta_1)\cdot v} \in \mathcal{G}_g$, which all intersect at the same point in $\widetilde{M}$. Then by the above discussion, since $\Phi(\gamma_v), \Phi(\gamma_{\theta_1\cdot v}), \Phi(\gamma_{(\theta_2+\theta_1)\cdot v}) \in \mathcal{G}_h$ form a geodesic triangle (see Figure 4.2):

$$f(v,\theta_1) + f(\theta_1\cdot v,\theta_2) + (\pi - f(v,\theta_1+\theta_2)) \leq \pi \ .$$

Hence

$$f(v,\theta_1) + f(\theta_1\cdot v,\theta_2) \leq f(v,\theta_1+\theta_2) \ .$$

Integrating as in the definition of F, and applying a change of variable $v \to \theta_1 \cdot v$ in one of the integrals:

$$\begin{aligned} F(\theta_1)+F(\theta_2) &= \frac{1}{\lambda(T^1M)}\left[\int_{T^1M} f(v,\theta_1)d\lambda(v) + \int_{T^1M} f(v,\theta_2)d\lambda(v)\right] \\ &= \frac{1}{\lambda(T^1M)}\left[\int_{T^1M} f(v,\theta_1)d\lambda(v) + \int_{T^1M} f(\theta_1\cdot v,\theta_2)d\lambda(v)\right] \\ &\leq \frac{1}{\lambda(T^1M)}\int_{T^1M} f(v,\theta_1+\theta_2)d\lambda(v) \\ &= F(\theta_1+\theta_2). \end{aligned}$$

□

The following result of Otal (see [Otal, 1990, Proposition 7]) also holds in our context:

Lemma 4.7. *Suppose that $\mathcal{L}_g = \mathcal{L}_h$. Then for any convex function $H\colon [0,\pi] \to \mathbb{R}$, we have:*

$$\int_0^\pi H(F(\theta)) \sin\theta d\theta \leq \int_0^\pi H(\theta)\sin\theta d\theta \ .$$

Proof. Starting on the left side:

$$\begin{aligned}
\int_0^\pi H(F(\theta))\sin\theta d\theta &= \int_0^\pi H\left(\frac{1}{\lambda(T^1M)}\int_{T^1M} f(v,\theta)d\lambda(v)\right)\sin\theta d\theta \\
&\leq \frac{1}{\lambda(T^1M)}\int_0^\pi H\left(\int_{T^1M} f(v,\theta)d\lambda(v)\right)\sin\theta d\theta && \text{(Convexity of } H\text{)} \\
&\leq \frac{1}{\lambda(T^1M)}\int_0^\pi \int_{T^1M} H\left(f(v,\theta)\right)d\lambda(v)\sin\theta d\theta && \text{(Jensen's Inequality)} \\
&= \frac{1}{\lambda(T^1M)}\int_{T^1M}\int_0^\pi H\left(f(v,\theta)\right)\sin\theta d\theta d\lambda(v) && \text{(Fubini's Theorem)} \\
&= \frac{1}{\lambda(T^1M)}\int_{T^1M} k(v)d\lambda(v)
\end{aligned}$$

where we have defined the function $k(v) := \int_0^\pi H(f(v,\theta))\sin\theta d\theta$. In the paper [Constantine et al., 2019, Theorem 3.2, Lemma 4.5], it is shown that the geodesic flow on CAT$(-k)$ spaces satisfies the *weak specification property*. In particular, [Constantine et al., 2019, Lemma 4.5] in place of [Sigmund, 1972, Lemma (1)] implies that the proofs of [Sigmund, 1972, Theorem 1 and Theorem 2] hold. This guarantees that for all $f\colon T^1M \to \mathbb{R}$ continuous functions, $\forall \varepsilon > 0$, $\exists \gamma_0$ closed geodesic in M such that:

$$\left|\frac{1}{\lambda(T^1M)}\int_{T^1M} f d\lambda - \frac{1}{l_g(\gamma_0)}\int_{\gamma_0} f dt\right| < \varepsilon.$$

Applying this to the function $k(v)$, for all $\varepsilon > 0$, there exists a closed geodesic $\gamma_g\colon [0,T] \to T^1M$ (with respect to g) such that:

$$\begin{aligned}
\varepsilon &> \left|\frac{1}{\lambda(T^1M)}\int_{T^1M} k(v)d\lambda(v) - \frac{1}{l_g(\gamma_g)}\int_{\gamma_g} k(\gamma_g(t))dt\right| \\
&= \left|\frac{1}{\lambda(T^1M)}\int_{T^1M} k(v)d\lambda(v) - \frac{1}{l_g(\gamma_g)}\int_{\gamma_g}\int_0^\pi G(f(\gamma_g(t),\theta))\sin\theta d\theta dt\right| \ .
\end{aligned}$$

Claim: Denoting $\gamma_h = \Phi(\gamma_g)$, the homeomorphism $\Phi : \mathcal{G}_g \to \mathcal{G}_h$ induces a homeomorphism:

$$\tilde{\varphi}\colon \gamma_g \times (0,\pi) \to \gamma_h \times (0,\pi)$$
$$\tilde{\varphi}(\gamma_g(t),\theta) = (\gamma_h(t'), f(\gamma_g(t),\theta))$$

whose definition we will explain below.

Proof of Claim: Choose parametrizations $t \mapsto \gamma_g(t)$ and $t' \mapsto \gamma_h(t')$. Denote $v = \gamma_g(t) \in T^1\widetilde{M}$, and let $\gamma_{\theta\cdot v} \in \mathcal{G}_g$ be a geodesic going through $\theta\cdot v$. We define $t' = t'(t,\theta)$ as the number such that γ_h and $\Phi(\gamma_{\theta\cdot v})$ intersect at the base point of $\gamma_h(t')$. (See the Figure 4.3 below). □

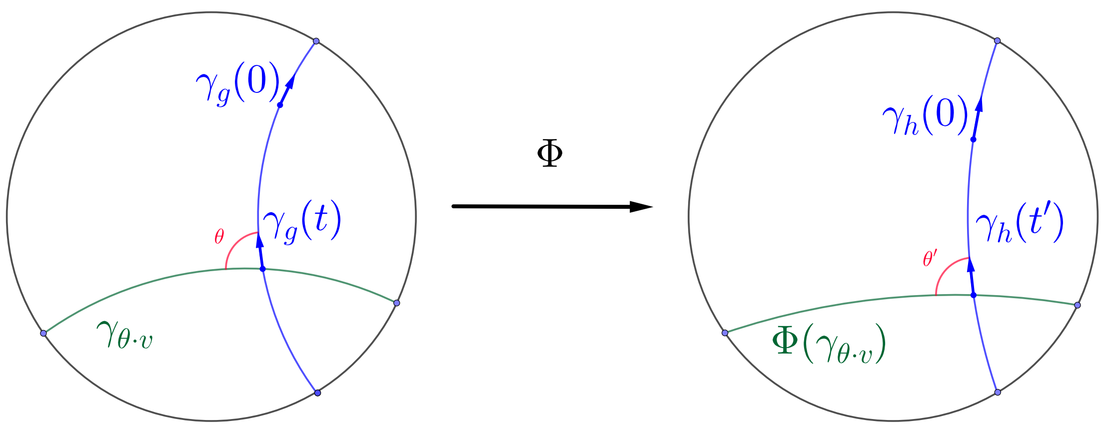

Figure 4.3: A sketch of the above definition

Note that the origins $\gamma_g(0) \in T^1\widetilde{M}_g$, $\gamma_h(0) \in T^1\widetilde{M}_h$ provide a base point for the special coordinate systems defined in the previous Chapter. Moreover, in the new coordinates (t',θ'), we have that $f(\gamma_h(t'),\theta') = \theta'$. Then, since Φ preserves the Liouville current (recall Proposition 4.4), we may perform a change of variable:

$$\begin{aligned}\int_{\gamma_g}\int_0^\pi G(f(\gamma_g(t),\theta))\sin\theta d\theta dt &= \int_{\gamma_h}\int_0^\pi G(f(\gamma_h(t'),\theta'))\sin\theta' d\theta' dt' \\ &= l_h(\gamma_h)\int_0^\pi G(\theta')\sin\theta' d\theta' \ .\end{aligned}$$

Hence, since $l_g(\gamma_g) = l_h(\gamma_h)$ (we are assuming that the marked length spectra are equal), we have that for all $\varepsilon > 0$:

$$\varepsilon > \left| \frac{1}{\lambda(T^1M)}\int_{T^1M} k(v)d\lambda(v) - \int_0^\pi G(\theta')\sin\theta' d\theta' \right| \ .$$

In other words,

$$\frac{1}{\lambda(T^1M)}\int_{T^1M} k(v)d\lambda(v) = \int_0^\pi G(\theta')\sin\theta' d\theta' = \int_0^\pi G(\theta)\sin\theta d\theta.$$

In particular, from what we started with above, this implies:

$$\int_0^\pi G(F(\theta))\sin\theta d\theta = \frac{1}{\lambda(T^1M)}\int_{T^1M} k(v)d\lambda(v) \le \int_0^\pi G(\theta)\sin\theta d\theta\ .$$

□

Proposition 4.8. $F\colon [0,\pi]\to[0,\pi]$ *is the identity.*

Proof. We make a sequence of claims to prove this fact.
Claim 1: Since F is super-additive, and hence monotone-increasing, there are three possible cases for the behavior of F near zero:

(I) $\exists a\in(0,\pi]$ s.t $F(a)=a$ and $F(\theta)>\theta\ \forall\theta\in(0,a)$.

(II) $\exists a\in(0,\pi]$ s.t $F(a)=a$ and $F(\theta)<\theta\ \forall\theta\in(0,a)$.

(III) $\exists a\in(0,\pi]$ s.t $F(\theta)=\theta\ \forall\theta\in[0,a]$.

Proof of Claim 1: Suppose that neither (I) nor (II) hold. Then, for all $\varepsilon>0$ there exists $y,z\in(0,\varepsilon)$ such that $F(y)>y$, $F(z)<z$; we may assume that $y<z$. By continuity, there must exist $x\in(y,z)\subset(0,\varepsilon)$ such that $F(x)=x$. Hence, we may construct a sequence $x_n\to\infty$ such that $F(x_n)=x_n$ for all $n\in\mathbb{N}$. It follows by superadditivity that, for all $k,n\in\mathbb{N}$:

$$F(kx_n)\ge kx_n\ .$$

For all $\varepsilon>0$, for all $x\in[0,\pi]$, there exists $k,n\in\mathbb{N}$ such that $|x-kx_n|\le\varepsilon$. By continuity of F, we have that for all $x\in[0,\pi]$:

$$F(x)\ge x.$$

Now, pick some $a=x_n$, so that $F(a)=a$. Then for all $s,t\in[0,a]$ such that $s+t=a$:

$$s+t=F(s+t)\ge F(s)+F(t)\ge s+t$$

$$\implies F(s+t)=F(s)+F(t)$$

In particular, it follows that:

$$F(a/N)=a/N$$

for all $N\in\mathbb{N}$, and further that

$$F(Ma/N)=Ma/N$$

for all $M \in \{1, \ldots, N\}$. Since the set $\{Ma/N \mid N \in \mathbb{N},\ M \in \{1, \ldots, N\}\} = \mathbb{Q} \cap [0, a]$ is dense in $[0, a]$, and F is continuous, we have that

$$F(x) = x \ \forall x \in [0, a]$$

implying that Case (III) holds. □

Claim 2: The following both hold:

(1) $\nexists a \in [0, \pi]$ s.t $F(\theta) > \theta \ \forall \theta \in (0, a)$.

(2) $\nexists a \in [0, \pi]$ s.t $F(\theta) < \theta \ \forall \theta \in (0, a)$.

Proof of Claim 2: For contradiction, suppose (1) holds for some $a \in (0, \pi]$, and choose $n > 0$ large enough such that $\pi/n < a$. Then, by the super-additivity of F:

$$F(\pi) = F(n \cdot \pi/n) \geq n \cdot F(\pi/n) > n \cdot \pi/n = \pi.$$

But this is in direct contradiction with the fact that F fixes the endpoint $F(\pi) = \pi$.

To prove (2), suppose for contradiction that there does exist an $a \in (0, \pi]$ such that $F(\theta) < \theta$ for all $\theta \in (0, a)$. Consider then the convex function $G\colon [0, \pi] \to \mathbb{R}$ defined by $G(\theta) = \sup(a - \theta, 0)$. It follows that $\forall \theta \in [0, \pi]$:

$$a - F(\theta) > a - \theta$$

and hence

$$G(F(\theta)) = \sup(a - F(\theta), 0) \geq \sup(a - \theta, 0) = G(\theta)$$

with equality if and only if $F(\theta) = \theta = 0$. Since $\sin\theta \geq 0$ for all $\theta \in [0, \pi]$, we obtain:

$$\int_0^\pi G(F(\theta)) \sin\theta d\theta > \int_0^\pi G(\theta) \sin\theta d\theta.$$

This is a contradiction of Lemma 4.7. □

Since the claim holds, we have that neither (I) nor (II) can hold; hence (III) holds: $\exists a \in (0, \pi]$ such that $F(\theta) = \theta$ for all $\theta \in [0, a]$.

Claim 3: $a \geq \pi/2$.

Proof of Claim 3: Suppose not. That is, suppose the maximal number $a \in [0, \pi]$ such that $F(\theta) = \theta$ for all $\theta \in [0, a]$ is strictly less than $\pi/2$. Then there exists some $b \in (a, \pi - a)$ such that either one of the following two hold:

(i) $F(b) < b$

(ii) $F(b) > b$

By super-additivity, (i) cannot hold. Indeed, let $n > 0$ such that $b/n < a$. Then $F(b) \geq nF(b/n) = b$, which is a contradiction.

Suppose instead that (ii) holds. We may choose $c \in [\pi - a, \pi]$ such that $c > 2a$, since $2a < \pi$ by assumption. This implies that $c - b < a$, hence $F(c-b) = c - b$. Meanwhile, by symmetry, $F(c) = c$. However, super-additivity implies:

$$F(c) \geq F(c-b) + F(b) > c - b + b = c$$

which is a contradiction. $\square$

Since the claim holds, $F(\theta) = \theta$ for all $\theta \in [0, \pi/2]$. The symmetry of F about $\pi - \theta$ guarantees that $F(\theta) = \theta$ for all $\theta \in [\pi/2, \pi]$. That is, $F(\theta) = \theta$ for all $\theta \in [0, \pi]$. $\square$

Corollary 4.9. *The map $\Phi \colon \mathcal{G}_g \to \mathcal{G}_h$ sends a triple of geodesics intersecting at a point in $(\widetilde{M}, g)$ to a triple of geodesics intersecting at a point in $(\widetilde{M}, h)$.*

Proof. Since F is the identity, it is additive. In particular, $\forall \theta_1, \theta_2 \in [0, \pi]$ with $\theta_1 + \theta_2 \leq \pi$, one has that:

$$\lambda(T^1M) \cdot F(\theta_1 + \theta_2) = \int_{T^1M} f(v, \theta_1 + \theta_2) d\lambda(v) = \int_{T^1M} f(v, \theta_1) d\lambda(v) + \int_{T^1M} f(v, \theta_2) d\lambda(v).$$

Since the Liouville measure is invariant under rotations in the fibers of T^1M, we further have

$$\int_{T^1M} \left[f(v, \theta_1 + \theta_2) - (f(v, \theta_1) + f(\theta_1 \cdot v, \theta_2)) \right] d\lambda(v) = 0 \ .$$

But recall that

$$f(v, \theta_1 + \theta_2) \geq f(v, \theta_1) + f(\theta_1 \cdot v, \theta_2)$$

for all $v \in T^1M$. Hence,

$$f(v, \theta_1 + \theta_2) = f(v, \theta_1) + f(\theta_1 \cdot v, \theta_2)$$

for all $v \in T^1M$, necessarily.

Let $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{G}_g$ intersect at a point. Choose $v \in T^1M$ and $\theta_1, \theta_2 \in [0, \pi]$ with $\theta_1 + \theta_2 \leq \pi$ such that the projections of $\gamma_1, \gamma_2, \gamma_3$ onto M pass through $v, \theta_1 \cdot v, \theta_2 \cdot \theta_1 \cdot v \in T^1M$, respectively. Without loss of generality we may assume $\theta_1 \leq \theta_2$. We may denote $\gamma_1 = \gamma_v, \gamma_2 = \gamma_{\theta_1 \cdot v}, \gamma_3 = \gamma_{\theta_1 \cdot \theta_2 \cdot v} \in \mathcal{G}_g$. Since Φ maps a pair of intersecting geodesics to a pair of intersecting geodesics, we have that $\Phi(\gamma_v), \Phi(\gamma_{\theta_1 \cdot v}), \Phi(\gamma_{\theta_2 \cdot \theta_1 \cdot v})$ intersect in a geodesics triangle, with internal angles (see Figure 4.2):

$$\begin{aligned} \eta_1 &= f(v, \theta_1) \\ \eta_2 &= f(\theta_1 \cdot v, \theta_2) \\ \eta_3 &= \pi - f(v, \theta_1 + \theta_2) \ . \end{aligned}$$

Hence, by the above, these internal angles add up to π:

$$\eta_1 + \eta_2 + \eta_3 = \pi - f(v, \theta_1 + \theta_2) + f(v, \theta_1) + f(\theta_1 \cdot v, \theta_2) = \pi$$

This is only possible when the triangle formed by the intersection of $\Phi(\gamma_v), \Phi(\gamma_{\theta_1 \cdot v}), \Phi(\gamma_{\theta_2 \cdot \theta_1 \cdot v})$ is degenerate; i.e. the new triple of geodesics intersects at a point. $\square$

In light of the above corollary, Φ defines a point map $\tilde{\varphi}\colon (\widetilde{M}, g) \to (\widetilde{M}, h)$ as follows. For any $p \in \widetilde{M}$, let $\gamma_1, \gamma_2 \in \mathcal{G}_g$ be geodesics that intersect at p, and define $\tilde{\varphi}(p) = p' \in \widetilde{M}$ to be the point of intersection of $\Phi(\gamma_1), \Phi(\gamma_2)$. This map is well-defined, since the corollary above guarantees that the new intersection point p' is independent of the choice of γ_1, γ_2. Indeed, if $\gamma_3 \in \mathcal{G}_g$ passes through p, then $\gamma_1, \gamma_2, \gamma_3$ all intersect at p, hence by the corollary, $\Phi(\gamma_1), \Phi(\gamma_2), \Phi(\gamma_3)$ all intersect at a common point, which must be p'.

The following result proves Theorem 1.3.

Proposition 4.10. *The map $\tilde{\varphi}$ is a $\pi_1(M)$-equivariant isometry of the universal covers. In particular, it descends to an isometry $\varphi\colon (M, g) \to (M, h)$.*

Proof. The map φ is $\pi_1(M)$-equivariant because Φ is $\pi_1(M)$-equivariant. To show this explicitly, let $\gamma \in \pi_1(M)$, and let γ_1, γ_2 be geodesics intersecting at some $p \in \widetilde{M}$. Then, since $\pi_1(M)$ acts by deck transformations, $\gamma \cdot \gamma_1$ and $\gamma \cdot \gamma_2$ intersect at $\gamma \cdot p$. On the other hand, by $\pi_1(M)$-equivariance, we have $\Phi(\gamma \cdot \gamma_1) = \gamma \cdot \Phi(\gamma_1)$ and $\Phi(\gamma \cdot \gamma_2) = \gamma \cdot \Phi(\gamma_2)$. Hence, the intersection point $\tilde{\varphi}(\gamma \cdot p)$ of $\Phi(\gamma \cdot \gamma_1)$ and $\Phi(\gamma \cdot \gamma_2)$ is exactly the intersection point $\gamma \cdot \tilde{\varphi}(p)$ of $\gamma \cdot \Phi(\gamma_1)$ and $\gamma \cdot \Phi(\gamma_2)$. That is, $\tilde{\varphi}(\gamma \cdot p) = \gamma \cdot \tilde{\varphi}(p)$.

Next, by the Myers-Steenrod theorem[7], it suffices to show that $\tilde{\varphi}$ is distance-preserving. For this purpose, let $p, q \in \widetilde{M}$, and let γ_{pq} be the unique geodesic segment in $(\widetilde{M}, g)$ connecting p to q; we further denote by $\tilde{\gamma}_{pq}$ the full geodesic in $\widetilde{M}$ of which γ_{pq} is a segment. Recall that, by the same argument seen in Lemma 4.2, the length of γ_{pq} is exactly the Liouville current of $G(\gamma_{pq}) \subset \mathcal{G}_g$, the set of all geodesics intersecting γ_{pq}. Hence:

$$d_g(p, q) = l_g(\gamma_{pq}) = m_g\left(G(\gamma_{pq})\right) \ .$$

Denote $p' = \tilde{\varphi}(p)$, $q' = \tilde{\varphi}(q)$ and let $\gamma_{p'q'} \in$ be the unique geodesic segment, with respect to h, that connects p' to q'. Once again:

$$d_h(p', q') = l_h(\gamma_{p'q'}) = m_h\left(G(\gamma_{p'q'})\right) \ .$$

Claim: $G(\gamma_{p'q'}) = \Phi(G(\gamma_{pq}))$.

[7]This theorem states that every distance-preserving map of connected Riemannian manifolds is a smooth-isometry. First proved in [Myers and Steenrod, 1939].

Proof of Claim: Suppose $\gamma' \in \mathcal{G}_g$ intersects γ_{pq} somewhere between p and q along the segment. It is clear the $\Phi(\gamma')$ and $\Phi(\tilde{\gamma}_{pq})$ intersect. Moreover, the intersection point is between p' and q' along the segment $\gamma_{p'q'}$, since Φ is a homeomorphism. Indeed, $G(\gamma_{pq})$ is connected and Φ sends connected sets to connected sets; if the intersection point were not between p' and q', then $\Phi(G(\gamma_{pq}))$ would be disconnected. Hence, we have that $\Phi(G(\gamma_{pq})) \subseteq G(\gamma_{p'q'})$. The converse is true using the inverse map Φ^{-1}. $\square$

Since the Liouville current is invariant under the map $\Phi \colon \mathcal{G}_g \to \mathcal{G}_h$, the above claim concludes the proof:

$$d_g(p,q) = m_g\left(G(\gamma_{pq})\right) = m_h\left(\Phi(G(\gamma_{pq}))\right) = m_h\left(G(\gamma_{p'q'})\right) = d_h(p',q').$$

$\square$

This finally proves Theorem 1.3, thus concluding the proof of the main Theorem 1.1.

Appendix A

Riemannian Geometry

The main reference for this section is [do Carmo, 1992]. A (C^k) **Riemannian manifold** is a (C^k) smooth manifold equipped with a (C^k) non-degenerate symmetric bilinear form $g \in T^*M \otimes T^*M$, called the **Riemannian metric**. Equivalently, the metric g can be characterized as a (C^k) smooth family of inner products on the tangent spaces of M:

$$\begin{aligned} g \colon M &\to \mathrm{Sym}(T^*M \otimes T^*M) \\ g \colon p &\mapsto g_p := \langle \cdot, \cdot \rangle_p \end{aligned}$$

An **isometry** is a diffeomorphism $f \colon (M, g) \to (N, h)$ of Riemannian manifolds that preserves the metrics, in the sense that

$$g = f^* h$$

where f^* denotes the pullback of the map f. In other words, for all $p \in M$, $u, v \in T_x M$, we have:

$$\langle u, v \rangle_p = \langle T_p f(u), T_p f(v) \rangle_{f(p)}$$

where by an abuse of notation $\langle \cdot, \cdot \rangle$ denotes either the Riemannian metric g or h, depending on which manifold the arguments are taken.

A Riemannian metric gives the notion of **length of a tangent vector** $v \in T_p M$ (i.e. induces a norm on each tangent space):

$$\|v\|_p := \sqrt{\langle v, v \rangle_p}$$

and hence the **length of a curve**. If $\gamma \colon I = [a, b] \subset \mathbb{R} \to M$ is a (at least C^1) curve in between the points $\gamma(a) = p, \gamma(b) = q$, the length of γ is defined by:

$$l(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt \ .$$

We denote the set of all curves between p and q by $\mathcal{C}_p^q$. This in turn gives a distance function:

$$d\colon M \times M \to \mathbb{R}$$
$$d(p,q) = \inf\left\{l(\gamma) \mid \gamma \in \mathcal{C}_p^q\right\}$$

and hence any Riemannian manifold has a canonical metric-space structure. We call **geodesics** those curves $\gamma\colon I = [a,b] \subset \mathbb{R} \to M$ that locally minimize distances; that is, for every $t \in [a,b]$, there exists a neighborhood $J \subset I$ of t such that for all $t_1, t_2 \in J$

$$d(\gamma(t_1), \gamma(t_2)) = \min\left\{l(\tilde{\gamma}) \mid \tilde{\gamma} \in \mathcal{C}_{\gamma(t_1)}^{\gamma(t_2)}\right\} \ .$$

Equivalently, there must exist some constant $v \geq 0$ such that for all $t_1, t_2 \in J$:

$$d(\gamma(t_1), \gamma(t_2)) = v|t_1 - t_2|.$$

Moreover, v is necessarily independent of the neighborhood J (it must agree on overlapping neighborhoods); it depends only on the parametrization of γ. We thus call v the **speed** of γ.

Covariant Derivative

A Riemannian metric g (of at least C^1 regularity) on a manifold M allows one to define a notion of derivatives of vector fields different from, but related to, the Lie derivative. The **Levi-Civita connection** is defined as the *unique* map

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$
$$\nabla\colon (X,Y) \to \nabla_X Y$$

satisfying the following properties $\forall X, Y, Z \in \mathfrak{X}(M)$, $\forall f, g \in C^1(M)$, $\forall \alpha, \beta \in \mathbb{R}$:

1. $C^1(M)$-linearity in the first entry: $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

2. $\mathbb{R}$-linearity in the second entry: $\nabla_X(\alpha Y + \beta Z) = \alpha\nabla_X Y + \beta\nabla_X Z$.

3. Leibniz Rule in the second entry: $\nabla_X(fZ) = X(f)Z + f\nabla_X Z$.

4. Compatibility with the metric: $\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

5. Torsion-free: $\nabla_X Y - \nabla_Y X = [X,Y] =: \mathcal{L}_X Y$.

The first three properties are the usual defining axioms of an **affine connection**; the last two properties are unique to the Levi-Civita (affine) connection. Indeed, the usual proof for the existence and uniqueness of the Levi-Civita (see for example [do Carmo, 1992]) derives an explicit expression from the defining properties; writing $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^j \frac{\partial}{\partial y^j}$ in local coordinates, we have:

$$\nabla_X Y = \left[X^i \frac{\partial Y^k}{\partial x^i} + \Gamma^k_{ij} X^i Y^j \right] \frac{\partial}{\partial x^k}$$

where Γ^k_{ij} denotes the **Christoffel symbols** of the Levi-Civita connection in the given coordinates[1]:

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left[\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right] .$$

The Christoffel symbols for an arbitrary affine connection are defined as the coefficients such that $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$.

The Levi-Civita connection can be used to define, exactly as above, the differentiation of vector fields along curves, often called the **covariant derivative**. Let $\gamma \colon [a, b] \to M$ be a (at least C^1) curve. Then the covariant derivative $\frac{D}{dt}$ along γ is defined as the unique map sending vector fields to vector fields (along γ), which satisfies for all vector fields (along γ) $V, W \colon t \in [a, b] \mapsto V(t) \in T_{\gamma(t)} M$:

1. $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$.
2. $\frac{D}{dt}(fV) = \frac{df}{dt} V + f \frac{DV}{dt}$.
3. If $V(t) = Y(\gamma(t))$ for some $Y \in \mathfrak{X}(M)$, then: $\frac{DV}{dt} = \nabla_{d\gamma/dt} Y$.

If $V(t) = v^i(t) \frac{\partial}{\partial x^i}$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ in local coordinates, then it follows from these properties that (at any point $\gamma(t)$):

$$\frac{DV}{dt} = \left[\frac{dv^k}{dt} + \Gamma^k_{ij} v^i \frac{d\gamma^j}{dt} \right] \frac{\partial}{\partial x^k} .$$

Structure of the Tangent Bundle

For any Riemannian manifold (M, g), the Levi-Civita connection induces a "natural" splitting of $T(TM)$, the tangent bundle of the tangent bundle. In turn, it is possible to define a "natural" Riemannian structure on TM, called the **Sasaki metric**.[2]

We first define how the Levi-Civita connection splits $T(TM)$. Let $a \in T(TM)$ be a tangent vector to TM. We may assume that a has base-point $(p, v) \in TM$, and that it is represented by a differentiable curve $t \mapsto (p(t), v(t)) \in TM$. Here $v(t) \in T_{p(t)} M$

[1] Note that the Γ^k_{ij} are not components of any tensor; they do not "transform-like-a-tensor" under change of coordinates.

[2] There are other "natural" metrics on TM (see [Musso and Tricerri, 1988, §4]).

for all t, with $p(0) = p$ and $v(0) = v$; a is the tangent vector to this curve at $t = 0$. We thus define the **connector map** using this representation:

$$\begin{cases} \kappa \colon T(TM) \to TM \\ \kappa(a) = \nabla_{\frac{dp}{dt}} v(t)\Big|_{t=0} \end{cases}$$

It should be clear that this map is fiberwise linear. At each fiber, we denote the corresponding map $\kappa_{(p,v)} \colon T_{(p,v)}(TM) \to T_pM$. Denoting further the natural projection by $\pi \colon TM \to M$, we define the **horizontal** and **vertical** components of $a \in T(TM)$ to be, respectively:

$$\begin{cases} a_{hor} := T\pi(a) = \dfrac{dp}{dt}(0) \\ a_{vert} := \nabla_{\frac{dp}{dt}} v(t)\Big|_{t=0} = \dfrac{Dv}{dt}(0) \end{cases}$$

We thus define the **horizontal subbundle** of $T(TM)$ as the (fiberwise) kernel of the connector map:

$$H(TM) = \ker(\kappa)$$

and the **vertical subbundle** of $T(TM)$ as the kernel of $T\pi$:

$$V(TM) = \ker(T\pi) \ .$$

This naming is justified by the fact that, for any $(p, v) \in TM$:

$$H_{(p,v)}(TM) \cap V_{(p,v)}(TM) = \ker(\kappa_{(p,v)}) \cap \ker(T_{(p,v)}\pi) = \{0\} \ .$$

The proof of this identity is an easy computation when coordinate descriptions of the maps $T\pi$ and κ are obtained (see below). By the rank-nullity theorem, a direct corollary of this fact is that the mapping

$$\begin{aligned} (T\pi, \kappa) \colon T(TM) &\to TM \oplus TM \\ a &\mapsto (a_{hor}, a_{vert}) = (T\pi(a), \kappa(a)) \end{aligned}$$

where $\oplus$ denotes the Whitney sum[3], is a vector bundle isomorphism (since $(T\pi, \kappa)$ is fiberwise linear and its kernel is zero, i.e injective and linear on the fibers). It thus gives what is called the **"horizontal-vertical splitting"** of the tangent bundle:

$$T(TM) \equiv H(TM) \oplus V(TM) \cong TM \oplus TM \ .$$

The **Sasaki metric** on the tangent bundle TM, denoted $\hat{g} := \langle\langle \cdot, \cdot \rangle\rangle$, is thus defined as the usual metric when restricted to either the *horizontal* or *vertical* subbundles, but which declares them to be orthogonal. That is, for any $V, W \in T(TM)$ at some $(p, v) \in TM$:

$$\langle\langle V, W \rangle\rangle_{(p,v)} = \langle T\pi(V), T\pi(W) \rangle_p + \langle \kappa(V), \kappa(W) \rangle_p \ .$$

[3]This is the fiber-wise direct sum of vector bundles.

If one chooses local coordinates $(x^1, \dots, x^n)$ on M, and induced coordinates $(x^1, \dots, x^n, v^1, \dots, v^n)$ on TM, then for any vector $V = a^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial v^i}$ at $(p,v) \in TM$, one can show that:

$$T_{(p,v)}\pi(V) = a^i \frac{\partial}{\partial x^i}\Big|_p$$

and

$$\kappa_{(p,v)}(V) = \left(\xi^k + \Gamma^k_{ij} a^i v^j\right) \frac{\partial}{\partial x^k}\Big|_p$$

where $v = v^j \frac{\partial}{\partial x^j}|_p \in T_pM$. In particular, for all $i = 1, \dots, n$:

$$T_{(p,v)}\pi\left(\frac{\partial}{\partial x^i}\Big|_{(p,v)}\right) = \frac{\partial}{\partial x^i}\Big|_p \qquad T_{(p,v)}\pi\left(\frac{\partial}{\partial v^i}\Big|_{(p,v)}\right) = 0$$
$$\kappa_{(p,v)}\left(\frac{\partial}{\partial x^i}\Big|_{(p,v)}\right) = \Gamma^k_{ij} v^j \frac{\partial}{\partial x^k}\Big|_p \qquad \kappa_{(p,v)}\left(\frac{\partial}{\partial v^i}\Big|_{(p,v)}\right) = \frac{\partial}{\partial x^i}\Big|_p$$

A straightforward computation then shows that:

$$\begin{aligned}
\hat{g}_{(p,v)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g_{ij} + g_{kl}\Gamma^k_{ir}\Gamma^l_{js} v^r v^s =: g_{ij} + A_{ij} \\
\hat{g}_{(p,v)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v^j}\right) &= \Gamma^k_{il} v^l g_{kj} =: B_{ij} \\
\hat{g}_{(p,v)}\left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial x^j}\right) &= \Gamma^k_{jl} v^l g_{ki} = B_{ji} \\
\hat{g}_{(p,v)}\left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j}\right) &= g_{ij}.
\end{aligned}$$

Hence, with the definitions of the matrices A, B above, the local representation of the Sasaki metric in block-matrix form is:

$$\hat{g} = \begin{bmatrix} g + A & B \\ B^T & g \end{bmatrix}.$$

Alternatively, if one defines $Dv^k := dv^k + \Gamma^k_{ij} v^i dx^j$, then from the above, the Sasaki metric may be written locally as:

$$\hat{g} = g_{ij} dx^i \otimes dx^j + g_{ij} Dv^i \otimes Dv^j.$$

Remark A.1. *It is possible to recursively define a Riemannian structure on increasingly higher order tangent bundles over M, as long as the Riemannian structure on M is sufficiently smooth. Denote by $T^m M = T(T(\cdots(TM)))$ the m^{th} order tangent bundle over M. If g is a class C^k Riemannian metric on M, with $k \geq m$, then recursively applying the definition of the Sasaki metric yields a class C^{k-m} Riemannian metric $\hat{g}$ on $T^m M$.*

Parallel Transport, Geodesics, and the Exponential Map

Consider a curve $t \mapsto p(t) \in M$. A vector field $t \mapsto v(t) \in T_{p(t)}M$ along $p(t)$ is said to be **parallel** if its covariant derivative is zero:

$$\frac{Dv}{dt} = \nabla_{\frac{dp}{dt}} v = 0.$$

A corresponding notion is that of parallel transport. Given a curve $t \mapsto p(t) \in M$ as above, there exists for each t_0, t a unique *linear isometry*:

$$P_{p;t_0;t} \colon T_{p(t_0)}M \to T_{p(t)}M$$

such that, for any $v_0 \in T_{p(t_0)}$, the resulting curve $t \mapsto v(t) = P_{p;t_0;t}(v_0)$ is parallel:

$$\frac{Dv}{dt} = \frac{D}{dt} P_{p;t_0;t}(v_0) = 0.$$

We call the map $P_{p;t_0;t}$ the **parallel transport** along the curve p from $p(t_0)$ to $p(t)$. The existence and uniqueness of such a map comes from the fact that the above equation defines an initial value problem for an ordinary differential equation. Indeed, choosing some arbitrary local coordinates $(x^1, \ldots, x^n)$, the problem is to find a curve $t \mapsto v(t) \in T_{p(t)}M$ such that:

$$\begin{cases} \dfrac{dv^k}{dt} + \Gamma^k_{ij} v^i \dfrac{dp^j}{dt} = 0 \\ v^k(0) = v_0 \end{cases}$$

for each $k = 1, \ldots, n$.

Although we have defined **geodesics** as locally length-minimizing curves with affine parametrization, in the smooth setting it is equivalent to define them as curves γ such that their velocity field $d\gamma/dt$ is parallel:

$$\frac{D}{dt}\frac{d\gamma}{dt} := \nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0.$$

In other words, a geodesic is a vector field along a curve that is "its own parallel transport". As in the case of the parallel transport, the above equation has a representation in local coordinates. Writing the the curve $t \mapsto \gamma(t) \in M$ in coordinate form as $(x^1(t), \ldots, x^n(t))$, we obtain the **geodesic equations**:

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

Given initial conditions $x^k(0) = x^k_0$, $dx^k/dt(0) = v^k_0$, this defines an initial value problem. Again, by the theory of ordinary differential equations, there always exists a unique solution on some neighborhood of an initial condition. Defining the corresponding coordinates x^k, v^k on the tangent bundle TM, the geodesic equations obtain the equivalent form for the curve:

$$\begin{cases} \dfrac{dx^k}{dt} = v^k \\ \dfrac{dv^k}{dt} = -\Gamma^k_{ij} v^i v^j \end{cases}$$

These equations introduce a well-defined vector field on the tangent bundle $X \in \mathfrak{X}(TM)$, which in coordinates has the form:

$$X = v^k \frac{\partial}{\partial x^k} - \Gamma^k_{ij} v^i v^j \frac{\partial}{\partial v^k}$$

and whose integral curves are geodesics in the tangent bundle $t \mapsto (\gamma(t), \dot{\gamma}(t)) \in TM$. We call this vector field the **geodesic field**. We denote its flow by $\phi^t \colon TM \to TM$ and call it the **geodesic flow**.

Geodesics defined in this manner have many properties, one of which is that they locally minimize the length/distance between points in M (see [do Carmo, 1992]).

Proposition A.2 (Properties of geodesics. See [do Carmo, 1992, §3].)**.** *We list some of the properties of geodesics.*

- *Geodesics are of constant speed:*

$$\frac{d}{dt} \left\| \frac{d\gamma}{dt} \right\| = 0.$$

- *If $\gamma \colon I \to M$ is a geodesic with $\left\| \frac{d\gamma}{dt} \right\| = c$, then the arc length on an interval $[t_0, t]$ is:*

$$s(t) = \int_{t_0}^{t} \left\| \frac{d\gamma}{dt} \right\| dt = c(t - t_0)\ .$$

- *Geodesics are homogeneous. If $\gamma_v \colon [-\delta, \delta] \to M$ is the unique geodesic such that $\gamma_v(0) = v \in TM$, then for any $a > 0$, $\gamma_{av} \colon [-\delta/a, \delta/a] \to M$ is such that:*

$$\gamma_{av}(t) = \gamma_v(at)\ .$$

The existence and uniqueness of solutions to the geodesic equations allows us to define the exponential map. Fix a point $p \in M$, and let $\varepsilon > 0$ be small enough such that $\gamma_v \colon [0, 1] \to M$, the unique geodesic such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$, is defined for all $v \in B_\varepsilon(0)$. Here $B_\varepsilon(0) \subset T_pM$ denotes an open ball of radius ε centered at 0. We define the **exponential map**:

$$\begin{cases} \exp_p \colon B_\varepsilon(0) \subset T_pM \to M \\ \exp_p(v) = \gamma_v(1) \end{cases}$$

It can be shown that for any $p \in M$, $\exp_p$ is a **local diffeomorphism**: there exists $\varepsilon > 0$, possibly very small, such that $\exp_p \colon B_\varepsilon(0) \to \exp_p(B_\varepsilon(0))$ is a diffeomorphism. The image $B_\varepsilon(p) = \exp_p(B_\varepsilon(0))$ is called a **normal ball** or **geodesic ball** of radius ε centered at p; that a normal ball has a radius is justified by the local minimizing property of geodesics, seen in the proposition further below.

It should be clear straight from the above definitions, that the geodesic flow can be retrieved from the exponential map. For any $p \in M$, $v \in T_pM$, we have:

$$\phi^t((p,v)) = (\gamma_v(t), \dot{\gamma}_v(t)) = \left(\exp_p(tv), \frac{d}{dt}\exp_p(tv)\right) \in TM$$

for all $t \in \mathbb{R}$ such that each of the above is defined at $tv \in T_pM$.

A Riemannian manifold is called **(geodesically) complete** if $\exp_p$ is defined on all of T_pM, for all $p \in M$. In other words, the quantities in the above equality are defined for all $v \in T_pM$ and for all $t \in \mathbb{R}$; the domain of every geodesic can be (maximally) extended to all of $\mathbb{R}$. In the case that M is geodesically complete, the exponential map can be further upgraded to a map on the entire tangent bundle, defined as:

$$\begin{cases} \exp \colon TM \to M \\ \exp(p,v) = \exp_p(v) \end{cases}$$

While the tangent of this map is defined on the tangent bundle of the tangent bundle, $T\exp \colon T(TM) \to TM$, we do have, as previously, the splitting into the horizontal and vertical subbundles $T(TM) \equiv H(TM) \oplus V(TM)$. In particular, we have that $TM \cong H(TM)$, and hence the interpretation $T_v \exp_p \colon T_pM \to T_{\exp_p(v)}M$ is valid. This leads to the following classical result.

Lemma A.3 (Gauss' Lemma)**.** *Let $p \in M$, and $v, w \in B_\varepsilon(0) \subset T_pM$. Identifying $T_pM \cong H_{(p,v)}(TM)$, we have:*

$$\langle T_v \exp_p(v), T_v \exp_v(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p \ .$$

Proposition A.4 (More Properties.)**.** *We list some properties related to geodesics which are a consequence of the introduction of the exponential map.*

- *Geodesics locally minimize arc length. If $B \subset M$ is a normal ball, and $\gamma \colon [a,b] \to B$ is a geodesic path from $p = \gamma(a)$ to $q = \gamma(b)$, then for any other (piecewise-smooth) path $c \colon [a,b] \to B$ from p to q:*

  $$l(\gamma) \leq l(c) \ .$$

  Moreover, if equality holds, then their images are the same $\gamma([a,b]) = c([a,b])$.

- *Every point $p \in M$ posses a totally normal neighborhood. $\exists W \subset M$ with $p \in W$, and $\exists \delta > 0$ such that $\forall q \in W$,*

  $$\exp_q \colon B_q(0) \subset T_qM \to \exp_q(B_\delta(0))$$

  is a diffeomorphism.

Curvature

On a Riemannian manifold (M, g) of class C^2 or above, we define the **curvature** of M as a map R which associates to any pair of vector fields $X, Y \in \mathfrak{X}^2(M)$ an operator:

$$\begin{cases} R(X,Y)\colon \mathfrak{X}^2(M) \to \mathfrak{X}^2(M) \\ R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \end{cases}$$

Note that $R(X,Y)Z$ is only well defined for C^2 vector fields X, Y, Z.

Proposition A.5 (See [do Carmo, 1992, §4 Propositions 2.2 and 2.4])**.** *The (Riemann) curvature has the following properties.*

1. *$C^2(M)$-Bilinearity of R:*

$$\begin{cases} R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1) \\ R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2) \end{cases}$$

   for any $f, g \in C^2(M)$, $X_1, Y_1, X_2, Y_2 \in \mathfrak{X}(M)$.

2. *$C^2(M)$-Linearity of the operator $R(X,Y)$:*

$$R(X,Y)(fZ_1 + gZ_2) = fR(X,Y)Z_1 + gR(X,Y)Z_2$$

   for any $f, g \in C^1(M)$, $X, Y, Z_1, Z_2 \in \mathfrak{X}(M)$.

3. *The Bianchi Identity:*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

   for all $X, Y \in \mathfrak{X}^2(M)$ and $Z \in \mathfrak{X}^1(M)$.

The tri-linearity of the map $(X, Y, Z) \mapsto R(X,Y)Z$ allows us to define the **Riemann curvature tensor**, with an abuse of notation:

$$\begin{cases} R(\cdot,\cdot,\cdot,\cdot)\colon \mathfrak{X}^2(M) \times \mathfrak{X}^2(M) \times \mathfrak{X}^2(M) \times \mathfrak{X}^r(M) \to C^2(M) \\ R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle \end{cases}$$

The linearity in the last argument follows from the bilinearity of the Riemannian metric $\langle \cdot, \cdot \rangle$. Also, while the last argument can be of any smoothness, say class C^r, the outputted function will in general only have class C^2 smoothness.[4]

As a corollary of the above properties of the curvature, we have the following:

[4] I admit, it may be somewhat overly pedantic to make the distinction between what classes of smoothness are allowed in any of the arguments of the Riemmann curvature.

Proposition A.6 (See [do Carmo, 1992, §4 Proposition 2.5])**.** *The Riemann curvature tensor has the following properties.*

1. *The other Bianchi identity:* $R(X,Y,Z,W)+R(Z,X,Y,W)+R(Y,Z,X,W) = 0$.
2. *Skew-symmetry in the first two arguments:* $R(X,Y,Z,W) = -R(Y,X,Z,W)$.
3. *Skew-symmetry in the last two arguments:* $R(X,Y,Z,W) = -R(X,Y,W,Z)$.
4. *Symmetry under swap of first two and last two:* $R(X,Y,Z,W) = R(Z,W,X,Y)$.

In local coordinates $(x^1,\dots,x^n)$ on M, one has that $[\partial_i,\partial_j] = 0$ for all $i,j = 1,\dots,n$. Hence, the components of the Riemann curvature tensor are computed as:

$$R_{ijkl} = \langle R(\partial_i,\partial_j)\partial_k,\partial_l\rangle = \langle \left(\Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik}\right)\partial_r,\partial_l\rangle = g_{rl}\left(\Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik}\right).$$

One then often writes the components with the index l up:

$$R^l_{ijk} = \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik}.$$

These are technically the components of the tensor obtained by performing the tensor contraction of the Riemannian metric g with the last component of the Riemann curvature tensor $R(\cdot,\cdot,\cdot,\cdot)$.

Given a 2-dimensional subspace of the tangent plane at a point $\sigma \subset T_pM$, we define the **sectional curvature** $K(\sigma)$ to be:

$$K(\sigma) := \frac{R(u,v,u,v)}{|u\wedge v|^2} = \frac{\langle R(u,v)u,v\rangle}{|u\wedge v|^2}$$

where $u,v\in\sigma$ are any two vectors spanning σ, and $|u\wedge v|^2 := \|u\|^2\|v\|^2 - \langle u,v\rangle^2$. It can be shown that K is well-defined, in the sense that any other basis u',v' will give the same value. Moreover, the Riemann curvature is determined entirely by the sectional curvatures of all 2-dimensional subspaces of all tangent spaces; see [do Carmo, 1992, §4 Lemma 3.3].

There are two more related notions of curvature. We define the **Ricci curvature tensor**

$$\begin{cases} \operatorname{Ric}(\cdot,\cdot)\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to C(M) \\ \operatorname{Ric}(X,Y) := \dfrac{1}{n-1}\displaystyle\sum_k R(X,\partial_k,Y,\partial_k) := \dfrac{1}{n-1}X^iY^jR_{ikjk}. \end{cases}$$

where $\partial_1,\dots,\partial_n$ is a local frame for TM, and X^i,Y^j are the corresponding components of X,Y. The components of this tensor are often denoted:

$$\operatorname{Ric}_{ij} = \frac{1}{n-1}\sum_k R_{ikjk}.$$

Averaging over all the coordinate planes containing $x = x^i\partial_i \in T_pM$, one obtains what is called the **Ricci Curvature** in the direction $x \in T_pM$ at p (dropping the Einstein summation convention):

$$\mathrm{Ric}_p(x) = \frac{1}{n-1}\sum_j R(x, \partial_j, x, \partial_j) = \frac{1}{n-1}\sum_{ij} x^i R_{ijij} \ .$$

Further, choosing an orthonormal basis $\partial_1, \ldots, \partial_n$ and averaging over the sectional curvatures of all coordinate planes, one obtains by computation what is called the **Scalar Curvature** at the point p:

$$\begin{cases} K\colon M \to C(M) \\ K(p) = \dfrac{1}{n(n-1)}\displaystyle\sum_{ij} R_{ijij}. \end{cases}$$

Remark A.7. *The Riemannian metric g needs to be at least class C^2 for any of the above notions of curvature to be well-defined. Additionally, recall that if g is class C^k, then any higher order tangent bundle, say $T^m(M) = T(T(\cdots(TM))$, has Sasaki metric $\hat{g}$ of class C^{k-m}. Hence, for the curvature to be defined on T^mM with respect $\hat{g}$, we require the metric g on M to be at least class C^{2+m}.*

Theorem of Cartan-Hadamard

We state the following theorem, without proof.

Theorem A.8 (See [do Carmo, 1992, §7 Theorem 3.1])**.** *If M is a (geodesically) complete, simply connected Riemannian manifold of everywhere non-positive sectional curvature, then for any $p \in M$, $\exp_p\colon T_pM \to M$ is a diffeomorphism; in particular M is diffeomorphic to $\mathbb{R}^n$ (where $n = dim(M)$).*

As a corollary, we have that the universal cover of such a manifold is $\mathbb{R}^n \cong T_pM$, with covering given by the exponential map.

Appendix B

Symplectic Geometry

The constructions here are pretty much standard, though the reader may wish to reference [Lee, 2013, Chapter 22]. A **symplectic manifold** (M, ω) is a smooth $2n-$dimensional manifold M with a closed, non-degenerate differential 2-form ω, called the **symplectic form**. A manifold of this type is said to have a **symplectic structure**. We give the special name of **canonical coordinates** to local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that

$$\omega = dp_\mu \wedge dq^\mu.$$

The top exterior power of the symplectic form is a volume form on M. In canonical coordinates:

$$\omega^n := \underbrace{\omega \wedge \cdots \wedge \omega}_{\text{n-times}} = n!(dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \cdots \wedge dq^n).$$

A **symplectomorphism** is a diffeomorphism $\phi\colon M \to M$ that preserves the symplectic form:

$$\phi^* \omega = \omega.$$

Symplectomorphisms also preserve the volume.

An important fact in symplectic geometry is the existence of (local) canonical coordinates. This is a consequence of **Darboux's Theorem** for differential 1-forms. We state the version of this theorem as seen in [Sternberg, 1964, Theorem 6.2].

Theorem B.1 (Darboux' Theorem)**.** *Let M be an n-dimensional manifold. Let $\theta \in \Omega^1(M)$ be a differential 1-form such that $d\theta \in \Omega^2(M)$ has constant rank k (i.e $(d\theta)^k \neq 0$ and $(d\theta)^{k+1} = 0$).*

- *If $\theta \wedge (d\theta)^k = 0$ everywhere, then there exist local coordinates $(p_1, \ldots, p_{n-k}, q^1, \ldots, q^k)$ such that:*

$$\theta = p_1 dq^1 + \cdots + p_k dq^k.$$

- *If* $\theta \wedge (d\theta)^k \neq 0$ *everywhere, then there exist local coordinates* $(p_1, \ldots, p_{n-k}, q^1, \ldots, q^k)$ *such that:*

$$\theta = p_1 dq^1 + \cdots + p_k dq^k + dp_{k+1}.$$

Corollary B.2. *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold. Then for any* $x \in M$*, there exists a local chart* $(U, \psi = (q^1, \ldots, q^n, p_1, \ldots, p_n))$ *around* x *such that:*

$$\omega = dp_1 \wedge dq^1 + \cdots + dp_n \wedge dq^n = dp_\mu \wedge dq^\mu$$

i.e. canonical coordinates can be locally defined anywhere on (M, ω)*.*

Proof. Let $x \in M$. By the manifold structure on M, there exists a neighborhood $V \subset M$ of x homeomophic to $\mathbb{R}^{2n}$; hence V is a star-shaped domain. Moreover, since ω is a closed form, its restriction ω_V to V is also closed. By the Poincaré Lemma, $\omega_V \in \Omega^2(V)$ is exact; i.e there exists a 1-form $\theta \in \Omega^1(V)$ such that $d\theta = \omega_V$.

Since $(\omega)^n$ is a top-form and $(d\theta)^n = (\omega_V)^n$, we have that $(d\theta)^n \neq 0$ and $(d\theta)^{n+1} = 0$; hence $d\theta$ is of constant rank n. Moreover, $d\theta \wedge (d\theta)^n = 0$ everywhere. Applying Darboux's Theorem (and taking an exterior derivative), there exists a local chart $(U, \psi = (q^\mu, p_\mu))$ on some subset $U \subset V$ such that on U:

$$\omega = d\theta = dp_\mu \wedge dq^\mu.$$

□

A consequence of Darboux's theorem applied to symplectic geometry is that any two symplectic manifolds of equal dimension are **locally symplectomorphic**. Hence, there are no "local invariants" in symplectic geometry. Any tensorial quantity[1] which is invariant under (local) symplectomorphism is necessarily globally invariant under symplectomorphism; indeed, around any two points on different symplectic manifolds of equal dimension, $p \in M$, $q \in N$ there exist neighborhoods $U_q \subset M, U_p \subset N$ which are symplectomorphic. This is in contrast to Riemannian geometry, where it is generically not possible to find local coordinates in which the Riemannian metric resembles the Euclidean metric (it is possible at a point, using geodesic normal coordinates, but not in any *neighborhood*, unless the metric is *flat*); in Riemannian geometry, the Riemann curvature tensor is an example of a local invariant (with respect to isometries).

Hamiltonian Vector Fields

The existence of a symplectic structure on a manifold yields a correspondence between vector fields and 1-forms. For any 1-form $\theta \in \Omega^1(M)$, we may uniquely define a corresponding vector field $X \in \mathfrak{X}(M)$ through the identity:

$$\theta = \iota_X \omega.$$

[1] Tensorial means well-defined on a manifold, i.e. coordinate-independent.

Here ι_X is the interior product. That is, for any tangent vector $\xi \in T_xM$, the vector field X is such that

$$\theta_x(\xi) = \omega_x(X(x), \xi).$$

Since ω is non-degenerate, this equality uniquely defines the vector $X(x)$. Moreover, since θ and ω are smooth, so is X.

This leads to the concept of **Hamiltonian vector fields**. For any (sufficiently smooth) "Hamiltonian" function $H\colon M \to \mathbb{R}$, one can define a unique vector field $X_H \in \mathfrak{X}(M)$ by

$$dH = \iota_{X_H}\omega.$$

Vector fields generated by (Hamiltonian) functions in this way are called Hamiltonian vector fields. The flow of a Hamiltonian vector field is called a **Hamiltonian flow**.

This terminology is inspired by Hamiltonian mechanics. Indeed, in canonical coordinates, the Hamiltonian vector field is

$$X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\frac{\partial}{\partial p_\mu}$$

and hence the Hamiltonian flow is exactly the solution to Hamilton's equations:

$$\begin{cases} \dfrac{dq^\mu}{dt} = \dfrac{\partial H}{\partial p_\mu} \\ \dfrac{dp_\mu}{dt} = -\dfrac{\partial H}{\partial q^\mu}. \end{cases}$$

Hamiltonian flows have several nice properties.

Proposition B.3. *Let X_H be the Hamiltonian flow associated to some smooth function $H :: M \to \mathbb{R}$. Denote by $\phi^t_{X_H}$ the corresponding Hamiltonian flow. Then:*

1. *$H \circ \phi^t_{X_H} = H$ for all t. The Hamiltonian flow preserves the Hamiltonian.*
2. *$(\phi^t)^*\omega = \omega$. The Hamiltonian flow preserves the symplectic structure (i.e ϕ^t is a 1-parameter family of symplectomorphisms). This is sometimes called **Liouville's theorem**.*
3. *$(\phi^t)^*\omega^n = \omega^n$. The Hamiltonian flow preserves the volume. (This is an immediate consequence of the previous fact).*

Proof. A useful tool here is the Lie derivative with respect to a vector field. Since the flow of a vector field forms a family of local diffeomorphisms, the Lie derivative (of an abitrary tensor A) has a definition in terms of the pullback of the flow:

$$\mathcal{L}_{X_H}A = \frac{d}{dt}\Big|_{t=0}(\phi^t_{X_H})^*A.$$

(1). We must show that the Hamiltonian is constant along trajectories of the Hamiltonian vector field. By the flow definition of the Lie derivative, it is equivalent to show that the Lie derivative w.r.t X_H is zero: $\mathcal{L}_{X_H}H = 0$. Simply following definitions:

$$\mathcal{L}_{X_H}H = dH(X_H) = \iota_{X_H}\omega(X_H) = \omega(X_H, X_H) = 0$$

where the last equality follows since ω is a 2-form.

(2). We follow a similar approach as above. In this case, we may use *Cartan's magic formula*:

$$\begin{aligned} \mathcal{L}_{x_H}\omega &= d(\iota_{X_H}\omega) + \iota_{X_H}d\omega \\ &= d(dH) = 0 \end{aligned}$$

where we have used the fact that ω is a closed form, and that $d^2 = 0$.

(3). Since the Lie derivative satisfies the Leibniz rule:

$$\mathcal{L}_{X_H}(\omega^2) = \mathcal{L}_{x_H}\omega \wedge \omega + \omega \wedge \mathcal{L}_{x_H}\omega = 0$$

by (2) above. Recursively, it can be shown that $\mathcal{L}_{x_H}\omega^k = 0$ for any k-th exterior power. □

Remark B.4. *One should note that the above proof of (1) requires that H is at least class C^1, while (2) (and hence (3)) requires that H is at least class C^2.*

Contact Structures

We briefly cover the definition of a contact structure, as there is some relation to symplectic geometry. Let M be an odd-dimensional manifold, say $\dim(M) = 2n+1$. A **contact structure** on M is a completely non-integrable smooth distribution $\Delta \subset TM$ of codimension 1. At any point $p \in M$, we call the subspace $\Delta_p \subset T_pM$ a **contact element** at p. The pair (M, Δ) is called a **contact manifold**. A **contactomorphism** between contact manifolds $f\colon (M, \Delta_M) \to (N, \Delta_N)$ is a diffeomorphism that preserves the contact structure:

$$\Delta_M = f^*\Delta_N.$$

If M is orientable, any contact structure can be equivalently defined as the kernel of some differential 1-form $\alpha \in \Omega^1(M)$:

$$\Delta_p = \ker(\alpha_p) \quad \forall p \in M$$

such that

$$\alpha \wedge (d\alpha)^n \neq 0$$

everywhere. It is standard to call α a **contact 1-form** on M. The pair (M, α) is called a **co-orientable contact manifold**. In this case, a **contactomorphism** between coorientable contact manifolds $f\colon (M, \alpha) \to (N, \beta)$ is characterized as a diffeomorphism that preserves the contact 1-form:

$$\alpha = f^*\beta.$$

We now justify that the condition $\alpha \wedge (d\alpha)^n \neq 0$ is equivalent to non-integrability.

Proposition B.5. *A codimension 1 distribution $\Delta = \ker(\alpha)$ on a $2n+1$-dimensional manifold is completely non-integrable if and only if $\alpha \wedge (d\alpha)^n \neq 0$ everywhere.*

Proof. It is equivalent to show that $\Delta = \ker(\alpha)$ is *completely integrable* if and only if $\alpha \wedge (d\alpha)^n = 0$ everywhere. Moreover, by the Frobenius theorem, complete integrability, integrability and involutivity of $\Delta = \ker(\alpha)$ are all equivalent. Hence, it is sufficient to prove the following two claims:

Claim 1: Let $\alpha \in \Omega^1(M)$. Then $\Delta = \ker(\alpha)$ is involutive if and only if $d\alpha|_\Delta = 0$ everywhere.
Claim 2: Let $\alpha \in \Omega^1(M)$, $\dim(M) = 2n+1$, and $\Delta = \ker(\alpha)$. Then $d\alpha|_\Delta = 0$ if and only if $\alpha \wedge (d\alpha)^n = 0$.

Proof of Claim 1. Locally, we may write $\Delta = \ker(\alpha) = \mathrm{span}(\{X_i\}_i)$ for some vector fields $\{X_i\}_i$.

If $\Delta = \ker(\alpha)$ is involutive, then $[X_i, X_j] \in \ker(\alpha)$ for all i, j. Hence, for all i, j:

$$d\alpha(X_i, X_j) = X_i(\alpha(X_j)) - X_j(\alpha(X_i)) - \alpha([X_i, X_j]) = 0 - 0 - 0 = 0$$
$$\implies d\alpha|_\Delta = 0.$$

If $d\alpha|_\Delta = 0$, then $d\alpha(X_i, X_j) = 0$ for all i, j. Hence, for all i, j:

$$\alpha([X_i, X_j] = X_i(\alpha(X_j)) - X_j(\alpha(X_i)) = 0 - 0 = 0$$
$$\implies [X_i, X_j] \in \ker(\alpha) = \Delta.$$

□

Proof of Claim 2. Let $p \in M$. Since $\Delta = \ker(\alpha)$ is codimension 1, we may write $\Delta = \ker(\alpha) = \mathrm{span}(X_1, \dots, X_{2n})$ for some local basis of vector fields $X_1, \dots, X_{2n}$, defined on a neighborhood $U \subset M$ of p. Moreover, if we let X_0 be some vector field (also locally defined around p on U) such that $X_0 \notin \ker(\alpha)$, with say $\alpha(X_0) = 1$, then:

$$TU = \ker(\alpha) \oplus \langle X_0 \rangle.$$

Hence, if $V_0, \dots, V_{2n}$ are any vector fields defined on U, there exist functions $\{v_k^i\}_{i,k}$ such that:

$$V_k = \sum_i v_k^i X_i.$$

A computation then shows:

$$\begin{aligned}
\alpha \wedge (d\alpha)^n(V_0, \dots, V_{2n}) &= \alpha(V_0) \cdot \prod_{k=1}^{n} [d\alpha(V_{2k-1}, V_{2k})] \\
&= \left[\sum_r v_0^r \alpha(X_r)\right] \cdot \prod_{k=1}^{n} \left[\sum_{i,j} v_{2k-1}^i v_{2k}^j d\alpha(X_i, X_j)\right] \\
&= v_0^0 \cdot \prod_{k=1}^{n} \left[v_{2k-1}^0 v_{2k}^0 d\alpha(X_0, X_0) + \sum_{i,j=1}^{2n} v_{2k-1}^i v_{2k}^j d\alpha(X_i, X_j)\right] \\
&= v_0^0 \cdot \prod_{k=1}^{n} \left[0 + \sum_{i,j=1}^{2n} v_{2k-1}^i v_{2k}^j \left(-\alpha([X_i, X_j])\right)\right] \\
&= -v_0^0 \cdot \prod_{k=1}^{n} \left[\sum_{i,j=1}^{2n} v_{2k-1}^i v_{2k}^j \alpha([X_i, X_j])\right].
\end{aligned}$$

($\Rightarrow$: Forward direction.) If $d\alpha|_\Delta = 0$, then by the previous claim, Δ is involutive, hence $[X_i, X_j] \in \Delta = \ker(\alpha)$ for all i, j. It follows that

$$\alpha \wedge (d\alpha)^n(V_0, \dots, V_{2n}) = 0$$

for all vector fields $V_0, \dots, V_{2n}$ defined on U.

($\Leftarrow$: Backward direction) Suppose $\alpha \wedge (d\alpha)^n = 0$. Then, for all i, j:

$$\begin{aligned}
0 = \alpha \wedge (d\alpha)^n(X_0, X_i, X_j, \dots, X_i, X_j) = -\prod_{k=1}^{n} \Big[\alpha([X_i, X_j])\Big] \\
\implies \alpha([X_i, X_j]) = 0 \\
\implies d\alpha(X_i, X_j) = 0.
\end{aligned}$$

It follows that $d\alpha|_\Delta = 0$ on all of U.

Since M can be covered by neighborhoods such as U, Claim 2 holds. $\square$

Hence $\Delta = \ker(\alpha)$ is completely integrable $\Leftrightarrow d\alpha|_\Delta = 0 \Leftarrow \alpha \wedge (d\alpha)^n = 0$ everywhere. $\square$

As in symplectic geometry, there is an application of Darboux's theorem to contact geometry. Let (M, α) be a coorientable contact manifold. Since $\alpha \wedge (d\alpha)^n \neq 0$, we have that $(d\alpha)^n \neq 0$ and $(d\alpha)^{n+1} = 0$ (since $(d\alpha)^n$ is a top-form), hence $d\alpha$ has constant rank n. Then there exists around any point $p \in M$ local coordinates $(x_1, \dots, x_{n+1}, y_1, \dots, y_n)$ such that

$$\alpha = dx_{n+1} + \sum_{i=1}^{n} x_i dy_i.$$

Relabelling $x_{n+1} = z$, one obtains the **standard contact form** on $\mathbb{R}^{2n+1} := \{(x_1, \dots, x_n, y_1, \dots, y_n, z) \mid x_i, y_i, z \in \mathbb{R}\ \forall i = 1, \dots, n\}$. A direct corollary of this is that any two contact manifolds are **locally contactomorphic**; hence there are no "local invariants" in contact geometry. Moreover, Darboux's theorem also implies that if a form θ of constant rank n satisfies the complete non-integrability condition $\theta \wedge (d\theta)^n \neq 0$, then locally it assumes the form of the standard contact 1-form, and is hence (globally) a contact 1-form.

Every contact 1-form defines a vector field R called the Reeb field, which is the unique vector field satisfying:

$$d\alpha(R, \cdot) = \iota_R d\alpha \equiv 0, \qquad \alpha(R) = \iota_R \alpha \equiv 1.$$

This follows from the fact that $d\alpha$ is necessarily non-degenerate, as $\alpha \wedge (d\alpha)^n \neq 0$. The Reeb field has the property that its flow $\phi_R^t \colon M \to M$ preserves the contact form:

$$(\phi_R^t)^* \alpha = \alpha \quad \forall t.$$

Indeed, the Lie derivative of α with respect to R is zero:

$$\mathcal{L}_R \alpha = d(\iota_R \alpha) + \iota_R d\alpha = 0 + 0 = 0.$$

Appendix C

$C^{k,\alpha}$ Regularity

Here we accumulate notes on the basic definitions and constructions of Hölder spaces and their topologies, in some generality. These are spaces of functions (between manifolds) which satisfy a particular kind of smoothness, called $C^{k,\alpha}$ regularity, appearing in the refinements of the Gromov Compactness theorem by [Pugh, 1987],[Greene and Wu, 1988]. In defining $C^{k,\alpha}$ regularity, we follow mostly [Ruelle, 1989] and somewhat [Bonic and Frampton, 1966]. In both references the discussion extends more generally to the setting of Banach manifolds.

Definition of $C^{k,\alpha}$ Manifolds

Roughly, $C^{k,\alpha}$ Manifolds are manifolds whose transition maps between local charts are of $C^{k,\alpha}$ regularity. We first specify the definition of α-Hölder continuity for maps between Euclidean spaces.

Definition C.1 (α-Hölder continuity)**.** *Let $n, m \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^n$. Let $0 < \alpha \leq 1$.*

- *A function $f\colon \Omega \to \mathbb{R}^m$ is called **α-Hölder continuous at** $x \in \Omega$ iff there exists a neighborhood $U_x \subseteq \Omega$ of x, and $\exists C = C(U_x) \geq 0$ such that $\forall y \in U_x$:*

$$\|f(x) - f(y)\|_{\mathbb{R}^m} \leq C(y) \cdot \|x - y\|_{\mathbb{R}^n}^{\alpha} \quad \forall x, y \in \Omega.$$

  *This is called the **Hölder condition**.*

- *$f\colon \Omega \to \mathbb{R}^m$ is called **locally α-Hölder continuous** on Ω if and only if it is α-Hölder continuous at every $x \in \Omega$.*

- *$f\colon \Omega \to \mathbb{R}^m$ is **(uniformly) α-Hölder continuous** on Ω if and only if there exists $0 < C < \infty$ such that the Hölder condition holds for all $x, y \in \Omega$.*

Remark C.2.

- *If $\alpha = 1$, then $f\colon \Omega \to \mathbb{R}^m$ is called **Lipchitz continuous**.*

- *If the closure* Ω *is compact, then* $f\colon \Omega \to \mathbb{R}^n$ *is locally Hölder continuous if and only if it is uniformly Hölder continuous. This conclusion comes from optimizing the constants* $\max\{C(y) \mid y \in U_\beta\}$ *over some finite cover* $\{U_\beta\}_\beta$ *of* Ω.

- *It should be clear how these definitions can be adapted to maps between metric spaces, replacing the norm with the distance function.*

Combining the α-Hölder condition with k^{th}-order differentiability yields the notion of $C^{k,\alpha}$-regularity.

Definition C.3 ($C^{k,\alpha}$ Regularity/Manifolds)**.** *Fix* $k \in \mathbb{N}$, $0 < \alpha \leq 1$. *Again let* $n, m \in \mathbb{N}$.

- *Let* $\Omega \subset \mathbb{R}^n$. *We say that a map* $f\colon \Omega \to \mathbb{R}^m$ *is of* $C^{k,\alpha}$ ***regularity*** *if and only if* f *has continuous derivatives up to order* k*, and the* k^{th} *derivative is locally*[1] α*-Hölder continuous.*

- *(*$C^{k,\alpha}$ *Manifolds [Bonic and Frampton, 1966]) A Banach manifold* (M, g) *is called* $C^{k,\alpha}$ *if and only if its transition functions are of* $C^{k,\alpha}$ *regularity. A function* $f\colon M \to N$ *between* $C^{k,\alpha}$ *manifolds is of* $C^{k,\alpha}$ *regularity if and only if its local representation in any local chart is of* $C^{k,\alpha}$ *regularity).*

The ***Hölder space*** $C^{k,\alpha}(M, N)$ *is defined to be the space of all* $C^{k,\alpha}$*-regular functions* $f\colon M \to N$.

Proposition C.4. *(Implicit from [Ruelle, 1989, Appendix B]) We have the following facts:*

(i) *With the above notation, we have the following inclusions, for all* $k \in \mathbb{Z}_{\geq 0}, \alpha \in (0, 1]$*:*
$$C^\infty(M, N) \subset C^{k+1}(M, N) \subset C^{k,\alpha}(M, N) \subset C^k(M, N).$$

(ii) *If* f *is* $C^{0,\alpha}$ *and* g *is* $C^{0,\beta}$*, then* $g \circ f$ *is* $C^{0,\alpha\beta}$.

(iii) *If* $k + \alpha \geq 1$ *or* $(k, \alpha) = (0, 0)$*, then the composition of* $C^{k,\alpha}$ *maps is* $C^{k,\alpha}$*:*
$$f \in C^{k,\alpha}(M, N), g \in C^{k,\alpha}(N, P) \implies g \circ f \in C^{k,\alpha}(M, P).$$

Remark C.5. *A few things to note:*

- *Having the Hölder condition on the* k^{th} *derivative, in the definition of* $C^{k,\alpha}$ *regularity, is equivalent to having the Hölder condition on derivatives of all orders up to and including* k*. This follows by (i) in the Proposition above.*

[1] This is the assumption as in [Ruelle, 1989]

- *The definition of $C^{k,\alpha}$ regularity of maps between $C^{k,\alpha}$ manifolds is only well-defined for $k+\alpha \geq 1$ and $(k,\alpha) = (0,0)$. Indeed, if $f\colon M \to N$ is a map between $C^{k,\alpha}$ manifolds, then its local representation is $C^{k,\alpha}$ in some local chart if and only if it is $C^{k,\alpha}$ in every other local chart; this follows immediately from the fact that the transition maps are defined to be $C^{k,\alpha}$, and that compositions of $C^{k,\alpha}$ maps are $C^{k,\alpha}$. On the other hand, compositions of $C^{0,\alpha}$ where $0 < \alpha < 1$ are only guaranteed to be C^{0,α^2} (by (ii) in the Proposition above.*

- *All C^∞ manifolds (and hence all C^∞ maps between C^∞ manifolds) are $C^{k,\alpha}$, for any $k \in \mathbb{Z}_{\geq 0}, \alpha \in (0,1]$. This follows from the fact that all C^∞ functions on Euclidean space are α-Hölder continuous (and thus so are all derivatives). The result for maps between manifolds follows by passing to local charts.*

Definition of $D^{k,\alpha}$-Riemannian Manifolds

Here we introduce $D^{k,\alpha}$-Riemannian manifolds following the definition of a $D^{1,1}$-Riemannian manifold appearing in the Gromov Compactness theorem, as stated in [Greene and Wu, 1988], and extending the definition to arbitrary $k \in \mathbb{Z}_{\geq 0}$, $\alpha > 0$ in the obvious way.

Definition C.6 ($D^{k,\alpha}$-Riemannian manifold)**.** *A $D^{k,\alpha}$-Riemannian manifold M is a $C^{k,\alpha}$ manifold with a Riemannian metric g whose induced distance function $d_g : M \times M \to \mathbb{R}_{\geq 0}$ is C^k (within the cut locus and off the diagonal), and the derivatives of d_g up to order k are locally α-Hölder continuous, i.e. d_g is class $C^{k,\alpha}$.*

In the work of Greene and Wu in [Greene and Wu, 1988], the refined version of the Gromov compactness theorem (or corollary thereof) states that if $\{(M_l, g_l)\}_{l\in\mathbb{N}}$ is a sequence of smooth Riemannian manifolds with bounded geometry, then there exists a subsequence $\{(M_k, g_k)\}_{k\in\mathbb{N}}$ and a C^∞ manifold M with Riemannian metric g which is $C^{1,\alpha}$ for all $\alpha \in (0,1)$, such that $M_k \to M$ in the Lipschitz distance. Note that the manifold M is *not* necessarily C^∞-Riemannian, since the metric is not C^∞.

It is noted in [Greene and Wu, 1988] that this refined result does imply the original statement of Gromov, which claims that the limiting Riemannian manifold is $D^{1,1}$.

Equivalent notions of $C^{k,\alpha}$ regularity

It is possible to defines maps between Riemannian manifolds $f\colon (M,g) \to (N,h)$ as Hölder continuous if and only if the Hölder condition holds with respect to the metric space structures induced by g and h on M and N, respectively. This definition coincides with the above definition on ordinary smooth manifolds, since the distance function d_g is locally bounded above and below by a Euclidean norm when expressed in local coordinates.

Proposition C.7 ($C^{k,\alpha}$ Regularity)**.** *If (M, g), (N, h) are a Riemannian manifolds, then a function $f\colon M \to N$ is of $C^{k,\alpha}$ regularity iff it is C^k and $T^k f\colon T^k M \to T^k N$ is (locally) α-Hölder continuous with respect to the k-th Sasaki metrics on $T^k M$ and $T^k N$.*

Proof. The proof is immediate since any two norms can locally be controlled above and below by each other. Choose a local chart on a subset $V \subset \mathbb{R}^n$ such that the closure $\bar{V}$ is compact. Then there exist constants $A, B \in \mathbb{R}_{>0}$ such that:

$$A^{-1}\|\cdot\|_{\mathbb{R}^n} \leq \|\cdot\|_g \leq A\|\cdot\|_{\mathbb{R}^n}$$

and

$$B^{-1}\|\cdot\|_{\mathbb{R}^m} \leq \|\cdot\|_h \leq B\|\cdot\|_{\mathbb{R}^m}.$$

Hence α-Hölder continuity holds with respect to the norms $\|\cdot\|_g, \|\cdot\|_h$ if and only if it holds with respect to $\|\cdot\|_{\mathbb{R}^n}, \|\cdot\|_{\mathbb{R}^m}$. It is clear how this fact proves the claimed result. $\square$

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