Poisson Resolution via Weighted Blowups

Boris Zupancic

McGill University

August, 2025

Poisson Resolution

(X, π) - smooth Poisson variety. $Y \subset X$ - singular Poisson subvariety.

(X, π) - smooth Poisson variety. $Y \subset X$ - singular Poisson subvariety.

A **Poisson resolution** is a Poisson map $b\colon (Y' \subset X', \pi') \to (Y \subset X, \pi)$ such that

Poisson Resolution

(X, π) - smooth Poisson variety. $Y \subset X$ - singular Poisson subvariety.

A **Poisson resolution** is a Poisson map $b\colon (Y' \subset X', \pi') \to (Y \subset X, \pi)$ such that

1. $b\colon Y' \subset X' \to Y \subset X$ is a proper morphism of pairs,

Poisson Resolution

(X, π) - smooth Poisson variety. $Y \subset X$ - singular Poisson subvariety.

A **Poisson resolution** is a Poisson map $b\colon (Y' \subset X', \pi') \to (Y \subset X, \pi)$ such that

1. $b\colon Y' \subset X' \to Y \subset X$ is a proper morphism of pairs,
2. $Y' \subset X'$ are smooth,

Poisson Resolution

(X, π) - smooth Poisson variety. $Y \subset X$ - singular Poisson subvariety.

A **Poisson resolution** is a Poisson map $b\colon (Y' \subset X', \pi') \to (Y \subset X, \pi)$ such that

1. $b\colon Y' \subset X' \to Y \subset X$ is a proper morphism of pairs,
2. $Y' \subset X'$ are smooth,
3. b is an isomorphism away from Y_{sing}.

Poisson Resolution

(X, π) - smooth Poisson variety. $Y \subset X$ - singular Poisson subvariety.

A **Poisson resolution** is a Poisson map $b\colon (Y' \subset X', \pi') \to (Y \subset X, \pi)$ such that

1. $b\colon Y' \subset X' \to Y \subset X$ is a proper morphism of pairs,
2. $Y' \subset X'$ are smooth,
3. b is an isomorphism away from Y_{sing}.

(When) does Poisson resolution of singularities exist?

Dream: Use an existing resolution algorithm.

Poisson Resolution: Dream vs Reality

Dream: Use an existing resolution algorithm. E.g. in the absence of a Poisson structure:

1. Hironaka [Hir64]: resolution via **blowups**,

Poisson Resolution: Dream vs Reality

Dream: Use an existing resolution algorithm. E.g. in the absence of a Poisson structure:

1. Hironaka [Hir64]: resolution via **blowups**,
2. Abramovich, Temkin, Włodarczyk [ATW24]: functorial embedded resolution via **weighted blowups**.

Poisson Resolution: Dream vs Reality

Dream: Use an existing resolution algorithm. E.g. in the absence of a Poisson structure:

1. Hironaka [Hir64]: resolution via **blowups**,
2. Abramovich, Temkin, Włodarczyk [ATW24]: functorial embedded resolution via **weighted blowups**.

Dream: Use an existing resolution algorithm. E.g. in the absence of a Poisson structure:

1. Hironaka [Hir64]: resolution via **blowups**,
2. Abramovich, Temkin, Włodarczyk [ATW24]: functorial embedded resolution via **weighted blowups**.

$$(Y^{(n)} \subset X^{(n)}) \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} (Y^{(1)} \subset X^{(1)}) \xrightarrow{\phi^{(1)}} (Y \subset X)$$

Poisson Resolution: Dream vs Reality

Dream: Use an existing resolution algorithm. E.g. in the absence of a Poisson structure:

1. Hironaka [Hir64]: resolution via **blowups**,
2. Abramovich, Temkin, Włodarczyk [ATW24]: functorial embedded resolution via **weighted blowups**.

$$(Y^{(n)} \subset X^{(n)}) \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} (Y^{(1)} \subset X^{(1)}) \xrightarrow{\phi^{(1)}} (Y \subset X)$$

Reality: Multivectors (e.g. Poisson structures) don't always pullback.

Poisson Resolution: Dream vs Reality

Dream: Use an existing resolution algorithm. E.g. in the absence of a Poisson structure:

1. Hironaka [Hir64]: resolution via **blowups**,
2. Abramovich, Temkin, Włodarczyk [ATW24]: functorial embedded resolution via **weighted blowups**.

$$(Y^{(n)} \subset X^{(n)}) \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} (Y^{(1)} \subset X^{(1)}) \xrightarrow{\phi^{(1)}} (Y \subset X)$$

Reality: Multivectors (e.g. Poisson structures) don't always pullback.

My thesis problem: How close can you get to Poisson resolution using (weighted) blowups?

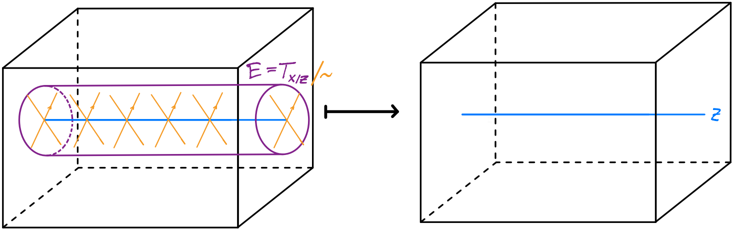
Aside: Blowups

Blowup: replace $Z \subset X$ with its bundle of transverse lines

Aside: Blowups

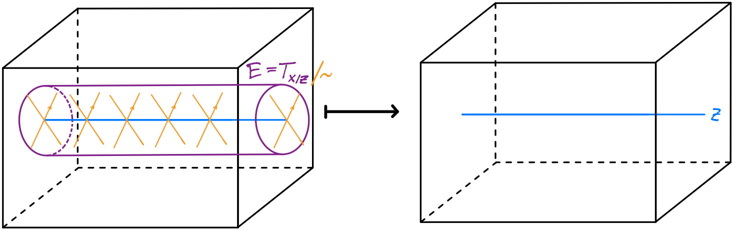
Blowup: replace $Z \subset X$ with its bundle of transverse lines

$$b : Bl(X, Z) \longrightarrow X$$



Blowup: replace $Z \subset X$ with its bundle of transverse lines

$$b: Bl(X, Z) \longrightarrow X$$



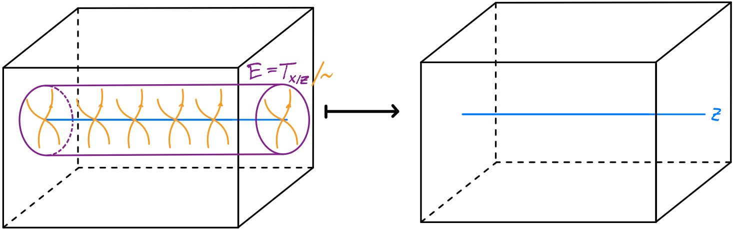
$$Z = \{x_1 = \cdots = x_k = 0\}$$

$$E = T_{X/Z}/\sim$$

$$(v_1, \ldots, v_k, x_{k+1}, \ldots, x_n) \sim (\lambda v_1, \ldots, \lambda v_k, x_{k+1}, \ldots, x_n)$$

Weighted blowup:* replace $Z \subset X$ with a bundle of transverse curves:

$$b : Bl_w(X, Z) \longrightarrow X$$



$$Z = \{x_1 = \cdots = x_k = 0\}$$

$$E = T_{X/Z}/\sim$$

$$(v_1, \ldots, v_k, x_{k+1}, \ldots, x_n) \sim (\lambda^{w_1} v_1, \ldots, \lambda^{w_k} v_k, x_{k+1}, \ldots, x_n)$$

*For precise definition see e.g: [Loizides-Meinrenken-2023], [Quek-Rydh].

Inflatable Poisson Structures

Lifting Criterion: A Poisson structure π on X lifts to a weighted blowup $Bl_{\mathcal{W}}(X, Z) \to X$ (with lift tangent to the exceptional divisor) if and only if

$$\mathrm{ord}_{\mathcal{W}}\pi \geq 0.$$

Lifting Criterion: A Poisson structure π on X lifts to a weighted blowup $Bl_{\mathcal{W}}(X, Z) \to X$ (with lift tangent to the exceptional divisor) if and only if

$$\mathrm{ord}_{\mathcal{W}}\pi \geq 0.$$

Example: a Jacobian structure on $X = \mathbb{C}^3$

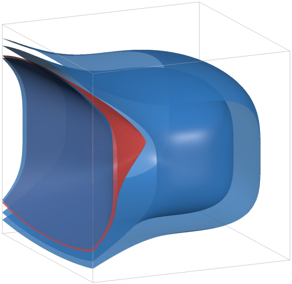
Inflatable Poisson Structures

Lifting Criterion: A Poisson structure π on X lifts to a weighted blowup $Bl_{\mathcal{W}}(X, Z) \to X$ (with lift tangent to the exceptional divisor) if and only if

$$\mathrm{ord}_{\mathcal{W}}\pi \geq 0.$$

Example: a Jacobian structure on $X = \mathbb{C}^3$

$$\begin{aligned} Y &= \{x_1^2 + x_2^3 + x_3^a = 0\},\ a \geq 3 \\ Z &= Y_{\mathsf{sing}} = \{0\}, \\ \pi &= [\partial_1 \wedge \partial_2 \wedge \partial_3, x_1^2 + x_2^3 + x_3^a] \\ &= 2x_1\partial_2 \wedge \partial_3 + 3x_2^2\partial_3 \wedge \partial_1 + ax_3^{a-1}\partial_1 \wedge \partial_2 \end{aligned}$$



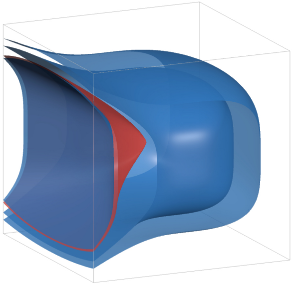
Inflatable Poisson Structures

Lifting Criterion: A Poisson structure π on X lifts to a weighted blowup $Bl_{\mathcal{W}}(X,Z) \to X$ (with lift tangent to the exceptional divisor) if and only if

$$\mathrm{ord}_{\mathcal{W}}\pi \geq 0.$$

Example: a Jacobian structure on $X = \mathbb{C}^3$

$$\begin{aligned} Y &= \{x_1^2 + x_2^3 + x_3^a = 0\}, \ a \geq 3 \\ Z &= Y_{\mathsf{sing}} = \{0\}, \\ \pi &= [\partial_1 \wedge \partial_2 \wedge \partial_3, x_1^2 + x_2^3 + x_3^a] \\ &= 2x_1\partial_2 \wedge \partial_3 + 3x_2^2\partial_3 \wedge \partial_1 + ax_3^{a-1}\partial_1 \wedge \partial_2 \end{aligned}$$

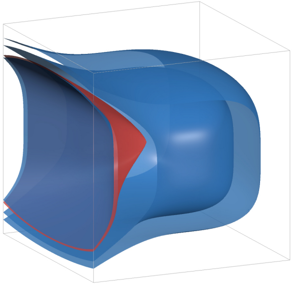


$$\mathrm{ord}_{\mathcal{W}}\pi = \min\{w_1 - w_2 - w_3, 2w_2 - w_3 - w_1, (a-1)w_3 - w_1 - w_2\}$$

Lifting Criterion: A Poisson structure π on X lifts to a weighted blowup $Bl_{\mathcal{W}}(X, Z) \to X$ (with lift tangent to the exceptional divisor) if and only if

$$\mathrm{ord}_{\mathcal{W}}\pi \geq 0.$$

Example: a Jacobian structure on $X = \mathbb{C}^3$



$$\begin{aligned} Y &= \{x_1^2 + x_2^3 + x_3^a = 0\}, \ a \geq 3 \\ Z &= Y_{\mathsf{sing}} = \{0\}, \\ \pi &= [\partial_1 \wedge \partial_2 \wedge \partial_3, x_1^2 + x_2^3 + x_3^a] \\ &= 2x_1 \partial_2 \wedge \partial_3 + 3x_2^2 \partial_3 \wedge \partial_1 + a x_3^{a-1} \partial_1 \wedge \partial_2 \end{aligned}$$

$$\mathrm{ord}_{\mathcal{W}}\pi = \min\{w_1 - w_2 - w_3, 2w_2 - w_3 - w_1, (a-1)w_3 - w_1 - w_2\}$$

A Poisson pair $(Y \subset X, \pi)$ is **inflatable** if there exists a weighted blowup $Bl_{\mathcal{W}}(X, Z) \to X$ such that $\mathrm{ord}_{\mathcal{W}}\pi \geq 0$ and the singularities of $Bl_{\mathcal{W}}(Y, Z)$ are strictly **better**[†] than the singularities of Y.

[†] In the sense that the singularity invariant of [ATW24] decreases.

Recall my thesis problem: *How close can you get to Poisson resolution using (weighted) blowups?*

Recall my thesis problem: *How close can you get to Poisson resolution using (weighted) blowups?*

Observation: One can always sequentially apply weighted blowups until arriving at a **non-inflatable** Poisson structure:

$$(Y^{(n)} \subset X^{(n)}, \pi^{(n)}) \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} (Y^{(1)} \subset X^{(1)}, \pi^{(1)}) \xrightarrow{\phi^{(1)}} (Y \subset X, \pi)$$

Classification Problem

Recall my thesis problem: *How close can you get to Poisson resolution using (weighted) blowups?*

Observation: One can always sequentially apply weighted blowups until arriving at a **non-inflatable** Poisson structure:

$$(Y^{(n)} \subset X^{(n)}, \pi^{(n)}) \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} (Y^{(1)} \subset X^{(1)}, \pi^{(1)}) \xrightarrow{\phi^{(1)}} (Y \subset X, \pi)$$

Can we classify the non-inflatables?

Complete classification of non-inflatables for $1 \leq \dim Y < \dim X \leq 3$:

Complete classification of non-inflatables for $1 \leq \dim Y < \dim X \leq 3$:

Theorem (Lapointe–Matviichuk–Pym–Z.)

Let (X, π) be a smooth Poisson variety. Let $Y \subset X$ be a singular Poisson subvariety.

1. *If $\dim Y = 1,\ \dim X = 2$, then π is inflatable.*

Complete classification of non-inflatables for $1 \leq \dim Y < \dim X \leq 3$:

Theorem (Lapointe–Matviichuk–Pym–Z.)

Let (X, π) be a smooth Poisson variety. Let $Y \subset X$ be a singular Poisson subvariety.

1. *If $\dim Y = 1,\ \dim X = 2$, then π is inflatable.*
2. *If $\dim Y = 1,\ \dim X = 3$, then π is non-inflatable iff near any singular point, Y is planar, say $Y \subset \{x_1 = 0\}$, and $\pi = x_1 \partial_1 \wedge \partial_2$.*

Complete classification of non-inflatables for $1 \leq \dim Y < \dim X \leq 3$:

Theorem (Lapointe–Matviichuk–Pym–Z.)

Let (X, π) be a smooth Poisson variety. Let $Y \subset X$ be a singular Poisson subvariety.

1. *If $\dim Y = 1,\ \dim X = 2$, then π is inflatable.*
2. *If $\dim Y = 1,\ \dim X = 3$, then π is non-inflatable iff near any singular point, Y is planar, say $Y \subset \{x_1 = 0\}$, and $\pi = x_1 \partial_1 \wedge \partial_2$.*
3. *If $\dim Y = 2,\ \dim X = 3$, then π is non-inflatable iff Y has only ADE singularities, where π is locally Jacobian.*

Proof Sketch $(\dim Y = 1, \dim X = 3)$

Y_{sing} is isolated and

$$\pi = \phi_1 \cdot \sigma_1 + \cdots + \phi_r \cdot \sigma_r$$

where $Y = \{\phi_1 = \cdots = \phi_r = 0\}$ and $\sigma_1, \ldots, \sigma_r \in \mathfrak{X}^2_X$.

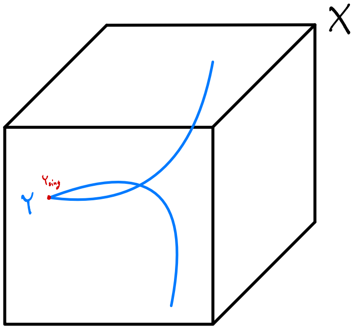
Proof Sketch $(\dim Y = 1, \dim X = 3)$

Y_{sing} is isolated and

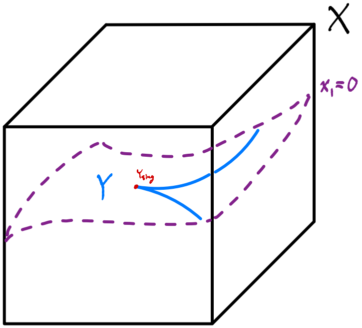
$$\pi = \phi_1 \cdot \sigma_1 + \cdots + \phi_r \cdot \sigma_r$$

where $Y = \{\phi_1 = \cdots = \phi_r = 0\}$ and $\sigma_1, \ldots, \sigma_r \in \mathfrak{X}_X^2$.

Space Curve

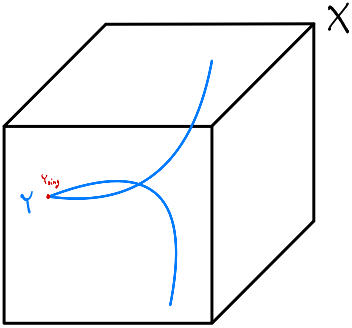


Plane Curve



Proof Sketch $(\dim Y = 1, \dim X = 3)$

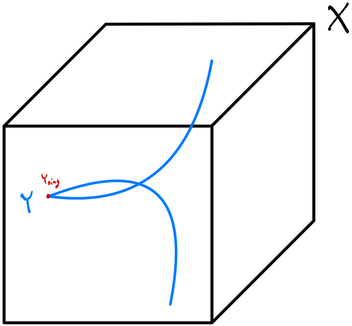
Space Curve



1. $\phi_1, \ldots, \phi_r$ must be at least quadratic.

Proof Sketch $(\dim Y = 1, \dim X = 3)$

Space Curve



1. $\phi_1, \ldots, \phi_r$ must be at least quadratic.

$$\Downarrow$$

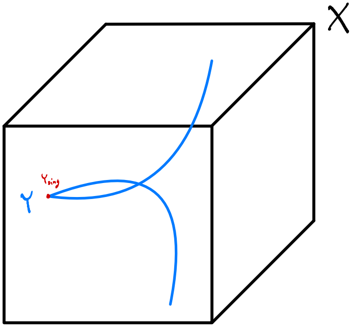
A straightforward computation shows that

$$\mathrm{ord}_{\mathcal{W}} \pi \geq 0$$

when $\mathcal{W}$ is the weighting of [ATW24].

Proof Sketch $(\dim Y = 1, \dim X = 3)$

Space Curve



1. $\phi_1, \ldots, \phi_r$ must be at least quadratic.

$$\Downarrow$$

A straightforward computation shows that

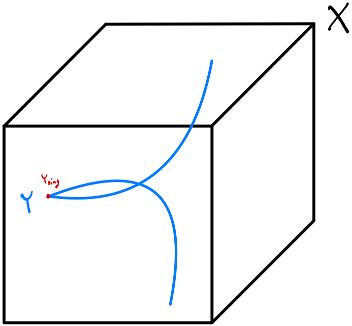
$$\text{ord}_{\mathcal{W}}\pi \geq 0$$

when $\mathcal{W}$ is the weighting of [ATW24].

2. The weighted blowup of [ATW24] always improves the singularities.

Proof Sketch $(\dim Y = 1, \dim X = 3)$

Space Curve



1. $\phi_1, \ldots, \phi_r$ must be at least quadratic.

$$\Downarrow$$

A straightforward computation shows that

$$\mathrm{ord}_{\mathcal{W}}\pi \geq 0$$

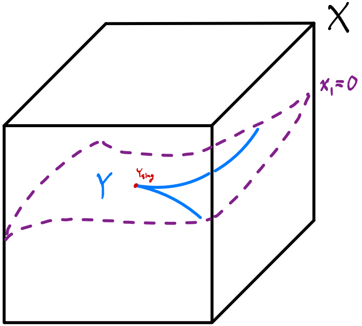
when $\mathcal{W}$ is the weighting of [ATW24].

2. The weighted blowup of [ATW24] always improves the singularities.

$$\implies \pi \text{ is inflatable.}$$

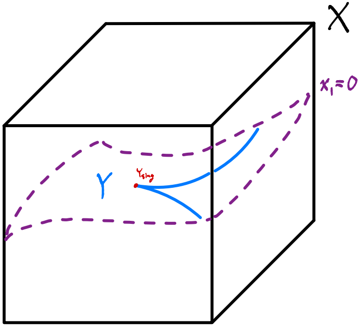
$Y = \{\phi_1 = \cdots = \phi_r = 0\}$ is (locally) trapped in a coordinate hypersurface $\{x_1 = 0\}$ where $x_1 = \phi_1$.

Plane Curve



Proof Sketch $(\dim Y = 1, \dim X = 3)$

Plane Curve

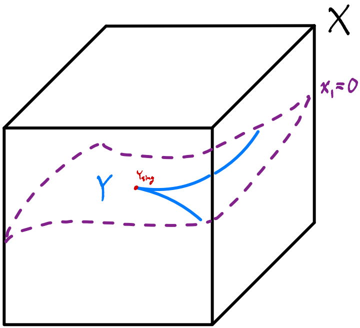


$Y = \{\phi_1 = \cdots = \phi_r = 0\}$ is (locally) trapped in a coordinate hypersurface $\{x_1 = 0\}$ where $x_1 = \phi_1$.

Case 1: If π is at least quadratic, then π is inflatable under the ordinary blowup.[a]

Proof Sketch ($\dim Y = 1, \dim X = 3$)

Plane Curve



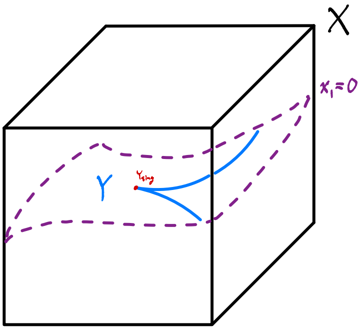
$Y = \{\phi_1 = \cdots = \phi_r = 0\}$ is (locally) trapped in a coordinate hypersurface $\{x_1 = 0\}$ where $x_1 = \phi_1$.

Case 1: If π is at least quadratic, then π is inflatable under the ordinary blowup.[a]

Case 2: If π has a linear term π_{lin}, then either

$$\pi_{\text{lin}} = x_1 \partial_1 \wedge \partial_2 \quad \text{or} \quad \pi_{\text{lin}} = x_1 \partial_2 \wedge \partial_3.$$

Plane Curve



$Y = \{\phi_1 = \cdots = \phi_r = 0\}$ is (locally) trapped in a coordinate hypersurface $\{x_1 = 0\}$ where $x_1 = \phi_1$.

Case 1: If π is at least quadratic, then π is inflatable under the ordinary blowup.[a]

Case 2: If π has a linear term π_{lin}, then either

$$\pi_{\text{lin}} = x_1\partial_1 \wedge \partial_2 \quad \text{or} \quad \pi_{\text{lin}} = x_1\partial_2 \wedge \partial_3.$$

By deformation theory:

$\pi = x_1\partial_1 \wedge \partial_2$	or $\pi = x_1\partial_2 \wedge \partial_3 + (\text{h.w.t})$
$\text{ord}_{\mathcal{W}}\pi = -w_2$	$\text{ord}_{\mathcal{W}}\pi = w_1 - w_2 - w_3$
non-inflatable	inflatable by [ATW24]

[a] A result of [Abhyankar-1983] guarantees that the singularity invariant decreases.