MA 110 NOTES

SEQUENCE AND SERIES

SEQUENCE

A Set of numbers or expressions called terms

A Sequence can be finite or infinite

Example:

(a) 3, 7 11, 15, $\Longrightarrow$ Finite sequence

(b) 2, 4, 6, 8, ... $\Longrightarrow$ Infinite sequence

EXAMPLES:

Find the four terms of the following Sequences given in form of a function.

(a) $a_n = 2n^2 - 3$ (b) $\begin{cases} a_1 = 7 \\ a_{n+1} = 2 - a_n, \ n \geq 1 \end{cases}$

(c) $f_0 = 1, \ f_n = n \cdot f_{n-1}$

Solution:

(a) The first four terms are generated by replacing n with 1, 2, 3, 4.

first term $= \boxed{a_1 = 2(1)^2 - 3 = -1}$ $\boxed{a_2 = 2(2)^2 - 3 = 5}$

Third term $= \boxed{a_3 = 2(3)^2 - 3 = 15}$ $\boxed{a_4 = 2(4)^2 - 3 = 29}$

$\therefore$ First Four terms are:

-1, 5, 15, 29

** 20^{th} term is $a_{20} = 2(20)^2 - 3 = 797$ ***

(b) We start with $a_1 = 7$ and use

$\boxed{a_{n+1} = 2 - a_n}$ for $n \geq 1$

$n=1: \quad a_{1+1} = \boxed{a_2} = 2 - a_1 = 2 - 7 = -5$

$n=2: \quad a_{2+1} = \boxed{a_3} = 2 - a_2 = 2 - (-5) = 7$

$n=3: \quad a_{3+1} = a_4 = 2 - a_3 = 2 - 7 = -5$

$n=4: \quad a_{4+1} = a_5 = 2 - a_4 = 2 - (-5) = 7$

$\therefore$ First Four terms are:

$a_1 = 7, \quad a_2 = -5, \quad a_3 = 7, \quad a_4 = -5$

$\Rightarrow \underline{7, -5, 7, -5, \ldots\ldots}$

(c) $f_0 = 1 \qquad f_n = n \cdot f_{n-1}$

$n=1, \quad f_1 = 1 \cdot f_{1-1} = 1 \cdot f_0 = 1(1) = 1$

$n=2 \quad f_2 = 2 f_{2+1} = 2 f_1 = 2(1) = 2$

$n=3 \quad f_3 = 3 f_{3-1} = 3 f_2 = 3(2) = 6$

$n=4 \quad f_4 = 4 f_{4-1} = 4 f_3 = 4(6) = 24$

First four terms are:

$\underline{1, 2, 6, 24, \ldots\ldots}$

SEQUENCES AND SERIES

A sequence is an ordered List of objects or events. Sequence of numbers are called elements or terms.

Two such sequences are Arithmetic and Geometric sequences.

General form of AP_s and GP

Given a sequence $a_1, a_2, a_3, \ldots a_n$, ($a_1$ is the first term).

AP	GP
1. Terms are obtained by Adding a fixed value d (Common difference)	Terms are obtained by multiplying a ratio Called Common ratio (r)
2. The n^{th} term of an AP is: $a_n = a_1 + (n-1) d$	The n^{th} term of a GP is $a_n = a_1 r^{n-1}$
3. Sum of AP with n terms is: $S_n = \frac{n}{2}[a_1 + a_n]$	Sum of a GP is: $S_n = \frac{a_1(1-r^n)}{1-r}$, $\boxed{r<1}$ and $S_n = \frac{a_1(r^n-1)}{r-1}$ $\boxed{r>1}$

SUM OF SEQUENCES

1. ARITHMETIC PROGRESSION (AP)

Consider an AP: $a_1, a_2, a_3, \cdots, a_n$ with common difference d.

The Sum (S_n) is

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)$$

$$S_n = \frac{n}{2}\left[a + (a_1 + (n-1)d\right]$$

$$\boxed{S_n = \frac{n}{2}\left[a_1 + a_n\right]}$$

Example

Find the first term of an A.P for which the fourth term is 26 and the nineth term is 61.

SOLUTION:

using $a_n = a_1 + (n-1)d$

$a_4 = 26$ and $a_9 = 61$

$26 = a_1 + (4-1)d$ and $61 = a_1 + (9-1)d$

$$\begin{cases} 26 = a_1 + 3d \\ 61 = a_1 + 8d \end{cases}$$

Solving Simutaneously; $\boxed{\begin{matrix} d = 7 \\ a_1 = 5 \end{matrix}}$ and

Examples.

(3)

1) Find the General term Expression for the AP $6, 2, -2, -6$

Solution:-

Common difference $d = 2 - 6 = -4$

n^{th} term: $a_n = a_1 + (n-1)d$

$a_n = 6 + (n-1)(-4)$

$\underline{a_n = 10 - 4n}$

2) Find the 40^{th} term of the AP $1, 5, 9, 13, \ldots$

n^{th} term $a_n = a_1 + (n-1)d$

$d = 5 - 1 = 4$

40^{th} term $a_{40} = 1 + (40-1)4$

$= 1 + 39(4)$

$= 1 + 156$

$= 157$

3) The first term of an AP is -4 and the 15^{th} term is double the 5^{th} term, find the 12^{th} term.

Solution:

4

first find d:

$a_n = a_1 + (n-1)d$

$a_{15} = -4 + 14d$ $\quad a_5 = -4 + 4d$

$a_{15} = 2(a_5)$

$-4 + 14d = 2(-4 + 4d)$

$-4 + 14d = -8 + 8d$

$6d = -4$

$d = -4/6 = -2/3$

12th term is:

$a_{12} = a_1 + (12-1)(-2/3)$

$= -4 + (11)(-2/3) = -\frac{34}{3}$

Example:- (7)

1. Find the sum of the first thirty terms of the AP 3, 7, 11, 15, ...

$S_n = \frac{n}{2}[a_1 + a_n]$ $\quad d = 7-3 = 4$

$S_{30} = \frac{30}{2}[a_1 + a_{30}]$

$a_{30} = a_1 + (30-1)d$

$= 3 + (29)4$

$= 3 + 116 = 119$

$S_{30} = \frac{30}{2}[3 + 119] = 15(122) = 1830$

2. Find the sum of $7 + 10 + 13 + \cdots + 157$

$d = 10 - 7 = 3$

$a_n = a_1 + (n-1)d$

$157 = 7 + (n-1)3$

$150 = 3n - 3$

$3n = 153$

$\underline{n = 51}$

$S_{51} = \frac{n}{2}[a_1 + a_{51}] = \frac{51}{2}[7 + 157]$

$= \frac{51}{2}[164)$

$= \underline{\underline{4182}}$

SUMMATION NOTATION (Σ)

(1) AP SUMMATION

We use Σ (Sigma) to indicate sum of certain number of terms of a sequence

e.g find the sum of

(a) $\sum_{i=1}^{50} (3i+4)$

Solution:-

$\sum_{i=1}^{50} (3i+4) = [3(1)+4] + [3(2)+4)] + \cdots + [3(50)+4]$

$= 7 + 10 + 13 + \cdots + 154$

$S_n = \frac{n}{2}(a_1 + a_n)$

$a_1 = 7,$
$a_n = 154$
$n = 50$

$S_{50} = \frac{50}{2}(7+154)$

$= 25(161) = 4025$

7

5) Express the Series $1+4+7+10+13+\cdots+298$

In form $\sum_{r=1}^{n} f(r)$ General term

Solution:

first find $d = 4-1 = 3$

first term $a_1 = 1$

n^{th} term $a_n = a_1 + (n-1)d$

$a_n = 1 + (n-1)3$

$= 1 + 3n - 3$

$= 3n - 2$

$\boxed{f(r) = 3r - 2}$

$\therefore \sum_{r=1}^{n} 3r-2$

$\boxed{\Rightarrow \sum_{r=1}^{100} 3r-2}$

find n

$a_n = 3n-2$

$298 = 3n-2$

$3n = 300$

$n = 100$

GEOMETRIC PROGRESSION:

(11)

When a sequence has a constant ratio between successive terms, it is called a Geometric progression (GP). The constant is called a common ratio r.

e.g a) $1, 3, 9, 27, 81, \ldots$ has common ratio

$3/1 = 3, \quad 9/3 = 3, \quad 81/27 = 3.$

(b) ~~16, 8~~ $16, 8, 4, 2, 1$ has common ratio

If a_1 is the first term and r the common ratio we generate the sequence as follows;

first Term : a_1

second Term : $a_1 \cdot r$

Third Term : $(a_1 \cdot r) r = a_1 r^2$

Fourth Term : $(a_1 r^2) r = a_1 r^3$

$\vdots$

n^{th} term : $a_1 r^{n-1}$

$\therefore$ The General term of a GP is

$\boxed{a_n = a_1 r^{n-1}}$ where $r = \dfrac{a_{k+1}}{a_k}$.

Example:

1. Find the general term for the GP
8, 16, 32, 64.

Solution:

Using $a_n = a_1 r^{n-1}$

$$r = \frac{16}{8} = 2$$

$$a_1 = 8$$

$$a_n = 8 \cdot 2^{n-1} = (2^3)(2^{n-1}) = 2^{n+2}$$

$$\therefore \boxed{a_n = 2^{n+2}}$$

2. Find the common ratios in the following:

(a) 2, 10, 50, 250, ...

(b) 90, −30, 10, $-3\frac{1}{3}$

(a) $r = \frac{a_{k+1}}{a_k} = \frac{a_2}{a_1} = \frac{10}{2} = 5$

(b) $r = \frac{a_3}{a_2} = \frac{10}{-30} = -\frac{1}{3}$

3. The second term of a GP is 4 and the 4th term is 8. Find the value of

(a) Common ratio

(b) First term

(c) The 10th term.

Solution:

(13)

Using $a_n = a_1 r^{n-1}$

4^{th} term is 8

$a_4 = a_1 r^{4-1} = 8$

$\boxed{= a_1 r^3 = 8}$ ....(i)

2^{nd} term is 4.

$a_2 = a_1 r^{2-1} = 4$

$\boxed{a_1 r = 4}$ ....(ii)

$$\begin{cases} a_1 r^3 = 8 & \text{(i)} \\ a_1 r = 4 & \text{(ii)} \end{cases}$$ divide (i) by (ii).

$r^2 = 2 \Rightarrow r = \sqrt{2}$

$a_1 r = 4$

$a_1 \sqrt{2} = 4 \Rightarrow a_1 = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = \underline{\underline{2\sqrt{2}}}$

The 10^{th} term:

$a_{10} = a_1 r^{10-1}$

$= a_1 r^9 = 2\sqrt{2}(\sqrt{2})^9$

$= 2(\sqrt{2})^{10} = 2(2^{1/2})^{10}$

$= 2(2^5) = 2^6 = \underline{\underline{64}}$

SUM OF A GEOMETRIC SERIES

Consider a general geometric sequence

$a_1, a_1r, a_1r^2, \dots a_1r^{n-1}$

let S_n denote the sum of the first n terms:

$$S_n = a_1 + a_1r + a_1r^2 + \cdots\cdots a_1r^{n-1} \quad \cdots (i)$$

We multiply both sides by r

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n \quad \cdots (ii)$$

Subtracting (2) from (1)

$$S_n - rS_n = a_1 - a_1r^n$$

$$S_n(1-r) = a_1(1-r^n)$$

$$S_n = \frac{a_1(1-r^n)}{1-r} \qquad r \neq 1$$

NOTE:

(a) If $r < 1$, $\quad S_n = \dfrac{a_1(1-r^n)}{1-r}$

(b) If $r > 1$ $\quad S_n = \dfrac{a_1(r^n-1)}{r-1}$

Example:-

1) Find the sum 15

$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{256}$

Solution:-

$a_1 = 1, \quad a_n = \frac{1}{256} \quad r = \frac{1/2}{1} = \frac{1}{2}$

$a_n = a_1 r^{n-1}$

$\frac{1}{256} = (1)\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{2^8} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow \left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{n-1}$

$8 = n - 1$

$\underline{\underline{n = 9}}$

$r = \frac{1}{2} < 1$

$\therefore S_n = \frac{a_1(1 - r^n)}{1 - r} = \cancel{1(1-(\tfrac{1}{2})}$

$= \frac{1(1 - (\frac{1}{2})^9)}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{512}}{\frac{1}{2}}$

$= \frac{511}{512} \div \frac{1}{2}$

$= \underline{\frac{511}{256}}$ or $\underline{1\frac{255}{256}}$

Solution B ***

(16)

$$S = 1 + \tfrac{1}{2} + \cdots + \frac{1}{256} \quad \cdots (1)$$

$$r = \tfrac{1}{2}$$

multiply both sides by 1/2.

$$\tfrac{1}{2}S = \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \tfrac{1}{512}. \quad \cdots (2)$$

subtract 2 from 1

$$S - \tfrac{1}{2}S = 1 - \frac{1}{512}$$

$$\frac{1}{2}S = \frac{511}{512}$$

$$S = \frac{511}{256} \text{ or } 1\frac{255}{256}$$

2. Find the Sum $\sum_{i=1}^{10} 2^i$

$$\sum_{i=1}^{10} 2^i = 2^1 + 2^2 + 2^3 + \cdots\cdots 2^{10}$$

$$= 2 + 4 + 8 + 16 + \cdots + 1024$$

Since the Indicated sum is a Geometric Series we have $a_1 = 2$, $n = 10$ $r = \frac{4}{2} = 2$

$$S_{10} = \frac{a_1(r^{10} - 1)}{r - 1} = \frac{2(2^{10} - 1)}{2 - 1} = \frac{2(1024 - 1)}{1}$$

$$= \underline{2046}$$

(3) Find $\sum_{i=1}^{10} (3\times 2^i)$ (17)

Solution: $= 3\times 2^1 + 3\times 2^2 + \cdots + 3\times 2^{10}$

$= 3(2^1 + 2^2 + 2^3 + \cdots + 2^{10})$

$= 3\left(\frac{a_1(r^n - 1)}{r-1}\right)$

$= 3\left[\frac{2(2^{10}-1)}{2-1}\right]$

$= 3[2046] = 6138$

THE SUM TO INFINITE OF GEOMETRIC SERIES

Consider $S_n = \frac{a_1(1-r^n)}{1-r}$, rewriting

$$S_n = \frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}$$

Examining r^n for $|r| < 1$

e.g ~~for~~ $r = 1/2$

$r^2 = (1/2)^2 = 1/4$ $\quad r^3 = (1/2)^3 = 1/8$

$r^4 = (\frac{1}{2})^4 = \frac{1}{16}$ $\quad r^5 = (\frac{1}{2})^5 = \frac{1}{32}$

for large values of n

$(\frac{1}{2})^n$ become close to zero

In general for $|r| < 1$, the expression r^n approach zero as n gets larger (18)

$$\therefore \quad S_n = \frac{a_1}{1-r}$$

The sum to infinite of a G.P is

$$S_\infty = \frac{a_1}{1-r}, \quad |r| < 1$$

Example:-

1) Find the sum of the Infinite Geometric series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

Solution:-

$a_1 = 1$ and $r = \frac{1/2}{1} = \frac{1}{2}$

$$S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \underline{2}$$

2) Change $0.\overline{14}$ to a/b form, where $a, b \in \mathbb{Z}$

(a) $b \neq 0$

$$0.\overline{14} = 0.14 + 0.0014 + 0.000014 + \cdots$$

$$r = \frac{a_2}{a_1} = \frac{0.0014}{0.14} = 0.01$$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.14}{1-0.01} = \frac{0.14}{0.99} = \frac{14}{99}$$

(b) $0.\overline{56}$

Find the Sum $\sum_{i=2}^{7} 2i^2$

Solution

$$\sum_{i=2}^{7} 2i^2 = 2(2)^2 + 2(3)^2 + 2(4)^2 + 2(5)^2 + 2(6)^2 + 2(7)^2$$

$$= 8 + 18 + 32 + 50 + 72 + 98$$

No common difference hence Not AP

Add usual way Sum is 278

3) Express the series $1 + 4 + 7 + 10 + 13 + \cdots + 298$ in the form $\sum_{r=1}^{n} f(r)$

Solution

$f(r)$ is the expression for r^{th} term of series.

Common difference $d = 3$

First term $a_1 = 1$

~~f(r)~~ r^{th} term: $f(r) = a_1 + (r-1)d$

$f(r) = 1 + (r-1)3$

$f(r) = 3r - 2$

Number of term n:

$298 = a_1 + (r-1)d$

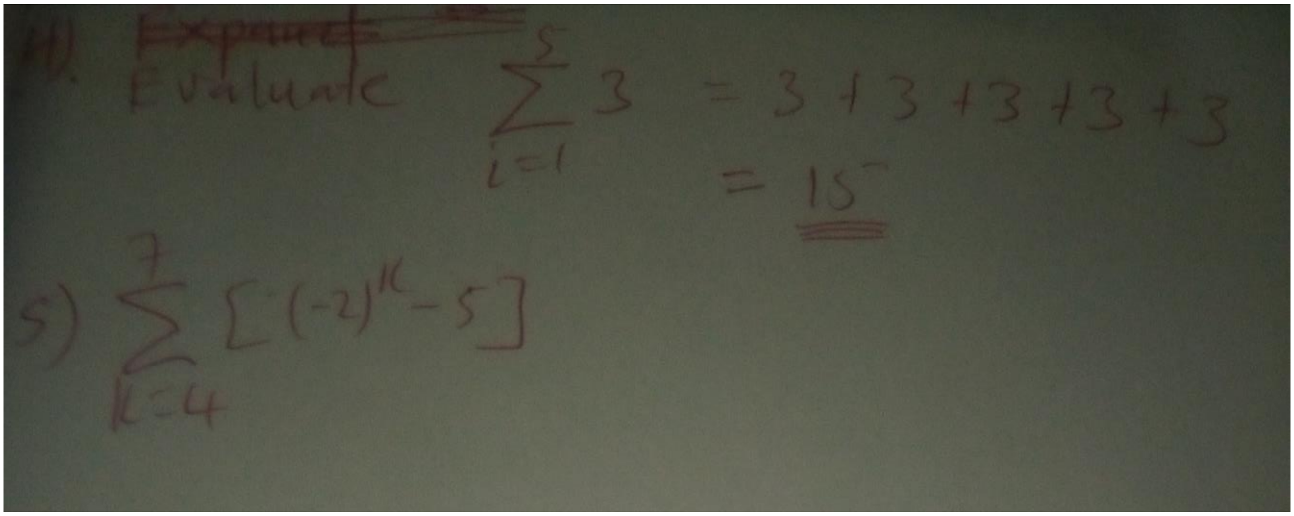
$298 = 1 + (r-1)3$

$3r - 2 = 298$

$3r = 300$

$r = 100$

$\therefore \sum_{r=1}^{100} 3r - 2$



PARTIAL FRACTIONS

PARTIAL FRACTIONS

1

A fraction $f(x)/g(x)$, $g(x) \neq 0$ and $f(x)$ and $g(x)$ are polynomials is called rational fraction. We can add two or more rational functions to form a Single rational function.

e.g $\frac{5}{x+2} + \frac{3}{x-4} = \frac{5(x-4)+3(x+2)}{(x+2)(x-4)} = \frac{8x-14}{(x+2)(x-4)}$

We can reverse the expression $\frac{8x-14}{(x+2)(x-4)}$ into partial fractions. $\frac{A}{x+2} + \frac{B}{x-4}$

First step is factorization of the denominator. Four cases may arise due to factorization of denominator:

1. Non repeated linear factors
2. Repeated linear factors
3. Non repeated and irreducible factors
4. Repeated Quadratic factors.

1. NON REPEATED LINEAR FACTORS.

Resolve the following into partial fractions

(a) $\frac{x+14}{(x-4)(x+2)} = \frac{A}{(x-4)} + \frac{B}{x+2}$

$\frac{x+14}{(x-4)(x+2)} = \frac{A(x+2)+B(x-4)}{(x-4)(x+2)}$

$A(x+2) + B(x-4) = x + 14$

let $x = -2$, $-6B = 12 \Rightarrow B = -2$

Let $x = 4$ $\quad 6A = 18$

$A = 3$

$$\therefore \frac{x+14}{(x-4)(x+2)} = \frac{3}{x-4} - \frac{2}{x+2}$$

(b) Express $\frac{2x+1}{2x^2-3x+1}$ in partial fractions

$$\frac{2x+1}{(x-1)(2x-1)} = \frac{A}{x-1} + \frac{B}{2x-1}$$

$A(2x-1) + B(x-1) = 2x+1$

Let $x = 1$, $\boxed{A = 2}$

Let $x = \frac{1}{2}$ $\quad -\frac{1}{2}B = 2 \Rightarrow \boxed{B = -4}$

(c) $$\frac{x}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+2)}$$

$$= \frac{A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1) = x$

Let $x = -1$, $B(-2)(1) = -1$

$B = \frac{1}{2}$

$x = 1$: $A(2)(3) = 1$

$A = \frac{1}{6}$

$x = -2$

$C(-2)(-1) = -2$

$C = -\frac{2}{3}$

(3)

2) DENOMINATOR WITH REPEAT LINEAR FACTORS

Find the partial fraction decomposition of

$$\frac{x-18}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$= \frac{A(x-3)^2 + B(x)(x-3) + Cx}{x(x-3)^2}$$

$$A(x-3)^2 + B(x)(x-3) + Cx = x-18$$

let $x=3$ $\quad 3C = 3-18$

$$\underline{\underline{C = -5}}$$

let $x=0$ $\quad A(-3)^2 = -18$

$$9A = -18$$

$$\underline{\underline{A = -2}}$$

Taking coefficients of x^2:

$$Ax^2 + Bx^2 = 0x^2$$

$$-2 + B = 0$$

$$\underline{\underline{B = 2}}$$

(3) NON REPEATED IRREDUCIBLE QUADRATIC FACTORS

$$\frac{2x+3}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$A(x^2+4) + (Bx+C)(x-1) = 2x+3$$

let $x=1$, $\quad A(5) = 5$

$$A = 1$$

Equating coefficients:

$$Ax^2 + 4A + Bx^2 - Bx + Cx - C = 2x + 3$$

Coefficients of x^2:

$$Ax^2 + Bx^2 = 0x^2$$

$$1 + B = 0$$

$$B = -1$$

Constants:

$$4A - C = 3$$

$$-C = 3 - 4$$

$$-C = -1$$

$$\underline{\underline{C = 1}}$$

NON FACTORISABLE QUADRATIC FACTOR THAT IS REPEATED n-Times.

$$\frac{P(x)}{(ax^2+bx+c)^n} = \frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^n} + \cdots$$

Example

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$(Ax+B)(x^2+1) + Cx + D = 5x^3 - 3x^2 + 7x - 3$$

$$(Ax^3 + Ax + Bx^2 + B) + Cx + D = 5x^3 - 3x^2 + 7x - 3$$

x^3: $Ax^3 = 5x^3 \Rightarrow A = 5$

x: $Ax + Cx = 7x$.

$$C = 7 - 5 = 2$$

x^2: $Bx^2 = -3x^2$

$\underline{\underline{B = -3}}$ (5)

constant! $B + D = -3$

$-3 + D = -3$

$D = 0$

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{5x - 3}{x^2+1} + \frac{2x}{(x^2+1)^2}.$$

IMPROPER FRACTIONS

A polynomial in which deg $P(x) \geq$ deg $Q(x)$, In this case, we divide the denominator into numerator to obtain:

$$\frac{P(x)}{Q(x)} = (\underset{\text{Quotient}}{\text{Polynomial}}) + \underset{\text{Remainder}}{\frac{P_1(x)}{Q_1(x)}}$$ and decompose

$\frac{P_1(x)}{Q_1(x)}$ in partial fractions.

e.g $\frac{4x^3}{x^2-2}$ is improper fraction.

$$\begin{array}{r} 4x \\ x^2 - 2 \,\big)\, 4x^3 \\ -(4x^3 - 8x) \\ \hline 8x \\ \hline \end{array}$$

$$\frac{4x^3}{x^2-2} = 4x + \frac{8x}{x^2-2}.$$

(6)

Since $x^2-2=(x-\sqrt{2})(x+\sqrt{2})$

$$\frac{8x}{(x-\sqrt{2})(x+\sqrt{2})}=\frac{A}{(x+\sqrt{2})}+\frac{B}{(x-\sqrt{2})}$$

$$A(x-\sqrt{2})+B(x+\sqrt{2})=8x.$$

let $x=\sqrt{2}$ $\quad B(2\sqrt{2})=8(\sqrt{2})$

$$\underline{\underline{B=4}}$$

$x=-\sqrt{2}$ $\quad A(-2\sqrt{2})=-8\sqrt{2}$

$$\underline{\underline{A=4}}$$

$$\therefore \frac{4x^3}{x^2-2}=4x+\frac{8}{x^2-2}$$

$$\underline{\underline{=4x+\frac{4}{x-\sqrt{2}}+\frac{4}{x+\sqrt{2}}.}}$$