MA 110 NOTES (GROUP B)

TERM ONE NOTES

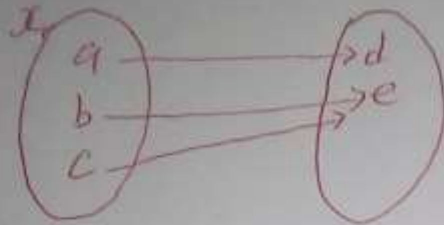
CONTENT

TYPES OF FUNCTIONS

1. INTO AND ONTO FUNCTIONS

A function $f: A \rightarrow B$ is said to be ONTO if every $y \in A$ is an image of some $x \in A$

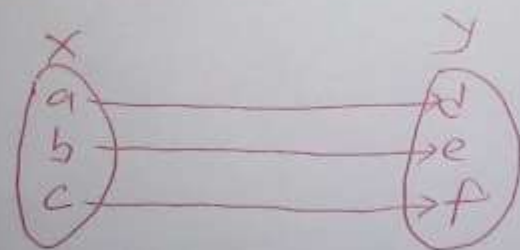
i.e



f is a function of X ONTO Y.

2. ONE TO ONE

$f: X \rightarrow Y$ is one to one if distinct elements in X have distinct images in Y



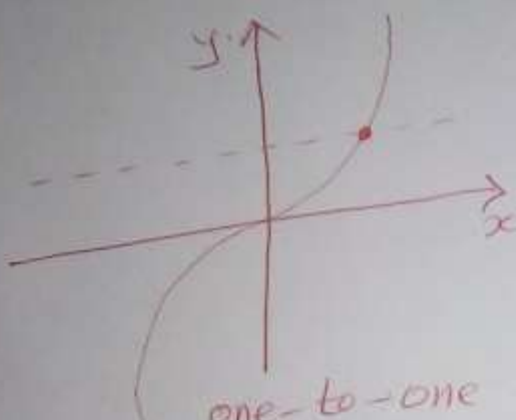
3. Many to ONE function.

Many elements in set X have one image in set Y

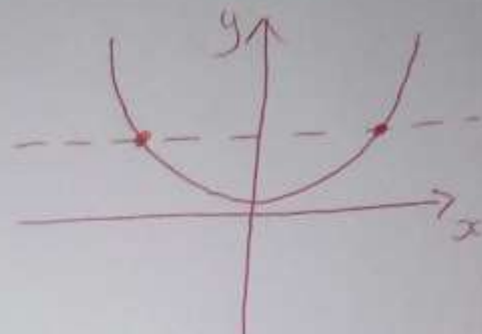
NOTE:

Any function whose graph is such that any Horizontal line (parallel to the x-axis) intersect the graph at only one point, the function is one-to-one

Example:-

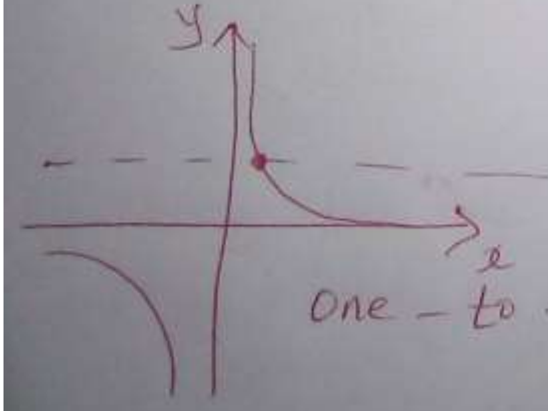


one-to-one function.



many values of x give one value of y

Hence it is a many-to-one function



One-to-one.

EVEN AND ODD FUNCTIONS

$f: A \to B$ is even if

$$f(-x) = f(x)$$

odd if $f(-x) = -f(x)$

Examples

Determine which of the following is even, odd or neither.

(a) $f(x) = x^3 + x$ (b) $g(x) = x^4 - 2x^2$

(c) $h(x) = x^2 + 2x + 1$

Solution:

(a) $f(x) = x^3 + x$

$$\begin{aligned} f(-x) &= (-x)^3 + (-x) \\ &= -x^3 - x \\ &= -(x^3 + x) \\ &= -f(x) \end{aligned}$$

$\therefore$ $f(x)$ is odd function

(b) $g(x) = x^4 - 2x^2$

$g(-x) = (-x)^4 - 2(-x)^2$

$= x^4 - 2x^2$

$= g(x)$

Hence $g(x)$ is an even function

(c) $h(x) = x^2 + 2x - 1$

$h(-x) = (-x)^2 + (-x) - 1$

$= x^2 - 2x - 1$

$= -(-x^2 + 2x + 1)$

$h(x)$ is Neither Even Nor odd.

COMPOSITION OF FUNCTIONS

Let $f(x)$ and $g(x)$ be functions with A and B being elements respectively. Then f and g can be combined to get new functions as follows:

1) $(f+g)(x) = f(x) + g(x)$

2) $(f-g)(x) = f(x) - g(x)$

3) $(f \cdot g)(x) = f(x) \cdot g(x)$

4 $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0$

Example:

If $f(x) = 3x - 1$ and $g(x) = x^2 - x + 2$, find

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$

Solution:

(a) $(f+g)(x) = f(x) + g(x)$

$= (3x - 1) + (x^2 - x + 2)$

$= x^2 + 2x + 1$

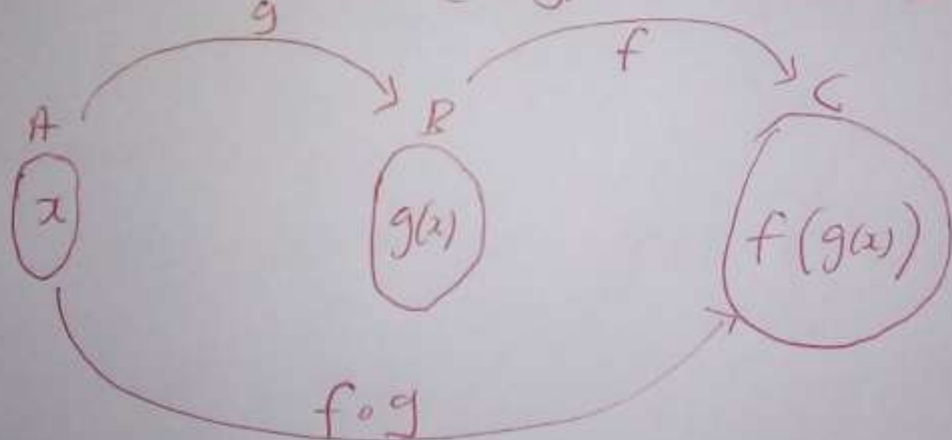
(b) $(f-g)(x) = f(x) - g(x)$

$$= (3x-1)-(x^2-x+2)$$
$$= 3x-1-x^2+x-2$$
$$= -x^2+4x-3$$

(c) $(f \cdot g)(x) = (3x-1)(x^2-x+2)$

$$= -x^2+4x-3.$$

The composition of functions f and g is defined by $(f \circ g)(x) = f[g(x)]$



Example

If $f(x) = x^2$ and $g(x) = 3x-4$, find:

(a) $(f \circ g)(x)$ and its Domain;

(b) $(f \circ g)(2)$

Solution :

(a) $(f \circ g)(x) = [3x-4)^2$

$= 9x^2 - 12x - 12x + 16$

$= 9x^2 - 24x + 16$

(b) $(f \circ g)(2) = [3(2) - 4]^2$

$= [6-4)^2 = \underline{\underline{4}}$

INVERSE OF A FUNCTION

Let f be a one-to-one function, with domain A and range B, then its inverse function f^{-1} has domain B and range A

Example :

Find the Inverse of the following

1) $f(x) = 3(x+2)$

Let $y = 3(x+2)$

$y = 3x + 6$

$3x = y - 6$

$x = \frac{y-6}{3}$

Now let x be ~~plus~~ $f^{-1}(x)$ and y be changed to x, we have

$$f^{-1}(x) = \frac{x-6}{3}$$

2) find inverse of $f(x) = x^2+2$

first let $y = x^2+2$

make x subject of formula

$$x^2+2 = y$$

$$x^2 = y-2$$

$$x = \sqrt{y-2}$$

$$\therefore \quad f^{-1}(x) = \sqrt{x-2}$$

CURVE SKETCHING

CURVE SKETCHING

To sketch a graph of a function, we

(i) Find intercepts

(ii) Express given function in more explicity form

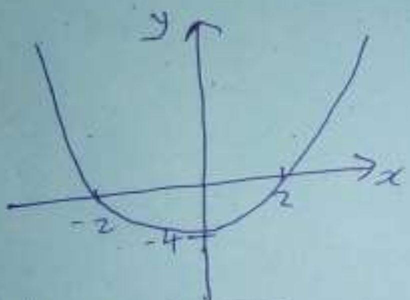
(iv) Set up table of values i.e ordered pairs.

Example

1. Sketch the graph $y = x^2 - 4$

Intercepts, $y = 0$, $0 = x^2 - 4$, $x = \sqrt{4} = \pm 2$

$x = 0$, $y = -4$



2. Sketch $y = \sqrt{x}$

$x \geq 0 \Rightarrow$ Domain $\{x \in R / x \geq 0\}$ or $[0, \infty)$

Range will be all non-negative numbers

$R_f = \{y \in \mathbb{R} / y \geq 0\}$

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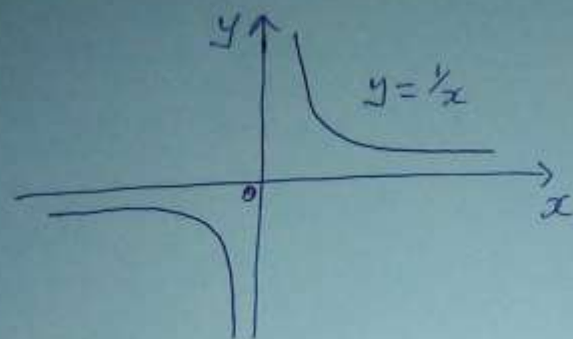
3. Sketch $y = \frac{1}{x}$.

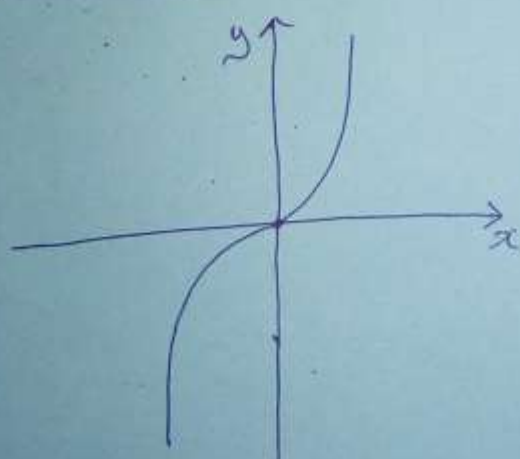
Here $x \neq 0$ Domain $D_f = \{x \in \mathbb{R} / x \neq 0\}$
$= (-\infty, 0) \cup (0, \infty)$



4. Sketch $y = x^3$

$x = 0,\ y = 0$

x	-2	-1	0	1	2
y	-8	-1	0	1	8



TRANSFORMATION OF CURVES.

1) $y = f(x) + a$ (Vertical ~~and horizontal~~ shift)
Shift up 'a' units if $a > 0$
Shift down "a" units if $a < 0$

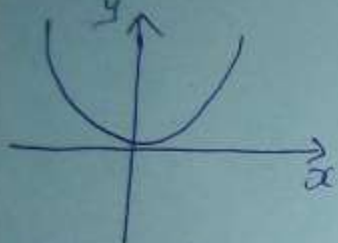
2) $y = f(x+a)$ (Horizontal shift)

Shift "a" units left if $a > 0$ and right if $a < 0$.

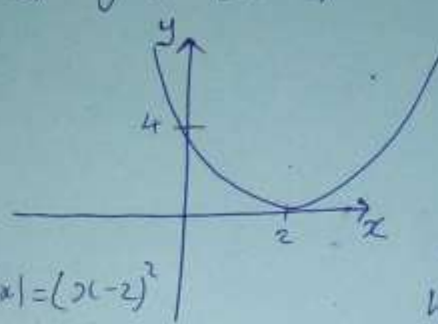
Example :-

Sketch the following

(a) $f(x) = x^2$ (b) $g(x) = (x-2)^2$ (c) $h(x) = (x+1)^2$



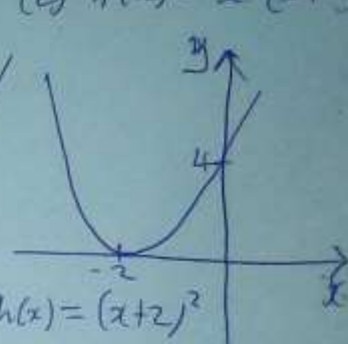
$f(x) = x^2$



$g(x) = (x-2)^2$

$x = 0,\ y = 4$

$y = 0,\ x = 2$



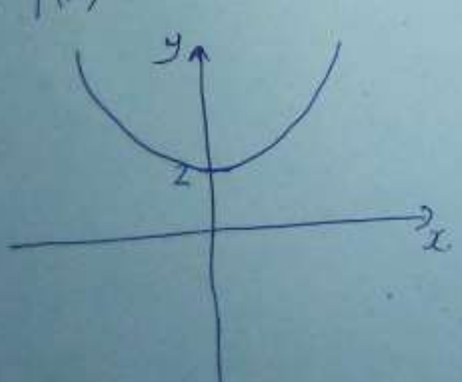
$h(x) = (x+2)^2$

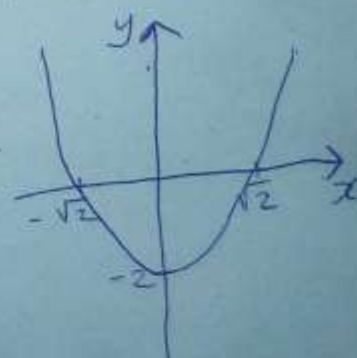
$x = 0,\ y = 4$

$y = 0\ x = -2$

3) Sketch

(a) $f(x) = x^2 + 2.$ (b) $f(x) = x^2 - 2.$





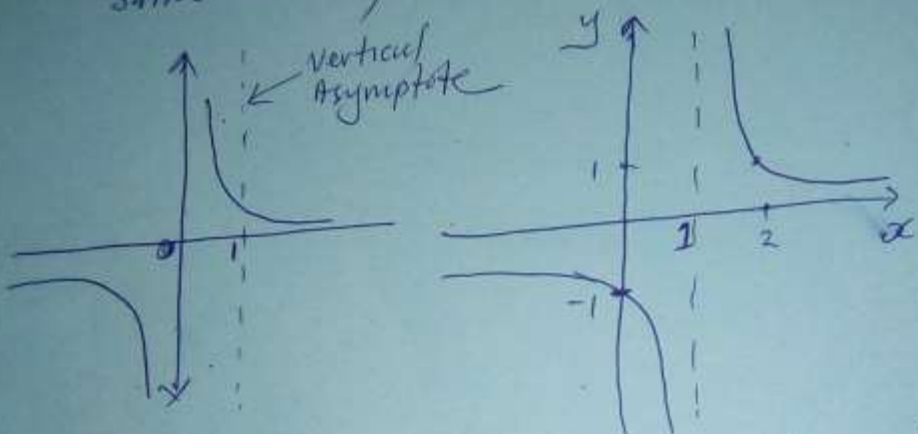
7) Sketch the graph $y = \frac{1}{x-1}$

If we let $g(x) = \frac{1}{x}$, $g(x-1) = \frac{1}{x-1}$

hence this is of the form $f(x+a)$ (Horizontal shift)

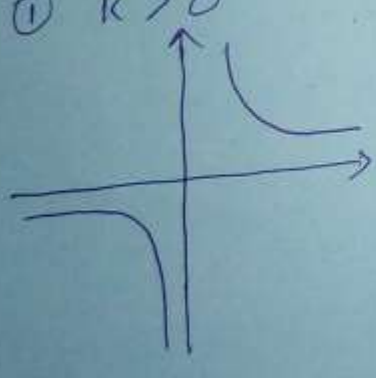
Since $a = -1$, -ve, we shift Right "1" unit

$x \neq 1$

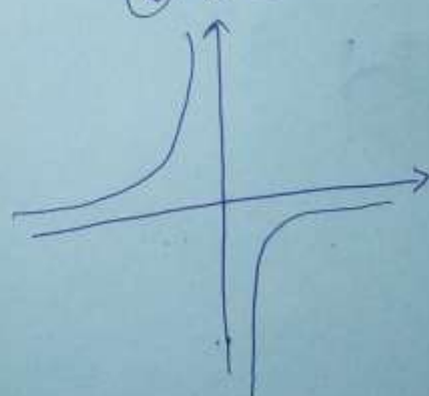


NB: for a curve $y = \frac{k}{x}$

① $k > 0$



② $k < 0$



2) Sketch (i) $y=\sqrt{2-x}$

(ii) $y=\sqrt{2x}$

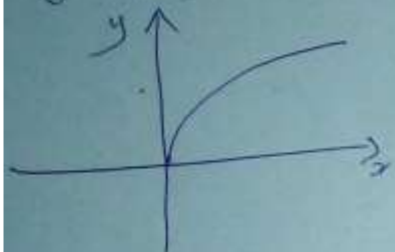
(iii) $1+\sqrt{x-2}$

$y=\sqrt{-x+2}$ [shift to left]

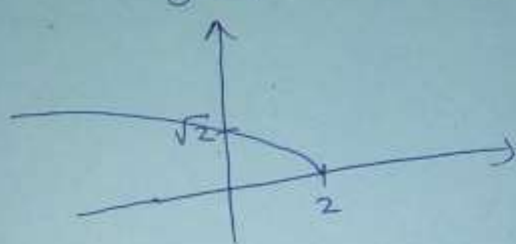
(i) Domain $2-x\geq 0 \Rightarrow x\leq 2$

$x=0,\ y=\sqrt{2}$

$y=\sqrt{x}$



$y=\sqrt{2-x}$



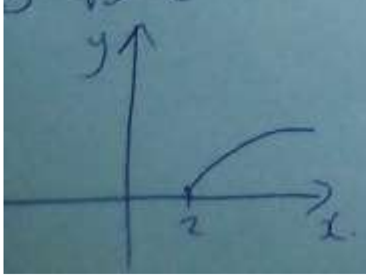
(ii) $y=\sqrt{2x}$

(iii) $y=1+\sqrt{x-2}$

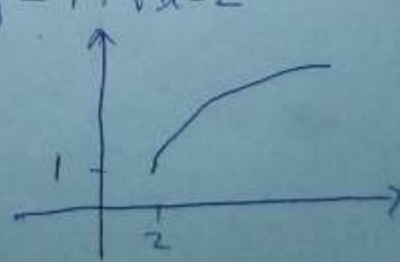
Domain: $x\geq 2$

at $x=2,\ y=1$

$y=\sqrt{x-2}$



$y=1+\sqrt{x-2}$



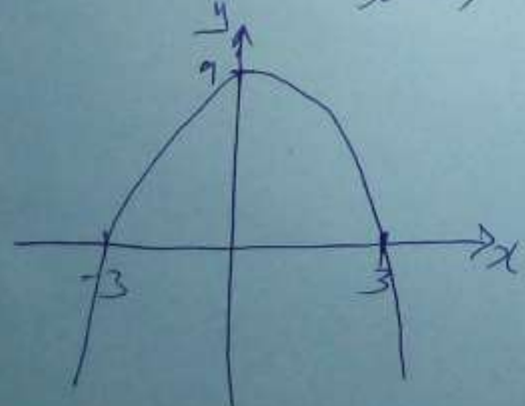
STRETCHING AND REFLECTING

1. If $a > 1$, $y = af(x)$ is the graph of $y = f(x)$ stretch by a scale factor a in vertical direction
2. for $0 < a < 1$, $y = \frac{1}{a} f(x)$, compress the graph of $f(x)$ by $1/a$.
3. $y = f(ax)$ is a horizontal stretch by a factor $1/a$
4. $y = f(\frac{x}{a})$, is a horizotal stretch by factor "a".
5. $f(ax)$, implies that $f(-x)$ is a reflection about the y-axis.

__Example:__

1. $f(x) = 9 - x^2$, sketch the curve ~~with equations~~
~~(a) $y = f(2x)$ (b) $y = 2f(x)$.~~

$f(x) = 9 - x^2$, $\Rightarrow$ $y = 0$, $x = \pm 3$
$x = 0$, $y = 9$.



PIECE - WISE FUNCTIONS

These are functions defined by formulae in different parts of the domain of formula

e.g. 1 $f(x) = \begin{cases} 1-x, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

Evaluate $f(0)$, $f(1)$ and $f(2)$

$f(0) \Rightarrow x = 0$ thus $x \leq 1$, $f(x) = 1 - x$.

$f(0) = 1 - 0$
$= 1$

$f(1) \Rightarrow x = 1$ thus $x \leq 1$ $f(x) = 1 - x$
$f(1) = 0$

$f(2) \Rightarrow x = 2$ thus $x > 1$ $f(x) = x^2$
$f(2) = 2^2 = 4$

2. Given

$$g(x) = \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

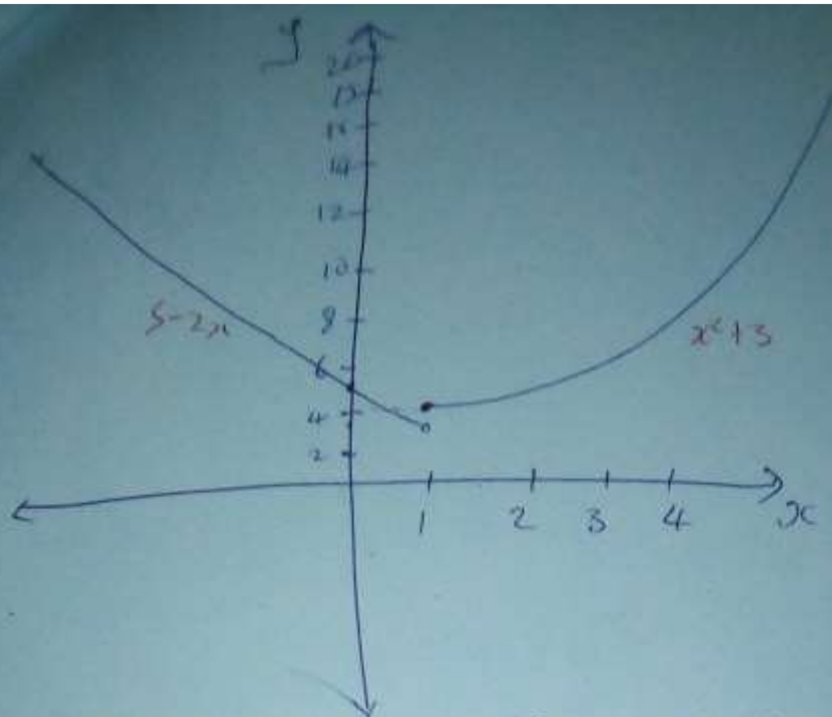
(a) Sketch $g(x)$ stating its range.

(b) Find the values of a such that $g(a) = 19$

Sol.

$g(x) = 5 - 2x$,
$y = 5, x = 0$
$y = 0, x = 5/2$
$y = 3, x = 1$

$g(x) = x^2 + 3$
$y = 4, x = 1$



Range : $\{y \mid y>3\}$ or $(3,\infty)$

(3) $g(a) = 19$

for $g(x) = 5-2x$, $x \le 1$

$g(a) \Rightarrow 5-2a = 19$

$-2a = 19-5$

$-2a = 14$

$a = -7$

for $g(x) = x^2+3$, $x > 1$

$g(a) = a^2+3 = 19$

$a^2 = 16$

$a = \pm 4 \Rightarrow a = 4$

$\underline{a \in \{4, -7\}}$

BINARY OPERATIONS

PRODUCTS OF SETS AND BINARY OPERATION

let B and D be any two sets, the product of B and D, denoted $B \times D$ consist of all ordered pairs (b, d) such that $b \in B$ and $d \in D$.

Example :

let $B = \{2, 3, 4\}$ and $D = \{a, c\}$

(i) Find $B \times D$ and (ii) $D \times B$.

$B \times D = \{(2,a), (2,c), (3,a), (3,c), (4,a), (4,c)\}$

$D \times B = \{(a,2), (a,3), (a,4), (c,2), (c,3), (c,4)\}$

$B \times D \neq D \times B$.

BINARY OPERATION :

A binary operation on a non-empty set S denoted by " $*$ " is a rule that associate a pair of elements a and b in S to a unique element $a * b$ of S.

~~Remarks~~ PROPERTIES

From the definition :

(i) The operator $*$ must be defined for every pair (a, b) where $a \in S$ and $b \in S$.

(ii) The order of a and b may be important ~~fat~~ since $a * b$ may not be equal to $b * a$

(iii) $a * b$ must be an element of S.

(IV) The set S is "closed (under)" with respect to the operator $*$, i.e $a * b \in S$.

1) Consider a set $A = \{1, 2, 3\}$. Is the set closed under $\overset{\text{addition}}{+}$ or $\overset{\text{multiplication}}{\times}$? (2)

Solution

Since $1+3 = 4 \notin A$, then A is not closed under $+$
also $(2 \times 3 = 6 \notin A$, " " " " " " $\times$

2) let $S = \{1, 2, 3, 4\}$ and $*$ be such that for any pair (a, b), $a * b = a + b$. is $*$ a binary operator S?

Solution:-
" $*$ " is not a binary operator on S since for $(1, 4)$, $1 * 4 = 1 + 4 = 5 \notin S$
$\therefore$ Addition is not binary on S.

PROPERTIES

A binary operation on a set S is said to be

(a) Commutative if for every pair of elements (a, b), $a * b = b * a$

(b) Associative if for all $a, b, c \in S$
$a * (b * c) = (a * b) * c$.

Examples:

1) let " $\circ$ " be an operator defined by ~~aob~~
$a \circ b = 2^{a^2 + b}$ where $a, b \in \mathbb{R}$

i) is the operator a binary operation of the set $\mathbb{R}$

(ii) Is it commutative

(iii) Evaluate $-1 \circ 4$ $*$ is the Binary

Solution

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i) Given $a \circ b = 2^{a^2+b}$ if $a, b \in \mathbb{R}$, then a^2+b is also a real number i.e $2^{a^2+b} \in \mathbb{R}$ so "o" is defined for all $a, b \in \mathbb{R}$, so $a \circ b = 2^{a^2+b}$ is a binary operation on $\mathbb{R}$

ii) The operator is commutative iff

$a * b = b * a$

$a \circ b = 2^{a^2+b}$

$b \circ a = 2^{b^2+a}$

$a \circ b = b \circ a$ if and only if $a^2+b = b^2+a$ i.e $a=b$

Therefor $2^{a^2+b} \neq 2^{b^2+a}$

Not commutative.

(iii) $-1 \circ 4 \Rightarrow a \circ b = 2^{a^2+b}$, $-1 \circ 4 = 2^{(-1)^2+4} = 2^5 = 32$

Q2

$*$ is an operation defined by $a * b = a^b + 1$

$a, b \in \mathbb{R}$

(i) is this operation binary on the set of real numbers

(ii) Evaluate $(2 * -1) * 5$

Solut:

(i) $a * b = a^b + 1$, Given $a = -1$ and $b = \frac{1}{2}$.

$(-1 * \frac{1}{2}) = (-1)^{1/2} + 1 = \sqrt{-1} + 1 = i + 1 \notin \mathbb{R}$

$\therefore a * b$ is not a binary operation on $\mathbb{R}$.

(ii) $(2 * -1) * 5 = (2^{-1} + 1) * 5 = (\frac{1}{2} + 1) * 5$

$= \frac{3}{2} * 5$

RADICAL EQUATIONS

RADICAL EQUATIONS

1) $\sqrt{3x+13} = x+1$

$\left[\sqrt{3x+13}\right]^2 = (x+1)^2$

$3x + 13 = x^2 + 2x + 1$

$x^2 + 2x - 3x + 1 - 13 = 0$

$x^2 - x - 12 = 0$

$(x+3)(x-4) = 0$

$x = -3$ and $x = 4$

2. $\sqrt[3]{7x - 1} = 3$

$\left[\sqrt[3]{7x-1}\right]^3 = 3^3$

$7x - 1 = 27$

$7x = 28$

$x = 4$

3.) $\sqrt{x} - \sqrt{x-5} = 1$

$\sqrt{x} - 1 = \sqrt{x-5}$

$\left[\sqrt{x} - 1\right]^2 = x - 5$

$x - 2\sqrt{x} + 1 = x - 5$

$$-2\sqrt{x}+1=-5$$
$$-2\sqrt{x}=-6$$
$$\sqrt{x}=3$$
$$x=3^2=9$$

4) $\sqrt{y+7}+3=\sqrt{y+4}$

Square both sides

$$(y+7)+6\sqrt{y+7}+9=y+4$$
$$6\sqrt{y+7}=y+4-9-y-7$$
$$6\sqrt{y+7}=-12$$
$$\sqrt{y+7}=-2$$
$$y+7=4$$
$$\underline{\underline{y=-3}}$$

5) $x-\sqrt{x}-6=0$

We Know that

$$(\sqrt{x})^2=x$$

$$\therefore (\sqrt{x})^2-\sqrt{x}-6=0$$

Let $y=\sqrt{x}$

$y^2 - y - 6 = 0$

$(y+2)(y-3) = 0$

$y = -2 \quad y = 3$

Since $y = \sqrt{x}$

Replacing y we have

$\sqrt{x} = -2$

$x = (-2)^2$

$\underline{\underline{x = 4}}$

$\sqrt{x} = 3$

$x = 3^2$

$\underline{\underline{x = 9}}$

6) $2x^{4/5} - 47 = 115$

$2x^{4/5} = 162$

$x^{4/5} = 81$

$\left[x^{4/5}\right]^{5/4} = [81]^{5/4}$

$x = (81^{1/4})^5$

$x = (\pm 3)^5$

$\underline{\underline{x = \pm 243}}$

INEQUALITIES

INEQUALITIES

Solve the following inequalities

i) $3(2x-1) \le 2$

$6x - 3 \le 2$

$6x \le 5$

$x \le 5/6$

(ii) $-3 < \frac{5-3x}{4} \le 2$

make x subject of formula

$-3(4) \le 5-3x \le 2(4)$

$-12 < 5-3x \le 8$

$-12-5 < -3x \le 8-5$

$-17 < -3x \le 3$

$\frac{17}{3} > x \ge -1$

(iii) $1 < 2x-5 \le 7$

$1+5 < 2x \le 7+5$

$6 < 2x \le 12$

$3 < x \le 6$ or $(3,6]$

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(iv) $\frac{x-3}{6} \leq \frac{2x+6}{3}$

$3(x-3) \leq 6(2x+6)$

$x-3 \leq 2(2x+6)$

$x-3 \leq 4x+12$

$-3x \leq 15$

$\underline{\underline{x \geq 5}}$

(v) Solve the following quadratic inequality

$x^2-2x-15 > 0$

$(x+3)(x-5) > 0$

Critical values are

$x=-3$ and $x=5$

	-4	-3	0	5	6
$x+3$	$-$		$+$		$+$
$x-5$	$-$		$-$		$+$
$f(x)$	$+$		$-$		$+$
	✓				✓

$\therefore\ x \in (-\infty,-3) \cup (5,\infty)$

Quotient Inequalities

④

1) Solve the following Inequalities

(a) $\dfrac{x+9}{x-6} \le 0$

write down the Critical values

$x = -9$ and $x = 6$

	-9	0	6	
$x+9$	$-$	$+$		$+$
$x-6$	$-$	$-$		$+$
$f(x)$	$+$	$-$		$+$

$\therefore$ ~~$x \in (-\infty, -9] \cup (6, \infty)$~~ $x \in [-9, 6)$

(b) $\dfrac{x+1}{x+3} \le 2$

$\dfrac{x+1}{x+3} - 2 \le 0$

$\dfrac{x+1-2(x+3)}{x+3} \le 0$

$\dfrac{x+1-2x-6}{x+3} \le 0$

$\dfrac{-x-5}{x+3} \le 0$

Critical values

$-x-5=0$

$-x = 5$

$x = -5$

$x = -3$

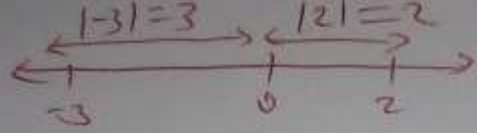
	-5		-3	
	$+$		$-$	$-$
	$-$		$-$	$+$
$f(x) =$	$-$		$+$	$-$
	✓		??	✓

$x \in$ ~~$(-3, \ldots$~~

$(-\infty, -5] \cup (-3, \infty)$

EQUATION OF ABSOLUTE VALUES

Absolute value of any number is the distance between the number and zero



~~In General~~

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

~~In general~~

~~$|x| = |-x| = x$~~

Example:

1) $|x+2| = 5$

$$\begin{cases} x+2 = 5 \Rightarrow x = 3 \\ x+2 = -5 \Rightarrow x = -7 \end{cases}$$

2) $|2x+5| - 3 = 8$

$|2x+5| = 11$

$$\begin{cases} 2x+5 = -11 \Rightarrow x = -8 \\ 2x+5 = 11 \Rightarrow x = 3 \end{cases}$$

(3) $|5x-7| = |4x+7|$

$$\begin{cases} 5x-7 = 4x+7 \Rightarrow \underline{\underline{x = 14}} \\ 5x-7 = -(4x+7) \Rightarrow \underline{\underline{x = 0}} \end{cases}$$

(7)

3) $\left|\frac{x-2}{x+3}\right| < 4$

$$\begin{cases} \frac{x-2}{x+3} < 4 \\ \frac{x-2}{x+3} > -4 \end{cases}$$

Solve each part:

$$\frac{x-2}{x+3} < 4$$

$$\frac{x-2}{x+3} - \frac{4}{1} < 0$$

$$\frac{x-2-4(x+3)}{x+3} < 0$$

$$\frac{x-2-4x-12}{x+3} < 0$$

$$\frac{-3x-14}{x+3} < 0$$

Critical values

$x = -14/3$ and $x = -3$

$$\frac{x-2}{x+3} > -4$$

$$\frac{x-2}{x+3} + \frac{4}{1} > 0$$

$$\frac{(x-2)+4(x+3)}{(x+3)} > 0$$

$$\frac{x-2+4x+12}{x+3} > 0$$

$$\frac{5x+10}{x+3} > 0$$

$x = -2$; $x = -3$

QUADRATIC EQUATIONS

MA 110 GROUP B ①

LESSON 1

QUADRATIC EQUATION

These are functions of the form $f(x) = ax^2 + bx + c$ where a, b, c are Real Numbers

A function of the form $ax^2 + bx + c = 0$ is called a quadratic Equation

The values of x which satisify $ax^2 + bx + c = 0$ are called roots, Zeroes or Solutions

METHOD OF SOLVING QUADRATIC EQUATION

(a) Factorization

Every quadratic Equation has two roots,

e.g if α and β are roots of the equation, then

$$(x - \alpha)(x - \beta) = 0$$

e.g The solution of the equation...

②

$x^2 + 7x + 12 = 0$

$(x+4)(x+3) = 0$

$\therefore\ x+4 = 0$ and $x+3 = 0$

$\underline{\underline{x = -4}}$ $\quad\quad$ $\underline{\underline{x = -3}}$

(b) COMPLETING THE SQUARE METHOD

Given the equation

$ax^2 + bx + c = 0$,

Complete the square in x.

Solution

Step 1

Divide through by "a" {so that the coefficient of x^2 is 1}

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2

Find the square of half of the coefficient of x.

* Coefficient of x is b/a
* half of $\frac{b}{a}$ is $\frac{1}{2}\left(\frac{b}{a}\right) = \frac{b}{2a}$
* Square of $\frac{b}{2a}$ $\Rightarrow$ $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$

Step 3 :- (3)

Add $\left(\frac{b}{2a}\right)^2$ on both sides of the equation

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = \left(\frac{b}{2a}\right)^2$$

* Take c/a to RHS :-

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Step 4

LHS is perfect square hence can be written as

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

write RHS as a single fraction

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

To solve for x, squareroot both sides of equation

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}} \quad (4)$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$\therefore$ The solution of equation is

$$x = \frac{-b + \sqrt{b^2-4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

Examples:-

Solve the following using completing the square method

a) $x^2 + 8x - 2 = 0$

Solution

$$x^2 + 8x = 2$$

Take half of coefficient of x and square it.

$$\tfrac{1}{2}(8) = 4 \quad 4^2 = 16$$

Add 4^2 both sides of equation

$$x^2 + 8x + 4^2 = 2 + 4^2$$

(5)

The LHS is a perfect square

$(x+4)^2 = 2 + 4^2$

$(x+4)^2 = 18$

$x+4 = \pm\sqrt{18}$

$\left\{ \begin{array}{l} \sqrt{18} = \sqrt{9}\cdot\sqrt{2} \\ \quad = 3\sqrt{2} \end{array} \right\}$

$x = -4 \pm \sqrt{18}$

$\underline{x = -4 \pm 3\sqrt{2}}$

(b) Solve $2x^2 + 6x - 3 = 0$

Solution

$2x^2 + 6x - 3 = 0$

NOTE:- Ensure that coefficient of x^2 is 1, hence divide through by 2.

$2x^2 + 6x - 3 = 0$

$x^2 + \frac{6}{2}x - \frac{3}{2} = 0$

$x^2 + 3x - \frac{3}{2} = 0$

$x^2 + 3x = \frac{3}{2}$

Find half of coefficient of x and square it:

$\frac{1}{2}(3) = \frac{3}{2}$, $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(6)

Add $(\frac{3}{2})^2$ both sides of equation

$$x^2+3x+(\tfrac{3}{2})^2 = \tfrac{3}{2}+(\tfrac{3}{2})^2$$

LHS is perfect square

$$(x+\tfrac{3}{2})^2 = \tfrac{3}{2}+\tfrac{9}{4}$$

$$(x+\tfrac{3}{2})^2 = \frac{15}{4}$$

$$x+\tfrac{3}{2} = \pm\sqrt{\frac{15}{4}}$$

$$x = -\tfrac{3}{2} \pm \sqrt{\frac{15}{4}}$$

$$\underline{\underline{x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{15}}}$$

2) Using the quadratic formular
Solve $4x^2-4x+3=0$

formular is:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a=4,\ b=-4,\ c=3$

(7)

$$x = \frac{4 \pm \sqrt{4^2 - 4(4)(3)}}{2(4)}$$

$$x = \frac{4 \pm \sqrt{16 - 48}}{8}$$

$$x = \frac{4 \pm \sqrt{-32}}{8}$$

$* \sqrt{32} = \sqrt{16} \cdot \sqrt{2}$
$= 4\sqrt{2}$

$$x = \frac{4 \pm i\, 4\sqrt{32}}{8}$$

$$x = \frac{1}{2} \pm i \frac{\sqrt{2}}{2}$$

DISCRIMINANT OF A QUADRATIC EQ.

If $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The expression $b^2 - 4ac$ is called the discriminat of $ax^2 + bx + c$

It determines the NATURE of roots of the equation.

(i) If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ have two distinct roots.

(ii) If $b^2 - 4ac = 0$, then we have two equal roots (Repeated roots)

(iii) If $b^2 - 4ac < 0$, then the equation have two distinct ~~roots~~ Complex roots (conjugate roots)

9

Example:

For what value of P will the equation $3x^2 + Px + 3 = 0$ have two distinct roots

Solution.

We have two distinct roots if

$b^2 - 4ac > 0$

for $3x^2 + Px + 3 = 0$

$a = 3, \quad b = P, \quad c = 3$

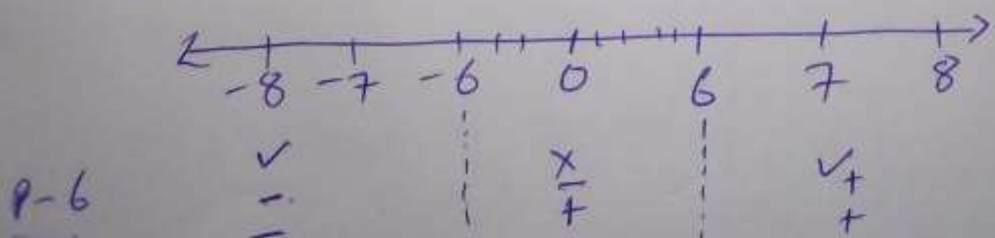
$\therefore \quad b^2 - 4ac > 0$

$p^2 - 4(3)(3) > 0$

$p^2 - 36 > 0$

$(P-6)(P+6) > 0$

Critical Values are 6 and -6



$\therefore \quad p \in (-\infty, -6) \cup (6, \infty)$

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The LHS is a perfect square

$(x+4)^2 = 2 + 4^2$

$(x+4)^2 = 18$

$x+4 = \pm\sqrt{18}$

$x = -4 \pm \sqrt{18}$

$x = -4 \pm 3\sqrt{2}$

$\underline{\underline{x = -4+3\sqrt{2}}}$ and $\underline{x = -4-3\sqrt{2}}$

(b) Solve $2x^2 + 6x - 3 = 0$

<u>Solution</u>

$2x^2 + 6x - 3 = 0$

$2x^2 + 6x = 3$

$\frac{1}{2}(6) = 3$, $3^2 = 9$

Add 3^2 both sides of equation

2) For what value of K will the equation $Kx^2 + (K+1)x + K = 0$ have real and equal roots. (II)

Solution

To have equal roots,

$$b^2 - 4ac = 0$$

$a = K, \quad b = (K+1), \quad c = K$

$$b^2 - 4ac = 0$$

$$(K+1)^2 - 4(K)(K) = 0$$

$$(K+1)^2 - 4K^2 = 0$$

$$(K^2 + 2K + 1) - 4K^2 = 0$$

$$-3K^2 + 2K + 1 = 0$$

$$K = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = -3$
$b = 2$
$c = 1$

$$K = \frac{-2 \pm \sqrt{2^2 - 4(-3)(1)}}{2(-3)}$$

$$K = \frac{-2 \pm \sqrt{16}}{-6} = \frac{-2 \pm 4}{-6}$$

$K = \frac{-2+4}{-6}$ or $K = \frac{-2-4}{-6}$

$= -\frac{1}{3}$ $\quad K = 1$

(12)

ROOTS OF A QUADRATIC EQUATION

Consider a quadratic equation whose solutions are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

let $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

be two roots of the equation

$ax^2+bx+c=0$.

(i) The sum of the two roots is

$$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2b}{2a}$$

$$= -\frac{b}{a}$$

$$\therefore \boxed{\alpha+\beta = -\frac{b}{a}}$$

ii) The product

$$\alpha \cdot \beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{b^2-(b^2-4ac)}{(2a)^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$$\therefore \boxed{\alpha \cdot \beta = \frac{c}{a}}$$

Since α and β are roots of $ax^2+bx+c=0$, then

This equation is equal to

$$(x-\alpha)(x-\beta)=0$$

$$x^2 - x\alpha - x\beta + \alpha\beta = 0$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$\therefore \quad ax^2 - bx + c = x^2 - (\alpha+\beta)x + \alpha\beta$$

Sum product

Example:-

Let α and β be roots of $7x^2+2x-5=0$
find

(a) $\frac{1}{\alpha}+\frac{1}{\beta}$ (b) $\alpha^2+\beta^2$ (c) $\alpha-\beta$

solution.

a) we know that

$$\begin{cases} \alpha+\beta = -\frac{b}{a} \\ \alpha\cdot\beta = \frac{c}{a} \end{cases}$$

Given $7x^2+2x-5=0$, then

$a=7$, $b=2$, $c=-5$

$$\begin{cases} \alpha+\beta = -b/a = -\frac{2}{7} \\ \alpha\beta = c/a = \frac{-5}{7} \end{cases}$$

$$1/\alpha + 1/\beta = \frac{\alpha+\beta}{\alpha\beta} = \frac{(-2/7)}{(-\frac{5}{7})} = \underline{\underline{\frac{2}{5}}}$$

b) $\alpha^2+\beta^2$

We need to write this expression in form of sum(s) or products of roots of the equation. To ~~achive~~ achieve that we make the following expressions

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The expansion of $(\alpha+\beta)^2$ is

$$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$

Now express the above equation in terms of $\alpha^2+\beta^2$

$$\alpha^2 + 2\alpha\beta + \beta^2 = (\alpha+\beta)^2$$

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$\therefore \alpha^2+\beta^2 = \left(-\tfrac{2}{7}\right)^2 - 2\left(-\tfrac{5}{7}\right)$$

$$= \frac{4}{49} + \frac{10}{7}$$

$$= \frac{4+70}{49} = \frac{74}{49}$$

(C) We need to write $\alpha-\beta$ in form of products and sum of α and β.

$$(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$$

$$(\alpha-\beta)^2 = \alpha^2+\beta^2 - 2\alpha\beta \quad \text{.....(i)}$$

from solution (b), we found

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta \quad \text{... ii}$$

(15)

Replacing (ii) in (i) we get

$$(\alpha-\beta)^2 = [(\alpha+\beta)^2 - 2\alpha\beta] - 2\alpha\beta$$

$$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$

$$\alpha-\beta = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

Since $\begin{cases} (\alpha+\beta) = -2/7 \\ \alpha\beta = -5/7 \end{cases}$ then

$$\alpha-\beta = \pm\sqrt{(-2/7)^2 - 4(-5/7)}$$

$$= \pm\sqrt{\frac{4}{49} + \frac{20}{7}}$$

$$= \pm\sqrt{\frac{144}{49}}$$

$$= \pm\frac{12}{7}$$

2. Find an equation whose roots are squares of the roots of the equation $2x^2 - x + 3 = 0$

<u>Solution.</u>

let α and β be roots of $2x^2 - x + 3 = 0$

$a = 2$
$b = -1$
$c = 3$

then $\begin{cases} \alpha + \beta = -\frac{b}{a} = \frac{1}{2} \\ \alpha\beta = \frac{c}{a} = \frac{3}{2} \end{cases}$

The roots of the required equation are square of roots of $2x^2 - x + 3 = 0$

Hence roots are α^2 and β^2

$\begin{cases} \text{The sum of the "New" roots is} \\ \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \\ \text{product is} \quad \alpha^2\beta^2 = (\alpha\beta)^2 \end{cases}$

$$\begin{cases} \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \\ \qquad = \left(\frac{1}{2}\right)^2 - 2\left(\frac{3}{2}\right) \\ \qquad = \frac{1}{4} - 3 = -\frac{11}{4} \\ (\alpha\beta)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} \end{cases}$$

(98)

The required equation is of the form

$$(x-\alpha^2)(x-\beta^2)=0$$

$$x^2-\alpha^2x-\beta^2x+\alpha^2\beta^2=0$$

$$x^2-\underbrace{(\alpha^2+\beta^2)}_{\text{sum}}x+\underbrace{(\alpha\beta)^2}_{\text{product}}=0$$

$$\therefore\ x^2-\left(-\frac{11}{4}\right)x+\frac{9}{4}=0$$

$$\underline{\underline{4x^2+11x+9=0}}$$

3) Given the equation $3x^2+8x+d=0$ find the value of d if the roots of this equation differ by 2.

Solution

Since the roots differ by 2, if one of the roots is α, the the other root is $\alpha+2$

Sum of the roots is

$$(\alpha+2)+\alpha=\frac{-b}{a}=\frac{-8}{3}$$

Solving for α we get (19)

$$2\alpha + 2 = -8/3$$

$$2\alpha = -\frac{8}{3} - 2$$

$$\alpha = -7/3$$

Since the two roots are

α and $\alpha + 2$,

$\alpha = -7/3$ therefor $\alpha + 2 = -\frac{7}{3} + 2 = -1/3$

The product of roots is

$$\alpha \cdot (\alpha + 2) = c/a = \frac{d}{3}$$

$$\left(-\frac{7}{3}\right)\left(-1/3\right) = \frac{d}{3}$$

$$\frac{7}{9} = \frac{d}{3}$$

$$\underline{\underline{d = \frac{7}{3}}}$$

GRAPHS OF QUADRATIC FUNCTIONS 20

Every quadratic function $f(x) = ax^2 + bx + c$ can be written in the "Standard Form"

$$f(x) = a(x+h)^2 + k \quad , \quad a \neq 0$$

Example: By completing the square write $f(x) = ax^2 + bx + c$ in standard form.

Solution:

$$f(x) = ax^2 + bx + c$$

*NOTE: make sure the coefficient of x^2 is one. Factorise "a".
DO NOTE DIVIDE THROUGH BY "a".

$$f(x) = ax^2 + bx + c$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

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$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{-b^2+4ac}{4a^2}\right]$$

multiply through by "a", we get

$$f(x) = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

let $h = \frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$

Then $f(x) = a(x+h)^2 + k$

Example:

Express the equation $f(x) = 2x^2 - 12x + 23$ in the form $a(x+p)^2 + q$

$$f(x) = 2x^2 - 12x + 23$$

$$f(x) = 2\left[x^2 - 6x + \frac{23}{2}\right]$$

coefficient of x is -6,

$\therefore \frac{1}{2}(-6) = -3$, $(-3)^2 = 9$

$$f(x) = 2\left[x^2 - 6x + (-3)^2 - (-3)^2 + \frac{23}{2}\right]$$

$$= 2\left[(x-3)^2 + \frac{23}{2} - 9\right]$$

$$f(x) = 2\left[(x-3)^2 + \tfrac{5}{2}\right]$$
$$= 2(x-3)^2 + 5$$

(b) $f(x) = 1 - 6x - x^2$

$$f(x) = -x^2 - 6x + 1$$
$$f(x) = -\left[x^2 + 6x - 1\right]$$
$$f(x) = -\left[x^2 + 6x + (3)^2 - (3)^2 - 1\right]$$
$$f(x) = -\left[(x+3)^2 - 9 - 1\right]$$
$$= -\left[(x+3)^2 - 10\right]$$
$$= -(x+3)^2 + 10$$

REMarks :-

Given a quadric function in the form $f(x) = a(x+h)^2 + K$, where

$$h = \frac{b}{2a}, \quad K = \frac{4ac - b^2}{4a}$$

The graph of $f(x)$ is a parabola with:

$$f(x) = 2\left[(x-3)^2 + \tfrac{5}{2}\right]$$
$$= 2(x-3)^2 + 5$$

(b) $f(x) = 1 - 6x - x^2$

$$f(x) = -x^2 - 6x + 1$$
$$f(x) = -[x^2 + 6x - 1]$$
$$f(x) = -[\underbrace{x^2 + 6x + (3)^2}_{} - (3)^2 - 1]$$
$$f(x) = -[(x+3)^2 - 9 - 1]$$
$$= -[(x+3)^2 - 10]$$
$$= -(x+3)^2 + 10$$

REMARKS :-

Given a quadric function in the form $f(x) = a(x+h)^2 + K$, where

$$h = \frac{b}{2a}, \quad K = \frac{4ac - b^2}{4a}$$

The graph of $f(x)$ is a parabola with:

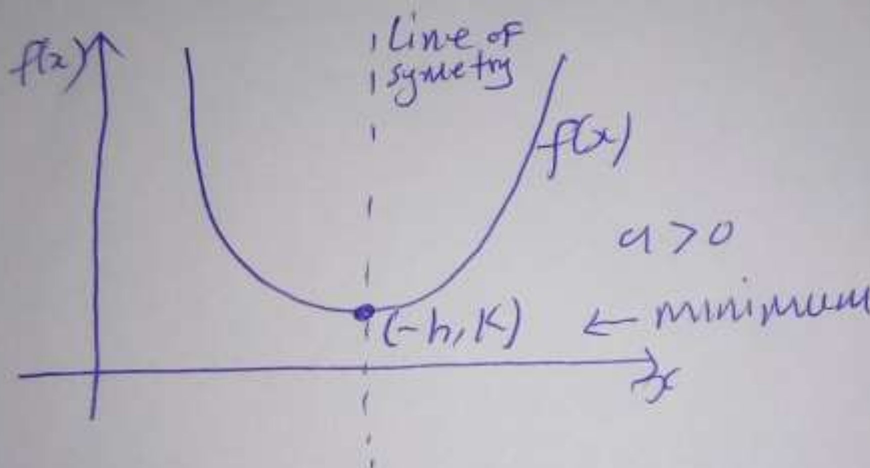
23

the vertex (turning point $(-h, K)$

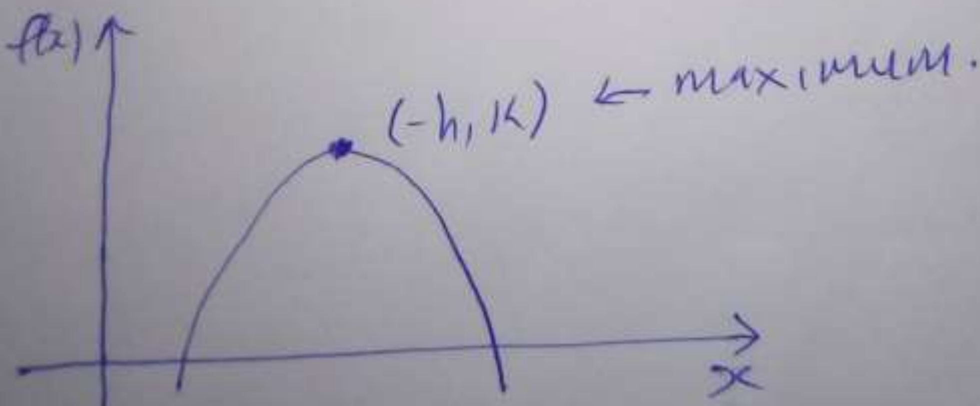
or $(-h, K) = \left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

for the function $f(x) = ax^2 + bx + c$

If $a > 0$, $(-h, K)$ is a minimum



If $a < 0$, then $(-h, K)$ is a maximum



Example :-

Sketch the graph of

(a) $f(x) = 2x^2 - 12x + 19$

in Standard form.

Solution :

$$f(x) = 2x^2 - 12x + 19$$

$$= 2\left[x^2 - 6x + \frac{19}{2}\right]$$

$$= 2\left[x^2 - 6x + (-3)^2 - (-3)^2 + \frac{19}{2}\right]$$

$$= 2\left[(x-3)^2 - 9 + \frac{19}{2}\right]$$

$$= 2\left[(x-3)^2 + \frac{1}{2}\right]$$

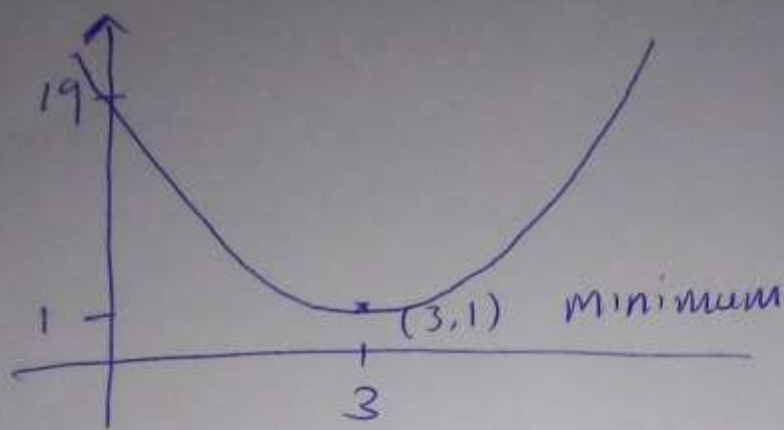
$$= 2(x-3)^2 + 1$$

$$\therefore \boxed{f(x) = 2(x-3)^2 + 1}$$

$a > 0$

$x - 3 = 0$
$x = 3$
(Graph shift 3 unit to right)

moves up one unit



turning point $= (-h, k) = (-(-3), 1)$

$= \underline{\underline{(3,1)}}$

(b) Graph $f(x) = -3x^2 + 12x - 7$

Line of symetry

$x = -\frac{b}{2a} = \frac{-12}{2(-3)} = 2$

$2 = -h$

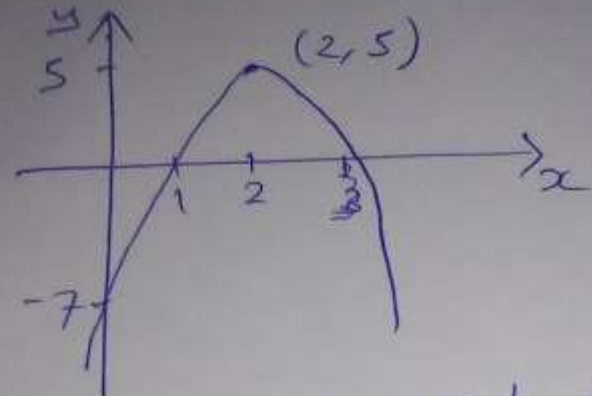
$k = f(-h) = f\left(-\frac{b}{2a}\right) = f(2) = 5$

Graph cut the x-axis at

$f(x) = 0$

Cut the y-axis at $x = 0$

$f(0) = -7$ or at $(0, -7)$



(26)

Alternative, we can write the equation in standard form

$$f(x) = -3x^2 + 12x - 7$$
$$= -3[x^2 - 4x + 7/3]$$
$$= -3[x^2 - 4x + (-2)^2 - (-2)^2 + 7/3]$$
$$= -3[(x-2)^2 - 4 + \frac{7}{3}]$$
$$= -3(x-2)^2 + 5$$

Vertex is $(2, 5)$, $a < 0$

(c) Sketch $f(x) = -\frac{1}{2}(x+1)^2 - 3$

POLYNOMIALS AND GRAPHS OF RATIONAL FUNCTIONS

1

ALGEBRA OF POLYNOMIALS

A polynomial function $P(x)$ is an algebraic expression that takes the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

where $a_n, a_{n-1}, \ldots a_1, a_0$ are real Numbers and the powers $n, n-1, \ldots$ are positive integers

The degree of a polynomial is the highest power of x in the expression

Example :

1. $P(x) = 5x^4 + 2x^2 + x - 7$ degree is 4
2. $4x^3 + 6x^5 - x^2 + \sqrt{2}$ degree is 5

DIVISION OF POLYNOMIALS

Recall : $\frac{11}{4} = 2 + \frac{3}{4} \Rightarrow 11 = 4 \times 2 + 3$

Here 11 is the dividend, 4 is the divisor, 2 is the quotient and 3 is the remainder.

In General if $P(x)$ and $D(x)$ are two polynomials with degree $P(x) \geq$ degree $D(x)$, There exist a polynomial $Q(x)$ and $R(x)$ such that

$$\underset{\text{dividend}}{P(x)} = \underset{\text{Divisor}}{D(x)} \times \underset{\text{Quotient}}{Q(x)} + \underset{\text{Remainder}}{R(x)}$$

↳ * The process of Polynomial division is equiva[lent] to that of long division of real numbers.

e.g

$$\begin{array}{r|l} & \;\;2 \leftarrow \text{quotient} \\ 4 & \;11 \\ & \underline{-8} \\ & \;\;3 \leftarrow \text{Remainder} \end{array}$$

Examples

1. Divide $P(x) = x^3 - 4x^2 + 5x - 1$ by $(x-2)$

$$
\begin{array}{r l}
 & x^2 - 2x + 1 \leftarrow \text{Quotient} \\
x-2 \,\big)\!\! & \overline{x^3 - 4x^2 + 5x - 1} \\
 & -(x^3 - 2x^2) \\
 & \overline{\quad -2x^2 + 5x - 1} \\
 & -(-2x^2 + 4x) \\
 & \overline{\qquad\quad x - 1} \\
 & \qquad -(x-2) \\
 & \overline{\qquad\quad 1} \leftarrow \text{Remainder}
\end{array}
$$

Therefore $P(x) = (x-2)(x^2-2x+1) + 1$

or $\dfrac{x^3-4x^2+5x-1}{x-2} = (x-2)(x^2-2x+1) + \dfrac{1}{x-2}$

Divide $2x^3 + 5x^2 - 13$ by $2x^2 + x - 2$

$$
\begin{array}{r l}
 & x + 2 \\
2x^2 + x - 2 \,\big)\!\! & \overline{2x^3 + 5x^2 - 13} \\
 & -(2x^3 + x^2 - 2x) \\
 & \overline{\qquad 4x^2 + 2x - 13} \\
 & \quad -(4x^2 + 2x - 4) \\
 & \overline{\qquad\qquad\quad -9}
\end{array}
$$

$$\frac{2x^3+5x^2-13}{2x^2+x-2} = (x+2) + \frac{-9}{2x^2+x-2}$$

SYNTHETIC DIVISION (3)

Suppose a polynomial $A(x)$ is divided by $(x-k)$ where k is a constant, The long division can be simplified in-to a process called synthetic division

Example! -

Divide $x^3 + 2x^2 - 3x + 4$ by $(x-2)$

Step 1

Write out the coefficients of the divident in decreasing powers of x

$(x-2) \Rightarrow x = 2$

	1	2	-3	4
2				

Step 2

place the leading coefficient in the last row.

	1	2	-3	4
2				
	1			

Step 3

multiply 1 by 2 i.e k and write the answer under second coefficient.

	1	2	-3	4
2		2	8	10
	1	4	5	14

then Add 2 + 2 = 4.

The last row indicate a quotient
$x^2 + 4x + 5$ remainder $\underline{14}$

Example 2: Divide $p(x) = 2x^3 + 5x^2 - 13x - 2$ by $D(x) = x + 4$

	2	5	-13	-2
-4		-8	12	4
	2	-3	-1	2

Last Row represent the Quotient
$2x^2 - 3x - 1$ remainder 2.

THE REMAINDER THEOREM.

For any polynomial $p(x)$, the remainder when divided by $(x - k)$ is $p(k)$.

Example:

1) If $p(x) = x^3 + 2x^2 - 5x - 1$. is divided by $x - 2$, Find the remainder

$$p(2) = (2)^3 + 2(2)^2 - 5(2) - 1$$
$$= 8 + 8 - 10 - 1 = 5$$

$\therefore$ Remainder is 5

) Find the value of k if the remainder of $(x) = x^3 + kx^2 - x + 2$ when divided by $x + 2$ is 20

$$(-2) = 20$$
$$\Rightarrow (-2)^3 + k(-2)^2 - (-2) + 2 = 20$$
$$-8 + 4k + 2 + 2 = 20$$
$$4k = 24$$
$$\underline{k = 6}$$

THE FACTOR THEOREM:

$(x-\alpha)$ is a factor of $p(x)$ if and only if $p(\alpha)=0$

That is to say if $(x-\alpha)$ is a factor of $p(x)$, the the remainder $P(\alpha)=0$

<u>Example</u>

Determine which of the following a factors of $p(x)=2x^3+7x^2+7x+2$

i) $(x-3)$, $P(3)=2(3)^3+7(3)^2+7(3)+2 \neq 0$

ii) $(x-1)$, $P(1)=2(1)^3+7(1)^2+7(1)+2 \neq 0$

iii) $(x+2)$, $P(-2)=2(-2)^3+7(-2)^2+7(-2)+2$

$= -16+28-14+2=0$

THEREFORE, $(x-3)$ and $(x-1)$ are NOT factors

$(x+2)$ is a factor.

THE RATIONAL ROOT THEOREM:

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If the Polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has any rational roots, then they must be of the form

$$\pm \left\{ \frac{\text{factors of } a_0}{\text{factors of } a_n} \right\}$$

Example:

Use the rational root theorem to find all rational solutions of;

$$3x^3 + 8x^2 - 15x + 4 = 0$$

$$\pm \left\{ \frac{\text{factors of } a_0}{\text{factors of } a_n} \right\} \qquad \frac{\text{Constant}}{\text{Leading Coefficient}}$$

$a_0 = 4$ factors are ± 1, ± 2, ± 4

$a_n = 3$ factors are ± 1 ± 3.

$$\frac{a_0}{a_n} \Rightarrow \pm 1, \pm 2 \ \pm 4, \pm \tfrac{1}{3}, \pm \tfrac{2}{3} \ \pm \tfrac{4}{3}$$

By Testing above numbers

$f(1) = 0$

Synthetic Division:

	3	8	-15	4
1		3	11	-4
	3	11	-4	0

Thus $(x-1)$ is a factor and $Q(x) = 3x^2 + 11x - 4$

$(x+4)(3x-1)$

$\therefore P(x) = (x-1)(x+4)(3x-1)$

or $(x-1)=0, \quad x+4=0, \quad 3x-1=0$

$x=1, \quad x=-4, \quad x=\frac{1}{3}$

2. Determine m and n so that $3x^3+mx^2-5x+n$ is divisible by both $(x-2)$ and $(x+1)$.

$P(2)=0$ and $P(-1)=0$

$P(2) = 3(2)^3 + m(2)^2 - 5(2) + n = 0$

$4m + n = -14 \quad --- \quad I$

$P(-1) = -3 + m + 5 + n = 0$

$m + n = -2 \quad --- \quad II$

Simultaneously

$3m = -12$ $\qquad$ $-4 + n = -2$

$\underline{\underline{m = -4}}$ $\qquad$ $\underline{\underline{n = 2}}$

GRAPHS OF POLYNOMIALS

Sketch the graph

(a) $f(x) = (x+1)(x-2)(x-4)$

Key points:

If $p(x) = a_n x^n + a_{n-1} x^{n-1} \ldots$ ~~and $x = r$ is a zero of p(x)~~

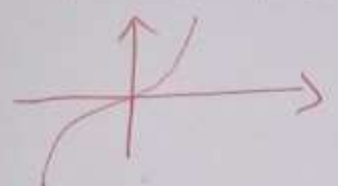
(i) If n is even and a_n is positive

The graph increase without bound positively at both ends.

i.e $y = x^2$

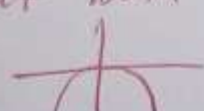
(ii) If n is odd and a_n is positive, then $p(x)$ increase without bound at the right end and decrease without bound at the left end.

i.e $y = x^3$



(iii) If n is even and $a_n < 0$ graph decrease without bound at both ends

$y = -x^2$



(iv) If n is odd and $a_n < 0$ then $p(x)$ will decrease without bound positively at the right end and increase at left end.

(v) Determine the multiplicity of zeroes

Examples

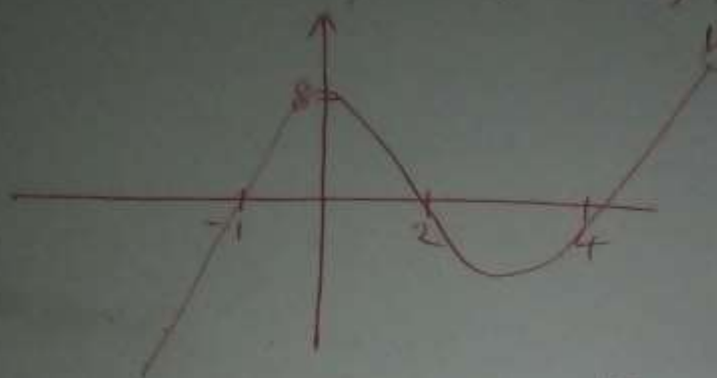
Sketch

(a) $f(x) = (x+1)(x-2)(x-4)$

Note: $a_n x^n \Rightarrow x^3$, a_n is +ve, n is odd

Zeroes are $x=-1$, $x=2$, $x=4$

y - intercept = 8



(b) $f(x) = (x+3)(x+2)(x-1)^3$

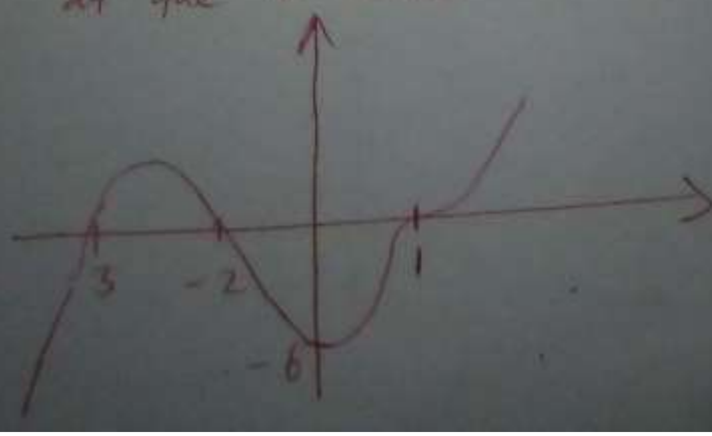
Zeroes are $x=-3$, $x=-2$

$x = 1$ (multiplicity = 3

NOTE * If multiplicity of a zero is even it just touch x-axis without crossing

* If it is odd it crosses and flates at the x-axis

Intercept $y = -6$

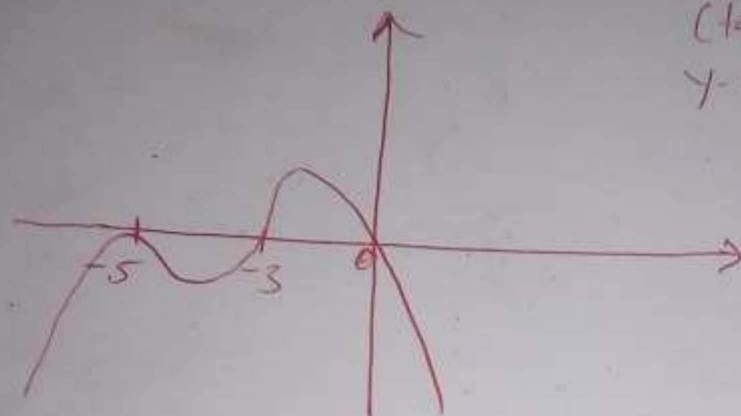


(c) $f(x) = -x(x+5)^2(x+3)$

$a_n x^n = -x^4$

Zeroes $x=0$, $x=-3$, $x=-5$ (multiplicity of 2) (touch x-axis)

y-intercept = 0



(d) $-x^3(x+2)^2(x-2)^2(x-5) = f(x)$

$a_n x^n = -x^3 \cdot x^2 \cdot x^2 \cdot x = -x^8$

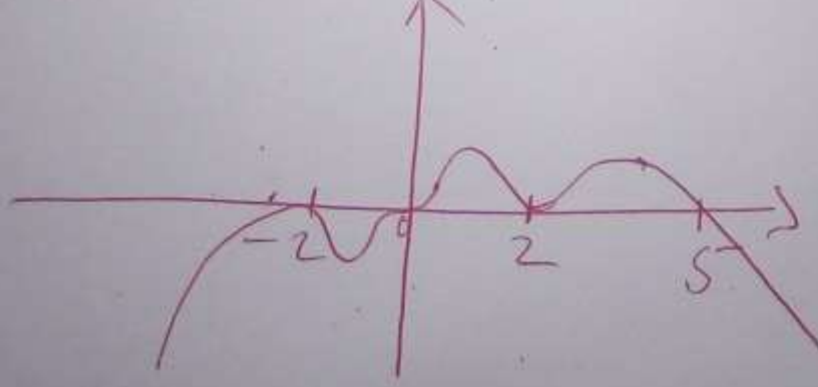
$x=0$ (multiplicity = 3)

$x=2$ (multiplicity = 2)

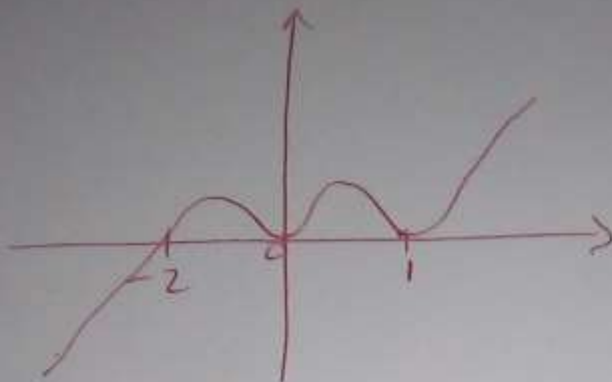
$x=-2$ (multiplicity = 2

$x=5$

y intercept = 0



(e) $f(x) = x^2(x-1)^2(2+x)$



$x = 0 \quad m = 2$

$x = 1 \quad m = 2$

$x = -2$

y intercept = 0

$a_n x^n = x^5$

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GRAPHS OF RATIONAL FUNCTIONS

A rational function is of the form

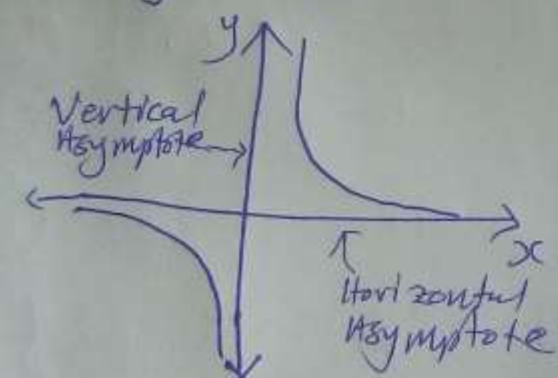
$$y = \frac{P(x)}{Q(x)}$$ where $Q(x) \neq 0$ and

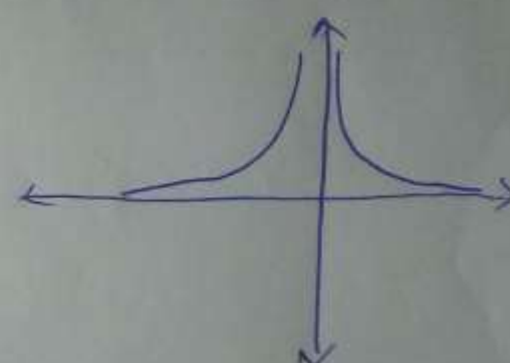
$P(x)$ and $Q(x)$ are polynomials.

Example:-

Sketch the graph of

(a) $y = 1/x$ (b) $y = \frac{1}{x^2}$





If the functions $P(x)$ and $Q(x)$ have NO COMMON FACTORS, the graph

$$y = \frac{P(x)}{Q(x)}$$ has a vertical

asymptote at the Line $x = a$ for each value "a" at which $Q(a) = 0$

∴ vertical asymptotes are found by solving the equation

$$\phi(x) = 0$$

HORIZONTAL ASYMPTOTES

If $y = \frac{P(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} \dots}{b_m x^m + b_{m-1} x^{m-1} \dots}$

where a_n and $b_m \neq 0$.

The graph of y has horizontal asymptotes:

(a) at $y = 0$ if $n < m$

(b) at $\frac{a_n}{b_m}$ if $n = m$

The graph have oblique asymptote if $n > m$

Example:

Find the vertical and Horizontal asymptotes of

(i) $y = \frac{x+1}{x-1}$ (ii) $y = \frac{x^2-1}{x^3+8}$

Solution :- (3)

(i) $y = \frac{x+1}{x-1}$

Since $(x+1)$ and $(x-1)$ have no common factors the VERTICAL Asymptote of y is at

$x - 1 = 0$

$\underline{x = 1}$

Graph have horizontal asymptote at $y = \frac{a_n}{b_m} = \frac{1}{1} = \underline{1}$

Since powers $\frac{x^1 + 1}{x^1 - 1}$ Equal powers

(ii) $y = \frac{x^2 - 1}{x^3 + 8}$

vertical Asymptote is at

$x^3 + 8 = 0$

$x^3 = -8$

$x^3 = (-2)^3$

$\underline{x = -2}$

HORIZONTAL ASYMPTOTE IS AT

$\underline{y = 0}$

Since x^2, x^3

$2 < 3$

Sketch the graph of the function

(i) $y = \frac{3}{x+1}$

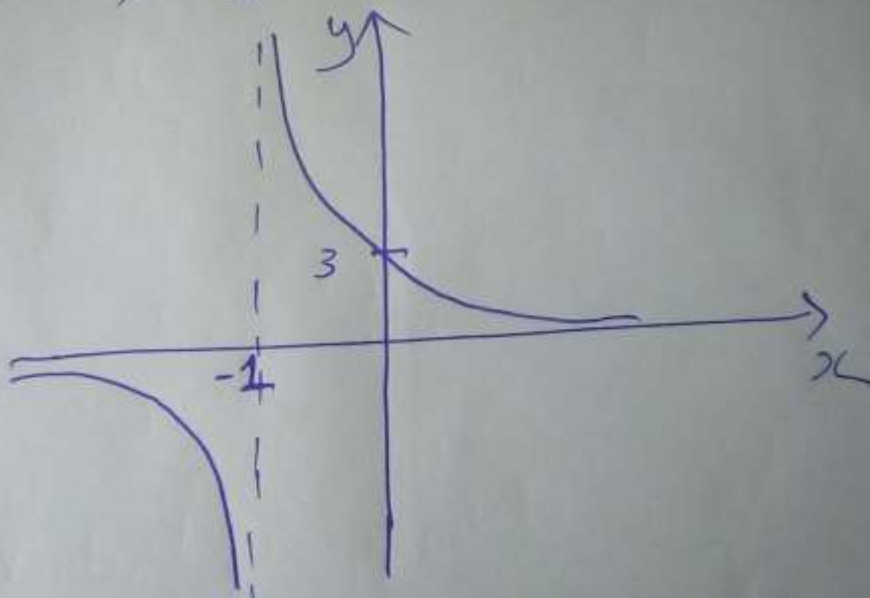
Vertical Asymptote: $x+1=0$

$\underline{x=-1}$

Horizontal Asymptote $\underline{y=0}$

cuts axes at

$x=0, \quad y = \frac{3}{0+1} = \frac{3}{1} = 3$



(2) $y = \frac{x+1}{x-1}$

Vertical Asymptote:

$x-1=0 \Rightarrow \underline{\underline{x=1}}$

Horizontal Asymptote.

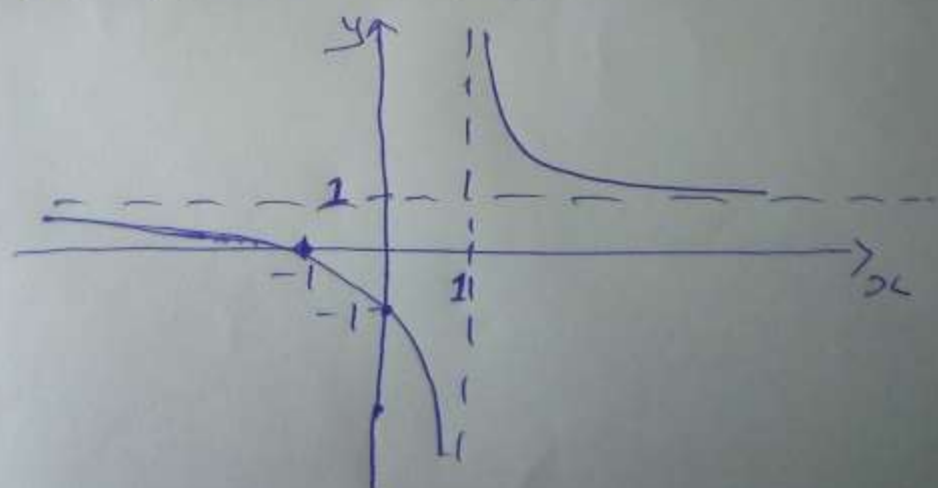
$y=1$ Since $y=\frac{(1)x^1+1}{(1)x^1-1}$, $\frac{1}{1}=1=y$

Cuts Axes at

$x=0$, $y=\frac{0+1}{0-1}=-1$

$y=0$ $\quad 0=\frac{x+1}{x-1} \Rightarrow x+1=0, x=-1$

$(0,-1)$ and $(-1,0)$



(3) Sketch $f(x) = \dfrac{3x^2}{x^2-4}$

Vertical Asymptote: $x^2 - 4 = 0$

$$x^2 = 4$$

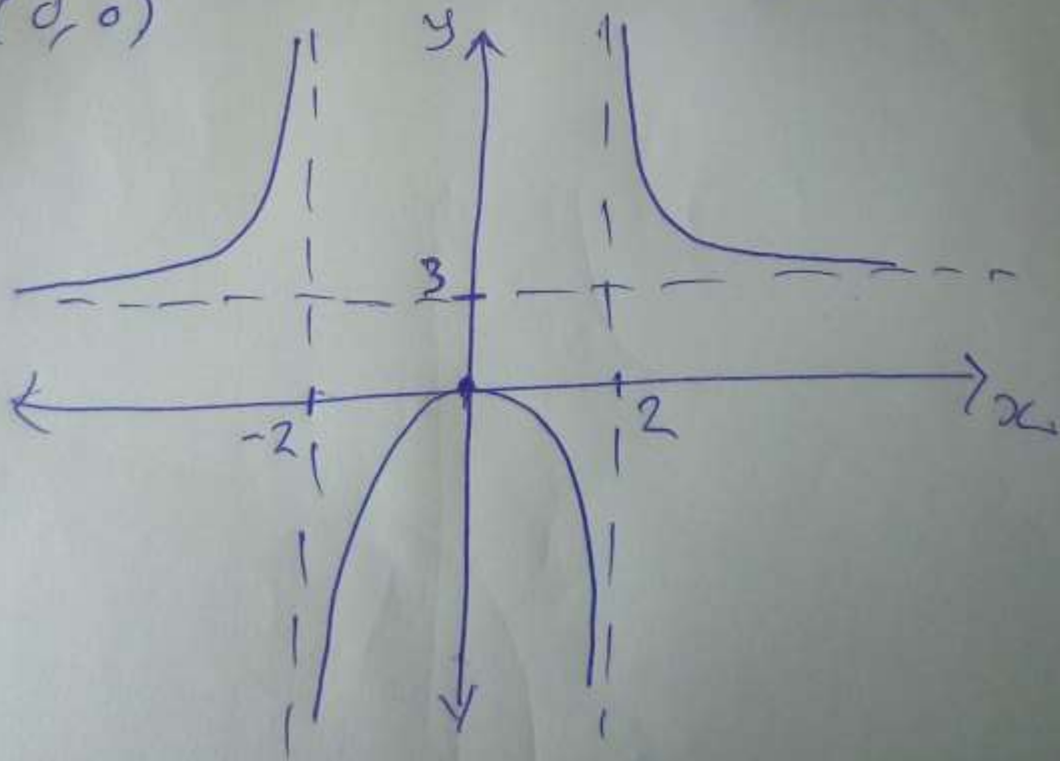
$$x = \pm 2$$

Horizontal Asymptote: $y = \dfrac{3}{1}$ Since $x^{(2)} = x^{(2)}$ Highest powers of x are equal in Numerator and Denominator

Cuts axes at

$x = 0$, $y = 0$

$(0, 0)$



(ii) $y = \dfrac{5x^2 - 2}{1 - x}$ (7)

Graph will have slant (oblique) and vertical Asymptotes.

It is an IMPROPER fraction so divide $(1-x)$ into $5x^2 - 2$

$$\begin{array}{r l}
 & -5x - 5 \\
-x+1 \,\big) & 5x^2 - 2 \\
 & -(5x^2 - 5x) \\
 & \quad 5x - 2 \\
 & -\quad (5x - 5) \\
 & \quad\quad 3 \leftarrow \text{Remainder}
\end{array}$$

$\therefore \dfrac{5x^2 - 2}{1 - x} = (-5x - 5) + \dfrac{3}{1-x}$

The oblique Asymptote is

$y = -5x - 5$

Vertical Asymptote is

$1 - x = 0$

$x = 1$

(c)

Intercepts

at $y = 0$, $0 = \dfrac{5x^2 - 2}{1 - x}$

$0 = 5x^2 - 2$

$x = \pm\sqrt{2/5}$

$(-\sqrt{2/5}, 0)$ and $(+\sqrt{2/5}, 0)$

$x = 0$, $y = -2$

$(0, -2)$

You need to sketch the oblique Asymptote ($y = -5x - 5$). Full graph of $y = \dfrac{5x^2 - 2}{1 - x}$ is