Exponential Spectral Deferred Correction Methods for Dynamical Cores

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Introduction

Solving shallow water equations (SWE) with high accuracy plays a crucial role in modern weather forecasting. Many research institutes and national labs work on refinement of solvers and their upscaling in the modern era of exascale computing. Recently emerged Parallel-in-Time solvers [1] stand high chances of becoming a new state-of-the-art approach when it comes to HPC, as they allow parallelization of time integration along with usual space integration. This research is an investigation of novel time integration approaches, which could be useful in the weather simulation problem domain: in specific, Exponential Time Integration and Spectral Deferred Correction.

Literature review

Spectral Deferred Correction (SDC) time integration methods [2] improve accuracy per function evaluation and provide additional degrees of parallelization in a specific formulation. This family of methods is appealing for climate modelling, since it provides a high order accuracy even with a first-order difference scheme in time, and no mathematical heavy lifting is required if the accuracy order needs to be improved. With suitable operator splitting, SDC methods are proven [3] to have good stability properties and cost, yet are easier to construct compared to non-spectral methods.

The idea of Exponential Time Integration (ETI) dates back to the early 1960s, and interestingly enough, was developed specifically for stiff systems. The approach of using exponential basis for ODE/PDEs is beneficial due to its differential properties. Compared to explicit schemes of the same order, they take less compute time, and can be further optimized for shallow water equations on a rotating sphere [4]. In this framework two specific approaches are to be studied: Lawson's method [5] and Exponential Time Differencing(ETD) [6].

Since 1982 [7] semi-Lagrangian (SL) schemes are known to fit weather forecasting applications. Due to vanishing primary source of nonlinear instability (advection terms in momentum equations) in the Lagrangian frame, SL schemes showcase larger stability regions.

All the aforementioned approaches were studied separately, and some attempts of merging have been made [8, 6]. However, there is no research covering ETI in combination with SDC on weather simulation specific problems (as an SLS scheme).

Research and design methods

Proposed thesis research would be focused on gradually combining approaches described above (see fig 1). Main goal is to establish, whether using EXPSDC/SL-EXPSDC schemes brings benefits to the weather simulation domain. Computational schemes would be developed and verified on the following sample problems:

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} \qquad \text{Advection 1D}$$

$$\frac{\partial u}{\partial t} = \sum_{d} \lambda_i u \qquad \text{Dahlquist}$$

$$\frac{DU}{Dt} = \mathcal{L}U + \tilde{\mathcal{N}}(U) \qquad \text{SWE [8]}$$

Implementation would be performed in Python, and later integrated into the SWEET software [9]. For each implemented solver, a sample problem would be simulated and compared to a direct schemes solutions with high time resolution.

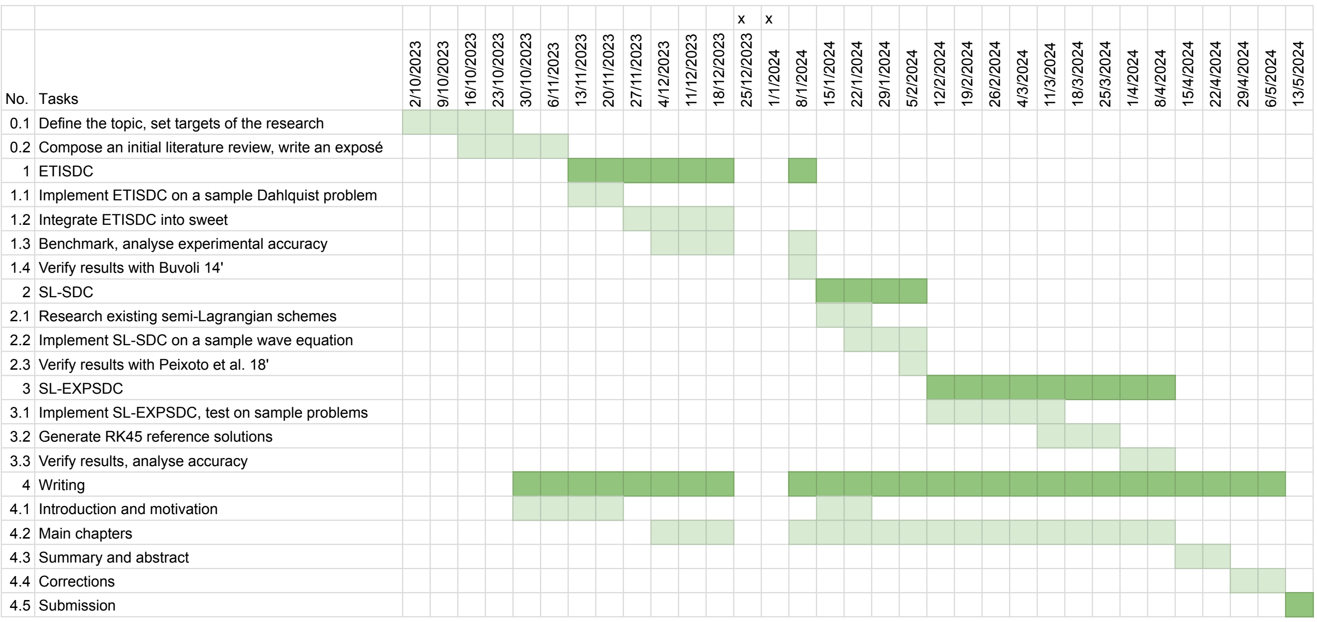

Figure 1: Preliminary schedule

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