An Introduction to Random DFA

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Deterministic Finite Automata (DFA)

DFA is $A = \langle \Sigma, Q, \tau, q_0, F \rangle$

- Σ finite alphabet
- Q set of states
- $\tau : Q \times \Sigma \to Q$ transitions
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final states

Computes $A : \Sigma^\star \to \{0, 1\}$

$$A(x) = \mathbb{I}[\tau(q_0, x) \in F]$$

$A^{-1}(1) = L_A \subseteq \Sigma^\star$ is the *language recognized* by A

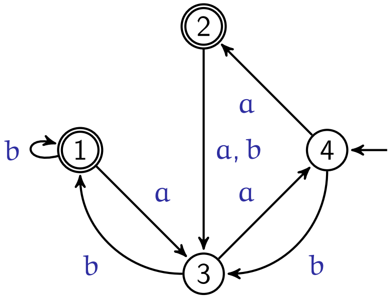
Example DFA

$\Sigma = \{a, b\}$ $\quad q_0 = 4$

$Q = \{1, 2, 3, 4\}$ $\quad \tau(1, a) = 3$

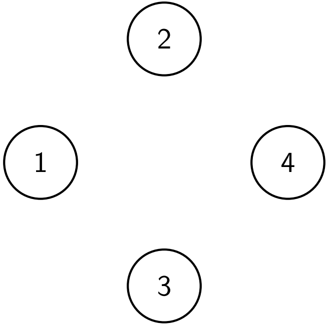
$F = \{1, 2\}$ $\quad \tau(2, b) = 3$

$$A(baa) = 1$$



Generating a DFA at Random

Fix $\Sigma = \{a, b\}$ and $Q = \{1, 2, 3, 4\}$, and choose at *random* τ, q_0, and F



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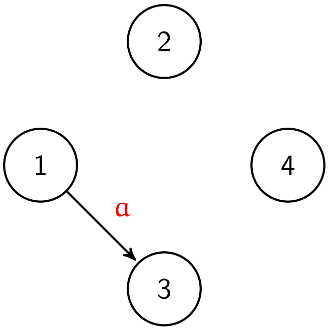

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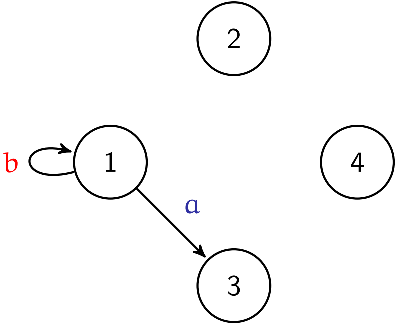

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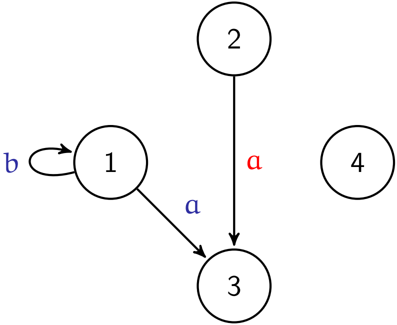

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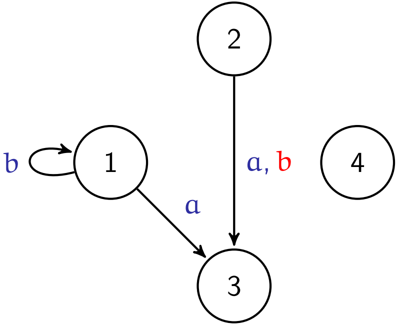

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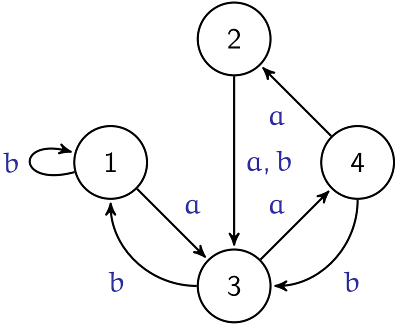

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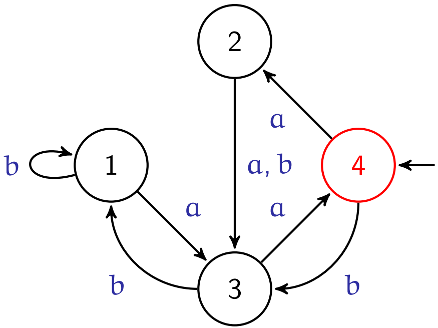

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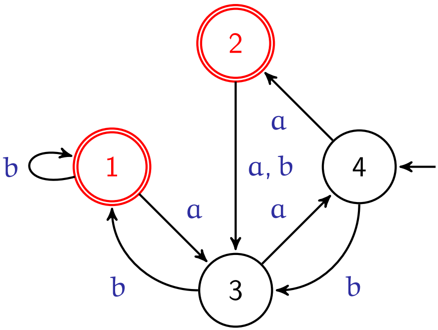

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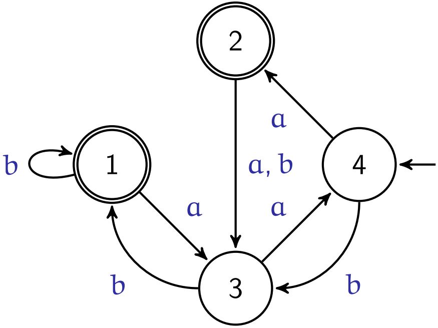

Note: All choices are independent

Outline

1. Motivation for Studying Random DFA
2. Some Properties of Generic DFA
3. Two Proofs of Grusho's Theorem

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Why Study Random DFA?

Consider the following assertions . . .

- Almost all DFA with n states are not minimal
- The average running time of Moore's DFA minimization algorithm is $O(n \log\log n)$ for DFA with n states
- Typical DFA with n states can be learned in time $\mathrm{poly}(n)$

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Leit Motif

Generic/Average Case Bounds *vs* Worst Case Bounds

PAC Learning DFA — Setup

- A minimal DFA with n states over Σ
- D probability distribution over $\Sigma^\star$
- $S = ((x^1, A(x^1)), \dots, (x^m, A(x^m))$ sample with i.i.d. $x^i \sim D$

Problem

Give algorithm L such that with probability $\geqslant 1 - \delta$ the output $\hat{A} = L(S)$ is computed in time $\mathrm{poly}(m)$ and satisfies $\mathbb{P}_{x \sim D}[A(x) \neq \hat{A}(x)] \leqslant \varepsilon$ whenever $m \geqslant \mathrm{poly}(n, |\Sigma|, 1/\varepsilon, 1/\delta)$

PAC Learning DFA — Some Bounds

Negative Results

[Pitt–Warmuth '93] Assuming P $\neq$ NP

No poly-time algorithm can approximate the minimum DFA/NFA consistent with S within a polynomial factor

[Kearns–Valiant '89] Assuming RSA is secure

No poly-time algorithm can PAC learn DFA

Positive Results

[Clark–Thollard '04]

Every DFA A with n states can be learned under distributions D_A with support L_A in time $\text{poly}(n, 1/\mu_{D_A})$

PAC Learning DFA — The Random Approach

Observations

- DFA used in lower bounds are *far from random*: acyclic or with single final state
- *Adapting* distribution to target is very restrictive

Questions

- Are random DFA easier to learn than arbitrary DFA?
- Are there distributions under which most DFA can be learned?

State of The Art (Based on [Jackson–Servedio '03, Sellie '09, Angluin et al. '10])

	Random DT	Random DNF	Random DFA
PAC (dist. free)	?	?	?
PAC/SQ (uniform)	✓	✓	?[1]
SQ (dist. free)	×	×	×

[1]Positive empirical evidence: [Lang '92] and competitions Abbadingo One, Gowachin, GECCO '04, Stamina, Zulu

Learning Random DFA under the Uniform Distribution

- A *random DFA* with n states over Σ
- D *uniform distribution* over Σ^T for some $T \geqslant 1$
- $S = ((x^1, A(x^1)), \dots, (x^m, A(x^m)))$ with i.i.d. $x^i \sim D$

Conjecture

The exists an algorithm L such that with probability $1 - o(1)$ over the choice of A, on input S produces an output $\hat{A}$ in time $\mathrm{poly}(m, T)$ such that $\mathbb{P}_{x \sim D}[A(x) \neq \hat{A}(x)] \leqslant \varepsilon$ with probability $\geqslant 1 - \delta$ whenever $m \geqslant \mathrm{poly}(n, |\Sigma|, T, 1/\varepsilon, 1/\delta)$

Note: The regime $T = O(\log n)$ is trivial

Outline

Definition of Generic Property

Given DFA $A = \langle \Sigma, Q, \tau, q_0, F \rangle$

- $|\Sigma| = r \geqslant 2$, usually a fixed constant
- $|Q| = n$, the regime of interest is $n \to \infty$
- $\mathcal{U}_{r,n}$ uniform distribution over $\mathcal{DFA}(r, n)$

Definition

We say that *generic DFA over* r *symbols satisfy property* P if the following holds when $n \to \infty$:

$$\mathbb{P}_{A \sim \mathcal{U}_{r,n}}[P(A)] = 1 - o(1)$$

Diameter of Random DFA

The diameter of a DFA is the minimum d such that:

if $q' \in \tau_\star(q) = \bigcup_{x \in \Sigma^\star} \{\tau(q, x)\}$, then
there exists $x \in \Sigma^{\leqslant d}$ such that $q' = \tau(q, x)$

Theorem (Trakhtenbrot–Barzdin '70)

Generic DFA have diameter at most $(1 + C_r + o(1)) \log_r n$, where C_r is a constant depending on r such that $C_r \to 0$ as $r \to \infty$

Reachability in Random DFA

The reachability is the number of states $|\tau_\star(q_0)|$ reachable from the initial state

Theorem (Grusho '73, Carayol–Nicaud '12, Berend–Kontorovich '13)

Generic DFA have reachability $n(c_r + o(1))$, where c_r is the positive solution of $c = 1 - e^{-rc}$

Examples of c_r

r	2	3	4	5	6	7
c_r	0.796	0.940	0.980	0.993	0.997	0.999

Note: Same result proved in the form of CLT [G73,CN12] and concentration bound [BK13]

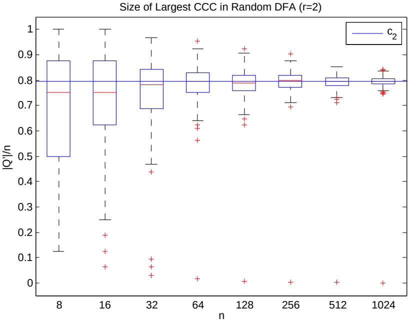
Communication Classes of Random DFA

A closed communication class (CCC) is a set of states $Q' \subseteq Q$ such that:

$$\tau_\star(Q') = \bigcup_{q \in Q'} \tau_\star(q) = Q'$$

Theorem (Grusho '73) (also follows from Berend–Kontorovich '13)

Generic DFA have a unique CCC and its size is $n(c_r + o(1))$, where c_r is the positive solution of $c = 1 - e^{-rc}$



Random Walks on Random DFA

By Grusho's theorem, the random walk starting at q_0 and ending in $\tau(q_0, x)$ with x uniform over Σ^T will end inside the CCC for large enough T

A CCC $Q' \subseteq Q$ is k-periodic if $Q' = Q_0 \sqcup \cdots \sqcup Q_{k-1}$ such that:

$$\text{for all } 0 \leqslant i \leqslant k-1 \text{ one has } \tau_\star(Q_i) = Q_{i+1 \bmod k}$$

Otherwise it is aperiodic

Theorem (Balle '13)

The unique CCC in a generic DFA is aperiodic

Note: This implies that random walks on random DFA are ergodic

Minimization of Random DFA

The minimal size of a DFA A is the size of a minimal automata accepting the same language as A

Theorem (Berend–Kontorovich '13)

Generic DFA have minimal size $n(c_r + o(1))$, where c_r is the positive solution of $c = 1 - e^{-rc}$

Running Time of DFA Minimization Algorithms

Algorithm	Worst-case	Average-case	
Hopcroft	$O(n \log n)$	$O(n \log \log n)$	[David '10]
Moore	$O(n^2)$	$O(n \log \log n)$	[David '10]
Brzozowski	exponential	super-polynomial	[De Felice–Nicaud '13]

Synchronization of Random DFA

A DFA has a reset word of length l if there exists $x \in \Sigma^l$ such that:

$$\text{for all } q' \in Q \text{ one has } \tau(q', x) = q \text{ for the same } q \in Q$$

A DFA with a reset word is called synchronizing

Conjecture (Černý '64)

Every synchronizing DFA has a reset word of length at most $(n-1)^2$

Theorem (Skvortsov–Zaks '10)

- Generic DFA on alphabets of size $r > 18 \log n$ have reset words of length less than $3n^2 \log n$.
- Generic DFA on alphabets of size $r > n^{1/2+\epsilon}$ satisfy Černý's conjecture.

Note: [SZ10] report experiments suggesting generic DFA with $r = 2$ have reset words of length $o(n)$

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3. Two Proofs of Grusho's Theorem

Proof 1: Kolmogorov Complexity

Whiteboard

— proof courtesy of Ricard Gavaldà —

Proof 2: Differential Equation Method

Whiteboard

Proof 2: Differential Equation Method

Find Reachable States

Input: DFA $A = \langle \Sigma, Q, \tau, q_0, F \rangle$
Output: $\tau_\star(q_0)$
$S \leftarrow \{q_0\}$
$T \leftarrow \{(q_0, \sigma_1), \ldots, (q_0, \sigma_r)\}$
while $T \neq \varnothing$ **do**
  Choose $(q, \sigma) \in T$
  Let $q' \leftarrow \tau(q, \sigma)$
  if $q' \notin S$ **then**
    Let $S \leftarrow S \cup \{q'\}$
    Let $T \leftarrow T \cup \{(q', \sigma_1), \ldots, (q', \sigma_r)\}$
  Let $T \leftarrow T \backslash \{(q, \sigma)\}$
return S

Conclusion

- Exciting research topic with many open problems
- Yields useful insights into analysis of DFA algorithms
- Almost nothing done between the 70's and 2010

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