(Co)-Algebraic and Analytical Aspects of Weighted Automata Minimisation and Equivalence (Part 2)

Borja Balle

Data Science | Lancaster University

Mathematics & Statistics | Lancaster University

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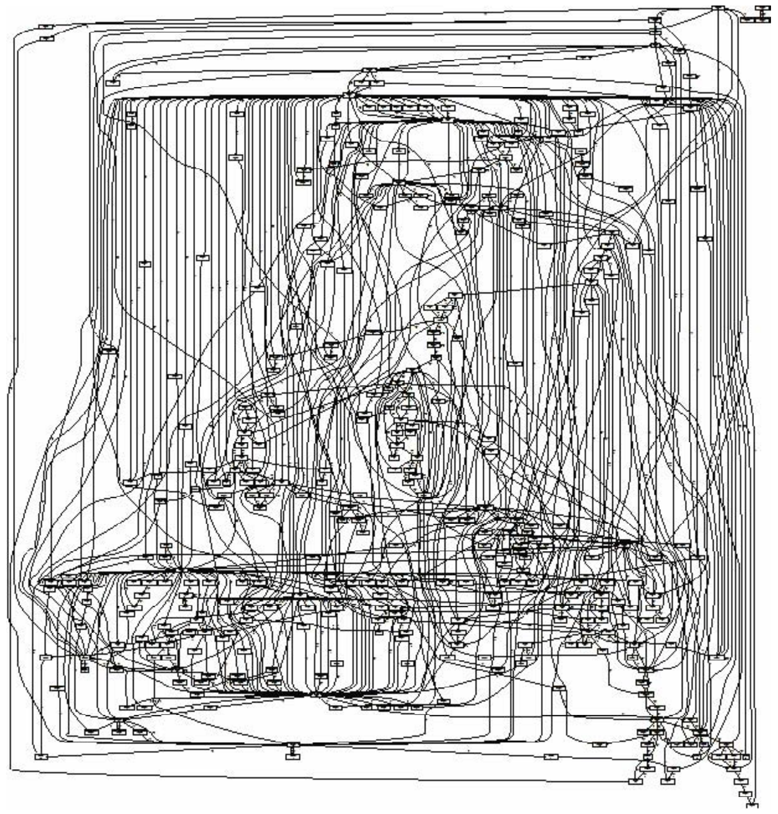
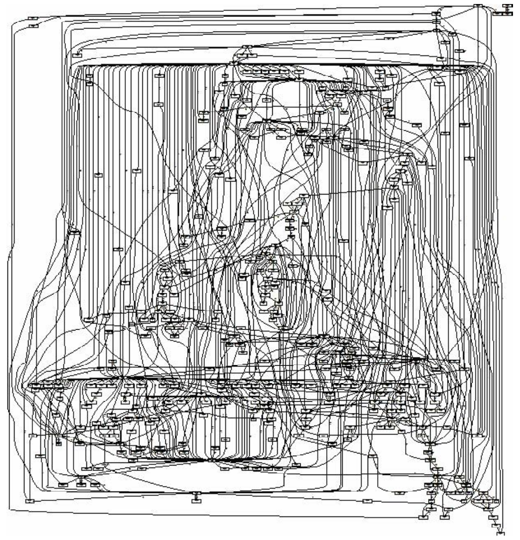
Minimisation vs Approximation



minimisation



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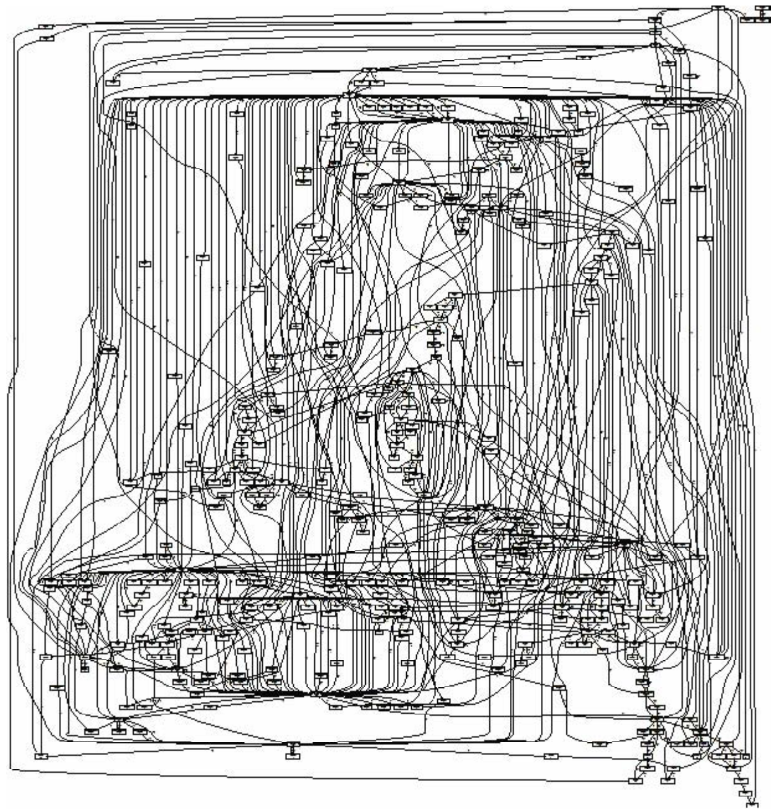
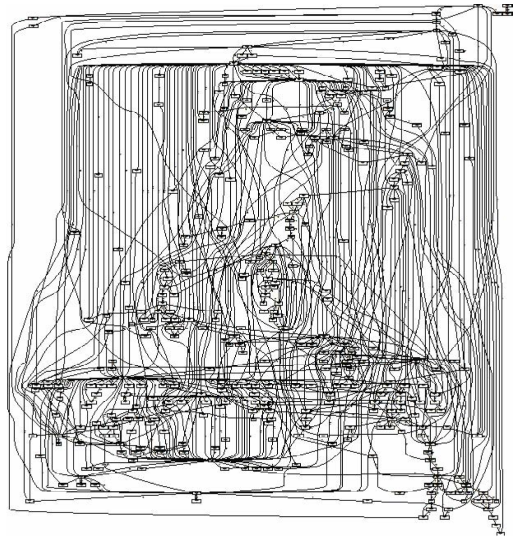
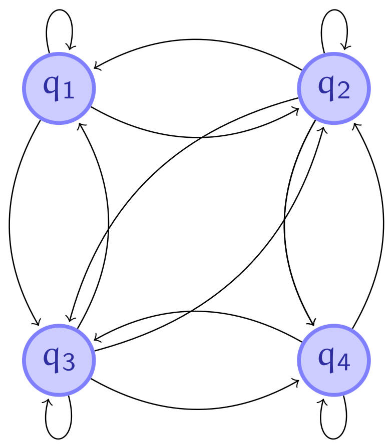


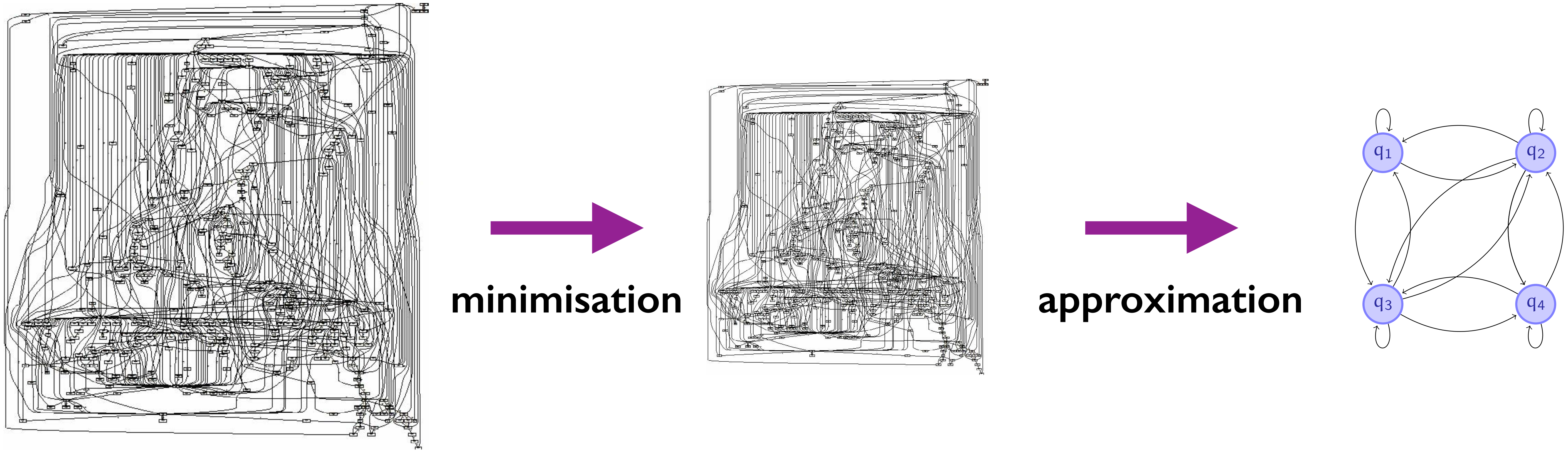
minimisation



approximation



Minimisation vs Approximation



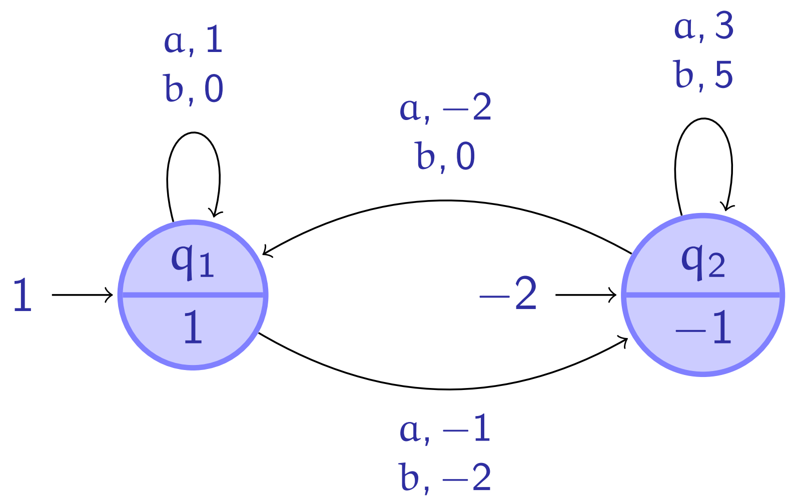
- Minimisation is naturally related to *(co-)algebraic* properties
- Approximation questions need *analytical* tools (eg. metrics)

Motivations for Approximation

1. Efficient (approximate) computation with WA
   - Compute faster and pay a controlled price in terms of accuracy
2. Machine learning for WA
   - Understand learning bias, design better algorithms
3. Interesting mathematical questions

WA and Rational Functions

$\mathcal{A}$

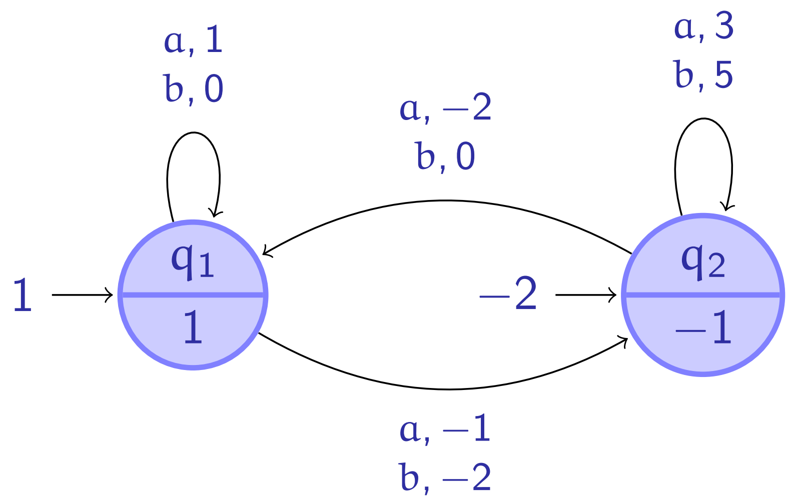


$$\mathcal{A} = \langle \alpha, \beta, \{T_a\}_{a \in \Sigma} \rangle$$
$$\alpha, \beta \in \mathbb{R}^n, T_a \in \mathbb{R}^{n \times n}$$

$$\alpha = \begin{bmatrix} 1 & -2 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad T_a = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \qquad T_b = \begin{bmatrix} 0 & -2 \\ 0 & 5 \end{bmatrix}$$

WA and Rational Functions

$\mathcal{A}$



$$\mathcal{A} = \langle \alpha, \beta, \{T_a\}_{a\in\Sigma} \rangle$$
$$\alpha, \beta \in \mathbb{R}^n, T_a \in \mathbb{R}^{n\times n}$$

$f_{\mathcal{A}} : \Sigma^\star \to \mathbb{R}$

sum of weights of all paths labelled by the input string

$$f_{\mathcal{A}}(x) = \alpha T_{x_1} \cdots T_{x_t} \beta$$
$$= \alpha T_x \beta$$

(Pseudo-)Metrics for Approximation

How do we measure the quality of WA approximations?

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$$\mathrm{dist}_p(\mathcal{A}, \mathcal{B}) = \|f_{\mathcal{A}} - f_{\mathcal{B}}\|_p \qquad \|f\|_p = \left(\sum_{x \in \Sigma^\star} |f(x)|^p \right)^{1/p}$$

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pseudo-metric

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Important linear subspaces of $\mathbb{R}^{\Sigma^\star}$

$$\mathsf{Rat} = \{f : \Sigma^\star \to \mathbb{R} \mid \textbf{rational}\}$$

$$\ell^p = \{f : \Sigma^\star \to \mathbb{R} \mid \|f\|_p < \infty\}$$

$$\ell^p_{\mathsf{Rat}} = \mathsf{Rat} \cap \ell^p = \{f : \Sigma^\star \to \mathbb{R} \mid \textbf{rational}, \|f\|_p < \infty\}$$

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Banach/Hilbert

$$\ell^p_{\mathsf{Rat}} = \mathsf{Rat} \cap \ell^p = \{f : \Sigma^\star \to \mathbb{R} \mid \textbf{rational}, \|f\|_p < \infty\}$$

not complete!

An Approximation Result

There is an algorithm that given a minimal WA $\mathcal{A}$ with n states and $k < n$, runs in time $O(|\Sigma| n^6)$ and returns a WA $\hat{\mathcal{A}}$ with k states such that

$$\mathrm{dist}_2(\mathcal{A}, \hat{\mathcal{A}}) \leqslant \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_n^2}$$

[Balle, Panangaden, Precup 2015, Kulesza, Jiang, Singh 2015]

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singular values of the *Hankel matrix* of $f_{\mathcal{A}}$

[Balle, Panangaden, Precup 2015, Kulesza, Jiang, Singh 2015]

The Plan

1. Hankel Matrices and Singular Value Automata (SVA)
2. Algorithms for Computing SVA
3. WA Approximation via SVA Truncation

The Hankel Matrix

$$f : \Sigma^\star \to \mathbb{R}$$

$$H_f \in \mathbb{R}^{\Sigma^\star \times \Sigma^\star}$$

$$H_f(p, s) = f(ps)$$

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$$\begin{array}{c@{\quad}c}
 & \begin{array}{cccccc} \varepsilon & a & b & \cdots & s & \cdots \end{array} \\
\begin{array}{c} \varepsilon \\ a \\ b \\ \vdots \\ p \\ \vdots \end{array} &
\left[\begin{array}{cccccc}
f(\varepsilon) & f(a) & f(b) & & \vdots & \\
f(a) & f(aa) & f(ab) & & \vdots & \\
f(b) & f(ba) & f(bb) & & \vdots & \\
 & & & & & \\
\cdots & \cdots & \cdots & & f(ps) & \\
 & & & & &
\end{array}\right]
\end{array}$$

The Hankel Matrix

$$f : \Sigma^\star \to \mathbb{R}$$

$$H_f \in \mathbb{R}^{\Sigma^\star \times \Sigma^\star}$$

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$$
\begin{array}{c|c}
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\begin{bmatrix}
f(\varepsilon) & f(a) & f(b) & & \vdots & \\
f(a) & f(aa) & f(ab) & & \vdots & \\
f(b) & f(ba) & f(bb) & & \vdots & \\
 & & & & & \\
\cdots & \cdots & \cdots & & f(ps) & \\
 & & & & &
\end{bmatrix}
\end{array}
$$

Theorem: $\mathrm{rank}(H_f) = n$ iff f computed by minimal WA with n states

[Kronecker 1881, Carlyle & Paz 1971, Fliess 1974]

Proof (upper bound $\mathrm{rank}(H_{f_{\mathcal{A}}}) \leqslant |\mathcal{A}|$)

$$H_f(p, s) = f(ps) = \alpha T_{p_1} \cdots T_{p_t} T_{s_1} \cdots T_{s_{t'}} \beta = \alpha_p \cdot \beta_s$$

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$$\begin{array}{cc} & \begin{array}{ccccc} & & s & & \end{array} \\ p & \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & \bullet & \cdots & \cdots \\ & & \vdots & & \end{bmatrix} \end{array} = \begin{array}{cc} p & \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \bullet & \bullet & \bullet \\ \cdot & \cdot & \cdot \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccccc} & & s & & \end{array} \\ \begin{bmatrix} \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \end{bmatrix} \end{array}$$

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$$\begin{array}{cc} & s \\ p & \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & \bullet & \cdots & \cdots \\ & & \vdots & & \end{bmatrix} \end{array} = \begin{array}{cc} & \\ & \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \bullet & \bullet & \bullet \\ \cdot & \cdot & \cdot \end{bmatrix} \\ p & \end{array} \begin{array}{c} s \\ \begin{bmatrix} \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \end{bmatrix} \end{array}$$

$$H_f = P \cdot S$$

Proof (equality $\mathrm{rank}(H_f) = \min_{f_{\mathcal{A}}=f} |\mathcal{A}|$)

1. Suppose $H_f = P \cdot S$ is rank factorisation
2. For $a \in \Sigma$ define $H_f^a(p,s) = f(pas)$ and note $H_f^a = R_a H_f$ where $(R_a f)(x) = f(xa)$
3. Observe that because S has full row rank $\mathrm{colspan}(R_a P) = \mathrm{colspan}(H_f^a) \subseteq \mathrm{colspan}(H_f) = \mathrm{colspan}(P)$
4. Hence there exists T_a such that $R_a P = P T_a$
5. Let $\mathcal{A} = \langle \alpha = P(\varepsilon, -), \beta = S(-, \varepsilon), \{T_a\} \rangle$
6. Note $f_{\mathcal{A}} = f$ since $P(\varepsilon,-)T_x S(-,\varepsilon) = R_x P(\varepsilon,-)S(-,\varepsilon) = P(x,-)S(-,\varepsilon) = H_f(x,\varepsilon)$

A Useful Correspondence

$\mathcal{A} = \langle \alpha, \beta, \{T_a\}_{a \in \Sigma} \rangle$ $\quad$ $Q \in \mathbb{R}^{n \times n}$ invertible

$$\mathcal{A}^Q = \langle \alpha Q, Q^{-1}\beta, \{Q^{-1} T_a Q\}_{a \in \Sigma} \rangle$$

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Theorem: For any rational $f : \Sigma^\star \to \mathbb{R}$

Minimal WA $\mathcal{A}$ computing f $\xleftrightarrow{\text{bijection}}$ Rank factorizations of the form $H_f = P \cdot S$

[Balle, Carreras, Luque, Quattoni 2014]

SVD of Hankel Matrices

$$H_f = UDV^\top = \begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} \cdots & v_1^\top & \cdots \\ & \vdots & \\ \cdots & v_n^\top & \cdots \end{bmatrix}$$

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Theorem:

H_f with f rational admits an SVD iff $\|f\|_2 < \infty$

[Balle, Panangaden, Precup 2015, 201?]

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Key Idea: see Hankel matrix as an operator in a Hilbert space

$$H_f : \ell^2 \to \ell^2$$

$$g \mapsto H_f g$$

$$(H_f g)(x) = \sum_{y \in \Sigma^\star} f(xy) g(y)$$

[Balle, Panangaden, Precup 2015, 201?]

The Singular Value Automaton

From Correspondence Theorem:

If $\mathrm{H_f}$ admits an SVD $\mathrm{H_f} = \mathrm{UDV}^\top$, then there exists a WA $\mathcal{A}$ for f inducing $\mathrm{P} = \mathrm{UD}^{1/2}$ and $\mathrm{S} = \mathrm{D}^{1/2}\mathrm{V}^\top$

[Balle, Panangaden, Precup 2015]

The Singular Value Automaton

<u>From Correspondence Theorem:</u> ***singular value automaton***

If $\mathrm{H_f}$ admits an SVD $\mathrm{H_f = UDV^\top}$, then there exists a WA $\mathcal{A}$ for f inducing $\mathrm{P = UD^{1/2}}$ and $\mathrm{S = D^{1/2}V^\top}$

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The Singular Value Automaton

From Correspondence Theorem: *singular value automaton*

If H_f admits an SVD $H_f = UDV^\top$, then there exists a WA $\mathcal{A}$ for f inducing $P = UD^{1/2}$ and $S = D^{1/2}V^\top$

$$\mathcal{A} = \langle \alpha, \beta, \{T_a\}_{a \in \Sigma} \rangle$$

WA computing sing. vect. can be obtained from SVA:

$$\mathcal{U}_i = \langle \alpha, e_i / \sqrt{\sigma_i}, \{T_a\} \rangle \qquad f_{\mathcal{U}_i} = u_i$$

$$\mathcal{V}_i = \langle e_i / \sqrt{\sigma_i}, \beta, \{T_a\} \rangle \qquad f_{\mathcal{V}_i} = v_i$$

[Balle, Panangaden, Precup 2015]

Consequences of SVA

1. For every $f \in \ell^2_{\mathsf{Rat}}$ the SVA provides a “unique” canonical WA representation

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1. For every $f \in \ell^2_{\mathsf{Rat}}$ the SVA provides a “unique” canonical WA representation

2. The vector subspace $\ell^2_{\mathsf{Rat}} \subset \mathbb{R}^{\Sigma^\star}$ is closed under taking left/right singular vectors of Hankel matrices

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Two Important Matrices

Reachability Gramian: $G_P = P^\top \cdot P$

Observability Gramian: $G_S = S \cdot S^\top$

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Observability Gramian: $\mathbf{G}_\mathbf{S} = \mathbf{S} \cdot \mathbf{S}^\top$

<u>**Theorem:**</u> [Balle, Panangaden, Precup 2015, 201?]

Assuming $\mathcal{A}$ is minimal, $\mathbf{G}_\mathbf{P}$ and $\mathbf{G}_\mathbf{S}$ are well-defined iff $\|f_\mathcal{A}\|_2 < \infty$

Two Important Matrices

Reachability Gramian: $G_P = P^\top \cdot P$

Observability Gramian: $G_S = S \cdot S^\top$

For SVA

$$P = UD^{1/2} \quad \Rightarrow \quad G_P = D$$

$$S = D^{1/2}V^\top \quad \Rightarrow \quad G_S = D$$

Two Important Matrices

Reachability Gramian: $$G_P = P^\top \cdot P$$

Observability Gramian: $$G_S = S \cdot S^\top$$

For SVA

$$P = UD^{1/2} \quad \Rightarrow \quad G_P = D$$
$$S = D^{1/2}V^\top \quad \Rightarrow \quad G_S = D$$

For $\mathcal{A}^Q$

$$G_P^Q = Q^\top G_P Q$$
$$G_S^Q = Q^{-1} G_S Q^{-\top}$$

Computing the SVA

Given minimal WA $\mathcal{A}$ computing $f \in \ell^2_{\mathsf{Rat}}$ do:

1. Compute Gramians G_P and G_S
2. Diagonalize $M = G_S G_P$
   (i.e. find Q s.t. $Q^{-1} M Q = D^2$)
3. Obtain SVA as $\mathcal{A}^Q$

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note:

Step 2 is a finite eigenvalue problem :-)

Step 1 involves products of infinite matrices :-o

Computing the Gramians

Fixed-point Equations for Gramians

$$G_S = \beta \cdot \beta^\top + \sum_{a \in \Sigma} T_a \cdot G_S \cdot T_a^\top$$

$$G_P = \alpha^\top \cdot \alpha + \sum_{a \in \Sigma} T_a^\top \cdot G_P \cdot T_a$$

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Closed-form Algorithm

Total time $O(n^6 + |\Sigma| n^4)$

$$vec(G_S) = \left(I - \sum_{a \in \Sigma} T_a \otimes T_a \right)^{-1} (\beta \otimes \beta)$$

Computing the Gramians

Fixed-point Equations for Gramians

$$G_S = \beta \cdot \beta^\top + \sum_{a \in \Sigma} T_a \cdot G_S \cdot T_a^\top$$

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Closed-form Algorithm

Total time $O(n^6 + |\Sigma| n^4)$

$$vec(G_S) = \left(I - \sum_{a \in \Sigma} T_a \otimes T_a \right)^{-1} (\beta \otimes \beta)$$

note: there is also an iterative algorithm with $O(|\Sigma| n^{2.4})$ flops per iteration

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Approximation (Attempt 1)

$$\underbrace{\begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} \cdots & v_1^\top & \cdots \\ & \vdots & \\ \cdots & v_n^\top & \cdots \end{bmatrix}}_{H_f} \approx \underbrace{\begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_{\color{red}k} \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{\color{red}k} \end{bmatrix} \begin{bmatrix} \cdots & v_1^\top & \cdots \\ & \vdots & \\ \cdots & v_{\color{red}k}^\top & \cdots \end{bmatrix}}_{\hat{H}}$$

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$\hat{H} \rightarrow$ **best rank k approximation**

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$\hat{H} \rightarrow$ **best rank k approximation**

Claim:

If $\hat{f} : \Sigma^\star \to \mathbb{R}$ is such that $H_{\hat{f}} = \hat{H}$, then $\mathrm{rank}(\hat{f}) = k$

and $\|f - \hat{f}\|_2 \leqslant \|H_f - \hat{H}\|_{op} = \sigma_{k+1}$

Approximation (Attempt 1)

$$\underbrace{\begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} \cdots & v_1^\top & \cdots \\ & \vdots & \\ \cdots & v_n^\top & \cdots \end{bmatrix}}_{H_f} \approx \underbrace{\begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_k \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix} \begin{bmatrix} \cdots & v_1^\top & \cdots \\ & \vdots & \\ \cdots & v_k^\top & \cdots \end{bmatrix}}_{\hat{H}}$$

best rank k approximation

Claim:

not Hankel!

If $\hat{f} : \Sigma^\star \to \mathbb{R}$ is such that $H_{\hat{f}} = \hat{H}$, then $\mathrm{rank}(\hat{f}) = k$ and $\|f - \hat{f}\|_2 \leqslant \|H_f - \hat{H}\|_{op} = \sigma_{k+1}$

Approximation (Attempt 2)

$$H_f = \begin{bmatrix} \vdots & & \\ f & & \\ \vdots & & \end{bmatrix} = \underbrace{\begin{bmatrix} \vdots & & \vdots \\ \sqrt{\sigma_1}u_1 & \cdots & \sqrt{\sigma_n}u_n \\ \vdots & & \vdots \end{bmatrix}}_{UD^{1/2}} \underbrace{\begin{bmatrix} \beta_1 & & \\ \vdots & & \\ \beta_n & & \end{bmatrix}}_{D^{1/2}V^\top}$$

Approximation (Attempt 2)

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Claim:

Taking $\hat{f} = \sum_{i=1}^{k} \beta_i \sqrt{\sigma_i} u_i$ we have a rational function sum of k orthogonal rational functions and $\|f - \hat{f}\|_2^2 = \sum_{i=k+1}^{n} \beta_i^2 \sigma_i \leqslant \sum_{i=k+1}^{n} \sigma_i^2$

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has rank n, but want k!

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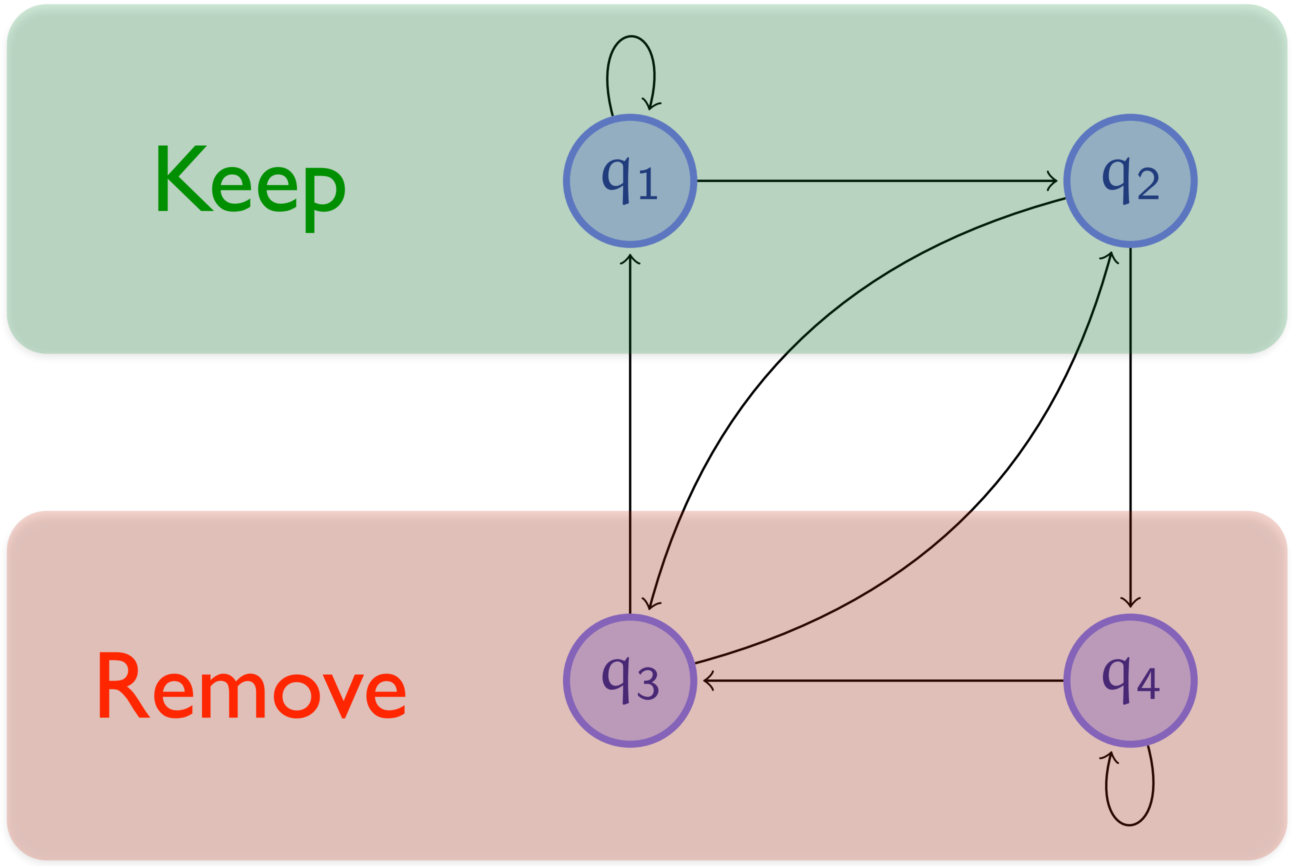
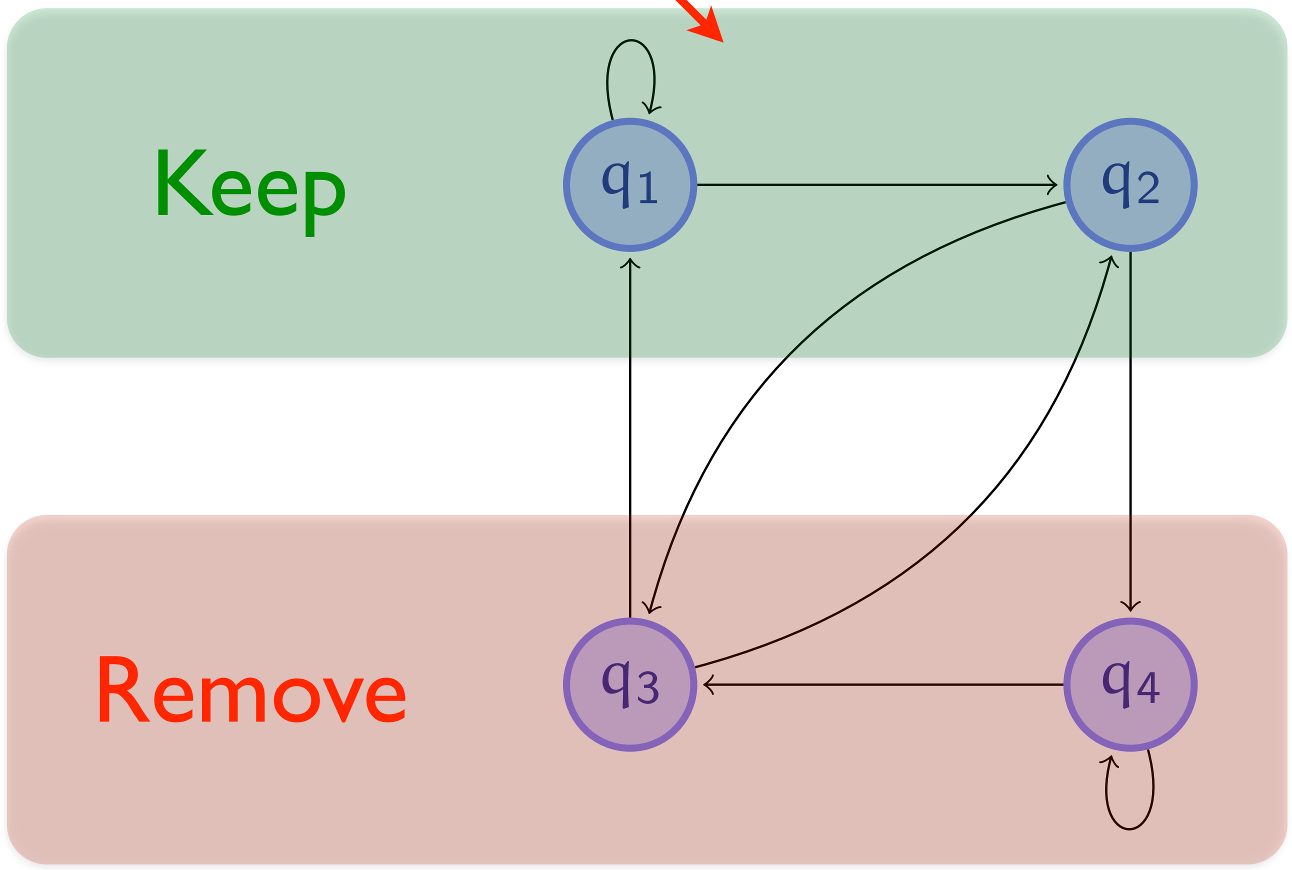
SVA Truncation

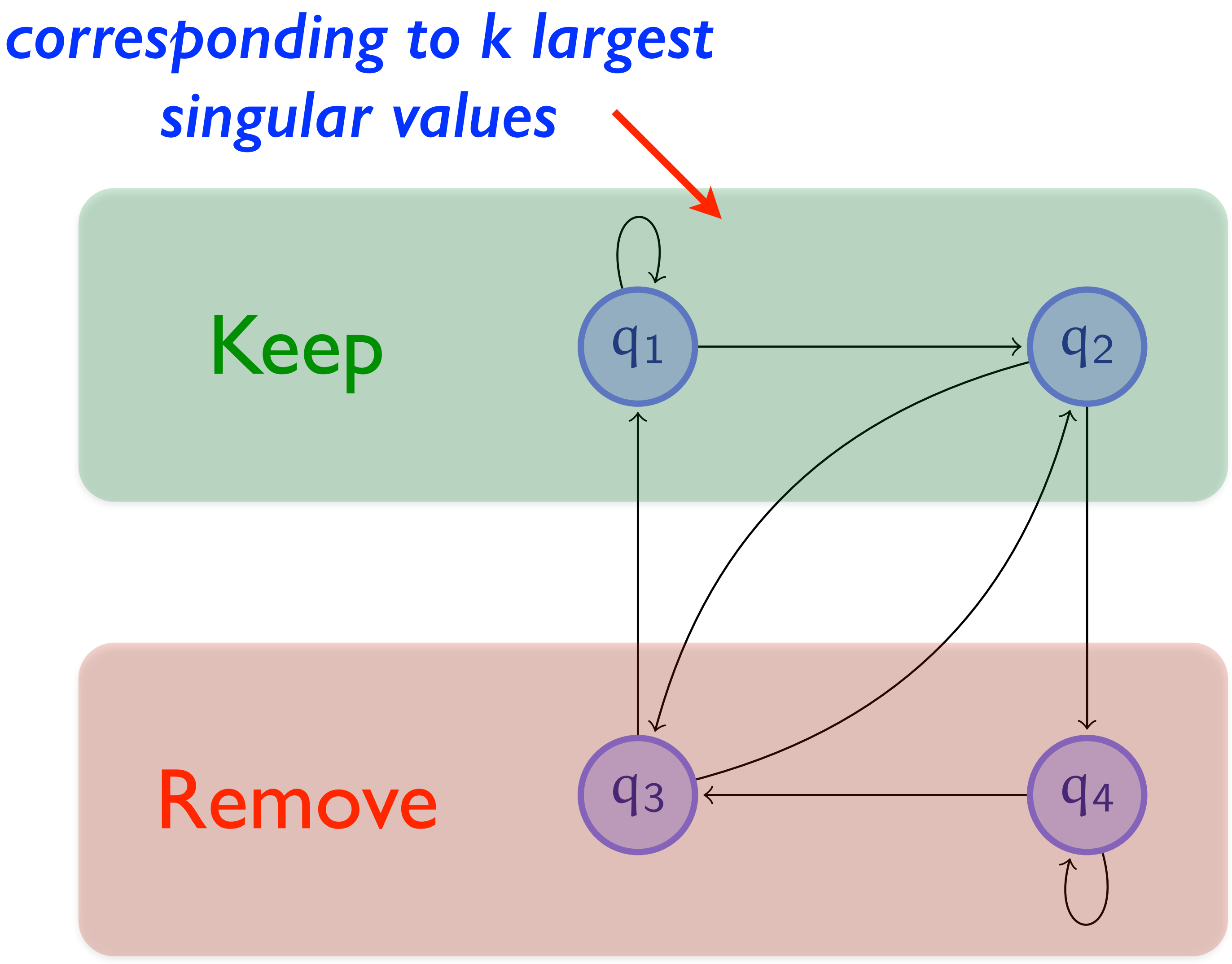


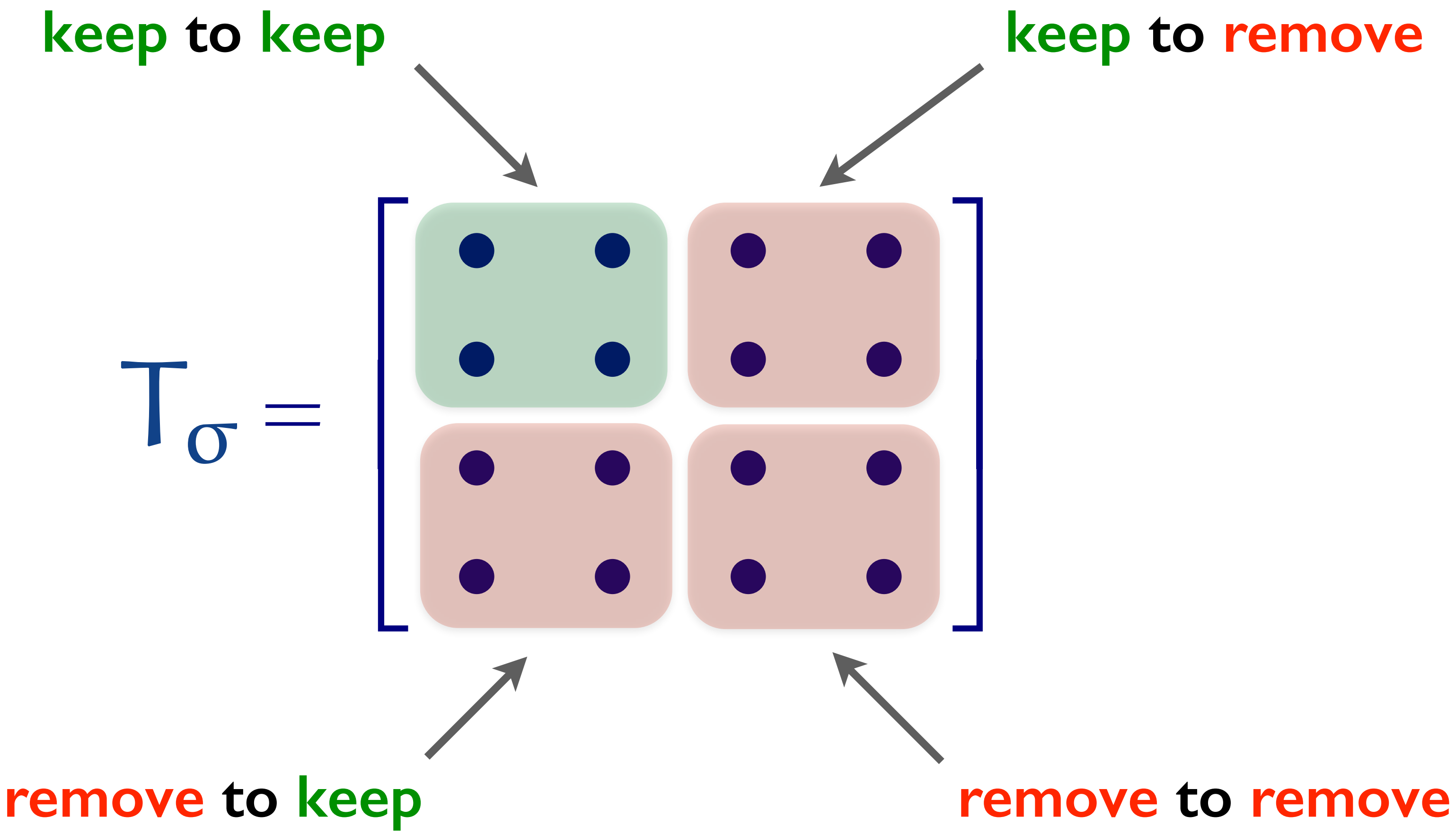
SVA Truncation

corresponding to k largest singular values



SVA Truncation

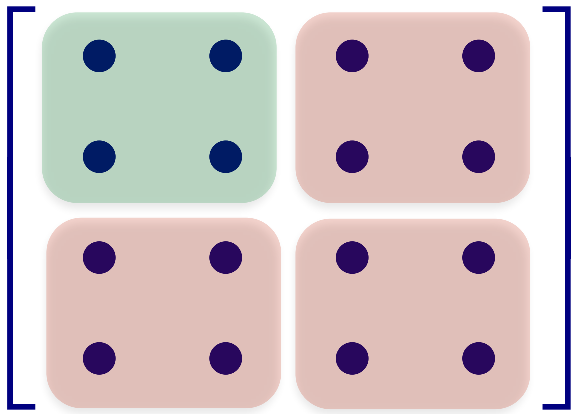




Error in SVA Truncation

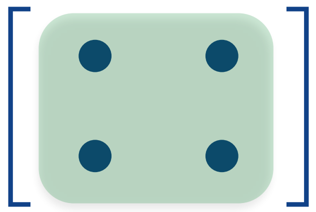
<u>*Original*</u>

$\mathcal{A}$, f, n states

$T_\sigma =$ 

<u>*Truncated*</u>

$\hat{\mathcal{A}}$, $\hat{f}$, k states

$\hat{T}_\sigma =$ 

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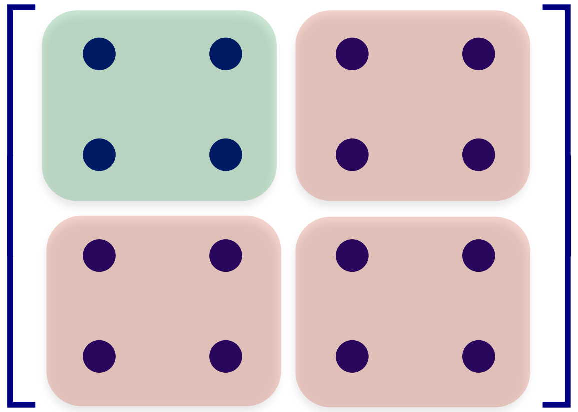
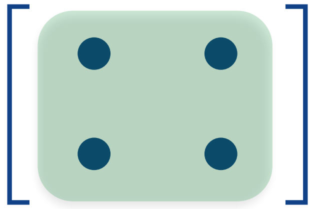
$T_\sigma =$ 

Truncated

$\hat{\mathcal{A}}$, $\hat{f}$, k states

$\hat{T}_\sigma =$ 

Error Bounds:

$$\|f - \hat{f}\|_2 = \text{big, ugly formula} \leqslant \begin{cases} C_f(\sigma_{k+1} + \cdots + \sigma_n)^{1/4} \\ (\sigma_{k+1}^2 + \cdots + \sigma_n^2)^{1/2} \end{cases}$$

[Balle, Panangaden, Precup 2015, 201?, Kulesza, Jiang, Singh 2015]

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- Change of basis in WA doesn't change the function but yields automata with different properties (eg. for truncation)

Open Problems

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- How important is it that WA are closed under reversal for the error analysis? [see Rabusseau, Balle, Cohen 2016]
- Can the AAK theorems in control theory be generalised to WA?
- Can we obtain efficient approximation algorithms with extra constraints (eg. sparsity, positivity)?

(Co)-Algebraic and Analytical Aspects of Weighted Automata Minimisation and Equivalence (Part 2)

Borja Balle

Data Science | Lancaster University

Mathematics & Statistics | Lancaster University

[CMCS Tutorial, Apr 2 2016]