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Generating random correlation matrices based on vines and extended onion method

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ABSTRACT

We extend and improve two existing methods of generating random correlation matrices, the onion method of Ghosh and Henderson [S. Ghosh, S.G. Henderson, Behavior of the norta method for correlated random vector generation as the dimension increases, ACM Transactions on Modeling and Computer Simulation (TOMACS) 13 (3) (2003) 276–294] and the recently proposed method of Joe [H. Joe, Generating random correlation matrices based on partial correlations, Journal of Multivariate Analysis 97 (2006) 2177–2189] based on partial correlations. The latter is based on the so-called *D*-vine. We extend the methodology to any regular vine and study the relationship between the multiple correlation and partial correlations on a regular vine. We explain the onion method in terms of elliptical distributions and extend it to allow generating random correlation matrices from the same joint distribution as the vine method. The methods are compared in terms of time necessary to generate 5000 random correlation matrices of given dimensions.

1. Introduction

In his recent work Joe [1] introduced a new method for generating random correlation matrices uniformly from the space of positive definite correlation matrices. The method is based on an appropriate transformation of partial correlations to ordinary product moment correlations. The partial correlations can be assigned to edges of a regular vine — an extension of the concept of Markov dependence trees. Joe based his method on the so-called *D*-vine. We show that his methodology can be applied to any regular vine and argue that another type of regular vine, namely the *C*-vine, is more suitable for generating random correlation matrices. They require less computational time since the transformation of a set of partial correlations on a *C*-vine to a corresponding set of unconditional correlations operates only on partial correlations that are already specified on that vine. Please see [2] for more details on dependence vines.

An alternative method of sampling correlation matrices called *onion* method has been proposed by Ghosh and Henderson [3]. This method can be explained in terms of elliptical distributions, and it does not involve partial correlations. We extend it to allow generating random correlation matrices with the joint density of the correlations being proportional to a power of the determinant of the correlation matrix.

The paper is organized as follows. Section 2 generalizes the method of generating correlation matrices proposed by Joe. In Section 3 we extend the onion method. We carry out a computational time analysis of both methods in Section 4. This is followed by the conclusions in Section 5.

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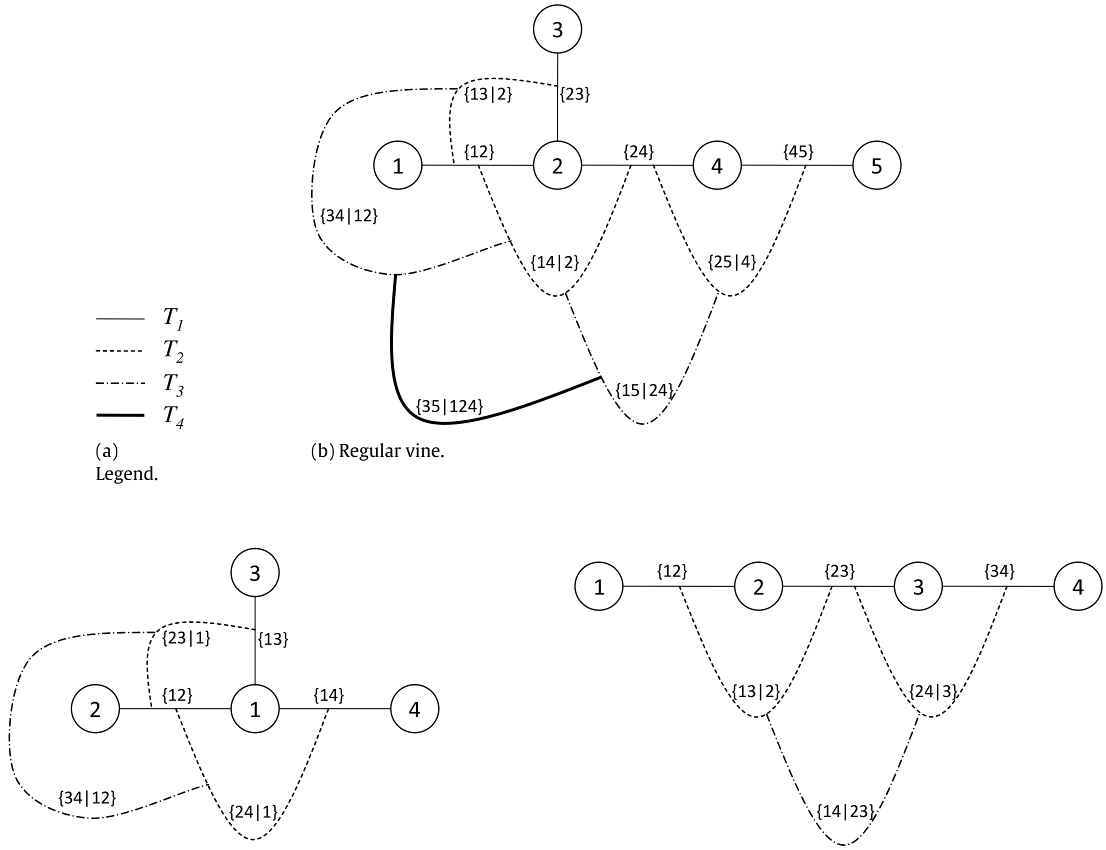

(a) Legend.

(b) Regular vine.

(c) C-vine.

(d) D-vine.

Fig. 1. Examples of various vine types.

2. Generating random correlation matrices with partial correlations regular vines

The main idea of Joe's method (see [1]) to generate a correlation matrix of size $d \times d$ is to sample values of $\binom{d}{2}$ appropriately chosen partial correlations. The distribution of a given partial correlation is a Beta $\left(\frac{d-k}{2}, \frac{d-k}{2}\right)$ distribution on $(-1, 1)$, where the value k is the cardinality of the set of conditioning variables for the partial correlation. For a 4-dimensional correlation matrix Joe's choice of partial correlations become the following

$$\rho_{12}, \rho_{23}, \rho_{34}, \rho_{13;2}, \rho_{24;3}, \rho_{14;23}. \tag{1}$$

However we extend the method to allow different choices for $\binom{d}{2}$ partial correlations. All choices of sets of partial correlations required for the method to work can be described using the notion of the partial correlation regular vine [2].

A vine $\mathcal{V}$ on d variables is a nested set of connected trees $\mathcal{V} = \{T_1, \dots, T_{d-1}\}$ where the edges of tree T_i are the nodes of tree T_{i+1}, $i = 1, \dots, d-2$. We denote the set of all edges in tree T_i by E_i. A *regular* vine is a vine in which two edges in tree T_i are joined by an edge in tree T_{i+1} only if these edges share a common node, $i = 1, \dots, d-2$. Fig. 1b shows an example of a regular vine on five variables. According to the regularity rule edges $\{1, 2\}$ and $\{4, 5\}$ of this vine cannot be joined by an edge in tree T_2, however this is possible for edges $\{2, 3\}$ and $\{2, 4\}$. For each edge e of a vine we define the *constraint* set U_e, the *conditioned* set $\{C_{1e}, C_{2e}\}$ and the *conditioning* set D_e of this edge as follows: the variables reachable from e are called the constraint set of this edge. When two edges are joined by an edge of the next tree, the intersection of the respective constraint sets form the conditioning set, and the symmetric difference of the constraint sets is the conditioned set. The regularity condition ensures the conditioned set to be a doubleton. In Fig. 1 symbol of the general form $\{L|K\}$ denotes a constraint set with conditioned set L and conditioning set K. The order of node e is $\#D_e$.

Two distinct subtypes of regular vines are the so-called C-vines (each tree T_i has a unique node of degree $d-i$; see Fig. 1c) and D-vines (each node in T_1 has degree at most 2, see Fig. 1d). This paper aims on employing the C-vine in further analysis to generate random correlation matrices. Theorems presented here will be illustrated on an example of a regular vine $\mathcal{V}_5$ shown in Fig. 1b.

We define two concepts allowing expressing some properties of regular vines.

Definition 1 (*m-child, m-descendent*). If node e of a regular vine is an element of node f, we say that e is an **m-child** of f; similarly, if e is reachable from f via the membership relation: $e \in e_1 \in \cdots \in f$, we say that e is an **m-descendent** of f.

A few of the properties of regular vines are (see [4]):

Property 1. There are $\binom{d}{2}$ edges in a regular vine on d variables.
Property 2. If $\mathcal{V}$ is a regular vine on d variables, then for all $i = 1, \ldots, d-1$ and all $e \in E_i$, the conditioned set associated with e is a doubleton and $\#D_e = i - 1$.
Property 3. If the conditioned sets of nodes e and f in a regular vine are equal, then $e = f$.
Property 4. For any node e in one of the trees $T_2, \ldots, T_{d-1}$ in a regular vine, if variable i is a member of the conditioned set of e, then i is a member of the conditioned set of exactly one of the m-children of e, and the conditioning set of an m-child of e is a subset of the conditioning set of e.

We add to this list one more property.

Lemma 1. *Let $e \in E_i$, $i > 1$, be the node with constraint set $\{1, \ldots, i+1\}$ and $\{s, t\} \subset D_e$. There exists $f \in E_j$, $j < i$, such that $\{C_{1f}, C_{2f}\} = \{s, t\}$.*

Proof. The cardinality of the constraint set U_e of e is $i+1$, thus there are $\binom{i+1}{2}$ distinct doubletons in this set. Note also that there are $\binom{i+1}{2}$ edges in the subvine on nodes $\{1, \ldots, i+1\}$ by Property 1. By Property 4 the conditioned sets of all m-descendants of e are subsets of the constraint set of e and by Property 3 these conditioned sets are all different. Therefore one of the m-descendants of e must have the conditioned set $\{s, t\}$. ■

As an example, Property 4 means that for node $\{35; 124\}$ of vine $\mathcal{V}_5$, variable 3 or 5 can occur only in the conditioned set of one of the m-children of this node, that is in either $\{34; 12\}$ or $\{15; 24\}$, never in both at the same time. According to Lemma 1 there should be three m-descendants of node $\{35; 124\}$ with conditioned sets being doubleton subsets of its conditioning set $\{124\}$. These are nodes $\{12\}$, $\{24\}$ and $\{14; 2\}$.

2.1. Partial and multiple correlations

We define two concepts crucial for vine method of generating random correlation matrices, namely the partial and the multiple correlation.

Definition 2 (*Partial Correlation*). The partial correlation of random variables X_1 and X_2 with $X_3, \ldots, X_d$ held constant is

$$\rho_{12;3,\ldots,d} = -\frac{C_{12}}{\sqrt{C_{11}C_{22}}},$$

where $C_{i,j}$ denotes the (i, j)th cofactor of the d-dimensional correlation matrix $\mathbf{R}$; that is, the determinant of the submatrix obtained by removing row i and column j and multiplied by $(-1)^{i+j}$. By permuting indices, other partial correlations in d variables are defined.

The partial correlation $\rho_{12;3,\ldots,d}$ can be interpreted as the correlation between the orthogonal projections of random variables X_1 and X_2 on the plane orthogonal to the space spanned by $X_3, \ldots, X_d$.

Partial correlations can be calculated recursively with the following formula [5]

$$\rho_{ij;kL} = \frac{\rho_{ij;L} - \rho_{ik;L}\rho_{jk;L}}{\sqrt{(1-\rho^2_{ik;L})(1-\rho^2_{jk;L})}}, \tag{2}$$

where L is a set of indices, possibly empty, distinct from $\{i, j, k\}$. Partial correlations can be assigned to the edges of a regular vine, such that conditioned and conditioning sets of the edges and those of partial correlations coincide. Every such assignment uniquely parameterizes a product moment correlation matrix. One can notice that Joe's choice of partial correlations in Eq. (1) corresponds to a partial correlation specification on the D-vine (compare with Fig. 1d). However the best choice for computing ordinary product moment correlations from partial correlations is a C-vine. For example, determining ρ_{34} from $\rho_{34;12}$ in the C-vine in Fig. 1c can be done recursively in two steps with Eq. (2) solved for $\rho_{ij;L}$ as follows:

$$\text{step1:} \quad \rho_{34;1} = \rho_{34;12}\sqrt{(1-\rho^2_{23;1})(1-\rho^2_{24;1})} + \rho_{23;1}\rho_{24;1},$$

$$\text{step2:} \quad \rho_{34} = \rho_{34;1}\sqrt{(1-\rho^2_{13})(1-\rho^2_{14})} + \rho_{13}\rho_{14}.$$

Notice that only partial correlations specified in the vine appear in the formulae. This is not the case with the partial correlations specified on a D-vine.

We adopt the notation $D(\{L\})$ for the determinant of the correlation matrix with random variables indexed by the set L.

Definition 3 (*Multiple Correlation*). The multiple correlation $R_{d\{d-1,\ldots,1\}}$ of variable X_d with respect to $X_{d-1},\ldots,X_1$ is given by:

$$1-R^2_{d\{d-1,\ldots,1\}}=\frac{D(\{1,\ldots,d\})}{C_{dd}},$$

where $D(\{1,\ldots,d\})$ is the determinant of the correlation matrix $\mathbf{R}$ and C_{dd} is the (d,d) cofactor of $\mathbf{R}$. By permuting indices, other multiple correlations in d variables are defined.

The multiple correlation satisfies (see [6]):

$$\begin{aligned}1-R^2_{d\{d-1,\ldots,1\}}&=(1-R^2_{d\{d-2,\ldots,1\}})(1-\rho^2_{d,d-1;d-2,\ldots,1})\\&=(1-\rho^2_{d,1})(1-\rho^2_{d,2;1})(1-\rho^2_{d,3;2,1})\ldots(1-\rho^2_{d,d-1;d-2,\ldots,1}).\end{aligned}\tag{3}$$

The determinant of a correlation matrix for d random variables can be expressed as a product of terms involving multiple correlations [4]:

$$\begin{aligned}D(\{1,\ldots,d\})&=(1-R^2_{d\{d-1,\ldots,1\}})(1-R^2_{d-1\{d-2,\ldots,1\}})\ldots(1-R^2_{2\{1\}})\\&=(1-R^2_{d\{d-1,\ldots,1\}})D(\{1,\ldots,d-1\}).\end{aligned}\tag{4}$$

Lemma 2. *Let $i,j\notin L$.*

$$1-\rho^2_{ij;L}=\frac{D(\{i,j,L\})D(\{L\})}{D(\{i,L\})D(\{j,L\})}.$$

Proof. From Eq. (3) with permuted indices we have

$$1-\rho^2_{ij;L}=\frac{1-R^2_{i\{j,L\}}}{1-R^2_{i\{L\}}}.$$

Use Eq. (4) to simplify the terms on the right-hand side to obtain the result. This simplifies the proof of Lemma 2 in [1]. ■

2.2. Jacobian of the transformation from unconditional correlations to the set of partial correlations

We investigate the Jacobian matrix for the transform T of a vector of ordinary product moment correlations $\mathbf{Q}$ (all cells of the upper triangle part of a correlation matrix $\mathbf{R}$ arranged in a row vector form) to a vector $\mathbf{P}$ of partial correlations on a regular vine. Both of these vectors have the same length by the construction of a regular vine. The elements of $\mathbf{P}$ are

$$P_i=\rho_{C_{1i},C_{2i};D_i},\quad i=1,\ldots,\binom{d}{2}.$$

Let the partial correlations in $\mathbf{P}$ be ordered lexigraphically as follows: first order partial correlations in the top tree T_1 lexigraphically, then order partial correlations in the tree T_2 lexigraphically, and so on. Reorder the product moment correlations in $\mathbf{Q}$ correspondingly simply by removing the conditioning sets from the partial correlations. Hence for the partial correlation specification on the regular vine $\mathcal{V}_5$ we have defined subsets $\mathbf{P}^{(i)}$ and $\mathbf{Q}^{(i)}$, $i=1,2,3,4$, of $\mathbf{P}$ and $\mathbf{Q}$ respectively as

$$\begin{aligned}&\mathbf{P}^{(1)}=\{\rho_{12},\rho_{23},\rho_{24},\rho_{45}\},\qquad \mathbf{Q}^{(1)}=\{\rho_{12},\rho_{23},\rho_{24},\rho_{45}\},\\&\mathbf{P}^{(2)}=\{\rho_{13;2},\rho_{14;2},\rho_{25;4}\},\qquad \mathbf{Q}^{(2)}=\{\rho_{13},\rho_{14},\rho_{25}\},\\&\mathbf{P}^{(3)}=\{\rho_{15;24},\rho_{34;12}\},\qquad \mathbf{Q}^{(3)}=\{\rho_{15},\rho_{34}\},\\&\mathbf{P}^{(4)}=\{\rho_{35;124}\},\qquad \mathbf{Q}^{(4)}=\{\rho_{35}\}.\end{aligned}$$

This order will be advantageous for deriving the Jacobian of the transformation T in a simple form. In the following pages we derive the appropriate conditions for this transformation to ensure the joint density of product moment correlations be proportional to a power of $\det(\mathbf{R})$ with the uniform distribution as a special case.

We show the relationship between the form of the determinant of the correlation matrix and the determinant of the Jacobian [7].

Theorem 1. *Let $\mathbf{R}$ be a d-dimensional correlation matrix and $\mathbf{P}$ the corresponding vector of partial correlations on a regular vine. One has then*

$$\det(\mathbf{R})=\prod_{i=1}^{\binom{d}{2}}(1-P_i^2)=\prod_{i=1}^{\binom{d}{2}}(1-\rho^2_{C_{1i},C_{2i};D_i}).\tag{5}$$

This is an important theorem as it allows us to express the determinant of a product moment correlation matrix as a product of 1 minus squared partial correlations on any regular vine. Joe [1] provides the special case of this formula for D-vines. We show that the Jacobian of the transformation T also includes the same partial correlations as in Eq. (5).

Lemma 3. *Let $\rho_{ij;L}$ be a partial correlation of order $|L|$. There is no other partial correlation $\rho_{st;D_{st}}$ of order $|L|$ in the regular vine, such that*

$$\frac{\partial \rho_{st;D_{st}}}{\partial \rho_{ij}} \neq 0.$$

Proof. Partial derivative $\partial \rho_{st;D_{st}} / \partial \rho_{ij} \neq 0$ if and only if set $\{i, j\}$ is in the constraint set $\{s, t, D_{st}\}$. By Property 3, $\{s, t\} \neq \{i, j\}$, thus either one of the elements, i or j, must be in $\{s, t\}$ and the other in D_{st}, or both $\{i, j\} \subset D_{st}$. In case of the first situation assume without loss of generality that $s = i$ and $j \in D_{st}$. That means that one of the m-children of $\rho_{st;D_{st}}$ has constraint set $\{i, j, D_{st} \setminus \{j\}\}$. This cannot happen because of Lemma 1. The second situation when $\{i, j\} \subset D_{st}$ also cannot happen because of Property 3 and Lemma 1. ■

Theorem 2. *The Jacobian matrix $\mathbf{J}$ of the transform from $\mathbf{Q}$ to $\mathbf{P}$ has the form*

$$\mathbf{J} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A} & \mathbf{B} \end{bmatrix},$$

where $\mathbf{I}$ is the identity matrix of size $(d-1) \times (d-1)$, $\mathbf{0}$ is the matrix of 0's of size $(d-1) \times (d-1)(d-2)/2$, $\mathbf{A}$ is a rectangular matrix of size $(d-1)(d-2)/2 \times (d-1)$ and $\mathbf{B}$ is a square lower triangular matrix of size $(d-1)(d-2)/2 \times (d-1)(d-2)/2$.

Proof. Let J_{ij} denote the partial derivative of P_i with respect to Q_j. The elements P_i and Q_i are equal, $i = 1, \ldots, d-1$, and are not functions of any correlations other than themselves, and hence for $i = 1, \ldots, d-1$ and $j = 1, \ldots, d(d-1)/1$

$$J_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

This gives the identity matrix $\mathbf{I}$ and the matrix of zeros $\mathbf{0}$ as the upper parts of the Jacobian matrix. By Definition 2 an element of $\mathbf{P}^{(i)}$ is a function of product moment correlations in $\cup_{k \leq i} \mathbf{Q}^{(k)}$ only. Combining this result with Lemma 3 gives matrices $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{B}$ is lower triangular. ■

Corollary 3. *The determinant* $\det(\mathbf{J})$ *of the Jacobian matrix $\mathbf{J}$ is*

$$\det(\mathbf{J}) = \prod_{i=1}^{\binom{d}{2}} \frac{\partial P_i}{\partial Q_i}. \tag{6}$$

The proof follows from $\mathbf{B}$ being lower triangular. For $i = 1, \ldots, d-1$ the partial derivative $\partial P_i / \partial Q_i = 1$, hence the product in Eq. (6) can start from $i = d$.

2.3. Partial derivatives

We derive the expression for the partial derivative of partial correlation $\rho_{ij;L}$ with respect to its corresponding unconditional correlation ρ_{ij}.

Lemma 4. *Let L be a nonempty set with indices distinct from $\{i, j\}$. Then*

$$\frac{\partial \rho_{ij;L}}{\partial \rho_{ij}} = \frac{1}{\sqrt{1 - R^2_{i\{L\}}}\sqrt{1 - R^2_{j\{L\}}}}. \tag{7}$$

Proof. The lemma will be proved by induction. If $L = \{l\}$ then from (2) we have

$$\begin{aligned} \frac{\partial \rho_{ij;l}}{\partial \rho_{ij}} &= \frac{\partial}{\partial \rho_{ij}} \left(\frac{\rho_{ij} - \rho_{il}\rho_{jl}}{\sqrt{(1 - \rho^2_{il})(1 - \rho^2_{jl})}} \right) \\ &= \frac{1}{\sqrt{(1 - \rho^2_{il})(1 - \rho^2_{jl})}} = \frac{1}{\sqrt{(1 - R^2_{i\{l\}})(1 - R^2_{j\{l\}})}} \end{aligned}$$

and the lemma holds. Assume that Eq. (7) holds for the conditioning set L containing d nodes. Extend now the conditioning set to include $d+1$ nodes, ie. $\{k, L\}$. The corresponding partial derivative thanks to the Chain Rule and the recursive formula (2) can be expressed as

$$\frac{\partial \rho_{ij;kL}}{\partial \rho_{ij}} = \frac{\partial \rho_{ij;kL}}{\partial \rho_{ij;L}} \frac{\partial \rho_{ij;L}}{\partial \rho_{ij}} = \frac{1}{\sqrt{1-\rho^2_{ik;L}}\sqrt{1-\rho^2_{jk;L}}} \frac{1}{\sqrt{1-R^2_{i\{L\}}}\sqrt{1-R^2_{j\{L\}}}}.$$

This can be expanded further by using Lemma 2

$$\frac{\partial \rho_{ij;kL}}{\partial \rho_{ij}} = \sqrt{\frac{1-R^2_{i\{L\}}}{1-R^2_{i\{kL\}}}} \sqrt{\frac{1-R^2_{j\{L\}}}{1-R^2_{j\{kL\}}}} \frac{1}{\sqrt{1-R^2_{i\{L\}}}\sqrt{1-R^2_{j\{L\}}}}.$$

Simplifying this equation yields

$$\frac{\partial \rho_{ij;kL}}{\partial \rho_{ij}} = \frac{1}{\sqrt{1-R^2_{i\{kL\}}}\sqrt{1-R^2_{j\{kL\}}}}. \quad \blacksquare$$

Joe [1] published a similar result:

$$\frac{\partial \rho_{1d;2\ldots d-1}}{\partial \rho_{1d}} = \frac{D(\{2,\ldots,d-1\})}{\sqrt{D(\{1,\ldots,d-1\})D(\{2,\ldots,d\})}} = \frac{1}{\sqrt{1-R^2_{1\{2,\ldots,d-1\}}}\sqrt{1-R^2_{d\{2,\ldots,d-1\}}}}$$

for one specific ordering of nodes using the properties of partial correlations on a D-vine. We gave a more general proof with no reference to any specific type of vine. This lemma shows that the partial derivative $\partial \rho_{35;124}/\partial \rho_{35}$ in case of $\mathcal{V}_5$ can be expressed as

$$\begin{aligned}\frac{\partial \rho_{35;124}}{\partial \rho_{35}} &= \left((1-R^2_{3\{124\}})(1-R^2_{5\{124\}})\right)^{-\frac{1}{2}} \\ &= \left((1-\rho^2_{34;12})(1-\rho^2_{13;2})(1-\rho^2_{23}) \cdot (1-\rho^2_{15;24})(1-\rho^2_{25;4})(1-\rho^2_{45})\right)^{-\frac{1}{2}}.\end{aligned}$$

Only partial correlations specified in $\mathcal{V}_5$ appear in this product.

Lemma 5. *Suppose variable d is in the conditioned set of the top node of a regular vine. Then there is a permutation $(j_1, \ldots, j_{d-1})$ of $(1, \ldots, d-1)$ such that the product of all partial derivatives involving variable d is equal to*

$$\left[D(\{d-1,\ldots,1\}) \prod_{i=2}^{d-1}\left(1-R^2_{d\{j_{i-1},\ldots,j_1\}}\right)\right]^{-\frac{1}{2}}.$$

Proof. Let $\{d, j_{d-1}; j_{d-2}, \ldots, j_1\}$ be the constraint set of the single node e of the top most tree T_{d-1}. Collect all m-descendants of e containing variable d. By Property 4, d occurs only in the conditioned set of m-descendent nodes of e and the conditioning set of a m-child is a subset of the conditioning set of its m-parent. By Property 3, variable d occurs exactly once with every other variable $\{d-1, \ldots, 1\}$ in the conditioned set of some node. Hence there is some permutation $(j_1, \ldots, j_{d-1})$ of $(1, \ldots, d-1)$, such that in tree T_i $(i = 1, \ldots, d-1)$ there is a partial correlation associated with one of the edges of the tree with the constraint set $\{d, j_i; j_{i-1}, \ldots, j_1\}$. By Lemma 4 the product of all partial derivatives of partial correlations involving node d can be expressed as

$$\begin{aligned}\prod_{i=2}^{d-1} \frac{\partial \rho_{dj_i;j_{i-1},\ldots,j_1}}{\partial \rho_{dj_i}} &= \prod_{i=2}^{d-1}\left[1-R^2_{d\{j_{i-1},\ldots,j_1\}}\right]^{-\frac{1}{2}} \cdot \prod_{i=2}^{d-1}\left[1-R^2_{j_i\{j_{i-1},\ldots,j_1\}}\right]^{-\frac{1}{2}} \\ &= \left[\prod_{i=2}^{d-1}\left(1-R^2_{d\{j_{i-1},\ldots,j_1\}}\right) \cdot D(\{d-1,\ldots,1\})\right]^{-\frac{1}{2}},\end{aligned}$$

where the last equality follows from the definition of the multiple correlation coefficient via $1 - R^2_{j_i\{j_{i-1},\ldots,j_1\}} = D(\{j_i, j_{i-1}, \ldots, j_1\})/D(\{j_{i-1}, \ldots, j_1\})$. If $i = 1$, then $\partial \rho_{dj_i;j_{i-1},\ldots,j_1}/\partial \rho_{dj_i} = 1$ and therefore there is no need to include this term in the above product. $\blacksquare$

The determinant $D(\{d-1, \ldots, 1\})$ does not depend on any particular way of indexing of the nodes $\{d-1, \ldots, 1\}$. Let $|\mathbf{J}_d|$ denote the determinant of the Jacobian of the transform of $\mathbf{Q}$ to $\mathbf{P}$ for a regular vine on d nodes.

Lemma 6. *Suppose variable d is in the conditioned set of the top node of a regular vine. Then there is a permutation $(j_1, \ldots, j_{d-1})$ of $(1, \ldots, d-1)$ such that the recursive formula for the determinant $|\mathbf{J}_d|$ of the Jacobian for the transform of $\mathbf{Q}$ to $\mathbf{P}$ is:*

$$|\mathbf{J}_d| = \left|\mathbf{J}_{d-1}\right| \left[D(\{d-1, \ldots, 1\}) \prod_{i=2}^{d-1} \left(1 - R^2_{d\{j_{i-1},\ldots,j_1\}}\right) \right]^{-\frac{1}{2}}.$$

Proof. By Corollary 3

$$|\mathbf{J}_d| = \prod_{i=1}^{\binom{d}{2}} \frac{\partial P_i}{\partial Q_i} = \prod_{i\in\mathcal{A}} \frac{\partial P_i}{\partial Q_i} \cdot \prod_{i\in\mathcal{B}} \frac{\partial P_i}{\partial Q_i},$$

where $\mathcal{A}$ is the set of all partial correlations on a regular vine without node d in the constraint set, and $\mathcal{B}$ is the set of all partial correlations with d in the conditioned set. By Corollary 3, the first product is $\left|\mathbf{J}_{d-1}\right|$. By Lemma 5, the second product simplifies and the claimed result is obtained. ■

Next is a main theorem.

Theorem 4. *The determinant $|\mathbf{J}_d|$ of the Jacobian for the transform of $\mathbf{Q}$ to $\mathbf{P}$ is*

$$|\mathbf{J}_d| = \left[\prod_{i=1}^{\binom{d}{2}-1} (1 - \rho^2_{C_{1i},C_{2i};D_i})^{d-\#D_i-2} \right]^{-\frac{1}{2}}. \tag{8}$$

Proof. Without loss of generality, assume variable d is in the conditioned set of the top node. Let $(j_1, \ldots, j_{d-1})$ be the permutation of $(1, \ldots, d-1)$ from Lemma 5.

The proof goes by induction. For $d = 3$, the P_i for $i = 1, 2, 3$ are $\rho_{j_1 j_2}$, $\rho_{3 j_1}$ and $\rho_{3 j_2; j_1}$, respectively. We have by Lemma 6

$$|\mathbf{J}_3| = \frac{|\mathbf{J}_2|}{\sqrt{1-\rho^2_{j_1 j_2}}\sqrt{1-\rho^2_{j_1 3}}} = \frac{1}{\sqrt{1-\rho^2_{j_1 j_2}}\sqrt{1-\rho^2_{j_1 3}}}$$

and the theorem is satisfied. Assume that Eq. (8) holds for $d-1$. Then again by Lemma 6 for d we have

$$|\mathbf{J}_d| = \left|\mathbf{J}_{d-1}\right| \left[D(\{d-1, \ldots, 1\}) \prod_{i=2}^{d-1} \left(1 - R^2_{d\{j_{i-1},\ldots,j_1\}}\right) \right]^{-\frac{1}{2}}.$$

However with Theorem 1 and induction,

$$\begin{aligned} \left|\mathbf{J}_{d-1}\right| D(\{d-1, \ldots, 1\})^{-\frac{1}{2}} &= \left[\prod_{i=1}^{\binom{d-1}{2}-1} (1 - \rho^2_{C_{1i},C_{2i};D_i})^{d-\#D_i-3} \cdot \prod_{i=1}^{\binom{d-1}{2}} (1 - \rho^2_{C_{1i},C_{2i};D_i}) \right]^{-\frac{1}{2}} \\ &= \left[\prod_{i=1}^{\binom{d-1}{2}} (1 - \rho^2_{C_{1i},C_{2i};D_i})^{d-\#D_i-2} \right]^{-\frac{1}{2}}. \end{aligned} \tag{9}$$

The above product contains all terms with partial correlation from the vine on nodes $\{d-1, \ldots, 1\}$ raised to the appropriate power. There are $d-2$ terms missing in order to obtain the claimed result. These are the terms involving all partial correlations with d in the conditioned set. They are obtained from $\prod_{i=2}^{d-1} \left(1 - R^2_{d\{j_{i-1},\ldots,j_1\}}\right)$. By Eq. (3)

$$\prod_{i=2}^{d-1} \left(1 - R^2_{d\{j_{i-1},\ldots,j_1\}}\right) = \prod_{i=2}^{d-1} (1 - \rho^2_{d,j_{i-1};j_{i-2},\ldots,j_1})^{d-(i-2)-2}. \tag{10}$$

Notice that $i-2$ in the exponent is the cardinality of the conditioning set. Hence by combining Eq. (9) with (10) we prove the theorem. ■

The product in Eq. (8) contains terms with all the partial correlations assigned to the edges of a regular vine taken to the appropriate power depending on the cardinality of the conditioning set. It does not explicitly include the term with the top most partial correlation with the highest cardinality of the conditioning set, i.e., for $i = \binom{d}{2}$, but its exponent according to the formula would be 0 anyway, hence index i can go safely from 1 to $\binom{d}{2}$ in Eq. (8).

The above calculations can also be carried out in a simplified form for C-vines. Let $\mathcal{V}$ be a C-vine on d nodes with node 1 as the root of the vine. Then one can introduce a partial correlation specification on the nodes of this vine and present them in the form of a matrix:

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,d-1} & \rho_{1,d} \\ & 1 & \rho_{2,3;1} & \cdots & \rho_{2,d-1;1} & \rho_{2,d;1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & 1 & \rho_{d-1,d;1\ldots d-2} \\ & & & & & 1 \end{bmatrix}.$$

The partial derivative of $\rho_{d-1,d;1\ldots d-2}$ with respect to $\rho_{d-1,d}$ is

$$\frac{\partial \rho_{d-1,d;1\ldots d-2}}{\partial \rho_{d-1,d}} = \prod_{i=1}^{d-2} \frac{\partial \rho_{d-1,d;1\ldots i}}{\partial \rho_{d-1,d;1\ldots i-1}} = \prod_{i=1}^{d-2} \left[(1-\rho^2_{d,i;1\ldots i-1})(1-\rho^2_{d-1,i;1\ldots i-1})\right]^{-\frac{1}{2}},$$

where we assume the conditioning set $\{1, \ldots, i-1\}$ for $i = 1$ to be an empty set. For the lower order partial correlations one has

$$\frac{\partial \rho_{j,j+n;1\ldots j-1}}{\partial \rho_{j,j+n}} = \prod_{i=1}^{j-1} \left[(1-\rho^2_{j,i;1\ldots i-1})(1-\rho^2_{j+n,i;1\ldots i-1})\right]^{-\frac{1}{2}}$$

for $1 \le j \le d-1$ and $2 \le j+n \le d$.

The determinant $|\mathbf{J}_d|$ of the Jacobian for the transform of $\mathbf{Q}$ to $\mathbf{P}$ for the partial correlations on a C-vine is

$$|J_d| = \left[\prod_{k=1}^{d-2} \prod_{i=k+1}^{d} \left(1-\rho^2_{k,i;1,\ldots,k-1}\right)^{d-k-1}\right]^{-\frac{1}{2}}. \tag{11}$$

All partial correlations from the correlation matrix $\mathbf{R}$ except $\rho_{d-1,d;1,\ldots,d-2}$ appear in the expression (11). However this term can also be added safely because its exponent would be 0 ($d-k-1$, where $k = d-1$). Therefore k in the first product in (11) can increase up to $d-1$ instead of $d-2$. We make this adjustment in the subsequent calculations.

2.4. Algorithm for generating correlation matrices with vines

We show how to use the theorems to generate random correlation matrices such that the density of the random correlation matrix is invariant under the choice of partial correlation vine. Following the calculations of Joe [1] we employ the linearly transformed Beta(α, α) distribution on the interval $(-1, 1)$ to simulate partial correlations. The density g of this random variable is

$$g(x;\alpha) = \frac{2^{-2\alpha+1}}{B(\alpha,\alpha)}(1-x^2)^{\alpha-1} = \frac{2^{-2\alpha+1}\Gamma(2\alpha)}{\Gamma^2(\alpha)}(1-x^2)^{\alpha-1}, \tag{12}$$

where B is the beta function.

Suppose $\rho_{C_{1i},C_{2i};D_i}$ has a Beta(β_i, β_i) density on $(-1, 1)$ and its realization is denoted by $p_{C_{1i},C_{2i};D_i}$. Similarly, let the realization of an ordinary product moment correlation $\rho_{C_{1i},C_{2i}}$ be denoted by $q_{C_{1i},C_{2i}}$. Then the joint density f of ordinary product moment correlations in $\mathbf{R}$ is proportional to

$$f(q_{C_{1i},C_{2i}}; 1 \le i \le d(d-1)/2) \propto \prod_{j=1}^{\binom{d}{2}} g(p_{C_{1j},C_{2j};D_j}; \beta_j) \cdot |\mathbf{J}_d| = \prod_{j=1}^{\binom{d}{2}} (1-p_{C_{1j},C_{2j};D_j})^{\beta_j - \frac{d-\#D_j}{2}}. \tag{13}$$

The exponent $\beta_j - \frac{d-\#D_j}{2}$ is a function of $\#D_j = n$ for a given d. In order to make this exponent equal to a constant $\eta - 1$, β_j will be replaced by α_n so that $\alpha_n - (d-n)/2 = \eta - 1$; thus $\alpha_n = \eta + \frac{d-n-2}{2}$. We replace β_j with α_n in Eq. (13) and use Theorem 1 to obtain

$$f(q_{C_{1i},C_{2i}}; 1 \le i \le d(d-1)/2) = c_d \prod_{j=1}^{\binom{d}{2}} (1-p_{C_{1j},C_{2j};D_j})^{\eta-1} = c_d \det(\mathbf{R})^{\eta-1}, \tag{14}$$

where c_d is the normalizing constant depending on the dimension d. The uniform density is obtained for $\eta = 1$, which means that the marginal densities for partial correlations $p_{C_{1i},C_{2i};D_i}$ are Beta $\left(\frac{d-\#D_i}{2}, \frac{d-\#D_i}{2}\right)$ on $(-1, 1)$, for $i = 1, \ldots, d(d-1)/2$,

For the C-vine the above reasoning has the following implications. By Eq. (11) the joint density f of the ordinary product moment correlations is

$$f(q_{ij}, 1 \le i < j \le d) = c_d \prod_{k=1}^{d-1} \prod_{l=k+1}^{d} (1 - p_{kl;1,\ldots,k-1}^2)^{\alpha_{k-1}-1-\frac{d-k-1}{2}}. \tag{15}$$

The exponent $\alpha_{k-1} - 1 - \frac{d-k-1}{2}$ is of the form $\beta_j - \frac{d-\#D_j}{2}$ as in Eq. (13) with $\#D_j = k-1$. Thus the density (15) is uniform if $\alpha_{k-1} = \frac{d-k+1}{2}$ and the marginal densities for partial correlations $\rho_{kl;1,\ldots,k-1}$ $(1 \le k \le d-1$ and $k+1 \le l \le d)$ in the matrix $\mathbf{R}$ are Beta $\left(\frac{d-k+1}{2}, \frac{d-k+1}{2}\right)$ on $(-1, 1)$.

The normalizing constant c_d for Eqs. (14) and (15) has the same formula as the one derived in [1] since it does not depend on the specific vine used in the calculations

$$c_d = 2^{\sum_{k=1}^{d-1}(2\eta-2+d-k)(d-k)} \prod_{k=1}^{d-1} \left[B\left(\eta + \frac{1}{2}(d-k-1), \eta + \frac{1}{2}(d-k-1)\right) \right]^{d-k}. \tag{16}$$

If $\eta = 1$ this equation simplifies to

$$2^{\sum_{k=1}^{d-1} k^2} \cdot \prod_{k=1}^{d-1} \left[B\left(\frac{k+1}{2}, \frac{k+1}{2}\right) \right]^k.$$

We denote the realization of random matrix $\mathbf{R}$ by $\mathbf{r}$. Elements of $\mathbf{r}$ are r_{ij}, $1 \le i, j \le d$. The algorithm for generating correlation matrices with density proportional to $[\det(\mathbf{r})]^{\eta-1}$, $\eta > 1$ is quite simple using the vine method based on a C-vine.

1. Initialization $\beta = \eta + (d-1)/2$.
2. Loop for $k = 1, \ldots, d-1$.
   (a) $\beta \leftarrow \beta - \frac{1}{2}$;
   (b) Loop for $i = k+1, \ldots, d$;
      (i) generate $p_{k,i;1,\ldots,k-1} \sim \text{Beta}(\beta, \beta)$ on $(-1, 1)$;
      (ii) use recursive formula (2) on $p_{k,i;1,\ldots,k-1}$ to get $q_{k,i} = r_{k,i} = r_{i,k}$.
3. Return $\mathbf{r}$, a $d \times d$ correlation matrix.

Because the partial correlations in a regular vine can independently take values in the interval $(-1, 1)$, one could more generally assign an arbitrary density g_i, supported on $(-1, 1)$, to $\rho_{C_{1i},C_{2i};D_i}$, and get a joint density for the correlation matrix by multiplying $\prod_{i=1}^{\binom{d}{2}} g_i(p_{C_{1i},C_{2i};D_i})$ by the Jacobian. This density in general is not invariant under the choice of partial correlation vine, but by choosing the vine and the g_i appropriately, one could get random correlation matrices that have larger correlations at a few particular pairs.

3. Onion method

Another interesting method of sampling uniformly from the set correlation matrices was the method proposed in [3]. We give a simpler explanation of their method, together with an extension to random correlation matrices with density proportional to $[\det(\mathbf{r})]^{\eta-1}$ for $\eta > 0$. With the derivation, we check that the normalization constant is the same as that given in [1].

3.1. Background results

We start with some background results on the elliptically contoured distributions. Consider the spherical density $c(1 - \mathbf{w}^{\mathrm{T}}\mathbf{w})^{\beta-1}$ for $\mathbf{w} \in \mathbb{R}^k$, $\mathbf{w}^{\mathrm{T}}\mathbf{w} \le 1$, where c is the normalizing constant. If $\mathbf{W}$ has this density, then it has the stochastic representation $\mathbf{W} = V\mathbf{U}$ where $V^2 \sim \text{Beta}(k/2, \beta)$ and $\mathbf{U}$ is uniform on the surface of the k-dimensional hypersphere. If $\mathbf{Z} = \mathbf{A}\mathbf{W}$, where $\mathbf{A}$ is a $k \times k$ nonsingular matrix, then the density of $\mathbf{Z}$ is

$$c[\det(\mathbf{A}\mathbf{A}^{\mathrm{T}})]^{-1/2}(1 - \mathbf{z}^{\mathrm{T}}[\mathbf{A}\mathbf{A}^{\mathrm{T}}]^{-1}\mathbf{z})^{\beta-1}$$

over $\mathbf{z}$ such that $\mathbf{z}^{\mathrm{T}}[\mathbf{A}^{\mathrm{T}}\mathbf{A}]^{-1}\mathbf{z} \le 1$.

Lemma 7. *The normalization constant c of the spherically contoured density $c(1 - \mathbf{w}^{\mathrm{T}}\mathbf{w})^{\beta-1}$ is*

$$c = \Gamma(\beta + k/2)\pi^{-k/2}/\Gamma(\beta).$$

Proof. From known results on elliptical densities (e.g. [8], page 129), the density of the radial direction V is

$$cS_k(1-v^2)^{\beta-1}v^{k-1}, \quad 0<v<1,$$

where $S_k = 2\pi^{k/2}/\Gamma(k/2)$. The density of $Y=V^2$ is

$$cS_k(1-y)^{\beta-1}y^{(k-1)/2}\cdot\frac{1}{2}y^{-1/2} = \frac{1}{2}cS_k y^{k/2-1}(1-y)^{\beta-1}, \quad 0<y<1.$$

This is a Beta$(k/2,\beta)$ density, so that

$$\frac{1}{2}cS_k = \frac{\Gamma(\beta+k/2)}{\Gamma(k/2)\,\Gamma(\beta)} \quad \text{or } c = \frac{\Gamma(\beta+k/2)}{\pi^{k/2}\Gamma(\beta)}. \quad \blacksquare$$

The onion method is based on the fact that any correlation matrix of size $(k+1)\times(k+1)$ can be partitioned as

$$\mathbf{r}_{k+1} = \begin{bmatrix}\mathbf{r}_k & \mathbf{z}\\ \mathbf{z}^{\mathrm{T}} & 1\end{bmatrix},$$

where $\mathbf{r}_k$ is an $k\times k$ correlation matrix and $\mathbf{z}$ is a k-vector of correlations. From standard results on conditional multivariate normal distributions we have $\det(\mathbf{r}_{k+1}) = \det(\mathbf{r}_k)\cdot\left(1-\mathbf{z}^{\mathrm{T}}\mathbf{r}_k^{-1}\mathbf{z}\right)$. Let the upper case letter of $\mathbf{r}_k$, $\mathbf{z}$, $\mathbf{r}_{k+1}$ denote random vectors and matrices and let $\beta,\beta_k>0$ be two known parameters. If $\mathbf{R}_k$ has density proportional to $[\det(\mathbf{r}_k)]^{\beta_k-1}$, and $\mathbf{Z}$ given $\mathbf{R}_k=\mathbf{r}_k$ has density proportional to $[\det(\mathbf{r}_k)]^{-1/2}(1-\mathbf{z}^{\mathrm{T}}\mathbf{r}_k^{-1}\mathbf{z})^{\beta-1}$ (hence it is elliptically contoured), then the density of $\mathbf{R}_{k+1}$ is proportional to $[\det(\mathbf{r}_k)]^{\beta_k-3/2}(1-\mathbf{z}^{\mathrm{T}}\mathbf{r}_k^{-1}\mathbf{z})^{\beta-1}$. If one sets $\beta_k=\beta+\frac{1}{2}$, then the density of $\mathbf{R}_{k+1}$ is proportional to $[\det(\mathbf{r}_{k+1})]^{\beta-1}$.

Because the density in Eq. (12) is proportional to $(1-u^2)^{\alpha-1}$, which is a power of $\det\begin{pmatrix}1 & u\\ u & 1\end{pmatrix} = 1-u^2$, it can be used to generate $\mathbf{r}_2$.

3.2. Algorithm for generating random correlation matrices

Combining the above results allows to provide the following algorithm for the extended onion method to get random correlation matrices in dimension d with density proportional to $[\det(\mathbf{r})]^{\eta-1}$, $\eta>1$

1. Initialization. $\beta=\eta+(d-2)/2$, $r_{12}\leftarrow 2u-1$, where $u\sim$ Beta(β,β), $\mathbf{r}\leftarrow\begin{pmatrix}1 & r_{12}\\ r_{12} & 1\end{pmatrix}$.
2. Loop for $n=2,\ldots,d-1$.
   (a) $\beta\leftarrow\beta-\frac{1}{2}$;
   (b) generate $y\sim$ Beta$(k/2,\beta)$;
   (c) generate $\mathbf{u}=(u_1,\ldots,u_k)^{\mathrm{T}}$ uniform on the surface of k-dimensional hypersphere;
   (d) $\mathbf{w}\leftarrow y^{1/2}\mathbf{u}$, obtain $\mathbf{A}$ such that $\mathbf{A}\mathbf{A}^{\mathrm{T}}=\mathbf{r}$, set $\mathbf{z}\leftarrow\mathbf{A}\mathbf{w}$;
   (e) $\mathbf{r}\leftarrow\begin{bmatrix}\mathbf{r} & \mathbf{z}\\ \mathbf{z}^{\mathrm{T}} & 1\end{bmatrix}$.
3. Return $\mathbf{r}$, a $d\times d$ correlation matrix.

In step (c), it should be numerically faster to use $\mathbf{A}$ from the Cholesky decomposition of $\mathbf{r}$ rather than $\mathbf{r}^{1/2}$ based on the singular value decomposition. The latter is indicated in [3].

3.3. Derivation of the normalizing constant

As in case of the vine method, every off-diagonal element of the random correlation matrix $\mathbf{R}$ has a marginal density Beta$(\eta+[d-2]/2,\ \eta+[d-2]/2)$ on $(-1,1)$. For the special case of $\eta=1$ leading to uniform over the space of correlation matrices, the marginal density of every correlation is Beta$(d/2,d/2)$ on $(-1,1)$.

In the kth step of the algorithm, $\beta=\eta+[d-1-k]/2$. Using Lemma 7 and Eq. (12), the reciprocal normalizing constant is

$$\begin{aligned} c_d{}' &= 2^{2\eta+d-3}\frac{\Gamma^2(\eta+\frac{d}{2}-1)}{\Gamma(2\eta+d-2)}\prod_{k=2}^{d-1}\frac{\pi^{\frac{k}{2}}\Gamma(\eta+\frac{d-1-k}{2})}{\Gamma(\eta+\frac{d-1-k}{2}+\frac{k}{2})}\\ &= 2^{2\eta+d-3}\frac{\Gamma^2(\eta+\frac{d}{2}-1)}{\Gamma(2\eta+d-2)}\prod_{k=2}^{d-1}\frac{\pi^{\frac{k}{2}}\Gamma(\eta+\frac{d-1-k}{2})}{\Gamma(\eta+\frac{d-1}{2})}. \end{aligned} \tag{17}$$

We show that the expressions for the normalizing constants (16) and (17) are equivalent. The proof makes use of the duplication formula (Legendre relation, [9])

$$\frac{\Gamma(2t)}{\Gamma(t)} = 2^{(2t-1)}\frac{\Gamma(t+\frac{1}{2})}{\Gamma(\frac{1}{2})} \Longrightarrow \frac{\Gamma^2(t)}{\Gamma(2t)}2^{2t-1} = \frac{\pi^{\frac{1}{2}}\Gamma(t)}{\Gamma(t+\frac{1}{2})}. \tag{18}$$

Proof. We start with Eq. (16). By the duplication formula (18) with $t = \eta + (d-1-k)/2$ we have

$$c_d = \prod_{k=1}^{d-1}\left[2^{2(\eta+(d-1-k)/2)-1}\frac{\Gamma^2(\eta+\frac{d-1-k}{2})}{\Gamma(2\eta+d-1-k)}\right]^{d-k} = \prod_{k=1}^{d-1}\left[\frac{\pi^{\frac{1}{2}}\Gamma(\eta+\frac{d-1-k}{2})}{\Gamma(\eta+\frac{d-1-k}{2}+\frac{1}{2})}\right]^{d-k}$$

$$= \frac{\prod_{k=1}^{d-1}\pi^{\frac{k}{2}}}{\Gamma^{d-1}\left(\eta+\frac{d-1}{2}\right)}\prod_{k=1}^{d-1}\Gamma^{d-k}\left(\eta+\frac{d-1-k}{2}\right)\cdot\prod_{k=2}^{d-1}\Gamma^{-(d-k)}\left(\eta+\frac{d-1-k}{2}+\frac{1}{2}\right)$$

$$= \frac{\prod_{k=1}^{d-1}\pi^{\frac{k}{2}}}{\Gamma^{d-1}\left(\eta+\frac{d-1}{2}\right)}\prod_{k=1}^{d-1}\Gamma\left(\eta+\frac{d-1-k}{2}\right) = \prod_{k=1}^{d-1}\frac{\pi^{\frac{k}{2}}\Gamma(\eta+\frac{d-1-k}{2})}{\Gamma(\eta+\frac{d-1}{2})}.$$

This is the expression for $c_d{}'$ with

$$2^{2\eta+d-3}\frac{\Gamma^2(\eta+\frac{d}{2}-1)}{\Gamma(2\eta+d-2)} = \frac{\pi^{\frac{1}{2}}\Gamma(\eta+\frac{d}{2}-1)}{\Gamma(\eta+\frac{d}{2}-\frac{1}{2})} = \frac{\pi^{\frac{k}{2}}\Gamma(\eta+\frac{d-1-k}{2})}{\Gamma(\eta+\frac{d-1}{2})}$$

where $k = 1$. ■

The expression for the normalizing constant can be further simplified for $\eta = 1$.

Theorem 5. *If $\eta = 1$ then the normalizing constant c_d can be expressed as*

$$c_d = \begin{cases} \pi^{(d^2-1)/4}\dfrac{\prod_{k=1}^{(d-1)/2}\Gamma(2k)}{2^{(d-1)^2/4}\,\Gamma^{d-1}\left(\frac{d+1}{2}\right)}, & \textit{if } d \textit{ is odd;}\\[2ex] \pi^{d(d-2)/4}\dfrac{2^{(3d^2-4d)/4}\Gamma^d\left(\frac{d}{2}\right)\prod_{k=1}^{(d-2)/2}\Gamma(2k)}{\Gamma^{d-1}(d)}, & \textit{if } d \textit{ is even.}\end{cases}$$

Proof. We rearrange terms in Eq. (17) with $\eta = 1$:

$$c_d' = \frac{\pi^{d(d-1)/4}}{\Gamma^{d-1}(\frac{d+1}{2})}\prod_{k=1}^{d-1}\Gamma\left(\frac{d-k+1}{2}\right) = \frac{\pi^{d(d-1)/4}}{\Gamma^{d-1}(\frac{d+1}{2})}\prod_{k=1}^{d-1}\Gamma\left(\frac{k}{2}+\frac{1}{2}\right). \tag{19}$$

If d is odd then by using the duplication formula (18) we obtain

$$\prod_{k=1}^{d-1}\Gamma\left(\frac{k}{2}+\frac{1}{2}\right) = \prod_{k=1}^{(d-1)/2}\Gamma(k)\,\Gamma\left(k+\frac{1}{2}\right)$$

$$= \prod_{k=1}^{(d-1)/2}\Gamma(k)\,\frac{\Gamma(2k)\,\pi^{\frac{1}{2}}}{\Gamma(k)\,2^{2k-1}} = \frac{\pi^{(d-1)/4}}{2^{\sum_{k=1}^{(d-1)/2}2k-1}}\prod_{k=1}^{(d-1)/2}\Gamma(2k). \tag{20}$$

Substituting Eq. (20) into Eq. (19) yields the claimed result. If d is even then

$$\prod_{k=1}^{d-1}\Gamma\left(\frac{k}{2}+\frac{1}{2}\right) = \Gamma\left(\frac{d}{2}\right)\prod_{k=1}^{(d-2)/2}\Gamma(k)\,\Gamma\left(k+\frac{1}{2}\right) = \frac{\Gamma\left(\frac{d}{2}\right)\pi^{(d-2)/4}}{2^{\sum_{k=1}^{(d-2)/2}2k-1}}\prod_{k=1}^{(d-2)/2}\Gamma(2k). \tag{21}$$

Substitute Eq. (21) into Eq. (19) gives

$$c_d' = \frac{\pi^{(d^2-2)/4}\,\Gamma\left(\frac{d}{2}\right)}{2^{(d-2)^2/4}\,\Gamma^{d-1}\left(\frac{d+1}{2}\right)}\prod_{k=1}^{(d-2)/2}\Gamma(2k).$$

Apply the duplication formula to $\Gamma^{d-1}(\frac{d+1}{2})$ and cancel common terms to obtain the final result. ■

All arguments of the gamma functions in the formulae presented in Theorem 5 are integers and hence can be replaced with factorials. Note that the exponent of π in Theorem 5 for an odd number d is the same as that for the next largest even number; for $d = 3, 4, \ldots$, the exponents are respectively $2, 2, 6, 6, 12, 12, 20, 20, \ldots$.

Table 1
Time in seconds required to generate 5000 correlation matrices of given dimension.

Dimension	Compiled C code with full optimization enabled (/0x)			*m*-script in Matlab 2007b		
	Onion	*C*-vine	*D*-vine	Onion	*C*-vine	*D*-vine
5	0.015	0.016	0.031	1.42	0.775	1.28
10	0.047	0.078	0.172	3.36	1.81	6.06
15	0.109	0.234	0.547	5.35	3.08	15.5
20	0.187	0.406	1.49	7.40	4.68	30.8
25	0.281	0.687	3.25	9.59	6.80	53.4
30	0.437	1.08	6.63	11.9	9.35	85.0
35	0.609	1.56	12.3	14.4	12.6	127
40	0.813	2.20	21.4	17.0	16.7	183
45	1.06	4.13	35.3	19.9	21.5	253
50	1.34	3.89	55.5	22.8	27.2	341
60	2.09	6.27	124	29.5	41.8	577
70	3.08	9.38	246	47.1	84.8	918
80	4.33	13.4	451	82.4	46.4	1404

4. Computational time analysis

Both the vine method and the onion method have been implemented in computer software and compared in terms of time required to generate a given number of random correlation matrices. Two different software platforms were used for this task, namely the scripting language of Matlab and a low level programming language C. We used the built-in functions of Matlab to generate Beta and Gaussian distributed random variables and to compute the Cholesky decomposition of correlation matrices required by the onion method. These functions of Matlab are compiled and cannot be edited. The onion method implemented in Matlab computes the full Cholesky decomposition at each iteration of the generating procedure. However the amount of calculations can be limited by implementing a Cholesky decomposition computed incrementally — that is a new row is added at each stage when a new $\mathbf{z}$ is generated. We took this approach when implementing the onion method in C; without the incremental Cholesky decomposition, the onion method was much slower than the vine method in the C programming language. It does not save any computational time in Matlab compared to the built-in Cholesky decomposition function because the advantage of having fewer operations is wasted on executing a noncompiled code. The programs have been run on a desktop computer with Intel Core 2 Duo (2×3.2 GHz) processor, 3 GB of RAM memory and Windows XP SP3 operating system. The source code of the software used for the analysis is available from the authors upon request.

Table 1 lists times necessary to complete the task of generating 5000 random correlation matrices of given dimension. The compiled code is faster as expected and the incremental Cholesky decomposition routine allows the onion method to be the clear winner in this case. The difference between the onion method and the vine method in terms of the required calculation time gets bigger as the dimension increases. We can see a different picture on the Matlab 2007b platform. The vine method is faster than the onion method for lower dimensions of correlation matrices ($d < 44$), but our tests showed that this also depends on the processor used for calculations. We have included the results for the vine method based on the *D*-vine for reference. Clearly, the *C*-vine-based method of generating correlation matrices performs better in terms of the execution time by a large margin.

5. Conclusions

The main goal of this paper was to study and improve existing methods of generating random correlation matrices. Two of such methods include the onion method of Ghosh and Henderson [3] and the vine method recently proposed by Joe [1]. Originally the vine method was based on the so-called *D*-vine. We extend this methodology to any regular vine with studying also the relationship between the multiple correlation and partial correlations on a regular vine. Computational advantage for generating random correlation matrices exhibits the *C*-vine, since the recursive formula (2) operates only on partial correlations that are already specified on a vine. It is the only vine with this property. This simplifies the generating algorithm and limits the number of necessary calculations.

We also give a simpler explanation of the onion method in terms of elliptical distributions. The generalization of this method yields a procedure to sample from the set of positive definite correlation matrices with joint densities of correlations proportional to $\det(\mathbf{r})^{\eta-1}$ with $\eta > 0$. This allows the choice of the method suited to the need. The efficiency of the algorithms for generating random correlation matrices depends heavily on programming language used for implementation. Preferably both methods would be implemented and benchmarked before the final decision is made on the usage of one or another, however the onion method with some heavy optimizations (like incremental Cholesky decomposition) seems to have an edge in this regard.

For the vine method, a particular regular vine should be used if the partial correlations associated with this vine are of main interest (i.e., the sequence of conditioning is most natural for the variables) and they are needed as part of the generation of the random correlation matrix.

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