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Generating random correlation matrices based on partial correlations

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Abstract

A d-dimensional positive definite correlation matrix $R = (\rho_{ij})$ can be parametrized in terms of the correlations $\rho_{i,i+1}$ for $i = 1, \ldots, d-1$, and the partial correlations $\rho_{ij|i+1,\ldots j-1}$ for $j - i \geqslant 2$. These $\binom{d}{2}$ parameters can independently take values in the interval $(-1, 1)$. Hence we can generate a random positive definite correlation matrix by choosing independent distributions F_{ij}, $1 \leqslant i < j \leqslant d$, for these $\binom{d}{2}$ parameters. We obtain conditions on the F_{ij} so that the joint density of (ρ_{ij}) is proportional to a power of $\det(R)$ and hence independent of the order of indices defining the sequence of partial correlations. As a special case, we have a simple construction for generating R that is uniform over the space of positive definite correlation matrices. As a byproduct, we determine the volume of the set of correlation matrices in $\binom{d}{2}$-dimensional space. To prove our results, we obtain a simple remarkable identity which expresses $\det(R)$ as a function of $\rho_{i,i+1}$ for $i = 1, \ldots, d-1$, and $\rho_{ij|i+1,\ldots j-1}$ for $j - i \geqslant 2$.

AMS 1991 subject classification: 62H20; 62E15

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1. Introduction

The main purpose of this paper is to propose new methods of generating a random d-dimensional correlation matrix $R = (\rho_{ij})$ based on the parametrization in terms of the

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correlations $\rho_{i,i+1}$ for $i = 1, \ldots, d-1$, and the partial correlations $\rho_{ij|i+1,\ldots j-1}$ for $j-i \geqslant 2$. With our approach, by specifying univariate densities for the $\rho_{i,i+1}$ and $\rho_{ij|i+1,\ldots j-1}$, and assuming these $\binom{d}{2}$ variables to be independent, we can obtain the joint density of $\{\rho_{ij} : 1 \leqslant i < j \leqslant d\}$ with the derivation of the determinant of the Jacobian of the transform of correlations to partial correlations. This approach means that the $\{\rho_{ij}\}$ are determined sequentially. Under some conditions, the joint density of $\{\rho_{ij}\}$ is proportional to a power of $\det(R)$ and hence independent of the order of indices defining the sequence of partial correlations. The uniform density over the space of positive definite correlation matrices is a special case.

Our approach of generating random correlation matrices is easy to implement in any mathematical or statistical software with a programming language. It has an advantage over existing methods in that we can get some exact distribution theory. Previous research in this area include the papers of Chalmers [4], Bendel and Mickey [2], Johnson and Welch [9], Marsaglia and Olkin [11], Lin and Bendel [10], and Holmes [5]. The main approach has a combination of random eigenvalues and random (orthogonal) matrices, or a random T matrix leading to $T'T$. In a Bayesian context, Brown et al. [3] have a sequential approach to generate a population covariance matrix; this is a previous method with some analogies to our approach.

We had several motivations to look into random correlation matrices. One application is for generation of random clusters for studying clustering methods (see [12]). Another application is for studying the performance of two-stage estimation methods [7] for something like the multivariate probit model; we might want the average relative efficiency over random correlation matrices. Johnson and Welch [9] used random correlation matrices for studying subset selection in multiple regression, and Bendel and Mickey mention their use in sampling experiments.

We summarize our main results in Section 2, and give the proofs in Section 3. To prove our results, we obtain a simple remarkable identity which expresses $\det(R)$ as a function of $\rho_{i,i+1}$ for $i = 1, \ldots, d-1$, and $\rho_{ij|i+1,\ldots j-1}$ for $j - i \geqslant 2$, see (2.2) in Theorem 1. Another byproduct is a simple formula for the volume of the set of correlation matrices in $\binom{d}{2}$-dimensional space.

2. Main results

Let $R = (\rho_{ij})$ be a d-dimensional positive definite correlation matrix. The diagonal elements are all 1. R can be parametrized in terms of $\rho_{i,i+1}$ for $i = 1, \ldots, d-1$ and the partial correlations $\rho_{j,j+k|j+1\ldots j+k-1}$ for $j = 1, \ldots, d-k$, $k = 2, \ldots, d-1$. We could also write $\rho_{j,j+1} = \rho_{j,j+k|j+1\ldots j+k-1}$ with $k = 1$ since the indices $j+1, \ldots, j+k-1$ form an empty set if $k = 1$.

From Anderson [1, p. 80], the partial correlation $\rho_{j,j+k|j+1,\ldots,j+k-1}$ $(2 \leqslant k \leqslant d-1)$ is

$$\frac{\rho_{j,j+k} - \mathbf{r}_1^T(j,k)(R_2(j,k))^{-1}\mathbf{r}_3(j,k)}{\left[1 - \mathbf{r}_1^T(j,k)(R_2(j,k))^{-1}\mathbf{r}_1(j,k)\right]^{1/2}\left[1 - \mathbf{r}_3^T(j,k)(R_2(j,k))^{-1}\mathbf{r}_3(j,k)\right]^{1/2}},$$

where

$$R[j:j+k]=\begin{pmatrix}1 & \mathbf{r}_1^T(j,k) & \rho_{j,j+k}\\ \mathbf{r}_1(j,k) & R_2(j,k) & \mathbf{r}_3(j,k)\\ \rho_{j+k,j} & \mathbf{r}_3^T(j,k) & 1\end{pmatrix},$$

with $\mathbf{r}_1^T(j,k) = (\rho_{j,j+1},\ldots,\rho_{j,j+k-1})$, $\mathbf{r}_3^T(j,k) = (\rho_{j+k,j+1},\ldots,\rho_{j+k,j+k-1})$, and $R_2(j,k)$ consisting of the middle $k-1$ rows and columns of $R[j:j+k]$. This leads to the equality on $\rho_{j,j+k}$:

$$\rho_{j,j+k}=\mathbf{r}_1^T(j,k)(R_2(j,k))^{-1}\mathbf{r}_3(j,k)+\rho_{j,j+k|j+1,\ldots,j+k-1}D_{jk},$$

where

$$D_{jk}^2=\left[1-\mathbf{r}_1^T(j,k)(R_2(j,k))^{-1}\mathbf{r}_1(j,k)\right]\left[1-\mathbf{r}_3^T(j,k)(R_2(j,k))^{-1}\mathbf{r}_3(j,k)\right].$$

These $d-1$ correlations and $(d-1)(d-2)/2$ partial correlations can independently vary in the interval $(-1,1)$. Hence to generate a random correlation matrix, one could generate $\rho_{j,j+k|j+1\ldots j+k-1}$ $(1\leqslant k\leqslant d-1)$ independently in the interval $(-1,1)$ and then transform to get the $\rho_{j,j+k}$ for $2\leqslant k\leqslant d-1$. If the density of $\rho_{j,j+k|j+1\ldots j+k-1}$ is $g_{j,j+k}$, then the joint density of the $\{\rho_{ij}:1\leqslant i<j\leqslant d\}$ (with constraint of positive definiteness) is

$$f_d(r_{ij},1\leqslant i<j\leqslant d)=\prod_{k=1}^{d-1}\prod_{j=1}^{d-k}g_{j,j+k}(r_{j,j+k|j+1\ldots j+k-1})\times|J_d|, \tag{2.1}$$

where $|J_d|$ is the determinant of the Jacobian of $\{\rho_{j,j+k|j+1\ldots j+k-1},\ j=1,\ldots,d-k,\ k=1,\ldots,d-1\}$ with respect to $\{\rho_{ij},1\leqslant i<j\leqslant d\}$ (see Theorem 4 below). In (2.1), we use r_{ij} as the arguments of the density because the ρ_{ij} are random variables.

To obtain a simple form for the Jacobian, a theorem with an identity of $\det(R)$ in terms of $\rho_{j,j+k|j+1\ldots j+k-1}$ $(1\leqslant k\leqslant d-1)$ is used. In this section, we will state the results with conditions on the densities $g_{j,j+k}$ so that the joint density f_d does not depend on the order of indexing of the variables in the correlation matrix. The results will be given for general d and then sometimes restated for $d=3,4$ so that the pattern is clearer.

Theorem 1.

$$\det(R)=\prod_{i=1}^{d-1}(1-\rho_{i,i+1}^2)\times\prod_{k=2}^{d-1}\prod_{j=1}^{d-k}(1-\rho_{j,j+k|j+1\ldots j+k-1}^2). \tag{2.2}$$

Remark. Note that for $d=3$ and $d=4$, (2.2) becomes respectively

$$\det(R)=(1-\rho_{12}^2)(1-\rho_{23}^2)(1-\rho_{13|2}^2)$$

and

$$\det(R)=(1-\rho_{12}^2)(1-\rho_{23}^2)(1-\rho_{34}^2)(1-\rho_{13|2}^2)(1-\rho_{24|3}^2)(1-\rho_{14|23}^2).$$

Of course, (2.2) is also valid for $d=2$ with $\det(R)=(1-\rho_{12}^2)$.

Now we introduce some notation and state some other lemmas needed to prove Theorem 1 and to derive $|J_d|$ in a simple form. For a subset L of $\{1, \ldots, d\}$, let $R[L] = (\rho_{ij})_{i,j\in L}$ be the subcorrelation matrix with indices in L, and let $D(L)$ be the determinant of $R[L]$. If i, j, k are indices not in L, then

$$R[i, j, k|L] \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \rho_{ij|L} & \rho_{ik|L} \\ \rho_{ij|L} & 1 & \rho_{jk|L} \\ \rho_{ik|L} & \rho_{jk|L} & 1 \end{pmatrix}$$

and $R[i, j|L]$, $R[i, k|L]$, $R[j, k|L]$ are principal 2×2 submatrices of $R[i, j, k|L]$.

Lemma 2. *Let i, j, k, be distinct integers in $1, \ldots, d$ and let L be a subset of $\{1, \ldots, d\}\backslash\{i, j, k\}$. Then*

$$1 - \rho^2_{ij|kL} = \frac{D(\{i, j, k, L\})\, D(\{k, L\})}{D(\{i, k, L\})\, D(\{j, k, L\})}.$$

Lemma 3. *The partial derivative of $\rho_{1d|2\ldots d-1}$ with respect to ρ_{1d} is*

$$\Big[(1-\rho^2_{12})(1-\rho^2_{d-1,d})(1-\rho^2_{13|2})(1-\rho^2_{d-2,d|d-1})(1-\rho^2_{14|23})(1-\rho^2_{d-3,d|d-2,d-1})\cdots (1-\rho^2_{1,d-1|2\ldots d-2})(1-\rho^2_{2,d|3\ldots d-1})\Big]^{-1/2}.$$

That is,

$$\frac{\partial\rho_{1d|2\ldots d-1}}{\partial\rho_{1d}} = \prod_{k=1}^{d-2}\Big[(1-\rho^2_{1,1+k|2\ldots k})(1-\rho^2_{d-k,d|d-k+1\ldots d-1})\Big]^{-1/2}. \tag{2.3}$$

By shifting indices,

$$\frac{\partial\rho_{j,j+m|j+1\ldots j+m-1}}{\rho_{j,j+m}} = \prod_{k=1}^{m-1}\Big[(1-\rho^2_{j,j+k|j+1\ldots j+k-1}) \times(1-\rho^2_{j+m-k,j+m|j+m-k+1\ldots j+m-1})\Big]^{-1/2}.$$

Remark. Note that for $d = 3$ and $d = 4$, (2.3) becomes respectively

$$\Big[(1-\rho^2_{12})(1-\rho^2_{23})\Big]^{-1/2}$$

and

$$\Big[(1-\rho^2_{12})(1-\rho^2_{34})(1-\rho^2_{13|2})(1-\rho^2_{24|3})\Big]^{-1/2}.$$

Theorem 4. *The determinant* $|J_d|$ *of the Jacobian for the transform of* $(\rho_{12}, \rho_{23}, \rho_{13}, \rho_{34}, \rho_{24}, \rho_{14}, \rho_{45}, \ldots, \rho_{1d})$ *to* $(\rho_{12}, \rho_{23}, \rho_{13|2}, \rho_{34}, \rho_{24|3}, \rho_{14|23}, \rho_{45}, \ldots, \rho_{1d|2\ldots d-1})$ *is*

$$\left[\prod_{i=1}^{d-1}(1-\rho_{i,i+1}^2)^{d-2} \times \prod_{k=2}^{d-2}\prod_{i=1}^{d-k}(1-\rho_{i,i+k|i+1\ldots i+k-1}^2)^{d-1-k}\right]^{-1/2}. \tag{2.4}$$

Note that only $(1-\rho_{1d|2\ldots d-1}^2)$ *is not in the above product.*

Remark. Note that for $d = 3$ and $d = 4$, (2.4) becomes respectively

$$\left[(1-\rho_{12}^2)(1-\rho_{23}^2)\right]^{-1/2}$$

and

$$\left[(1-\rho_{12}^2)^2(1-\rho_{23}^2)^2(1-\rho_{34}^2)^2(1-\rho_{13|2}^2)(1-\rho_{24|3}^2)\right]^{-1/2}.$$

The power of $(1-\rho_{i,i+k|i+1\ldots i+k-1}^2)$ is $d-1-k$ and it depends just on k and decreases by 1 as the separation k increases by 1.

From the form of $|J_d|$ in Theorem 4, we can see how to choose $g_{j,j+k}$ in (2.1) to achieve a simpler form for f_d. We state the beta density on the interval $(-1, 1)$; this is an example of a Pearson Type I density (see [8]).

Definition. The linearly transformed Beta(α, β) on the interval $(-1, 1)$ has density

$$g(u) = \tfrac{1}{2}[B(\alpha,\beta)]^{-1}\left(\frac{1+u}{2}\right)^{\alpha-1}\left(\frac{1-u}{2}\right)^{\beta-1}.$$

If $\alpha = \beta$, this is

$$g(u) = \tfrac{1}{2}[B(\alpha,\alpha)]^{-1}\left(\frac{1+u}{2}\right)^{\alpha-1}\left(\frac{1-u}{2}\right)^{\alpha-1} = 2^{-2\alpha+1}[B(\alpha,\alpha)]^{-1}(1-u^2)^{\alpha-1}.$$

From Theorem 4, the form of (2.1) and the form of the Beta density on $(-1, 1)$, f_d has simpler form if $g_{j,j+k}(r_{j,j+k|j+1\ldots j+k-1})$ is chosen to be a symmetric Beta density on $(-1, 1)$, with both parameters equal and depending on k only, say equal to α_k (with $\alpha_k > 0$). Then (2.1) is proportional to:

$$\prod_{k=1}^{d-1}\prod_{j=1}^{d-k}(1-r_{j,j+k|j+1\ldots j+k-1}^2)^{\alpha_k-1-(d-1-k)/2}$$
$$= \prod_{i=1}^{d-1}(1-r_{i,i+1}^2)^{\alpha_1-1-(d-2)/2} \times \prod_{k=2}^{d-1}\prod_{j=1}^{d-k}(1-r_{j,j+k|j+1\ldots j+k-1}^2)^{\alpha_k-1-(d-1-k)/2}.$$

If furthermore $\alpha_k = \alpha_{d-1} + \frac{1}{2}(d-1-k)$, $k = 1, \ldots, d-1$, then (2.1) is proportional to:

$$\prod_{i=1}^{d-1} (1 - r_{i,i+1}^2)^{\alpha_{d-1}-1} \times \prod_{k=2}^{d-1} \prod_{j=1}^{d-k} (1 - r_{j,j+k|j+1\ldots j+k-1}^2)^{\alpha_{d-1}-1}$$
$$= \left[\det\left\{(r_{ij})_{1 \leqslant i,j \leqslant d}\right\}\right]^{\alpha_{d-1}-1}, \tag{2.5}$$

where the last equality comes from Theorem 1. In this special case, the same density would arise if the indices of the correlation matrices were permuted before generating correlations and partial correlations along the kth diagonal, and each ρ_{ij} $(i < j)$ is marginally Beta$(\alpha_{d-1} + \frac{1}{2}[d-2], \alpha_{d-1} + \frac{1}{2}[d-2])$ on $(-1, 1)$. This is uniform on $(-1, 1)$ if $d = 3$ and $\alpha_{d-1} = \frac{1}{2}$; for $d \geqslant 4$, this marginal distribution cannot be uniform. For example, for $d = 3$, if $\alpha_1 = \alpha_2 + \frac{1}{2}$, then the density proportional to $[\det\{(r_{ij})\}]^{\alpha_2 - 1}$ could be obtained with ρ_{12}, ρ_{23} being independent Beta$(\alpha_2 + \frac{1}{2}, \alpha_2 + \frac{1}{2})$ random variables on $(-1, 1)$ and $\rho_{13|2}$ independently Beta(α_2, α_2) on $(-1, 1)$, or it could obtained with ρ_{12}, ρ_{13} being independent Beta$(\alpha_2 + \frac{1}{2}, \alpha_2 + \frac{1}{2})$ random variables on $(-1, 1)$ and $\rho_{23|1}$ independently Beta(α_2, α_2) on $(-1, 1)$. Hence, one implication of the symmetric joint density is that marginally $\rho_{12}, \rho_{23}, \rho_{13}$ are each Beta$(\alpha_2 + \frac{1}{2}, \alpha_2 + \frac{1}{2})$ on $(-1, 1)$.

Theorem 5. *If* $\alpha_k = \alpha_{d-1} + \frac{1}{2}(d-1-k)$, $k = 1, \ldots, d-1$, *and* $\rho_{i,i+k|i+1\ldots i+k-1}$ *is* Beta(α_k, α_k) *on* $(-1, 1)$ *for* $1 \leqslant i < i+k \leqslant d$, *then the joint density in* (2.1) *becomes*

$$c_d^{-1} \left[\det\left\{(r_{ij})_{1 \leqslant i,j \leqslant d}\right\}\right]^{\alpha_{d-1}-1},$$

where the normalizing constant c_d *is*

$$2^{\sum_{k=1}^{d-1} (2\alpha_{d-1}-2+d-k)(d-k)} \times \prod_{k=1}^{d-1} \left[B\left(\alpha_{d-1} + \tfrac{1}{2}(d-1-k),\ \alpha_{d-1} + \tfrac{1}{2}(d-1-k)\right)\right]^{d-k}.$$

If $\alpha_{d-1} = 1$ *and* $\alpha_k = \frac{1}{2}(d+1-k)$ *for* $k = 1, \ldots, d-2$, *leading to uniform joint density for* $\{\rho_{ij}, i < j\}$, *then the normalizing constant is*

$$c_d = 2^{\sum_{k=1}^{d-1} (d-k)^2} \times \prod_{k=1}^{d-1} \left[B\left(\tfrac{1}{2}(d-k+1), \tfrac{1}{2}(d-k+1)\right)\right]^{d-k}$$

$$= 2^{\sum_{k=1}^{d-1} k^2} \times \prod_{k=1}^{d-1} \left[B\left(\tfrac{1}{2}(k+1), \tfrac{1}{2}(k+1)\right)\right]^{k}, \tag{2.6}$$

and the recursion is

$$c_d = c_{d-1} \times 2^{(d-1)^2} \times \left[B(\tfrac{1}{2}d, \tfrac{1}{2}d)\right]^{d-1}.$$

Proof. From the definition on a Beta density on $(-1, 1)$, the normalizing constant is

$$\prod_{k=1}^{d-1} \left[2^{2\alpha_k - 1}[B(\alpha_k, \alpha_k)]\right]^{d-k} = 2^{\sum_{k=1}^{d-1}(2\alpha_k-1)(d-k)} \times \prod_{k=1}^{d-1} [B(\alpha_k, \alpha_k)]^{d-k},$$

because there are $d-k$ (partial) correlations that have indices that are k apart for $k = 1, \ldots, d-1$. Now substitute for α_k to get the result. □

Now we say more about two special cases of (2.5), with the first case leading to a uniform density.

(a) If $\alpha_{d-1} = 1$, $\alpha_{d-2} = 3/2, \ldots, \alpha_1 = d/2$, then the joint density of $\{\rho_{ij}, i < j\}$ is constant over the d-dimensional positive definite correlation matrices. In this case, the marginal density of each ρ_{ij} is proportional to $(1-r^2)^{\alpha_1 - 1} = (1-r^2)^{d/2-1}$, that is, Beta$(d/2, d/2)$ on $(-1, 1)$.
The normalizing constant c_d in (2.6) for this case is the volume of the set of d-dimensional positive definite correlation matrices in $\binom{d}{2}$-dimensional space. Some values are

d	c_d
2	2
3	4.934802
4	11.69731
5	22.53256
6	31.11388
7	27.85823
8	14.87740
9	4.411544
10	0.682269

(b) If $\alpha_{d-1} = \frac{1}{2}$ and $\alpha_k = \alpha_{d-1} + \frac{1}{2}(d-1-k)$ for $k = 1, \ldots, d-2$ (so that $\alpha_1 = (d-1)/2$), then f_d is proportional to $[\det\{(r_{ij})\}]^{-1/2}$ and the marginal density of each ρ_{ij} is proportional to $(1-r^2)^{(d-3)/2}$. This is the same as the marginal distribution from the vector $(U_1, \ldots, U_d)$ that is uniform on the surface of a d-dimensional hypersphere; compare (4.57) on p. 128 of Joe [6].

Only the special cases in Theorem 5 will lead to a joint density of $\{\rho_{ij} : 1 \leqslant i < j \leqslant d\}$ that is invariant to permutation transforms (or permutation of indices).

In general, one can get a density that is symmetric (in the indices) by generating a correlation matrix as in (2.1) and then permuting the rows and columns of R by a random permutation. Other than a special case with $d = 3$ (in following subsection), we cannot derive the marginal density of ρ_{ij}.

2.1. $d = 3$

In this subsection, we state additional results for the $d = 3$ case, in particular when $\rho_{12}, \rho_{23}, \rho_{13|2}$ are independently uniform on the interval $(-1, 1)$. For this special case, the marginal distribution of $\rho_{13} = \rho_{12}\rho_{23} + \rho_{13|2}\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}$ is

$$\Pr(\rho_{13} \leqslant z) = \tfrac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \Pr\left(\rho_{13|2} \leqslant \frac{(z - r_{12}r_{23})}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}\right) dr_{12}\, dr_{23}.$$

After applying transforms $r_{12} = \cos(\theta_1)$, $r_{23} = \cos(\theta_2)$, and defining the region of integration appropriately, and using symbolic manipulation software, the cumulative distribution function of ρ_{13} is

$$\begin{aligned} G_{13}(z) = {} & \tfrac{1}{2}(z+1) + \tfrac{1}{4}\Big[2\sqrt{1-z^2}\,\cos^{-1}(z) - z(\cos^{-1}(z))^2 \\ & + \pi z \cos^{-1}(z) - \pi\sqrt{1-z^2}\,\Big], \quad -1 \leqslant z \leqslant 1. \end{aligned}$$

Using the random permutation mentioned above, the distribution of any of $\rho_{12}, \rho_{23}, \rho_{13}$ is

$$H(z) = \tfrac{2}{3} \cdot \tfrac{1}{2}(z+1) + \tfrac{1}{3}G_{13}(z), \quad -1 \leqslant z \leqslant 1.$$

3. Proofs

The proofs of Lemmas 2, 3 and Theorems 1, 4 in the preceding section require some known results: the recursion formula for partial correlations and the determinant of a partitioned covariance matrix.

Result 1. Recursion for partial correlations (e.g., [1, p. 34]). Let i, j, k, be distinct integers in $1, \ldots, d$ and let L be a subset of $\{1, \ldots, d\}\backslash\{i, j, k\}$. Then

$$\rho_{ij|kL} = \frac{\rho_{ij|L} - \rho_{ik|L}\rho_{jk|L}}{[(1-\rho_{ik|L}^2)(1-\rho_{jk|L}^2)]^{1/2}} \tag{3.1}$$

Result 2. Let $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ be a positive definite covariance matrix. Then $\det(\Sigma) = \det(\Sigma_{11})\det(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$. If $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T \sim N(\mathbf{0}, \Sigma)$ with Σ_{jj} being the covariance matrix of $\mathbf{X}_j$, $j = 1, 2$, then $\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ is the covariance matrix of $\mathbf{X}_2 \mid \mathbf{X}_1 = \mathbf{x}$ for any $\mathbf{x}$ with the dimension of $\mathbf{X}_1$.

Proof of Lemma 2. From (3.1),

$$1-\rho^2_{ij|kL} = \frac{(1-\rho^2_{ik|L})(1-\rho^2_{jk|L})-(\rho_{ij|L}-\rho_{ik|L}\rho_{jk|L})^2}{(1-\rho^2_{ik|L})(1-\rho^2_{jk|L})}$$
$$= \frac{1-\rho^2_{ik|L}-\rho^2_{jk|L}-\rho^2_{ij|L}+2\rho_{ik|L}\rho_{jk|L}\rho_{ij|L}}{(1-\rho^2_{ik|L})(1-\rho^2_{jk|L})}$$
$$= \frac{\det(R[ijk|L])}{\det(R[ik|L])\det(R[jk|L])}.$$

If $L=\emptyset$, the above is

$$\frac{\det(R[ijk])}{\det(R[ik])\det(R[jk])} = \frac{D(\{i,j,k\})\,D(\{k\})}{D(\{i,k\})\,D(\{j,k\})}$$

since by definition $D(\{k\})=1$.

Otherwise for $L\neq\emptyset$, Let $(X_i,X_j,X_k,\mathbf{X}_L)$ be a mean zero normal random vector with correlation matrix $R[\{i,j,k,L\}]$ and unit variances. Let $V_{ijk} =$ diag $\big(\mathrm{Var}\,(X_i|\mathbf{X}_L), \mathrm{Var}\,(X_j|\mathbf{X}_L), \mathrm{Var}\,(X_k|\mathbf{X}_L)\big)$ so that $V_{ijk}^{1/2}\,R[ijk|L]\,V_{ijk}^{1/2}$ is the covariance matrix of $(X_i,X_j,X_k)|\mathbf{X}_L$. By Result 2,

$$\det\left(V_{ijk}^{1/2}\,R[ijk|L]\,V_{ijk}^{1/2}\right) = \det(R[\{i,j,k,L\}])\big/\det(R[L]) = D(\{i,j,k,L\})/D(L)$$

so that

$$\det(R[ijk|L]) = \frac{D(\{i,j,k,L\})}{D(L)\,\mathrm{Var}\,(X_i|\mathbf{X}_L)\,\mathrm{Var}\,(X_j|\mathbf{X}_L)\,\mathrm{Var}\,(X_k|\mathbf{X}_L)}.$$

Similarly,

$$\det(R[ik|L])\ \det(R[jk|L]) = \frac{D(\{i,k,L\})\,D(\{j,k,L\})}{D^2(L)\,\mathrm{Var}\,(X_i|\mathbf{X}_L)\,\mathrm{Var}\,(X_j|\mathbf{X}_L)\,\{\mathrm{Var}\,(X_k|\mathbf{X}_L)\}^2}.$$

Hence

$$\frac{\det(R[ijk|L])}{\det(R[ik|L])\ \det(R[jk|L])} = \frac{D(\{i,j,k,L\})\,D(L)\,\mathrm{Var}\,(X_k|\mathbf{X}_L)}{D(\{i,k,L\})\,D(\{j,k,L\})}.$$

By another application of Result 2, $D(L)\,\mathrm{Var}\,(X_k|\mathbf{X}_L) = D(\{k,L\})$, which completes the proof. □

Proof of Theorem 1. The result is known for $d=2$. To start the induction proof of Theorem 1, we prove it for $d=3$. As a special case of partial correlation defined at the beginning of Section 2,

$$\rho_{13|2} = \frac{\rho_{13}-\rho_{12}\rho_{23}}{\left[(1-\rho^2_{12})(1-\rho_{23})\right]^{1/2}},$$

so that

$$1-\rho^2_{13|2} = \frac{1-\rho^2_{12}-\rho^2_{23}-\rho^2_{13}+2\rho_{12}\rho_{23}\rho_{13}}{(1-\rho^2_{12})(1-\rho^2_{23})} = \frac{\det(R)}{(1-\rho^2_{12})(1-\rho^2_{23})}.$$

Hence

$$\det(R) = (1-\rho_{12}^2)(1-\rho_{23}^2)(1-\rho_{13|2}^2).$$

We proceed by induction and go from $d-1$ to d. The induction hypothesis gives us

$$\det(R[\{1,\dots,d-1\}]) = D(\{1,\dots,d-1\}) = \prod_{i=1}^{d-2}(1-\rho_{i,i+1}^2) \times \prod_{k=2}^{d-2}\prod_{i=1}^{d-1-k}(1-\rho_{i,i+k|i+1\dots i+k-1}^2),$$

and we want to show that $\det(R)$ for dimension d is

$$\begin{aligned}&\prod_{i=1}^{d-1}(1-\rho_{i,i+1}^2) \times \prod_{k=2}^{d-1}\prod_{i=1}^{d-k}(1-\rho_{i,i+k|i+1\dots i+k-1}^2)\\ &\quad = D(\{1,\dots,d-1\}) \times (1-\rho_{d-1,d}^2)(1-\rho_{d-2,d|d-1}^2)\cdots(1-\rho_{1d|2\dots d-1}^2).\end{aligned}$$

By Lemma 2, this is:

$$\begin{aligned}&D(\{1,\dots,d-1\}) \times D(\{d-1,d\}) \times \frac{D(\{d-2,d-1,d\})D(\{d-1\})}{D(\{d-2,d-1\})D(\{d-1,d\})}\\ &\quad\times \frac{D(\{d-3,d-2,d-1,d\})\,D(\{d-2,d-1\})}{D(\{d-3,d-2,d-1\})D(\{d-2,d-1,d\})}\\ &\quad\times\cdots\times \frac{\det(R)\,D(\{2,\dots,d-1\})}{D(\{1,\dots,d-1\})\,D(\{2,\dots,d\})}\\ &= D(\{1,\dots,d-1\}) \times D(\{d-1,d\})\\ &\quad\times \prod_{k=2}^{d-1}\frac{D(\{d-k,\dots,d\})\,D(\{d-k+1,\dots,d-1\})}{D(\{d-k,\dots,d-1\})\,D(\{d-k+1,\dots,d\})}\\ &= D(\{1,\dots,d-1\}) \times D(\{d-1,d\}) \times \frac{D(\{d-1\})}{D(\{1,\dots,d-1\})} \times \frac{D(\{1,\dots,d\})}{D(\{d-1,d\})}\\ &= \det(R)\,D(\{d-1\}) = \det(R). \qquad \square\end{aligned}$$

Proof of Lemma 3. By applying Result 1 recursively, with the index of the product decreasing

$$\begin{aligned}\frac{\partial\rho_{1d|2\dots d-1}}{\partial\rho_{1d}} &= \left\{\prod_{i=d-2}^{2}\frac{\partial\rho_{1d|2\dots i+1}}{\partial\rho_{1d|2\dots i}}\right\} \times \frac{\partial\rho_{1d|2}}{\partial\rho_{1d}}\\ &= \left\{\prod_{i=d-2}^{2}[(1-\rho_{1,i+1|2\dots i}^2)(1-\rho_{i+1,d|2\dots i}^2)]^{-1/2}\right\}\\ &\quad\times[(1-\rho_{12}^2)(1-\rho_{2d}^2)]^{-1/2}.\end{aligned}$$

Hence, by Lemma 2,

$$\begin{aligned}\left[\frac{\partial\rho_{1d|2\ldots d-1}}{\partial\rho_{1d}}\right]^{-2} &= \left\{\prod_{i=d-2}^{2}\frac{D(\{1,\ldots,i+1\})D(2,\ldots,i\}}{D(\{1,\ldots,i\})D(2,\ldots,i+1\}}\right.\\ &\quad\left.\times\frac{D(\{d,2,\ldots,i+1\})D(2,\ldots,i\}}{D(\{d,2,\ldots,i\})D(2,\ldots,i+1\}}\right\}\times(1-\rho_{12}^2)(1-\rho_{2d}^2)\\ &= \frac{D(\{1,\ldots,d-1\})}{D(\{1,2\})}\times\frac{D(2)}{D(\{2,\ldots,d-1\})}\times\frac{D(\{2,\ldots,d\})}{D(\{d,2\})}\\ &\quad\times\frac{D(2)}{D(\{2,\ldots,d-1\})}\times(1-\rho_{12}^2)(1-\rho_{2d}^2)\\ &= \frac{D(\{1,\ldots,d-1\})\,D(\{2,\ldots,d\})}{D^2(\{2,\ldots,d-1\})}.\end{aligned}$$

Applying Theorem 1 and cancelling the common terms from the numerator and denominator results in the claimed result. □

Proof of Theorem 4. With the order of the variables as given in the statement of the theorem, the Jacobian matrix is lower triangular. Hence its determinant is

$$\prod_{k=1}^{d-1}\prod_{i=1}^{d-k}\frac{\partial\rho_{i,i+k|i+1\ldots i+k-1}}{\partial\rho_{i,i+k}}.$$

For $d=3$, (2.4) is the same as

$$\frac{\partial\rho_{13|2}}{\partial\rho_{13}}=\left[(1-\rho_{12}^2)(1-\rho_{23}^2)\right]^{-1/2}.$$

We complete the proof by induction. Suppose (2.4) is valid for $d\geqslant 3$. Then in going up a dimension to $d+1$, the determinant of the Jacobian is

$$\begin{aligned}&\left[\prod_{i=1}^{d-1}(1-\rho_{i,i+1}^2)^{d-2}\times\prod_{k=2}^{d-2}\prod_{i=1}^{d-k}(1-\rho_{i,i+k|i+1\ldots i+k-1}^2)^{d-1-k}\right]^{-1/2}\\ &\quad\times\prod_{j=1}^{d-1}\frac{\partial\rho_{d-j,d+1|d-j+1\ldots d}}{\partial\rho_{d-j,d+1}}.\end{aligned}\tag{3.2}$$

By Lemma 3, the square reciprocal of the second term in (3.2) is

$$\begin{aligned}&\prod_{j=1}^{d-1}\prod_{k=1}^{j}(1-\rho_{d-j,d-j+k|d-j+1\ldots d-j+k-1}^2)(1-\rho_{d+1-k,d+1|d+2-k\ldots d}^2)\\ &=\prod_{j=1}^{d-1}\prod_{k=1}^{j}(1-\rho_{d-j,d-j+k|d-j+1\ldots d-j+k-1}^2)\times\prod_{k=1}^{d-1}(1-\rho_{d+1-k,d+1|d+2-k\ldots d}^2)^{d-k}\\ &=\prod_{i=1}^{d-1}\prod_{k=1}^{d-i}(1-\rho_{i,i+k|i+1\ldots i+k-1}^2)\times\prod_{k=1}^{d-1}(1-\rho_{d+1-k,d+1|d+2-k\ldots d}^2)^{d-k}\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{d-1}(1-\rho^2_{i,i+1}) \times \prod_{i=1}^{d-1}\prod_{k=2}^{d-i}(1-\rho^2_{i,i+k|i+1\ldots i+k-1}) \times \prod_{k=1}^{d-1}(1-\rho^2_{d+1-k,d+1|d+2-k\ldots d})^{d-k} \\
&= \prod_{i=1}^{d-1}(1-\rho^2_{i,i+1}) \times \prod_{k=2}^{d-1}\prod_{i=1}^{d-k}(1-\rho^2_{i,i+k|i+1\ldots i+k-1}) \times \prod_{k=1}^{d-1}(1-\rho^2_{d+1-k,d+1|d+2-k\ldots d})^{d-k} \\
&= \prod_{i=1}^{d-1}(1-\rho^2_{i,i+1}) \times \prod_{k=2}^{d-2}\prod_{i=1}^{d-k}(1-\rho^2_{i,i+k|i+1\ldots i+k-1}) \times (1-\rho^2_{1d|2\ldots d-1}) \\
&\quad \times \prod_{k=1}^{d-1}(1-\rho^2_{d+1-k,d+1|d+2-k\ldots d})^{d-k}.
\end{aligned}
$$

Substitution of this into (3.2) means that all terms $(1-\rho^2_{ij})$ with $1 \leqslant i < j \leqslant d$ have an exponent increased by 1, and the new term $(1-\rho^2_{d+1-k,d+1|d+2-k\ldots d})$ with second index $d+1$ and indices k apart has exponent $(d+1)-1-k$. Hence (3.2) has the same form as (2.4) with d replaced by $d+1$. □

4. Discussion

We have outlined a method of generating random correlation matrices based on partial correlations, such that the joint density of the correlations can be obtained. We obtained conditions so that the joint density is invariant to permutations; in these cases, the joint density is proportional to a power of the determinant of the correlation matrix.

As an intermediate result, we have a simple remarkable identity (2.2) for the determinant of a positive definite correlation matrix. The identity is also valid with index j replaced by a_j everywhere, where $(a_1, \ldots, a_j)$ is a permutation of $(1, \ldots, d)$. Of course, (2.2) is also valid for a singular correlation matrix since the determinant is 0 and at least one partial correlation is ± 1 with singularity. In checking the literature, this determinant identity appears to be new.

Based on the results for the $d = 3$ case, we had conjectured (2.2) as an identity and then conjectured the results in and between Theorems 4 and 5. The identity was verified numerically and the univariate margin for (2.5) was also checked numerically before proving the results for general d.

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