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NOTES AND COMMENTS

RANGE RESTRICTIONS FOR PRODUCT-MOMENT CORRELATION MATRICES

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It is well-known that for a trivariate distribution if two correlations are fixed the remaining one is constrained. Indeed, if one correlation is fixed, then the remaining two are constrained. Both results are extended to the case of a multivariate distribution. The results are applied to some special patterned matrices.

Key words: constrained correlations, multiple correlation, partial correlation, patterned correlation matrices.

It is well-known that if a subset of correlations is given, that the ranges of the remaining correlations are restricted. Suppose that the correlation matrix R of p variables $X = (X^{(1)}, X^{(2)}, X^{(3)})$, where $X^{(i)}$ is of order p_i, $p_1 + p_2 + p_3 = p$, is given in partitioned form by

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \tag{1}$$

The problem is to find restrictions on a submatrix of R when the complementary submatrices are given. Two distinct subproblems emerge: (i) the case where all submatrices but R_{11} are fixed, (ii) the case where all submatrices but R_{12} are fixed.

The trivariate case $p = 3$ ($p_1 = 1$, $p_2 = 2$) has received considerable attention. If r_{13} and r_{23} are given, then r_{12} is restricted by the inequalities

$$r_{13}r_{23} - [(1 - r_{13}^2)(1 - r_{23}^2)]^{1/2} \le r_{12} \le r_{13}r_{23} + [(1 - r_{13}^2)(1 - r_{23}^2)]^{1/2}. \tag{2}$$

McCornack [1956] discusses the use of (2) when different weighing methods are applied to a set of tests or test items. Stanley and Wang [1969] note that condition (2) is obtained from the fact that the partial correlation $r_{12 \cdot 3}$ is in the interval $[-1, 1]$. Actually the two conditions are equivalent, and this equivalence is useful in the more general case. Novel geometric proofs of (2) are provided by Glass and Collins [1970] and by Leung and Lam [1975].

As a first extension consider the case $p_1 = 1$, $p_2 = p - 1$. If,

$$R = \begin{pmatrix} 1 & r_{12} \cdots r_{1p} \\ & R_{22} \end{pmatrix} \equiv \begin{pmatrix} 1 & r_1 \\ & R_{22} \end{pmatrix}, \tag{3}$$

and R_{22} is given, then how free is r_1? The answer is provided by Hubert [1972] in the form of a quadratic inequality:

$$r_1 R_{22}^{-1} r_1' \le 1. \tag{4}$$

A more intuitive interpretation of (4) is the equivalent statement that the squared multiple

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correlation satisfies

$$r^2_{1(2,\ldots,p)} \le 1. \tag{5}$$

Hubert's [1972] proof of (4) is based on the positive-definiteness of R. Indeed, this condition is fundamental in more general contexts.

To resolve case (i) consider

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

with R_{12} and R_{22} given. Positive-definiteness of R is equivalent to positive-definiteness of R_{22} and the condition

$$R_{11} > R_{12} R_{22}^{-1} R_{21}, \tag{6}$$

where $A > B$ means that the matrix $A - B$ is positive definite.

In the particular case that

$$R_{11} = \begin{pmatrix} 1 & r_{12} \\ & 1 \end{pmatrix},$$

condition (6) is equivalent to the inequalities

$$q_{12\cdot 3} - [(1 - q_{11\cdot 3})(1 - q_{22\cdot 3})]^{1/2} \le r_{12} \le q_{12\cdot 3} + [(1 - q_{11\cdot 3})(1 - q_{22\cdot 3})]^{1/2}, \tag{7}$$

where

$$Q = \begin{pmatrix} q_{11\cdot 3} & q_{12\cdot 3} \\ q_{21\cdot 3} & q_{22\cdot 3} \end{pmatrix} = \begin{pmatrix} r_{13} & \cdots & r_{1p} \\ r_{23} & \cdots & r_{2p} \end{pmatrix} R_{(3)}^{-1} \begin{pmatrix} r_{13} & r_{23} \\ \vdots & \vdots \\ r_{1p} & r_{2p} \end{pmatrix},$$

and $R_{(3)} = (r_{ij})$, $i, j = 3, 4, \ldots, p$. As in the trivariate case, inequalities (7) are equivalent to the assertion that the partial correlation $r_{12(3,\ldots,p)}$ must lie in the interval $[-1, 1]$.

The resolution of case (ii) is more cumbersome. In (1) all matrices but R_{12} are fixed. For simplicity of notation, write Z for R_{12}. Positive-definiteness of R implies positive-definiteness of

$$\begin{pmatrix} R_{11} & Z \\ Z' & R_{22} \end{pmatrix} - \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} R_{33}^{-1} (R_{31} \quad R_{32})$$
$$= \begin{pmatrix} R_{11} - R_{13} R_{33}^{-1} R_{31} & Z - R_{13} R_{33}^{-1} R_{32} \\ Z' - R_{23} R_{33}^{-1} R_{31} & R_{22} - R_{23} R_{33}^{-1} R_{32} \end{pmatrix}, \tag{8}$$

which is equivalent to

$$R_{22\cdot 3} > (Z - R_{13} R_{33}^{-1} R_{32})' R_{11\cdot 3}^{-1} (Z - R_{13} R_{33}^{-1} R_{32}), \tag{9}$$

or

$$R_{11\cdot 3} > (Z - R_{13} R_{33}^{-1} R_{32}) R_{22\cdot 3}^{-1} (Z - R_{13} R_{33}^{-1} R_{32})', \tag{10}$$

where

$$R_{jj\cdot 3} = R_{jj} - R_{j3} R_{33}^{-1} R_{3j}, \qquad j = 1, 2.$$

In the special case that $p_1 = 1$, so that $R_{11} = 1$, condition (10) reduces to a quadratic inequality that is a generalization of (4).

Special Cases: Patterned Matrices

Priest [1968] and Stanley and Wang [1969] discuss bounds for the mean of correlations. It is well-known that $\bar{r} \equiv \sum_{i<j} r_{ij}/[k(k-1)/2] \geq -1(k-1)$. We now extend this idea when means of some correlations are fixed.

For notational simplicity, write

$$P(a) = \begin{pmatrix} 1 & a & \cdots & a \\ & 1 & \cdots & a \\ & & \vdots & \\ & & & 1 \end{pmatrix}, \quad Q(b) = \begin{pmatrix} b & \cdots & b \\ & \cdots & \\ & \cdots & \\ & \cdots & \\ b & \cdots & b \end{pmatrix} \tag{11}$$

If $R_{11} = P(\bar{r}_1)$, $R_{22} = P(\bar{r}_2)$, $R_{12} = Q(t)$, and if $\bar{r}_2$ and t are given, then $\bar{r}_1$ must satisfy the inequalities

$$\frac{p_1 p_2 t^2}{(p_1 - 1)[1 + (p_2 - 1)\bar{r}_2]} - \frac{1}{(p_1 - 1)} \leq \bar{r}_1 \leq 1, \tag{12}$$

whereas if $\bar{r}_1$ and $\bar{r}_2$ are given, then t must satisfy

$$t^2 \leq \frac{[1 + (p_1 - 1)\bar{r}_1][1 + (p_2 - 1)\bar{r}_2]}{p_1 p_2}. \tag{13}$$

Conditions comparable to (12) and (13) can be extended to more general patterns. If $R_{11} = P(\bar{r}_1)$, $R_{22} = P(\bar{r}_2)$, $R_{33} = P(\bar{r}_3)$, $R_{12} = Q(t_{12})$, $R_{13} = Q(t_{13})$, $R_{23} = Q(t_{23})$, then for $\bar{r}_2, \bar{r}_3, t_{13}, t_{13}, t_{23}$ fixed, $\bar{r}_1$ must satisfy the inequalities

$$\frac{p_1}{p_1 - 1} [(p_2)^{1/2} t_{12} (p_3)^{1/2} t_{13}] \times \begin{pmatrix} 1 + (p_2 - 1)\bar{r}_2 & (p_2 p_3)^{1/2} t_{23} \\ (p_2 p_3)^{1/2} t_{23} & 1 + (p_3 - 1)\bar{r}_3 \end{pmatrix}^{-1} \begin{pmatrix} (p_2)^{1/2} t_{12} \\ (p_3)^{1/2} t_{13} \end{pmatrix} - \frac{1}{(p_1 - 1)} \leq \bar{r}_1 \leq 1,$$

If $\bar{r}_1, \bar{r}_2, \bar{r}_3, t_{13}, t_{23}$ are fixed, then t_{12} must satisfy (14)

$$\left(t_{12} - \frac{p_3 (p_1 p_2)^{1/2} t_{13} t_{23}}{1 + (p_3 - 1)\bar{r}_3}\right)^2 \leq \left(1 + (p_1 - 1)\bar{r}_1 - \frac{p_1 p_3 t_{13}^2}{1 + (p_3 - 1)\bar{r}_3}\right)\left(1 + (p_2 - 1)\bar{r}_2 - \frac{p_2 p_3 t_{23}^2}{1 + (p_3 - 1)\bar{r}_3}\right). \tag{15}$$

Remark. Although (14) and (15) can be obtained from the development given in this paper, they can also be derived from the fact that the characteristic roots of the correlation matrix must be positive.

We close by noting that there is a literature on bounds for correlations in ipsatively scored tests, i.e., when the total score $X_1 + X_2 + \cdots + X_n$ is fixed. For references to this literature see Radcliffe [1963] and Gleser [1972].

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