Basic stray-light analysis

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1 Introduction

This analysis is the basic deduction for the surface brightness of the Earth and the Moon, on their Dark and Bright faces.

2 Surface brightness of the sun-illuminated face of the Earth

Let $f_{\odot,\lambda}$ be the flux of the Sun at the orbit of the Earth-Moon system, also referred as the "Solar Constant". This can be numerically derived from the Solar spectrum as a spectral flux density (W m^{-2} Hz^{-1}) or integrated over a certain band, as a radiative flux (W m^{-2}). To estimate this from the photometric absolute AB magnitudes of the Sun ($m_{\odot,\lambda}$) in SI units, we can use the following expression:

$$f_{\odot,\lambda} = 10^{-\frac{56.1+m_{\odot,\lambda}}{2.5}} \tag{1}$$

The irradiance (W Hz^{-1} or W, depending on the units of $f_{\odot,\lambda}$) received by the cross-section of the Earth is:

$$I_{\oplus,\lambda} = \pi R_{\oplus}^2 f_{\odot,\lambda}, \tag{2}$$

where $R_{\oplus}$ is the radius of the Earth in m. Assuming that the Earth has a constant albedo ($\alpha_{\oplus}$), the radiance reflected by the surface of illuminated hemisphere of the Earth at a certain altitude from the surface h (m) is:

$$f_{\oplus,\lambda}(h) = \frac{\alpha_{\oplus} R_{\oplus}^2 f_{\odot,\lambda}}{2(R_{\oplus}+h)^2}. \tag{3}$$

To calculate the surface brightness, we need to divide this flux by the total area of the Earth at an altitude h. The apparent radius on the Earth at h is:

$$r_{\oplus}(h) = \frac{180}{\pi} \arctan \frac{R_{\oplus}}{R_{\oplus}+h}. \tag{4}$$

so the total area is:

$$A_\oplus(h) = \pi r_\oplus(h)^2 \tag{5}$$

and the surface brightness intensity (W m^{-2} Hz^{-1} arcsec^{-2} or W m^{-2} arcsec^{-2}) is:

$$\Sigma_{\oplus,\odot,\lambda}(h) = \frac{\pi \alpha_\oplus f_{\odot,\lambda} R_\oplus^2}{2(R_\oplus + h)^2 (180 \arctan \frac{R_\oplus}{R_\oplus + h})^2}. \tag{6}$$

In AB magnitudes (mag arcsec^{-2}), this translates to:

$$\mu_{\oplus,\odot,\lambda} = -2.5 \log_{10} \Sigma_{\oplus,\lambda} - 56.1 \tag{7}$$

Note that when x is close to one, $\arctan x$ is $\sim$ to x. For this reason, the arctan term approximately cancels the $(R_\oplus + h)^2$ in eq. 9, so the equation almost no dependent with h. This demostrates that the surface brightness does not depend with the distance to an extended object.

3 Surface brightness of the Moon-illuminated face of the Earth

Assuming that the distance of the Moon and the Earth to the Sun is similar, following the same method above, the flux received by the Earth from a full Moon is:

$$f_{☾,\lambda}(h) = \frac{\alpha_☾ R_☾^2 f_{\odot,\lambda}}{2 d_{\oplus-☾}^2}. \tag{8}$$

where $\alpha_☾$ is the average albedo of the Moon, $R_☾$ is the radius of the Moon, and $d_{\oplus-☾}$ is the distance between the Earth and the Moon. Following the same procedure as above, the surface brightness intensity of the moonlight that reflects over the Earth and reaches a satellite in orbit at a distance h from the surface of the Earth is:

$$\Sigma_{\oplus,☾,\lambda}(h) = \frac{\pi \alpha_\oplus \alpha_☾ R_\oplus^2 R_☾^2 f_{\odot,\lambda}}{4(R_\oplus + h)^2 d_{\oplus-☾}^2 (180 \arctan \frac{R_\oplus}{R_\oplus + h})^2}. \tag{9}$$

So for a full Moon, the ratio between the sunlit side of the Earth and the moon-iluminated side of the Earth should be simply:

$$\frac{\Sigma_{\oplus,\odot,\lambda}}{\Sigma_{\oplus,☾,\lambda}} = 2 \frac{d_{\oplus-☾}^2}{R_☾^2 \alpha_☾} \tag{10}$$