Integer Divisibility

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Fall of 2005

Lecture 5 (out of seven)

■ Plan

1. Introduction to Diophantine Equations
2. Linear Diophantine Equations
3. Positive solutions to LDE

■ Introduction

Definition. Let $P(x, y, \ldots)$ is a polynomial with integer coefficients in one or more variables. A Diophantine equation is an algebraic equation

$$P(x, y, z, \ldots) = 0$$

for which integer solutions are sought.

For example,

$$2x + 3y = 11$$

$$7x^2 - 5y^2 + 2x + 4y - 11 = 0$$

$$y^3 + x^3 = z^3$$

The problem to be solved is to determine whether or not a given Diophantine equation has solutions in the domain of integer numbers.

In 1900 Hilbert proposed 23 most important unsolved problems of 20th century. His 10th problem was about solvability a general Diophantine equation. Hilbert asked for a *universal method* of solving all Diophantine equations.

What is the notion of *solvable*? What is the notion of an *algorithm*?

1930. Godel, Kleene, Turing developed the notion of computability.

1946. Turing invented Universal Turing Machine and discovered basic unsolvable problems

1970 Y. Matiyasevich proved that the Diophantine problem is unsolvable.

Theorem (Y. Matiyasevich) *There is no algorithm which, for a given arbitrary Diophantine equation, would tell whether the equation has a solution or not.*

By the way, Goldbach's conjecture (which was mentioned a few lectures back) is Hilbert's 8th problem.

Linear Diophantine Equations

Definition.

A linear Diophantine equation (in two variables x and y) is an equation

$$a x + b y = c$$

with integer coefficients $a, b, c \in \mathbb{Z}$ to which we seek integer solutions.

It is not obvious that all such equations solvable. For example, the equation

$$2 x + 2 y = 1$$

does not have integer solutions.

Some linear Diophantine equations have finite number of solutions, for example

$$2 x = 4$$

and some have infinite number of solutions.

Thereom.

The linear equation $a, b, c \in \mathbb{Z}$

$$a x + b y = c$$

has an integer solution in x and $y \in \mathbb{Z} \Longleftrightarrow \gcd(a, b) \mid c$

Proof.

$\Longrightarrow$)

$$\gcd(a, b) \mid a \ \wedge \ \gcd(a, b) \mid b \Longrightarrow$$

$$\gcd(a, b) \mid (x\,a + y\,b) \Longrightarrow \gcd(a, b) \mid c$$

$\Longleftarrow$)

Given

$$\gcd(a, b) \mid c \Longrightarrow \exists \ z \in \mathbb{Z}, \ c = \gcd(a, b) * z$$

On the other hand

$$\exists \ x_1, \ y_1 \in \mathbb{Z}, \quad \gcd(a, b) = x_1\,a + y_1\,b.$$

Multiply this by z:

$$z * \gcd(a, b) = a * x_1 * z + b * y_1 * z$$

$$c = a * x_1 * z + b * y_1 * z$$

Then the pair $x_1 * z$ and $y_1 * z$ is the solution

QED.

How do you find a particular solution?

$$a\,x + b\,y = c$$

By extended Euclidean algorithm we find gcd and such n and m that

$$a * n + b * m = \gcd(a, b)$$

Multiply this by c

$$a * n * c + b * m * c = \gcd(a, b) * c$$

Divide it by gcd

$$a \frac{n * c}{\gcd(a, b)} + b \frac{m * c}{\gcd(a, b)} = c$$

Compare this with the original equation

$$a\,x + b\,y = c$$

It follows that a particular solution is

$$x_0 = \frac{n * c}{\gcd(a,\, b)};\;\; y_0 = \frac{m * c}{\gcd(a,\, b)}$$

Question. Are x_0 and y_0 integer?

Exercise. Find a particular solution of

$$56\,x + 72\,y = 40$$

Solution. Run the EEA to find GCD, n and m

$$\text{GCD}(56,\, 72) = 8 = 4 * 56 + (-3) * 72$$

Then one of the solutions is

$$x_0 = \frac{4 * 40}{8};\;\; y_0 = \frac{(-3) * 40}{8}$$

$$x_0 = 20;\;\; y_0 = -15$$

How do you find all solutions?

$$a\,x + b\,y = c$$

By the extended Euclidean algorithm we find gcd and such n and m that

$$\gcd(a,\, b) = a * n + b * m$$

$$\gcd(a,\, b) * c = a * n * c + b * m * c$$

Next we add and subtract $a * b * k$, where $\forall\, k \in \mathbb{Z}$

$$\gcd(a,\, b) * c = a * n * c + b * m * c + a * b * k - a * b * k$$

Collect terms with respect a and b

$$a * (n\,c + b\,k) + b * (m\,c - a\,k) = \gcd(a,\, b) * c$$

Divide this by $\gcd(a,\, b)$

$$a * \frac{(n\,c + b\,k)}{\gcd(a,\,b)} + b * \frac{(m\,c - a\,k)}{\gcd(a,\,b)} = c$$

It can be rewritten as

$$c = a * \left(\frac{n\,c}{\gcd(a,\,b)} + \frac{b\,k}{\gcd(a,\,b)}\right) + b * \left(\frac{m\,c}{\gcd(a,\,b)} - \frac{a\,k}{\gcd(a,\,b)}\right)$$

or

$$c = a * \left(x_0 + \frac{b * k}{\gcd(a,\,b)}\right) + b * \left(y_0 - \frac{a * k}{\gcd(a,\,b)}\right)$$

$$k = 0,\ \pm 1,\ \pm 2,\ \ldots$$

since $(x_0,\ y_0)$ is a particular solution.

Therefore, all integers solutions are in the form

$$x = x_0 + \frac{b\,k}{\gcd(a,\,b)} \quad y = y_0 - \frac{a\,k}{\gcd(a,\,b)}$$

$$k = 0,\ \pm 1,\ \pm 2,\ \ldots$$

Exercise. Find all integer solutions of

$$56\,x\ +\ 72\,y\ =\ 40$$

Solution. Run the EEA to find GCD, n and m

$$\mathrm{GCD}(56,\,72) = 8 = 4 * 56 + (-3) * 72$$

All solutions are in the form

$$x\ =\ \frac{n\,c}{\gcd(a,\,b)} + \frac{b\,k}{\gcd(a,\,b)}$$

$$y\ =\ \frac{m\,c}{\gcd(a,\,b)} - \frac{a\,k}{\gcd(a,\,b)}$$

Hence

$$x\ =\ \frac{4 * 40}{8} + \frac{72\,k}{8} = 20\ +\ 9 * k$$

$$y = \frac{-3*40}{8} - \frac{56\,k}{8} = -15 - 7*k$$

■ Positive solutions of LDE

In some applications it might required to find all positive solutions $x,\ y \in \mathbb{Z}^+$.

We take a general solution

$$x = \frac{n\,c}{\gcd(a,\,b)} + \frac{b\,k}{\gcd(a,\,b)}$$

$$y = \frac{m\,c}{\gcd(a,\,b)} - \frac{a\,k}{\gcd(a,\,b)}$$

from which we get two inequalities

$$n\,c + b\,k > 0$$

$$m\,c - a\,k > 0$$

To find out how many positive solutions a given equation has let us consider two cases

1. $a\,x + b\,y = c, \quad \gcd(a,\,b) = 1,\ a,\,b > 0$

2. $a\,x - b\,y = c, \quad \gcd(a,\,b) = 1,\ a,\,b > 0$

It follows that in the first case, the equation has a finite number of solutions

$$-\frac{n\,c}{|\,b\,|} < k < \frac{m\,c}{|\,a\,|}$$

In the second case, there is an infinite number of solutions

$$n\,c - |\,b\,|\ k > 0$$

$$m\,c - |\,a\,|\ k > 0$$

Exercise. Determine the number of solutions in positive integers

$$4\,x + 7\,y = 117$$

Solution.

$$\text{GCD}(4, 7) = 1 = 2 * 4 + (-1) * 7$$

The number of solutions in positive integers can be determined from the system

$$n\,c + b\,k > 0$$

$$m\,c - a\,k > 0$$

which for our equation transforms to

$$2 * 117 + 7 * k > 0$$

$$(-1) * 117 - 4 * k > 0$$

This gives

$$-\frac{2 * 117}{7} < k < \frac{-117}{4}$$

There 4 such k, namely $k = -33, \ -32, \ -31, \ -30$.

■ LDEs with three variables

Consider

$$3\,x + 6\,y + 5\,z = 7$$

$$\text{GCD}(3, 6)\,(x + 2\,y) + 5\,z = 7$$

Let

$$w = x + 2\,y$$

The equation becomes

$$3\,w + 5\,z = 7$$

Its general solution is

$$w = 2 * 7 + 5\,k$$

$$z = (-1) * 7 - 3\,k$$

since

$$\text{GCD}(3, 5) = 1 = 2 * 3 + (-1) * 5$$

Next we find x and y

$$x + 2y = 14 + 5k$$

Since $\text{GCD}(1, 2) \mid (14 + 5k)$, the equation is solvable and the solution is

$$x = 1 * (14 + 5k) + 2 * l$$

$$y = 0 * (14 + 5k) - 1 * l$$

where $l \in \mathbb{Z}$ is another parameter. Here are all triple-solutions

$$x = 5k + 2l + 14$$

$$y = -l$$

$$z = -7 - 3k$$

where

$$k, \ l = 0, \pm 1, \pm 2, \ \ldots$$