STOCHASTIC PROCESS-TEMPORAL CHARACTERISTICS

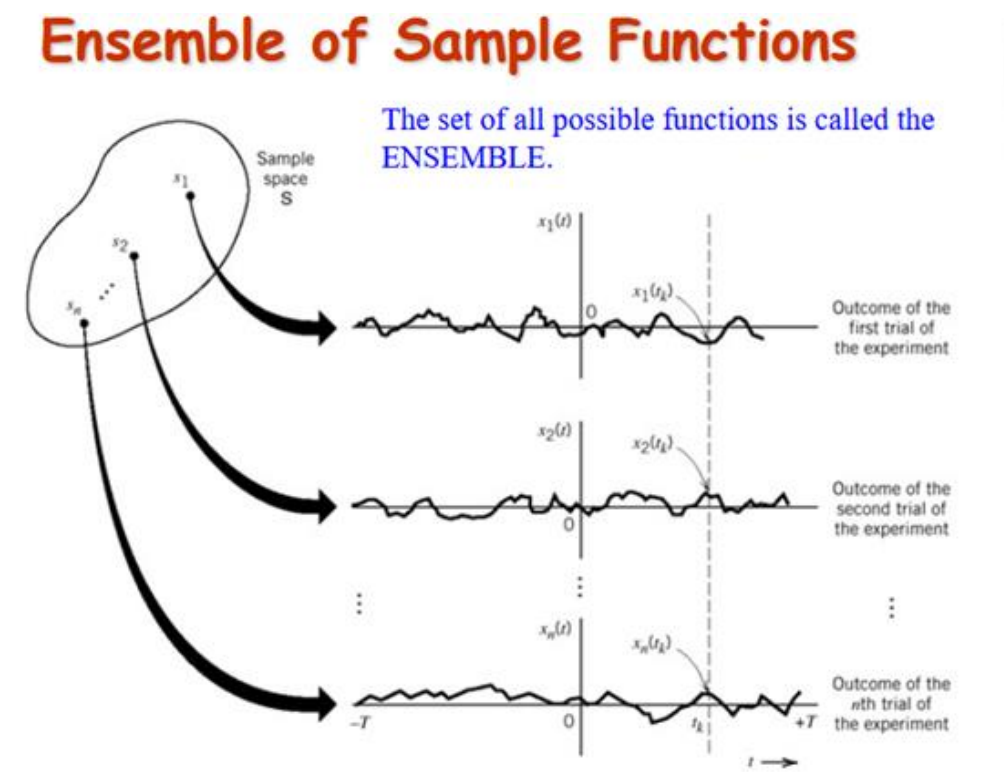
INTRODUCTION

A random process is an ensemble of (collection) number of time varying functions. Further it is nothing but a random variable with time added.

A random variable is a real valued function that assigns numerical values to the outcomes of physical experiment. If time is added to a random variable then it is called random process.

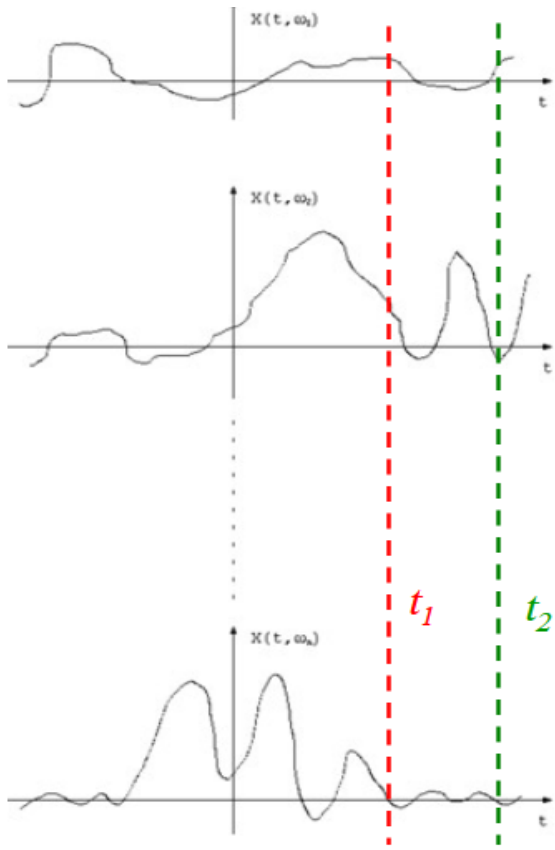
Random processes are used to describe the time varying nature of random variable. They describe the statistical behavior of various real time signals like speech, noise, atmosphere, etc…

Random processes are denoted by $X(t,s)$ or $X(t)$. If time is fixed i.e; if any specific time instant is taken then random process becomes random variable.



Random Processes

- A general Random or Stochastic Process can be described as:
  - Collection of time functions (signals) corresponding to various outcomes of random experiments.
  - Collection of random variables observed at different times.
- Examples of random processes in communications:
  - Channel noise,
  - Information generated by a source,
  - Interference.



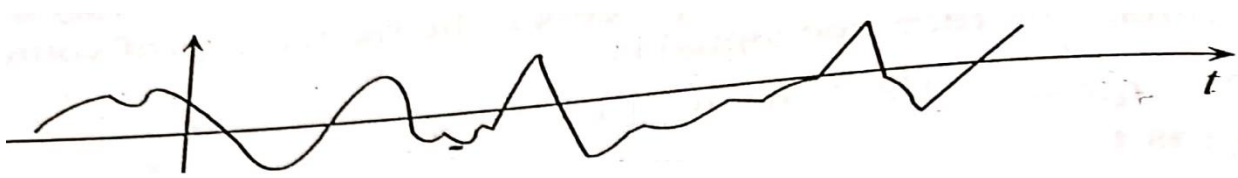
CLASSIFICATION OF RANDOM PROCESS

Based on the characteristics sample space of a random variable and time t random process are classified as

- ✓ Continuous Random Process
- ✓ Discrete Random Process
- ✓ Continuous Random Sequence
- ✓ Discrete Random Sequence

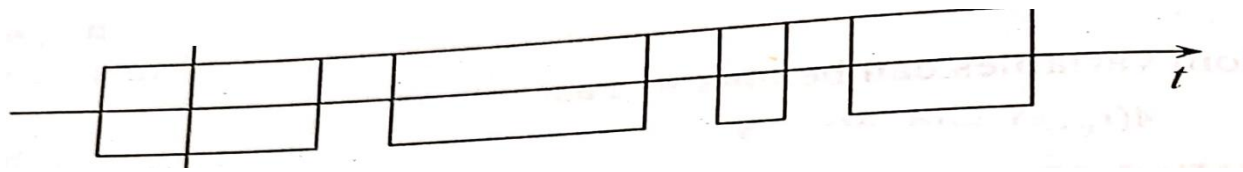
Continuous Random Process

A random process is said to be continuous, if random variable X and time t are continuous. It means that they can take any value



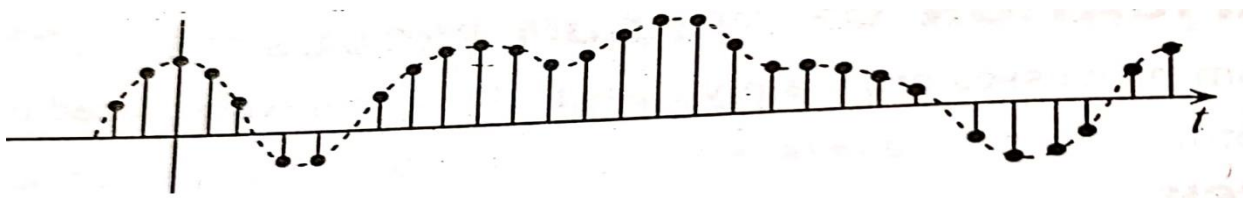
Discrete Random Process

A random process is said to be discrete, if random variable X is discrete and time t is continuous.



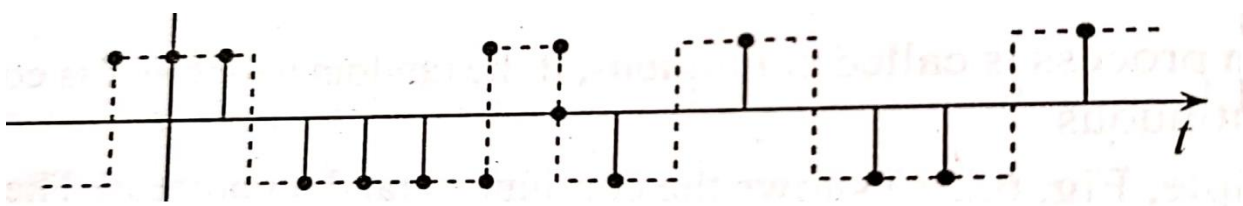
Continuous Random Sequence

A random sequence is said to be continuous, if random variable X is continuous and time t is discrete.



Discrete Random Sequence

A random sequence is said to be discrete, if random variable X and time t is discrete.



DETERMINISTIC RANDOM PROCESSES & NON DETERMINSTIC PROCESSES

A random process is said to be deterministic if its future values can be predicted from observed past values.

Ex: $X(t) = A\ \mathrm{Cos}(\omega_0 t + \theta)$

A random process is said to be non-deterministic if its future values cannot predicted from observed past values.

Ex: Random noise signal

DISTRIBUTION AND DENSITY FUNCTIONS OF RANDOM PROCESS

It is known that a random process becomes a random variable at specific time instant. Hence all the statistical properties of random variables are applicable to random processes. Based on the number of random variables various distribution and density functions are defined.

First order Distribution and Density Function

The first order distribution function of a random process is defined as

$$F_X(x_1;t_1) = \mathrm{P}\{X(t_1) \le x_1\}$$

Similarly the first order density function of random process is

$$f_X(x_1;t_1) = \frac{dF_X(x_1;t_1)}{dx_1}$$

Second order Distribution and Density Function

For two random variables at time instant $t_1 and\ t_2\ X(t_1) = X_1\ and\ X(t_2) = X_2$, the second order distribution (joint distribution) function of a random process is defined as

$$F_X\left(x_{1,}x_2;t_1,t_2\right) = \mathrm{P}\{X(t_1) \le x_1; X(t_2) \le x_2\}$$

The second order probability density function of random process is

$$f_X\left(x_{1,}x_2;t_1,t_2\right) = \frac{\partial^2 F_X\left(x_{1,}x_2;t_1,t_2\right)}{\partial x_1\,\partial x_2}$$

n^{th} order Distribution and Density Function

In general for N random variables n^{th} order joint distribution function is

$$F_X\left(x_{1,}x_2 \dots\dots x_n;t_1,t_2 \dots\dots t_n\right) = \mathrm{P}\{X(t_1) \le x_1; X(t_2) \le x_2 \dots\dots X(t_n) \le x_n\}$$

The nth order probability density function of random process is

$$f_X\left(x_{1,}x_2 \dots\dots x_n;t_1,t_2 \dots\dots t_n\right) = \frac{\partial^n F_X\left(x_{1,}x_2 \dots\dots x_n;t_1,t_2 \dots\dots t_n\right)}{\partial x_1\,\partial x_2 \dots\dots \partial x_n}$$

STATIONARY AND STATISTICAL INDEPENDENCE

Random process is generally characterized by two parameters

- Whether it is Stationary or not
- Whether the random variables involved are statistically independent or not (random process also)

First order Stationary Process

A random process X(t) is said to be first order stationary if its first order density function does not change with time.

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$$

where Δ is the minute increment or decrement of time

Whenever a random process is first order stationary then its average value or mean is constant over time.

$$E[X(t_1)] = E[X(t_2)]$$

$$= E[X(t_1 + \Delta)]$$

$$= constant$$

$$E[X(t)] = E[X(t + \Delta)]$$

$$= constant$$

Second order Stationary Process

A random process X(t) is said to be second order stationary if its second order density function does not change with time.

$$f_X(x_{1,}x_2; t_1, t_2) = f_X(x_{1,}x_2; t_1 + \Delta, t_2 + \Delta)$$

Let $E[X(t_1)\, X(t_2)]$ denote the correlation between two random variables X_1 *and* X_2 taken at time instants t_1 *and* t_2 then

$$R_{X_1X_2}(t, t + \Delta) = E[X(t)\, X(t + \Delta)]$$

$$= R_{XX}(\tau)$$

Where $R_{XX}(\tau)$ is auto correlation function of random process X(t).

If this auto correlation function is constant i.e.; independent on time such a random process is called second order stationary process.

Wide Sense Stationary Process

A random process is said to be wide sense stationary process if

$$E[X(t)] = constant$$

$$R_{XX}(\tau) = E[X(t)\, X(t+\tau)] = independent\ on\ time$$

n^{th} order Stationary Process

A random process X(t) is said to be n^{th} order stationary if its nth order density function does not change with time.

$$f_X(x_{1,}x_2 \dots \dots x_n; t_1, t_2 \dots \dots t_n) = f_X(x_{1,}x_2 \dots \dots x_n; t_1 + \Delta, t_2 + \Delta \dots \dots t_n + \Delta)$$

A random process is said to be n^{th} order stationary is also called strict sense stationary.

TIME AVERAGES:

The random process is also characterized by time average functions along with statistical averages. Statistical average of a random process is calculated by considering all sample functions at given time.

Time averages are calculated for any sample function. The time average of a random process is defined as a

$$A[\blacksquare] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\blacksquare]\ dt$$

Here A is used to denote time average in a manner analogous to E for the statistical average.

The time average of a random process $x(t)$ is given as

$$\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\ dt$$

Similarly the time average of $x(t)x(t+\tau)$ is called as time auto correlation function and is given by

$$\Re_{xx}(\tau) = A[x(t)\, x(t+\tau)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\ dt$$

The time auto correlation function is used to calculate the similarity between two random variables within a single random process.

The time cross correlation function measures the similarity between two different random processes.

$$\Re_{xy}(\tau) = A[x(t)\, y(t+\tau)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\ dt$$

ERGODIC THEOREM

This theorem stated that the all time averages $\bar{x}$ *and* $\Re_{xx}(\tau)$ of a random process are equal to Statistical averages $\bar{X}$ *and* R_{XX}

A random process is said to be ergodic if its satisfies ergodic theorem.

$$E[X(t)] = \ A[x(t)]$$

$$\bar{X} = \ \bar{x}$$

$$E[X(t)\, X(t+\tau)] = A[x(t)\, y(x+\tau)]$$

$$R_{XX}(\tau) = \Re_{xx}(\tau)$$

Mean Ergodic Random Process

A random process is said to be mean ergodic (or) ergodic in mean if the time average of $x(t)$ is equal to statistical average of X(t)

$$E[X(t)] = \ A[x(t)]$$

$$\bar{X} = \ \bar{x}$$

Auto Correlation Ergodic Random Process

A random process is said to be ergodic in auto correlation if the time auto correlation function is equal to statistical auto correlation function.

$$E[X(t)\,X(t+\tau)] = A[x(t)\,x(t+\tau)]$$

$$R_{XX}(\tau) = \Re_{xx}(\tau)$$

Cross Correlation Ergodic Random Process

A random process is said to be ergodic in cross correlation if the time cross correlation function is equal to statistical cross correlation function.

$$E[X(t)\,Y(t+\tau)] = A[x(t)\,y(t+\tau)]$$

$$R_{XY}(\tau) = \Re_{xy}(\tau)$$

AUTO CORRELATION FUNCTION

It is a measure of similarity between two random variables for a given random process. It is defined as the expected value of $x(t)\,x(t+\tau)$

$$R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)]$$

Properties of auto correlation function

1. $R_{XX}(\tau)$ cannot have an arbitrary shape.
2. The value of auto correlation function at origin $i.e;\ \tau = 0$ is equal to mean square value of the process (Average Power)

$$R_{XX}(0) = \overline{X^2(t)}$$

Proof:

We know that

$$R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$$

$let\ \tau = 0$

$$R_{XX}(0) = E[X(t)\,X(t)]$$
$$= E[X^2(t)]$$
$$R_{XX}(0) = \overline{X^2(t)}$$

3. The maximum value of auto correlation function occurs at origin

$$|R_{XX}(\tau)\ | \leq\ R_{XX}(0)$$

Proof: let X(t) be a wide sense stationary process such that $X(t_1) = X_1\ and\ X(t_2) = X_2$ then select a positive quantity such that

$$[X(t_1) \pm X(t_2)]^2 \geq 0$$

Apply expectation on both sides then

$$E[X(t_1) \pm X(t_2)]^2 \geq 0$$

$$E[X^2(t_1) + X^2(t_2) \pm 2\ X(t_1)\ X(t_2)] \geq 0$$

$$E[X^2(t_1)] + E[X^2(t_2)] \pm 2\ E[X(t_1)\ X(t_2)] \geq 0$$

Let $t_1 = t$ and $t_2 = t_1 + \tau = t + \tau$

$$E[X^2(t)] + E[X^2(t+\tau)] \pm 2\ E[X(t)\ X(t+\tau)] \geq 0$$

As statistical properties does not change with time, then

$$R_{XX}(0) + R_{XX}(0) \pm 2\ R_{XX}(\tau) \geq 0$$

$$2\ R_{XX}(0) \pm 2\ R_{XX}(\tau) \geq 0$$

$$|R_{XX}(\tau)|\ \leq\ |R_{XX}(0)|$$

4. Autocorrelation function is an even function

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

Proof:

We know that

$$R_{XX}(\tau) = E[X(t)\ X(t+\tau)]$$

Let $\tau = -\tau$

$$R_{XX}(-\tau) = E[X(t)\ X(t-\tau)]$$
$$t - \tau = u$$
$$t = u + \tau$$
$$R_{XX}(-\tau) = E[X(u+\tau)\ X(u)]$$
$$R_{XX}(-\tau) = R_{XX}(\tau)$$

5. When a random process X(t) is periodic with period T then the Autocorrelation function is also periodic.

$$R_{XX}(\tau + T) = R_{XX}(\tau)$$

We know that

$$R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$$

$$R_{XX}(\tau + T) = E[X(t)\, X(t+\tau+T)]$$

As X(t) is periodic $X(t+\tau+T) = X(t+\tau)$

$$R_{XX}(\tau + T) = E[X(t)\, X(t+\tau)]$$

$$R_{XX}(\tau + T) = R_{XX}(\tau)$$

6. If $E[X(t)] = \bar{X} \neq 0$ and X(t) is Ergodic with no periodic components, then the auto correlation function is given as

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

Proof:

We know that

$$R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$$

Since the process has no periodic components as $|\tau| \to \infty$, the random variables becomes independent.

$$\lim_{|\tau|\to\infty} E[X(t)\, X(t+\tau)] = E[X(t)\,]\, E[\, X(t+\tau)]$$

The given random process is Ergodic, then

$$\lim_{|\tau|\to\infty} E[X(t)\, X(t+\tau)] = E[X(t)\,]\, E[\, X(t)]$$

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

7. If X(t) is Ergodic, zero mean, and has no periodic components, then the auto correlation function is given as

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

Proof: from the above property

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

It is given that zero mean random process.

$$then \lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

8. Let there be a random process w(t) such that $w(t) = X(t) + Y(t)$. Then the auto correlation function of sum of random process is equal to

$$R_{WW}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

Proof:

We know that

$$R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$$

Given $w(t) = X(t) + Y(t)$

$$R_{WW}(\tau) = E[W(t)\,W(t+\tau)]$$

$$= E\big[\big(X(t) + Y(t)\big)\,\big(X(t+\tau) + Y(t+\tau)\big)\big]$$

$$= E\big[\big(X(t)\big)\,\big(X(t+\tau)\big)\big] + E\big[\big(Y(t)\big)\,\big(Y(t+\tau)\big)\big] + E\big[\big(X(t)\big)\,\big(Y(t+\tau)\big)\big] + E\big[\big(Y(t)\big)\,\big(X(t+\tau)\big)\big]$$

$$R_{WW}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

CROSS CORRELATION FUNCTION

It is a measure of similarity between two random processes X(t) and Y(t).

$$R_{XY}(t, t+\tau) = E[X(t)\,Y(t+\tau)]$$

Consider two random processes X(t) and Y(t) that are at least wide sense stationary.

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$$

Properties of auto correlation function

1. The cross correlation function is an even function

$$R_{XY}(-\tau) = R_{YX}(\tau)$$

Proof:

We know that

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$$

Let $\tau = -\tau$

$$R_{XY}(-\tau) = E[X(t)\,Y(t-\tau)]$$

$$t - \tau = u$$

$$t = u + \tau$$

$$R_{XY}(-\tau) = E[X(u+\tau)\,Y(u)]$$

$$R_{XY}(-\tau) = E[Y(u)X(u+\tau)\,]$$

$$R_{XY}(-\tau) = R_{YX}(\tau)$$

2. The cross correlation function of a random process is always less than or equal to the geometric mean of individual auto correlation functions.

$$R_{XY}(\tau) \leq \sqrt{R_{XX}(0)\,R_{YY}(0)}$$

Proof:

Let X(t) and Y(t) be two random processes such that

$$[Y(t+\tau) \pm \alpha\,X(t)]^2 \geq 0$$

Apply expectation on both sides then

$$E[Y(t+\tau) \pm \alpha\,X(t)]^2 \geq 0$$

$$E[Y^2(t+\tau) + \alpha^2 X^2(t) \pm 2\,\alpha X(t)\,Y(t+\tau)] \geq 0$$

$$E[Y^2(t+\tau)] + \alpha^2 E[X^2(t)] \pm 2\,\alpha\,E[X(t)\,Y(t+\tau)] \geq 0$$

Given processes are stationary, hence

$$\alpha^2\, E[X^2(t)] + E[Y^2(t)] \pm 2\alpha\, E[X(t)\, Y(t+\tau)] \geq 0$$

$$R_{XX}(0)\, \alpha^2 \pm 2\alpha\, R_{XY}(\tau) + R_{YY}(0) \geq 0$$

The above equation of the form $ax^2 + bx + c$. Hence the roots of the above equation are given as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{\mp 2\, R_{XY}(\tau) \pm \sqrt{4\, R^2{}_{XY}(\tau) - 4R_{XX}(0)\, R_{YY}(0)}}{2R_{XX}(0)} \geq 0$$

$$= \frac{\mp\, R_{XY}(\tau) \pm \sqrt{R^2{}_{XY}(\tau) - R_{XX}(0)\, R_{YY}(0)}}{2R_{XX}(0)} \geq 0$$

$$R^2{}_{XY}(\tau) - R_{XX}(0)\, R_{YY}(0) \geq 0$$

$$R^2{}_{XY}(\tau) \leq\ R_{XX}(0)\, R_{YY}(0)$$

$$R_{XY}(\tau) \leq\ \sqrt{R_{XX}(0)\, R_{YY}(0)}$$

3. The cross correlation function of a random process is always less than or equal to the arthematic mean of individual auto correlation functions.

$$R_{XY}(\tau) \leq \frac{R_{XX}(0) +\ R_{YY}(0)}{2}$$

Proof:

We know that

$$R_{XY}(\tau) = E[X(t)\, Y(t+\tau)]$$

$$R_{XY}(\tau) \leq\ \sqrt{R_{XX}(0)\, R_{YY}(0)}$$

The geometric mean of any given series is always less than or equal to arthematic mean

$$\sqrt{R_{XX}(0)\, R_{YY}(0)}\ \leq \frac{R_{XX}(0) +\ R_{YY}(0)}{2}$$

$$R_{XY}(\tau) \leq \frac{R_{XX}(0) +\ R_{YY}(0)}{2}$$

4. For two random processes X(t) and Y(t) having non-zero mean and are statically independent

$$R_{XY}(\tau) = \bar{X}\ \bar{Y}$$

Proof:

We know that

$$R_{XY}(\tau) = E[X(t)\, Y(t+\tau)]$$

As X(t) and Y(t) having non-zero mean and are statically independent

$$R_{XY}(\tau) = E[X(t)\,]\, E[\, Y(t+\tau)]$$

$$as \text{ X(t) and Y(t) } are\ WSS\ then\ \ E[\, Y(t+\tau)] = E[\, Y(t)]$$

$$R_{XY}(\tau) = E[X(t)\,]\, E[\, Y(t)]$$

$$R_{XY}(\tau) = \bar{X}\ \bar{Y}$$

COVARIANCE FUNCTION

The Covariance function is a measure of interdependence between two random variables of the random process X(t).

Auto Covariance function

$$C_{XX}(t, t+\tau) = E[\{X(t) - E[X(t)]\} - \{X(t+\tau) - E[X(t+\tau)]\}]$$

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)]\, E[\, X(t+\tau)]$$

Cross-Covariance function

$$C_{XY}(t, t+\tau) = E[\{X(t) - E[X(t)]\} - \{Y(t+\tau) - E[Y(t+\tau)]\}]$$

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E[X(t)]\, E[\, Y(t+\tau)]$$

NOTE:

1. If X(t) is at least wide sense stationary random process then

$$C_{XX}(\tau) = R_{XX}(\tau) - (\bar{X})^2$$

2. At $\tau = 0$

$$C_{XX}(0) = R_{XX}(0) - (\bar{X})^2 = {\sigma_X}^2$$

3. If X(t) and Y(t) is at least jointly wide sense stationary random process then

$$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\,\bar{Y}$$

DESCRPTIVE QUESTIONS

1. Explain numerous categories of random processes with examples.
2. Explain stationarity of random processes.
3. Interpret about ergodic random processes.
4. Interpret the significance of time averages and ergodicity.
5. Choose necessary expressions to verify the properties of Auto correlation function.
6. Choose relevant expressions to verify the properties of cross correlation function.
7. Interpret the concepts of covariance with relevance to random processes.

PROBLEMS

1. A random process is described by X(t) = A, where A is a continuous random variable and is uniformly distributed on (0,1). Show that X(t) is wide sense stationary.

2. Verify the Sine wave process X(t) = B sin ((ω_0t), where B is uniform random variable on (-1,1) is wide sense stationary or not.

3. A random process is given as X(t) = A Cos(ω_0t + θ), where A and ω_0 are constants and θ is uniformly distributed random variable on the interval (0, 2π). Verify whether given random process is wide sense stationary or not.
4. Two random process X(t) & Y(t) are defined as
   X(t) = A Cos(ω_0t) + B Sin(ω_0t) and Y(t) = B Cos(ω_0t) – A Sin(ω_0t), where A, B are uncorrelated, zero mean random variables with same variances and ω_0 is constant. Verify whether X(t),Y(t) are Jointly wide sense stationary or not

5. A random process is defined as X(t) = A Cos(ω_0t), where ω_0 is a constant and A is a random variable uniformly distributed over (0,1).Estimate the autocorrelation function.

6. A random process is given as X(t) = A Cos(ω_0t) + B Sin(ω_0t), where A & B are uncorrelated, zero mean random variables having same variance σ^2 then appraise whether X(t) is wide sense stationary or not.

7. A Random Process Y(t) = X(t) – X(t+τ) is defined in terms X(t) that is at least wide sense stationary
   (i) Deduce the mean value of Y(t) if E[X(t)] ≠ 0.
   (ii) Justify that the variance $\sigma_Y{}^2 = 2[R_{XX}(0) - R_{XX}(\tau)]$.
   (iii) If Y(t) = X(t) + X(t+τ), estimate E[Y(t)] and $\sigma_Y{}^2$.
8. Two statistically independent zero mean random processes X(t), Y(t) have auto correlation functions $R_{XX}(\tau) = \exp(-|\tau|)$, $R_{YY}(\tau) = \cos(2\pi\tau)$ respectively. Evaluate the
   (i) Autocorrelation of the Sum $W_1(t) = X(t)+Y(t)$
   (ii) Autocorrelation of the Difference $W_2(t) = X(t)-Y(t)$
   (iii) Cross correlation of $W_1(t)$ & $W_2(t)$

9. Given $\overline{X} = 6$ and $R_{XX}(t, t+\tau) = 36 + 25\exp(-\tau)$ for a random process X(t). Indicate which of the following statements are true and give the reason.
   (i) Is first order stationary?
   (ii) Has total average power of 61W
   (iii) Is wide sense stationary?
   (iv) Has a periodic component
   (v) Has an AC power of 36W

10. Show that X(t) & Y(t) are Jointly WSS, if random processes, $X(t) = ACos(\omega_1 t + \theta)$, $Y(t) = BCos(\omega_2 t + \Phi)$, where A,B, ω_1 & ω_2 are constants, while Φ, θ are Statistically independent uniform random variables on (0,2Π).

11. If $X(t) = A\,Cos(\omega_0 t + \theta)$, where A, ω_0 are constants, and θ is a uniform random variable on (-π, π). A new random process is defined by $Y(t) = X^2(t)$.
   (i) Obtain the Mean and Auto Correlation Function of X(t).
   (ii) Obtain the Mean and Auto Correlation Function of Y(t).
   (iii) Find the Cross Correlation Function of X(t) & Y(t).
   (iv) Are X(t) and Y(t) are WSS?
   (v) Are X(t) & Y(t) are Jointly WSS.

1. A random process is described by X(t) = A, where A is a continuous random variable and is uniformly distributed on (0,1). Show that X(t) is wide sense stationary.

Sol:

A random process is said to be wide sense stationary process if

$$mean\ value\ E[X(t)] = constant$$

$$correlation\ R_{XX}(t, t+\tau) = E[X(t)\ X(t+\tau)] = independent\ on\ time$$

$$E[X(t)] = \int_{-\infty}^{\infty} x(t).f_X(x)\ dx$$

Given A is uniformly distributed random variable in the interval (0, 1).

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\ f_A(A)\ dA$$

Density function of uniformly distributed random variable is

$$f_X(x) = \frac{1}{b-a} \qquad a \le X \le b$$

$$f_A(A) = \frac{1}{1-0} = 1$$

$$E[X(t)] = \int_{0}^{2\pi} A\ dA$$

$$= \left[\frac{A^2}{2}\right]_0^1$$

$$= \frac{1}{2} = constant$$

$$R_{XX}(t, t+\tau) = E[X(t)\ X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} x(t)\ \ x(t+\tau)\ f_A(A)\ dA$$

$$= \int_0^1 A\,A\;dA$$

$$= \int_0^1 \mathrm{A}^2\;\;dA$$

$$= \left[\frac{\mathrm{A}^3}{2}\right]_0^1 = \frac{1}{2} \text{ Independent on time}$$

Hence given RP is WSS.

2. A random process is given as X(t) = A Cos(ω_0t + θ), where A and ω_0 are constants and θ is uniformly distributed random variable on the interval (0, 2π). Verify whether given random process is wide sense stationary or not.

Sol:

A random process is said to be wide sense stationary process if

$$E[X(t)] = constant$$

$$R_{XX}(\tau) = E[X(t)\,X(t+\tau)] = independent\ on\ time$$

$$E[X(t)] = \int_{-\infty}^{\infty} x(t).f_X(x)\;\;dx$$

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\,f_\theta(\theta)\;\;d\theta$$

Given θ is uniformly distributed random variable on the interval (0, 2π).

$$f_X(x) = \frac{1}{b-a} \qquad a \le X \le b$$

$$f_\theta(\theta) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$$

$$E[X(t)] = \int_0^{2\pi} \mathrm{A}\cos(w_0 t + \theta)\;\frac{1}{2\pi}\;d\theta$$

$$= A\left[\frac{\sin(w_0 t+\theta)}{2\pi}\right]_0^{2\pi}$$

$$= \frac{\mathrm{A}}{2\pi}[\,\sin(w_0 t+2\pi) - \sin(w_0 t+0)]$$

$$= \frac{\mathrm{A}}{2\pi}[\,\sin(w_0 t) - \sin(w_0 t)] = 0$$

$$= constant$$

$$R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} x(t)\;\; x(t+\tau)\, f_\theta(\theta)\;\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (A\;\cos(w_0 t+\theta)\quad \mathrm{A}\cos(w_0(t+\tau)+\theta)\,)\;\; d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\,A\;\cos(w_0 t+\theta)\;\mathrm{A}\cos(w_0 t+\theta+w_0\tau)\,)\;\; d\theta$$

$$2\cos\mathrm{A}\cos\mathrm{B} = \cos(\mathrm{A}+\mathrm{B}) + \cos(\mathrm{A}-\mathrm{B})$$

$$= \frac{\mathrm{A}^2}{4\pi}\int_0^{2\pi} (\,\cos(2w_0 t+2\theta+\tau) \;+\; \cos(w_0\tau)\,)\;\; d\theta$$

$$= \frac{\mathrm{A}^2}{4\pi}\left[\int_0^{2\pi} (\,\cos(2w_0 t+2\theta+\tau)\,)d\theta + \int_0^{2\pi} (\,\cos(w_0\tau)\,)d\theta\right]$$

$$= \frac{\mathrm{A}^2}{4\pi}[\,2w_0\,\sin(2w_0 t+2\theta+\tau)\,]_0^{2\pi} + \cos(w_0\tau)\;\;\theta]_0^{2\pi}\,]$$

$$= \frac{\mathrm{A}^2}{4\pi}\;\cos(w_0\tau)\;\;2\pi$$

$$= \frac{\mathrm{A}^2}{2}\;\cos(w_0\tau)\;\;\text{Independent on time}$$

Hence given RP is WSS.

3. Verify the Sine wave process X(t) = B sin ((ω_0t), where B is uniform random variable on (-1,1) is wide sense stationary or not.

Sol:

A random variable is said to be wide sense stationary process if

$$E[x(t)] = constant$$

$$R_{XX}(\tau) = E[x(t)x(t+\tau)] = independent\ on\ time$$

$$E[x(t)] = \int_{-\infty}^{\infty} x(t)\, f_x(x)dx$$

$$E[x(t)x(t,t+\tau)] = \int_{-\infty}^{\infty} x(t)x(t+\tau)\, f_x(x)dx$$

B is a uniform random variable on (-1,1) with the density function

$$f_B(B) = \frac{1}{b-a}$$

$$f_B(B) = \frac{1}{1-(-1)} = \frac{1}{2}$$

$$E[x(t)] \ = \int_{-1}^{1} (Bsin\omega_0 t)\frac{1}{2}dB$$

$$= \frac{sin\omega_0 t}{2}\int_{-1}^{1} B\, dB$$

$$= \frac{sin\omega_0 t}{2}\left[\frac{B^2}{2}\right]_{-1}^{1}$$

$$= \frac{sin\omega_0 t}{2}\left[\frac{1}{2}-\frac{1}{2}\right] = 0 = Constant$$

$$R_{XX}(t,t+\tau) = E[x(t)x(t+\tau)]$$

$$= \int_{-1}^{1} (Bsin(\omega_0 t))(Bsin(\omega_0(t+\tau))\frac{1}{2}dB$$

$$= \frac{(sin(\omega_0 t))(sin(\omega_0(t+\tau))}{2} \int_{-1}^{1} B^2 \, dB$$

$$sinAsinB = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$= \frac{(\cos(\omega_0\tau) - \cos(2\omega_0 t - \omega_0\tau))}{4} \left[\frac{B^3}{3}\right]_{-1}^{1}$$

$$= \frac{(\cos(\omega_0\tau) - \cos(2\omega_0 t - \omega_0\tau))}{4} \left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right]$$

$$= \frac{(\cos(\omega_0\tau) - \cos(2\omega_0 t - \omega_0\tau))}{4} \left[\frac{2}{3}\right]$$

$$= \frac{(\cos(\omega_0\tau) - \cos(2\omega_0 t - \omega_0\tau))}{6}$$

$$= R_{XX}(t, t+\tau) = dependent\ of\ time(t)$$

Hence the given random process is not wide sense stationary

4. A random process $Y(t) = X(t) - X(t+\tau)$ is defined in terms of $X(t)$ that is at least wide sense stationary.

   (i) Deduce the mean value of $Y(t)$ if $E[X(t)] \neq 0$

   (ii) Justify that the variance ${\sigma_Y}^2 = 2[R_{XX}(0) - R_{XX}(\tau)]$

   (iii) If $Y(t) = X(t) - X(t+\tau)$, estimate $E[Y(t)]$ and ${\sigma_Y}^2$

Sol: Given,

$$Y(t) = X(t) - X(t+\tau)$$

Given $X(t)$ is wide sense stationary and $E[X(t)] \neq 0$

(i) Mean of $Y(t)$:

$$E[Y(t)] = E[X(t) - X(t+\tau)]$$

$$= E[X(t)] - E[X(t+\tau)]$$

$$= E[X(t)] - E[X(t)][\because X(t) \; is \; WSS]$$

$$\therefore E[Y(t)] = 0$$

(ii) Variance of $Y(t)$:

$$\sigma_Y{}^2 = E[Y^2(t)] - (E[Y(t)])^2$$

$$= E\left[\left(X(t) - X(t+\tau)\right)^2\right] - 0[\because E[Y(t)] = 0]$$

$$= E[X^2(t) + X^2(t+\tau) - 2X(t)X(t+\tau)]$$

$$= E[X^2(t)] + E[X^2(t+\tau)] - 2E[X(t)X(t+\tau)]$$

$$= E[X^2(t)] + E[X^2(t)] - 2R_{XX}(\tau)[\because X(t) \; is \; WSS]$$

$$= R_{XX}(0) + R_{XX}(0) - 2R_{XX}(\tau)\left[\because R_{XX}(0) = \overline{X^2(t)}\right]$$

$$= 2R_{XX}(0) - 2R_{XX}(\tau)$$

$$\therefore \sigma_Y{}^2 = 2[R_{XX}(0) - R_{XX}(\tau)]$$

(iii) Given,

$$Y(t) = X(t) + X(t+\tau)$$

Now,

$$E[Y(t)] = E[X(t) + X(t+\tau)]$$

$$= E[X(t)] + E[X(t+\tau)]$$

$$= E[X(t)] + E[X(t)][\because X(t) \; is \; WSS]$$

$$= 2\,E[X(t)]$$

$$\therefore E[Y(t)] = 2\,E[X(t)]$$

Now,

$$\sigma_Y{}^2 = E[Y^2(t)] - (E[Y(t)])^2$$

$$= E\left[\left(X(t) - X(t+\tau)\right)^2\right] - (2\,E[X(t)])^2 \left[\because E[Y(t)] = 2\,E[X(t)]\right]$$

$$= E[X^2(t) + X^2(t+\tau) + 2X(t)X(t+\tau)] - 4(E[X(t)])^2$$

$$= E[X^2(t)] + E[X^2(t+\tau)] + 2E[X(t)X(t+\tau)] - 4(E[X(t)])^2$$

$$= E[X^2(t)] + E[X^2(t)] + 2R_{XX}(\tau) - 4(E[X(t)])^2 [\because\ X(t)\ is\ WSS\ ]$$

$$= R_{XX}(0) + R_{XX}(0) + 2R_{XX}(\tau) - 4(E[X(t)])^2 \left[\because R_{XX}(0) = \overline{X^2(t)}\right]$$

$$= 2R_{XX}(0) + 2R_{XX}(\tau) - 4\bar{X}^2$$

$$= 2[R_{XX}(0) + R_{XX}(\tau) - 2\bar{X}^2]$$

$$\therefore\ {\sigma_Y}^2 = 2[R_{XX}(0) + R_{XX}(\tau) - 2\bar{X}^2]$$

5. A random process is given as X(t) = A Cos(ω_0t) + B Sin(ω_0t), where A & B are uncorrelated, zero mean random variables having same variance σ^2 then appraise whether X(t) is wide sense stationary or not.

Sol:

$$Given,$$

$$X(t) = A\cos(\omega_0 t) + B\ \sin(\omega_0 t)$$

$$If\ A\ \&\ B\ are\ uncorrelated$$

$$E(AB) = E(A)\,E(B)$$

$$But,$$

$$E(A) = E(B) = 0$$

$$E(AB) = 0$$

$$Now,$$

$$\sigma_A^2 = E[A^2] - (E[A])^2$$

$$= E[A^2]$$

$$\sigma_B^2 = E[B^2] - (E[B])^2$$

$$= E[B^2]$$

$$\therefore E[A^2] = \sigma^2 \; and \; E[B^2] = \sigma^2$$

A random process is said to be wide sense stationary process if

$$\boxed{\begin{array}{c} E[X(t)] = constant \\ R_{XX}\,(t, t+\tau) = E[X(t)\,X(t\;+\;\tau) = independent\;of\;time \end{array}}$$

Now,

$$E[X(t)] \;=\; E[A\;\cos(\omega_0 t) +\; B\;Sin(\omega_0 t)]$$

$$= E[A]\;Cos(\omega_0 t) \;+\; E[B]\;Sin(\omega_0 t)$$

$$= 0 \qquad\qquad (Since\;E[A] = E[B] = 0)$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= \boldsymbol{E[\,(A\cos(\omega_0 t) + B\;\sin(\omega_0 t))\;(\,A\cos(\omega_0 t + \omega_0\tau) + B\;\sin(\omega_0 t + \omega_0\tau))]}$$

$$= E[\,(\,A^2\;\cos(\omega_0 t)\;\cos(\omega_0 t + \omega_0\tau)\;+\;AB\;\;\cos(\omega_0 t)$$

$$+ AB\;\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)\;+ B^2\sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)]$$

$$= E[A^2]\;\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)\;+\;E[AB]\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)$$

$$+\;\;E[AB]\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) + E[B^2]\;\sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)$$

$$= \sigma^2\;\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) + 0 + 0 + \sigma^2\sin(\omega_0 t)\;\sin(\omega_0 t + \omega_0\tau)$$

$$\boldsymbol{(Since\;E[A^2] = E[B^2] = \sigma^2\;E[AB] = 0)}$$

$$= \sigma^2\;[\,\cos(\omega_0 t)\;\cos(\omega_0 t + \omega_0\tau) + \sin(\omega_0 t)\;\sin(\omega_0 t + \omega_0\tau)]$$

$$[Since\ CosA\ CosB\ +\ SinA\ SinB = Cos(A-B)]$$

$$= \sigma^2\ \cos[\omega_0 t - (\omega_0 t + \omega_0 \tau)]$$

$$=\ \sigma^2 \cos(\omega_0 \tau)$$

$$=\ R_{XX}(\tau)$$

$$Independent\ of\ time$$

$$Hence\ given\ Random\ proces\ is\ Wide\ Sense\ Stationary$$

6. Two random processes $X(t)$& $Y(t)$ are defined as

$X(t) = A\ Cos(\omega_0 t) + B\ Sin(\omega_0 t)$ and $Y(t) = B\ Cos(\omega_0 t) - A\ Sin(\omega_0 t)$, where A, B are uncorrelated, zero mean random variables with same variances and ω_0 is constant. Verify whether$X(t), Y(t)$ are Jointly wide sense stationary or not.

Sol:Given,

$$X(t) = A\ cos(\omega_0 t) + B\ sin(\omega_0 t)$$

$$Y(t) = B\ cos(\omega_0 t) - A\ sin(\omega_0 t)$$

<u>Conditions to be satisfied to be Jointly WSS</u>:

Mean of X(t): $\overline{X(t)} =\ constant$
Mean of Y(t):$\overline{Y(t)} =\ constant$
Auto Correlation of X(t)&Y(t): $R_{XX} and\ R_{YY} should\ be\ independent\ on\ time$
Cross Correlation of X(t) & Y(t):$R_{XY} or\ R_{YX} should\ be\ independent\ on\ time$

Given that,

$E[A] = E[B] = 0$ and A, B are uncorrelated.

$$\therefore E[AB] = E[A]E[B] = 0$$

$$\text{Now, } \sigma_A{}^2 = E[A^2] - (E[A])^2$$

$$= E[A^2] - 0$$

$$\therefore \sigma_A{}^2 = E[A^2]$$

Let $\sigma_A{}^2 = E[A^2] = \sigma^2$

Then, $\sigma_B{}^2 = E[B^2] - (E[B])^2$

$$= E[B^2] - 0$$

$$= E[B^2]$$

$$\therefore \sigma_B{}^2 = E[B^2] = \sigma^2$$

[$\because$ Given A and B have same variance]

<u>*To verify Jointly WSS or not*</u>:

Calculation of Mean:

$$E[X(t)] = E[A\, cos(\omega_0 t) + B\, sin(\omega_0 t)]$$

$$= E[A]cos(\omega_0 t) + E[B]sin(\omega_0 t)[\because E[A] = E[B] = 0\,]$$

$$\therefore \overline{\boldsymbol{X(t)}} = \mathbf{0} \Rightarrow \boldsymbol{constant}$$

$$E[Y(t)] = E[B\, cos(\omega_0 t) - A\, sin(\omega_0 t)]$$

$$= E[B]cos(\omega_0 t) - E[A]sin(\omega_0 t)[\because E[A] = E[B] = 0\,]$$

$$\therefore \overline{\boldsymbol{Y(t)}} = \mathbf{0} \Rightarrow \boldsymbol{constant}$$

Calculation of Auto Correlation Functions:

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E\big[\big(A\, cos(\omega_0 t) + B\, sin(\omega_0 t)\big)\big(A\, cos(\omega_0 t + \omega_0 \tau) + B\, sin(\omega_0 t + \omega_0 \tau)\big)\big]$$

$$= \boldsymbol{E[A^2\, cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + AB\, cos(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)}$$
$$\boldsymbol{+\ AB\, sin(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + B^2\ sin(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)]}$$

$$= \boldsymbol{E[A^2] cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + E[AB] cos(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)}$$
$$\boldsymbol{+ E[AB] sin(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + E[B^2] sin(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)}$$

$$= \sigma^2 cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + 0 + 0 + \sigma^2 sin(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)$$

$$= \sigma^2 \left[cos\left(\omega_0 t - (\omega_0 t + \omega_0 \tau)\right)\right] [\because Cos(A - B) = CosACosB + SinASinB]$$

$$= \sigma^2\ cos(\omega_0 \tau)$$

$$\boldsymbol{\therefore R_{XX}(t, t + \tau) = R_{XX}(\tau) \Rightarrow Independent\ on\ time}$$

$$R_{YY}(t, t + \tau) = E[Y(t) Y(t + \tau)]$$

$$= E\left[\left(B\ cos(\omega_0 t) - A\ sin(\omega_0 t)\right)\left(B\ cos(\omega_0 t + \omega_0 \tau) - A\ sin(\omega_0 t + \omega_0 \tau)\right)\right]$$

$$= E[B^2\ cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) - AB\ cos(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)$$
$$-\ AB\ sin(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + A^2\ sin(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)]$$

$$= E[B^2] cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) - E[AB] cos(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)$$
$$- E[AB] sin(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) + E[A^2] sin(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)$$

$$= \sigma^2 cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) - 0 - 0 + \sigma^2 sin(\omega_0 t) sin(\omega_0 t + \omega_0 \tau)$$

$$= \sigma^2 \left[cos\left(\omega_0 t - (\omega_0 t + \omega_0 \tau)\right)\right] [\because Cos(A - B) = CosACosB + SinASinB]$$

$$= \sigma^2\ cos(\omega_0 \tau)$$

$$\boldsymbol{\therefore R_{YY}(t, t + \tau) = R_{YY}(\tau) \Rightarrow Independent\ on\ time}$$

Calculation of Cross Correlation Function:

$$R_{XY}(t, t + \tau) = E[X(t) Y(t + \tau)]$$

$$= E\left[\left(A\ cos(\omega_0 t) + B\ sin(\omega_0 t)\right)\left(B\ cos(\omega_0 t + \omega_0 \tau) - A\ sin(\omega_0 t + \omega_0 \tau)\right)\right]$$

$$= E[AB\ cos(\omega_0 t)cos(\omega_0 t + \omega_0 \tau) - A^2\ cos(\omega_0 t)sin(\omega_0 t + \omega_0 \tau)$$
$$+\ \ B^2\ sin(\omega_0 t)cos(\omega_0 t + \omega_0 \tau) - AB\ sin(\omega_0 t)sin(\omega_0 t + \omega_0 \tau)]$$

$$= E[AB]cos(\omega_0 t)cos(\omega_0 t + \omega_0 \tau) - E[A^2]cos(\omega_0 t)sin(\omega_0 t + \omega_0 \tau)$$
$$+ E[B^2]sin(\omega_0 t)cos(\omega_0 t + \omega_0 \tau) - E[AB]sin(\omega_0 t)sin(\omega_0 t + \omega_0 \tau)$$

$$= 0 - \sigma^2 cos(\omega_0 t)sin(\omega_0 t + \omega_0 \tau) + \sigma^2 sin(\omega_0 t)cos(\omega_0 t + \omega_0 \tau) - 0$$

$$= \sigma^2\big[sin\big(\omega_0 t - (\omega_0 t + \omega_0 \tau)\big)\big][\because Sin(A - B) = SinACosB + CosASinB]$$

$$= \sigma^2[sin(-\omega_0 \tau)]$$

$$= -\sigma^2 sin(\omega_0 \tau)$$

$\therefore \boldsymbol{R_{XY}(t, t + \tau) = R_{XY}(\tau) \Rightarrow Independent\ on\ time}$

Hence, $X(t)$and$Y(t)$ are Jointly wide sense stationary.

7. A random process is defined as $X(t) = A\ Cos(\omega_0 t)$, where ω_0 is a constant and A is a random variable uniformly distributed over (0,1). Estimate the autocorrelation function.

Sol: Given,

$X(t) = A\ cos(\omega_0 t)$, where ω_0 is a constant.

Auto Correlation Function: $R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$

Now, $E[X(t)X(t + \tau)] = \int_{-\infty}^{\infty} x(t)\,.\,x(t + \tau) f_X(x)\, dx$

Given A is a uniformly distributed random variable in the interval (0,1).

$$E[X(t)X(t + \tau)] = \int_{-\infty}^{\infty} x(t).x(t + \tau) f_A(A)\, dA$$

Density function of a uniformly distributed random variable is

$$f_X(x) = \frac{1}{b-a} \;\; ; \; a \le X \le b$$

$$f_A(A) = \frac{1}{1-0} = 1$$

$$\Rightarrow E[X(t)X(t+\tau)] = \int_0^1 A\, cos(\omega_0 t) A\, cos(\omega_0 t + \omega_0 \tau)\,(1) dA$$

$$= cos(\omega_0 t) cos(\omega_0 t + \omega_0 \tau) \int_0^1 A^2 dA$$

$$= \frac{1}{2}\big(cos(2\omega_0 t + \omega_0 \tau) + cos(\omega_0 \tau)\big)\left[\frac{A^3}{3}\right]_0^1$$

$[\because 2CosACosB = Cos(A+B) + Cos(A-B)]$

$$= \frac{1}{2}\big(cos(2\omega_0 t + \omega_0 \tau) + cos(\omega_0 \tau)\big)\left[\frac{1}{3}\right]$$

$$\therefore E[X(t)X(t+\tau)] = \frac{1}{6}\big(cos(2\omega_0 t + \omega_0 \tau) + cos(\omega_0 \tau)\big)$$

$$\therefore \boldsymbol{R_{XX}(t, t+\tau) = \frac{1}{6}\big(cos(2\omega_0 t + \omega_0 \tau) + cos(\omega_0 \tau)\big) \Rightarrow Dependent\ on\ time}$$

8. Two statistically independent zero mean random processes $X(t)$, $Y(t)$ have autocorrelation functions $R_{XX}(\tau) = exp(-|\tau|)$, $R_{YY}(\tau) = \cos(2\pi\tau)$ respectively. Evaluate the
   (i) Autocorrelation of the Sum$W_1(t) = X(t) + Y(t)$
   (ii) Autocorrelation of the Difference $W_2(t) = X(t) - Y(t)$
   (iii) Crosscorrelation of $W_1(t)$ and $W_2(t)$

Sol: Given,

$E[X(t)] = 0$ and $E[Y(t)] = 0$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = exp(-|\tau|)$$

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = \cos(2\pi\tau)$$

Now, cross correlation of $X(t)$ and $Y(t)$ is

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

$$= E[X(t)]E[Y(t+\tau)][\because X(t)\ and\ Y(t)\ are\ statistically\ independent\ ]$$

$$= (0)(E[Y(t+\tau)])$$

$\therefore R_{XY}(\tau) = 0 \& R_{YX}(\tau) = 0$

(i) *Auto correlation of* $W_1(t)$:

Given, $W_1(t) = X(t) + Y(t)$

Now, $R_{W_1W_1}(t, t+\tau) = E[W_1(t)W_1(t+\tau)]$

$$= E\big[\big(X(t) + Y(t)\big)\big(X(t+\tau) + Y(t+\tau)\big)\big]$$

$$= E[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)]$$

$$= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

$$= exp(-|\tau|) + 0 + 0 + cos(2\pi\tau)$$

$$\therefore\ \boldsymbol{R_{W_1W_1}(t, t+\tau) = exp(-|\tau|) + cos(2\pi\tau)}$$

(ii)*Auto correlation of* $W_2(t)$:

Given, $W_2(t) = X(t) - Y(t)$

Now,

$$R_{W_2W_2}(t, t+\tau) = E[W_2(t)W_2(t+\tau)]$$

$$= E\big[\big(X(t) - Y(t)\big)\big(X(t+\tau) - Y(t+\tau)\big)\big]$$

$$= E[X(t)X(t+\tau) - X(t)Y(t+\tau) - Y(t)X(t+\tau) + Y(t)Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)] - E[X(t)Y(t+\tau)] - E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)]$$

$$= R_{XX}(\tau) - R_{XY}(\tau) - R_{YX}(\tau) + R_{YY}(\tau)$$

$$= exp(-|\tau|) - 0 - 0 + cos(2\pi\tau)$$

$$\therefore \boldsymbol{R_{W_2W_2}(t, t+\tau) = exp(-|\tau|) + cos(2\pi\tau)}$$

(iii)*Cross correlation of* $W_1(t)$ and $W_2(t)$:

$$R_{W_1W_2}(t, t+\tau) = E[W_1(t)W_2(t+\tau)]$$

$$= E\big[\big(X(t) + Y(t)\big)\big(X(t+\tau) - Y(t+\tau)\big)\big]$$

$$= E[X(t)X(t+\tau) - X(t)Y(t+\tau) + Y(t)X(t+\tau) - Y(t)Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)] - E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] - E[Y(t)Y(t+\tau)]$$

$$= R_{XX}(\tau) - R_{XY}(\tau) + R_{YX}(\tau) - R_{YY}(\tau)$$

$$= exp(-|\tau|) - 0 + 0 - cos(2\pi\tau)$$

$$\therefore \boldsymbol{R_{W_1W_2}(t, t+\tau) = exp(-|\tau|) - cos(2\pi\tau)}$$

9. Given $\bar{X} = 6$ and $R_{XX}(t, t+\tau) = 36+25\, exp(-\tau)$ for a random process $X(t)$. Indicate which of the following statements are true and give the reason.
   (i) Is first order stationary?
   (ii) Has total average power of 61W
   (iii) Is wide sense stationary?
   (iv) Has a periodic component
   (v) Has an AC power of 36W

Sol: Given,

$$\bar{X} = 6 \Rightarrow constant$$

(i) *First order stationary*:

It is first order stationary because the mean value $[\bar{X} = 6]$ is constant.

Hence, the given statement is true.

(ii) *Total Average Power*:

The average power is the mean square value of the process$X(t)$.

Average Power: $\boldsymbol{E[X^2(t)] = \overline{X^2(t)} = R_{XX}(0)}$

Given,

$R_{XX}(t, t+\tau) = 36+25\ exp(-\tau)$

$\Rightarrow R_{XX}(\tau) = 36+25\ exp(-\tau)$

Now,

$R_{XX}(0) = 36+25\ exp(-0) = 36 + 25(1) = 61$

∴ Average Power = 61W

Hence, the given statement is true.

(iii) *Wide Sense Stationary*:

Here,

Mean: $\bar{X} = 6 \Rightarrow constant$

Auto Correlation: $R_{XX}(t, t+\tau) = 36+25\ exp(-\tau) \Rightarrow$ $Independent\ on\ time$

∴ $X(t)$ is wide sense stationary random process.

Hence, the given statement is true.

(iv) *Periodic component*:

If the given RP $X(t)$ has no periodic components then, it satisfies the condition

$$\boxed{\lim_{|\tau|\to\infty} \boldsymbol{R_{XX}(\tau) = (\bar{X})^2}}$$

Now,

$$\lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{\tau\to\infty}[36 + 25\, exp(-\tau)]$$

$$= 36 + 25\, exp(-\infty)$$

$$= 36 + 25(0)$$

$$= 36$$

$$= 6^2$$

$$\therefore \boldsymbol{\lim_{\tau\to\infty} R_{XX}(\tau) = (\bar{X})^2}$$

Hence, the given RP $X(t)$ has no periodic components.

<u>(v) *AC Power*:</u>

The value of AC Power is given by the variance of the RP $X(t)$.

<u>AC Power</u>: $\boldsymbol{\sigma^2{}_{X(t)} = E[X^2(t)] - (E[X(t)])^2}$

$$\Rightarrow \sigma^2{}_{X(t)} = R_{XX}(0) - (\bar{X})^2$$

$$= 61 - 6^2$$

$$= 61 - 36$$

$$\therefore \sigma^2{}_{X(t)} = 25$$

∴ AC Power = 25W

Hence, the given statement is not true.

Show that $X(t)$&$Y(t)$ are Jointly WSS, if random processes, $X(t) = A\ Cos(\omega_1 t + \theta)$,

$Y(t) = B\ Cos(\omega_2 t + \Phi)$, where A, B, ω_1&ω_2 are constants, while Φ, θ are statistically

independent uniform random variables on $(0,2\pi)$.

Sol: Given,

$$X(t) = A\ cos(\omega_1 t + \theta)$$

$$Y(t) = B\ cos(\omega_2 t + \Phi)$$

<u>Conditions to be satisfied to be Jointly WSS</u>:

Mean of X(t): $\overline{X(t)} = constant$
Mean of Y(t):$\overline{Y(t)} = constant$
Auto Correlation of X(t) &Y(t): $R_{XX} and\ R_{YY} should\ be\ independent\ on\ time$
Cross Correlation of X(t) & Y(t):$R_{XY} or\ R_{YX} should\ be\ independent\ on\ time$

Mean of$X(t)$:

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_X(x)\, dx$$

Given, θ is a uniformly distributed random variable on the interval $(0,2\pi)$.

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_\theta(\theta)\, d\theta$$

Density function of a uniformly distributed random variable is

$$f_X(x) = \frac{1}{b-a}\ ;\ a \le X \le b$$

$$f_\theta(\theta) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$$

$$\Rightarrow E[X(t)] = \int_0^{2\pi} A\, cos(\omega_1 t + \theta)\, f_\theta(\theta)\, d\theta$$

$$= A \int_0^{2\pi} cos(\omega_1 t + \theta) \left(\frac{1}{2\pi}\right) d\theta$$

$$= A \left[\frac{sin(\omega_1 t + \theta)}{2\pi}\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}[sin(2\pi + \omega_1 t) - sin(\omega_1 t)]$$

$$= \frac{A}{2\pi}[sin(\omega_1 t) - sin(\omega_1 t)]$$

$$= \frac{A}{2\pi}[0]$$

$\therefore\ \boldsymbol{E[X(t)] = 0 \Rightarrow constant}$

Auto Correlation of $X(t)$:

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} x(t)\, x(t+\tau) f_X(x)\, dx$$

$$= \int_0^{2\pi} A\, cos(\omega_1 t + \theta) A\, cos(\omega_1 t + \omega_1 \tau + \theta) f_\theta(\theta)\, d\theta$$

$$= A^2 \int_0^{2\pi} cos(\omega_1 t + \theta) cos(\omega_1 t + \omega_1 \tau + \theta) \left(\frac{1}{2\pi}\right) d\theta$$

$$= \frac{A^2}{4\pi} \int_0^{2\pi} [cos(2\omega_1 t + \omega_1 \tau + 2\theta) + cos(\omega_1 \tau)]\, d\theta$$

$[\because 2CosACosB = Cos(A+B) + Cos(A-B)]$

$$= \frac{A^2}{4\pi}\left[\frac{sin(2\omega_1 t + \omega_1\tau + 2\theta)}{2} + cos(\omega_1\tau)[\theta]\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[\frac{sin(2\pi + 2\omega_1 t + \omega_1\tau)}{2} + 2\pi cos(\omega_1\tau) - \left(\frac{sin(2\omega_1 t + \omega_1\tau)}{2} + 0\right)\right]$$

$$= \frac{A^2}{4\pi}\left[\frac{sin(2\omega_1 t + \omega_1\tau)}{2} + 2\pi cos(\omega_1\tau) - \frac{sin(2\omega_1 t + \omega_1\tau)}{2}\right]$$

$$= \frac{A^2}{4\pi}[2\pi cos(\omega_1\tau)]$$

$$\therefore\ \boldsymbol{R_{XX}(t, t+\tau) = \frac{A^2}{2}\big(\,cos(\omega_1\tau)\big) \Rightarrow Independent\ on\ time}$$

Mean of $Y(t)$:

$$E[Y(t)] = \int_{-\infty}^{\infty} y(t)\, f_Y(y)\, dy$$

Given, Φ is a uniformly distributed random variable on the interval $(0,2\pi)$.

$$E[Y(t)] = \int_{-\infty}^{\infty} y(t)\, f_\Phi(\Phi)\, d\Phi$$

Density function of a uniformly distributed random variable is

$$f_Y(y) = \frac{1}{b-a}\ \ ;\ a \le Y \le b$$

$$f_\Phi(\Phi) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$$

$$\Rightarrow E[Y(t)] = \int_0^{2\pi} B\, cos(\omega_2 t + \Phi)\, f_\Phi(\Phi)\, d\Phi$$

$$= B \int_0^{2\pi} cos(\omega_2 t + \Phi) \left(\frac{1}{2\pi}\right) d\Phi$$

$$= B \left[\frac{sin(\omega_2 t + \Phi)}{2\pi}\right]_0^{2\pi}$$

$$= \frac{B}{2\pi}[sin(2\pi + \omega_2 t) - sin(\omega_2 t)]$$

$$= \frac{B}{2\pi}[sin(\omega_2 t) - sin(\omega_2 t)]$$

$$= \frac{B}{2\pi}[0]$$

$\therefore\ \boldsymbol{E[Y(t)] = 0 \Rightarrow constant}$

Auto Correlation of $Y(t)$:

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= \int_{-\infty}^{\infty} y(t)\, y(t+\tau) f_Y(y)\, dy$$

$$= \int_0^{2\pi} B\, cos(\omega_2 t + \Phi) B\, cos(\omega_2 t + \omega_2 \tau + \Phi) f_\Phi(\Phi)\, d\Phi$$

$$= B^2 \int_0^{2\pi} cos(\omega_2 t + \Phi) cos(\omega_2 t + \omega_2 \tau + \Phi) \left(\frac{1}{2\pi}\right) d\Phi$$

$$= \frac{B^2}{4\pi} \int_0^{2\pi} [cos(2\omega_2 t + \omega_2 \tau + 2\Phi) + cos(\omega_2 \tau)]\, d\Phi$$

$[\because 2CosACosB = Cos(A+B) + Cos(A-B)]$

$$= \frac{B^2}{4\pi}\left[\frac{sin(2\omega_2 t + \omega_2\tau + 2\Phi)}{2} + cos(\omega_2\tau)[\Phi]\right]_0^{2\pi}$$

$$= \frac{B^2}{4\pi}\left[\frac{sin(2\pi + 2\omega_2 t + \omega_2\tau)}{2} + 2\pi cos(\omega_2\tau) - \left(\frac{sin(2\omega_2 t + \omega_2\tau)}{2} + 0\right)\right]$$

$$= \frac{B^2}{4\pi}\left[\frac{sin(2\omega_2 t + \omega_2\tau)}{2} + 2\pi cos(\omega_2\tau) - \frac{sin(2\omega_2 t + \omega_2\tau)}{2}\right]$$

$$= \frac{B^2}{4\pi}[2\pi cos(\omega_2\tau)]$$

$$\therefore\ \boldsymbol{R_{YY}(t, t+\tau) = \frac{B^2}{2}\left(cos(\omega_2\tau)\right) \Rightarrow Independent\ on\ time}$$

Cross Correlation of $X(t)$ *and* $Y(t)$:

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

Given θ and Φ are statistically independent random variables then,

the random processes $X(t)$ and $Y(t)$ also would be statistically independent.

$$\Rightarrow R_{XY}(t, t+\tau) = E[X(t)]\ E[Y(t+\tau)]$$

$$= (0)\ E[Y(t+\tau)][\because\ E[X(t)] = 0]$$

$$\therefore\ \boldsymbol{R_{XY}(t, t+\tau) = 0 \Rightarrow Independent\ on\ time}$$

Hence, $X(t)$ and $Y(t)$ are Jointly wide sense stationary.

10. If $X(t) = A\ Cos(\omega_0 t + \theta)$, where A, ω_0 are constants and θ is a uniform random variable on $(-\pi, \pi)$. A new random process is defined by $Y(t) = X^2(t)$.

(i) Obtain the Mean and Auto Correlation Function of $X(t)$.
(ii) Obtain the Mean and Auto Correlation Function of $Y(t)$.
(iii) Find the Cross Correlation Function of $X(t)$ and $Y(t)$.
(iv) Are $X(t)$ and $Y(t)$ are WSS ?
(v) Are $X(t)$ and $Y(t)$ are Jointly WSS ?

Sol: Given,

$X(t) = A\ cos(\omega_0 t + \theta)$, where A, ω_0 are constants.

(i) *Mean of* $X(t)$:

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_X(x)\, dx$$

Given, θ is a uniformly distributed random variable on the interval $(-\pi, \pi)$.

$$E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_\theta(\theta)\, d\theta$$

Density function of a uniformly distributed random variable is

$$f_X(x) = \frac{1}{b-a}\ ;\ a \le X \le b$$

$$f_\theta(\theta) = \frac{1}{\pi - (-\pi)} = \frac{1}{2\pi}$$

$$\Rightarrow E[X(t)] = \int_{-\pi}^{\pi} A\ cos(\omega_0 t + \theta) f_\theta(\theta)\, d\theta$$

$$= A \int_{-\pi}^{\pi} cos(\omega_0 t + \theta) \left(\frac{1}{2\pi}\right) d\theta$$

$$= A\left[\frac{sin(\omega_0 t+\theta)}{2\pi}\right]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}[sin(\pi+\omega_0 t) - sin(-\pi+\omega_0 t)]$$

$$= \frac{A}{2\pi}[-sin\omega_0 t + sin\omega_0 t]$$

$$\therefore \boldsymbol{E[X(t)] = 0 \Rightarrow constant}$$

Auto Correlation Function of $X(t)$:

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)f_\theta(\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi} A\,cos(\omega_0 t+\theta)\,A\,cos(\omega_0 t+\omega_0\tau+\theta)\left(\frac{1}{2\pi}\right)d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}[cos(2\omega_0 t+\omega_0\tau+2\theta) + cos(\omega_0\tau)]\,d\theta$$

$[\because 2CosACosB = Cos(A+B) + Cos(A-B)]$

$$= \frac{A^2}{4\pi}\left[\frac{sin(2\omega_0 t+\omega_0\tau+2\theta)}{2} + cos(\omega_0\tau)[\theta]\right]_{-\pi}^{\pi}$$

$$= \boldsymbol{\frac{A^2}{4\pi}\left[\frac{sin(2\pi+2\omega_0 t+\omega_0\tau)}{2} + \pi cos(\omega_0\tau) - \left(\frac{sin(-2\pi+2\omega_0 t+\omega_0\tau)}{2} - \pi cos(\omega_0\tau)\right)\right]}$$

$$= \frac{A^2}{4\pi}\left[\frac{sin(2\omega_0 t+\omega_0\tau)}{2} + \pi cos(\omega_0\tau) - \left(\frac{sin(2\omega_0 t+\omega_0\tau)}{2} - \pi cos(\omega_0\tau)\right)\right]$$

$$= \frac{A^2}{4\pi}[\pi cos(\omega_0\tau) + \pi cos(\omega_0\tau)]$$

$$= \frac{A^2}{4\pi}[2\pi cos(\omega_0\tau)]$$

$$\therefore \boldsymbol{R_{XX}(t, t+\tau) = R_{XX}(\tau) = \frac{A^2}{2} cos(\omega_0\tau) \Rightarrow Independent\ on\ time}$$

$\therefore$ The random process $X(t)$ is Wide Sense Stationary.

(ii) *Mean of* $Y(t)$:

Given,

$$Y(t) = X^2(t)$$

$$\Rightarrow E[Y(t)] = E[X^2(t)]$$

$$= R_{XX}(0)\left[\because R_{XX}(0) = \overline{X^2(t)}\right]$$

$$= \frac{A^2}{2} cos\big(\omega_0(0)\big)$$

$$= \frac{A^2}{2}(1)$$

$$\therefore\ \boldsymbol{E[Y(t)] = \frac{A^2}{2} \Rightarrow constant}$$

Auto Correlation Function of $Y(t)$:

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= \int_{-\infty}^{\infty} y(t)y(t+\tau)f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} x^2(t)x^2(t+\tau)\,f_\theta(\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi} [A^2 cos^2(\omega_0 t + \theta) A^2 cos^2(\omega_0 t + \omega_0 \tau + \theta)] \left(\frac{1}{2\pi}\right) d\theta$$

$$= \frac{A^4}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1 + cos(2\omega_0 t + 2\theta)}{2}\right)\left(\frac{1 + cos(2\omega_0 t + 2\omega_0 \tau + 2\theta)}{2}\right) d\theta$$

$$= \frac{A^4}{8\pi} \int_{-\pi}^{\pi} 1\, d\theta + \frac{A^4}{8\pi} \int_{-\pi}^{\pi} cos(2\omega_0 t + 2\theta) d\theta + \frac{A^4}{8\pi} \int_{-\pi}^{\pi} cos(2\omega_0 t + 2\omega_0 \tau + 2\theta) d\theta$$

$$+ \frac{A^4}{8\pi} \int_{-\pi}^{\pi} cos(2\omega_0 t + 2\theta) cos(2\omega_0 t + 2\omega_0 \tau + 2\theta) d\theta$$

$$= \frac{A^4}{8\pi} [\theta]_{-\pi}^{\pi} + \frac{A^4}{8\pi} \left[\frac{sin(2\omega_0 t + 2\theta)}{2}\right]_{-\pi}^{\pi} + \frac{A^4}{8\pi} \left[\frac{sin(2\omega_0 t + 2\omega_0 \tau + 2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$+ \frac{A^4}{16\pi} \int_{-\pi}^{\pi} [cos(4\omega_0 t + 2\omega_0 \tau + 4\theta) + cos(2\omega_0 \tau)]\, d\theta$$

$$= \frac{A^4}{8\pi} [\pi - (-\pi)] + \frac{A^4}{16\pi} [sin(2\pi + 2\omega_0 t) - sin(-2\pi + 2\omega_0 t)] +$$

$$\frac{A^4}{16\pi} [sin(2\pi + 2\omega_0 t + 2\omega_0 \tau) - sin(-2\pi + 2\omega_0 t + 2\omega_0 \tau)] +$$

$$\frac{A^4}{16\pi} \left[\left[\frac{sin(4\omega_0 t + 2\omega_0 \tau + 4\theta)}{4}\right]_{-\pi}^{\pi} + cos(2\omega_0 \tau)[\theta]_{-\pi}^{\pi}\right]$$

$$= \frac{A^4}{8\pi} [2\pi] + \frac{A^4}{16\pi} [sin(2\omega_0 t) - sin(2\omega_0 t)] + \frac{A^4}{16\pi} [sin(2\omega_0 t + 2\omega_0 \tau) - sin(2\omega_0 t + 2\omega_0 \tau)]$$

$$+ \frac{A^4}{16\pi} \left[\frac{sin(4\pi + 4\omega_0 t + 2\omega_0 \tau) - sin(-4\pi + 4\omega_0 t + 2\omega_0 \tau)}{4} + cos(2\omega_0 \tau)[\pi - (-\pi)]\right]$$

$$= \frac{A^4}{4} + \frac{A^4}{16\pi}[0] + \frac{A^4}{16\pi}[0] + \frac{A^4}{16\pi}\left[\frac{sin(4\omega_0 t + 2\omega_0\tau) - sin(4\omega_0 t + 2\omega_0\tau)}{4} + cos(2\omega_0\tau)[2\pi]\right]$$

$$= \frac{A^4}{4} + 0 + 0 + \frac{A^4}{16\pi}[0 + 2\pi\, cos(2\omega_0\tau)]$$

$$= \frac{A^4}{4} + \frac{A^4}{8}\big(cos(2\omega_0\tau)\big)$$

$$= \frac{A^4}{8}[2 + cos(2\omega_0\tau)]$$

$$\therefore \boldsymbol{R_{YY}(t, t+\tau) = R_{YY}(\tau) = \frac{A^4}{8}[2 + cos(2\omega_0\tau)] \Rightarrow Independent\ on\ time}$$

$\therefore$ The new random process $Y(t)$ is also Wide Sense Stationary.

(iii) *Cross Correlation Function of* $X(t)$ *and* $Y(t)$:

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

$$= \int_{-\infty}^{\infty} x(t)y(t+\tau)f_\theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} [A\, cos(\omega_0 t + \theta)A^2 cos^2(\omega_0 t + \omega_0\tau + \theta)]\left(\frac{1}{2\pi}\right) d\theta$$

$$= \frac{A^3}{2\pi}\int_{-\pi}^{\pi}\big(cos(\omega_0 t + \theta)\big)\left(\frac{1 + cos(2\omega_0 t + 2\omega_0\tau + 2\theta)}{2}\right) d\theta$$

$$= \frac{A^3}{4\pi}\int_{-\pi}^{\pi}[cos(\omega_0 t + \theta) + cos(\omega_0 t + \theta)cos(2\omega_0 t + 2\omega_0\tau + 2\theta)]\, d\theta$$

$$= \frac{\boldsymbol{A^3}}{\boldsymbol{4\pi}} \int_{-\pi}^{\pi} \left[\boldsymbol{cos(\omega_0 t + \theta)} + \frac{\boldsymbol{1}}{\boldsymbol{2}} \left(\boldsymbol{cos(3\omega_0 t + 2\omega_0 \tau + 3\theta)} + \boldsymbol{cos(\omega_0 t + 2\omega_0 \tau + \theta)} \right) \right] \boldsymbol{d\theta}$$

$[\because 2CosACosB = Cos(A+B) + Cos(A-B)]$

$$= \frac{\boldsymbol{A^3}}{\boldsymbol{4\pi}} \left[\frac{\boldsymbol{sin(\omega_0 t + \theta)}}{\boldsymbol{1}} \right]_{-\pi}^{\pi} + \frac{A^3}{8\pi} \left[\frac{\boldsymbol{sin(3\omega_0 t + 2\omega_0 \tau + 3\theta)}}{\boldsymbol{3}} \right]_{-\pi}^{\pi} + \frac{\boldsymbol{A^3}}{\boldsymbol{8\pi}} \left[\frac{\boldsymbol{sin(\omega_0 t + 2\omega_0 \tau + \theta)}}{\boldsymbol{1}} \right]_{-\pi}^{\pi}$$

$$= \frac{A^3}{4\pi} [sin(\pi + \omega_0 t) - sin(-\pi + \omega_0 t)]$$

$$+ \frac{A^3}{24\pi} [sin(3\pi + 3\omega_0 t + 2\omega_0 \tau) - sin(-3\pi + 3\omega_0 t + 2\omega_0 \tau)]$$

$$+ \frac{A^3}{8\pi} [sin(\pi + \omega_0 t + 2\omega_0 \tau) - sin(-\pi + \omega_0 t + 2\omega_0 \tau)]$$

$$= \frac{A^3}{4\pi} [-sin\omega_0 t + sin\omega_0 t] + \frac{A^3}{24\pi} [-sin(3\omega_0 t + 2\omega_0 \tau) + sin(3\omega_0 t + 2\omega_0 \tau)]$$

$$+ \frac{A^3}{8\pi} [-sin(\omega_0 t + 2\omega_0 \tau) + sin(\omega_0 t + 2\omega_0 \tau)]$$

$$= \frac{A^3}{4\pi} [0] + \frac{A^3}{24\pi} [0] + \frac{A^3}{8\pi} [0]$$

$$= 0$$

$$\therefore \boldsymbol{R_{XY}(t, t+\tau) = 0 \Rightarrow Independent\ on\ time}$$

(iv) Yes, $X(t)$and$Y(t)$are Wide Sense Stationary.

(v) Yes, $X(t)$ and $Y(t)$ are Jointly wide sense stationary.