Random Processes

CO3: RANDOM PROCESSES –TEMPORAL CHARACTERISTICS

The Random Process Concept,
Classification of Processes,
Deterministic and Non deterministic Processes,
Distribution and Density Functions,
Concept of Stationarity and Statistical Independence.
First-Order Stationary Processes, Second-order and Wide-Sense Stationarity, Nth-order and Strict- Sense Stationarity,
Time Averages and Ergodicity,
Autocorrelation Function and its Properties,
Cross-Correlation Function and its Properties,
Covariance Functions,
Gaussian Random Processes,
Poisson Random Process

Random Processes

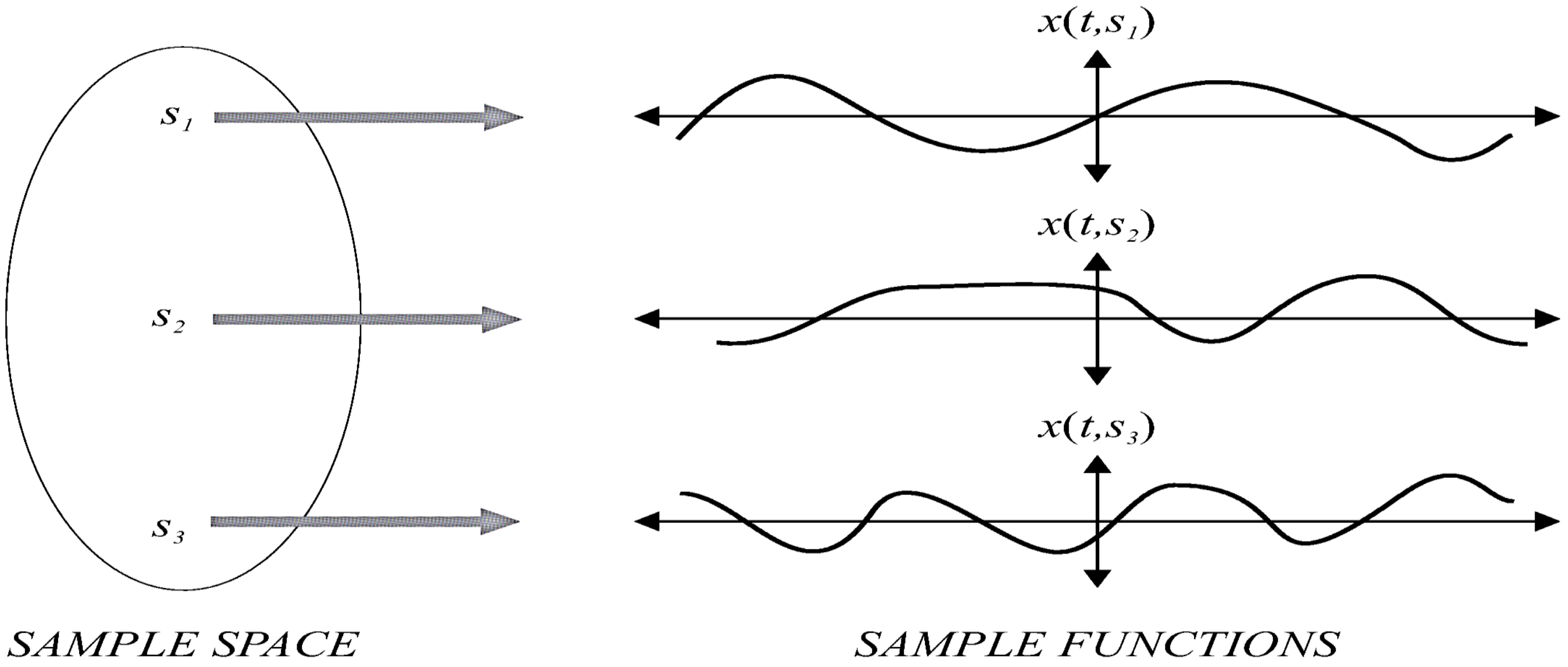
- A **RANDOM VARIABLE** X, is a rule for assigning to every outcome, ω, of an experiment a number $X(\omega)$.
  - Note: X denotes a random variable and $X(\omega)$ denotes a particular value.
- A **RANDOM PROCESS** $X(t)$ is a rule for assigning to every ω, a function $X(t,\omega)$.
  - Note: for notational simplicity we often omit the dependence on ω.

	Random variable	Random process
1.	A function of the possible outcomes of an experiment. i.e., $X(s)$	A function of the possible outcomes of an experiment and also time. i.e., $X(s, t)$.
2.	Outcome is mapped into a number 'x'	Outcomes are mapped into wave form which is a function of time 't'.

Random Processes

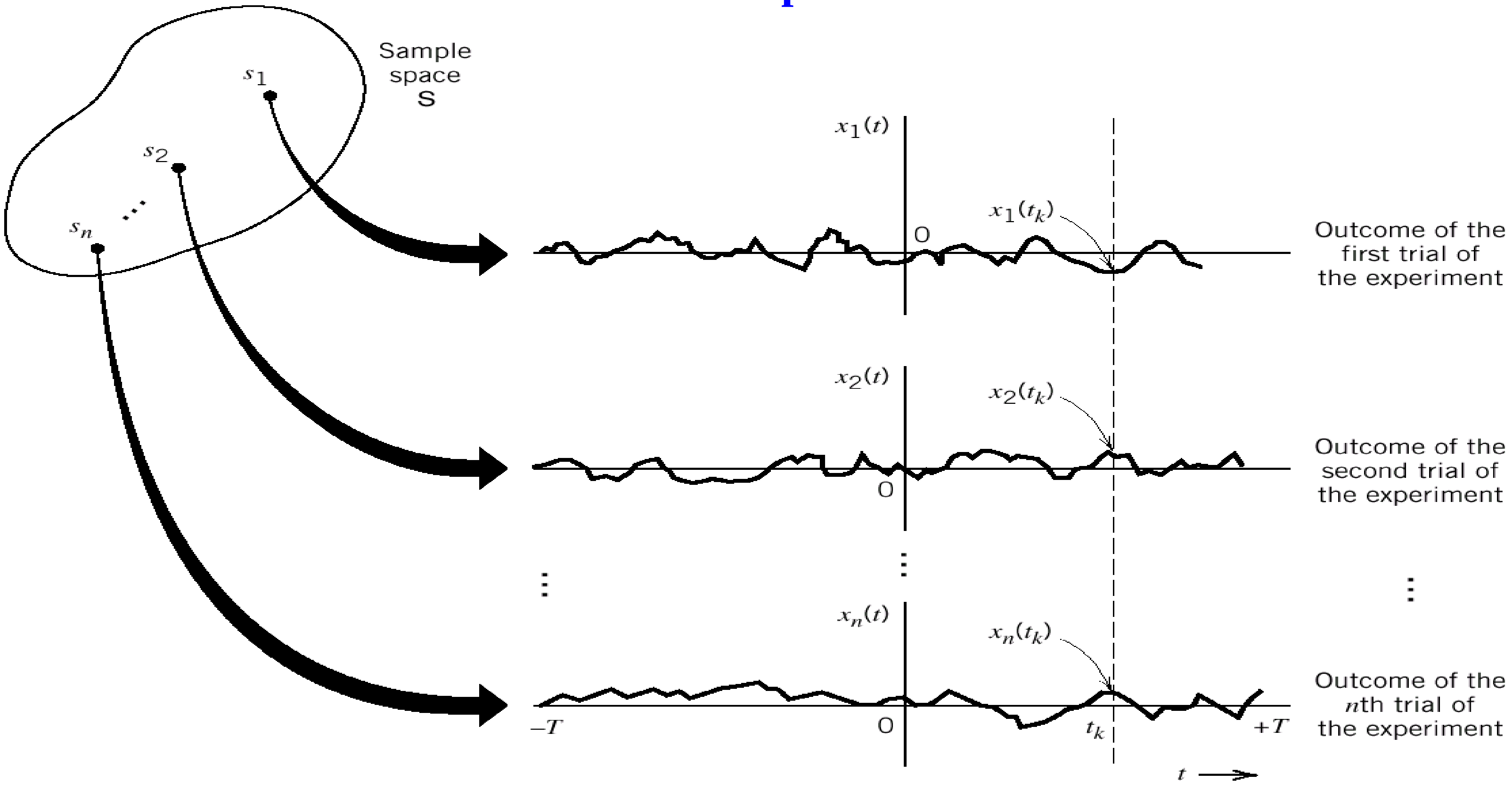
- The random processes are also called as stochastic processes which deal with randomly varying time wave forms such as any message signals and noise.
- They are described statistically since the complete knowledge about their origin is not known. So statistical measures are used.
- Probability distribution and probability density functions give the complete statistical characteristics of random signals. A random process is a function of both sample space and time variables. And can be represented as: $\{X \leq x(s, t)\}$
- Examples of random processes in communications:
  - ✓ Channel noise,
  - ✓ Information generated by a source,
  - ✓ Interference.

Conceptual Representation of Random Process



Ensemble of Sample Functions

The set of all possible functions is called the ENSEMBLE.



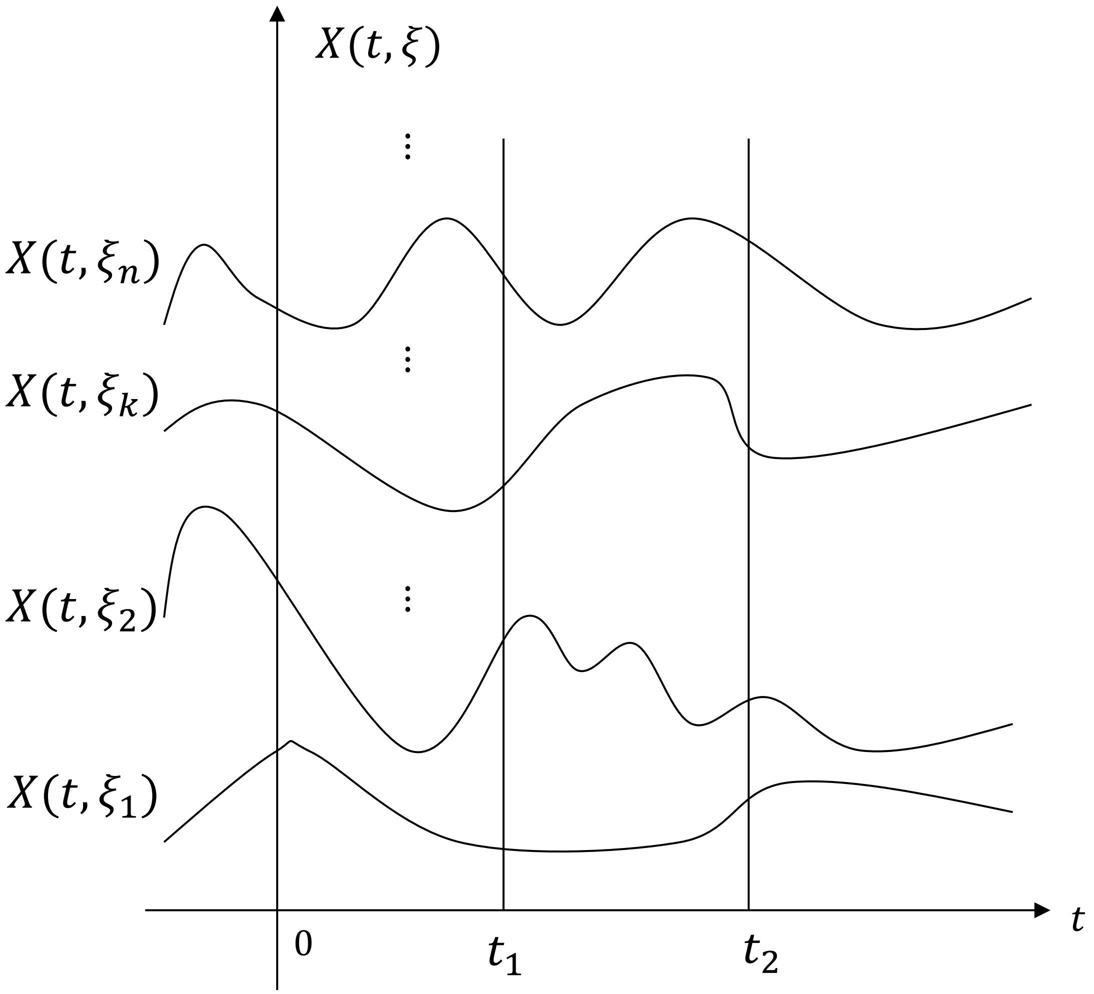
Mathematical representation of Random Processes

Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t,\xi)$ is assigned.

The collection of such waveforms form a stochastic process. The set of $\{\xi_k\}$ and the time index t can be continuous or discrete (countably infinite or finite) as well.

For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t,\xi)$ is a specific time function.

For fixed t, $X_1 = X(t_1, \xi_i)$ is a random variable. The ensemble of all such realizations $X(t,\xi)$ over time represents the stochastic



Classification of Random process

It is convenient to classify random processes according to the characteristics of t and the random variable $X = X(t)$ at time t. We shall consider only four cases based on t and X having values in the ranges $-\infty < t < \infty$ and $-\infty < x < \infty$

1. Continuous random process
2. Continuous random sequence
3. Discrete random process
4. Discrete random sequence

Classification of Random process

X(t) \ t	Continuous t	Discrete t
Continuous $X(t)$	**1. Continuous random process** If both X and t are continuous, the random process is called as continuous random process. Example : $X(t)$ represents the maximum temperature at a place in the interval $(0, t)$	**2. Continuous random sequence** If X is continuous and t is discrete, the random process is called as continuous random sequence. Example : X_n represents the temperature at the end of the n^{th} hour of a day, in the interval (1, 24).
Discrete $X(t)$	**3. Discrete random process.** If X is discrete and t is continuous, the random process is called as discrete random process. Example : $X(t)$ represents the number of telephone calls received in the interval $(0, t)$ $S = \{0, 1, 2, 3, \ldots\}$	**4. Discrete random sequence.** If both X and t are discrete, then the random process is called as discrete random sequence. Example : If X_n represents the outcome of the n^{th} toss of a fair die, then $\{X_n : n \geq 1\}$ is a discrete random sequence. Since $T = \{1, 2, 3, \ldots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$

Classification of Random process

Continuous Random Process: A random process is said to be continuous if both the random variable X and time t are continuous over the entire interval.

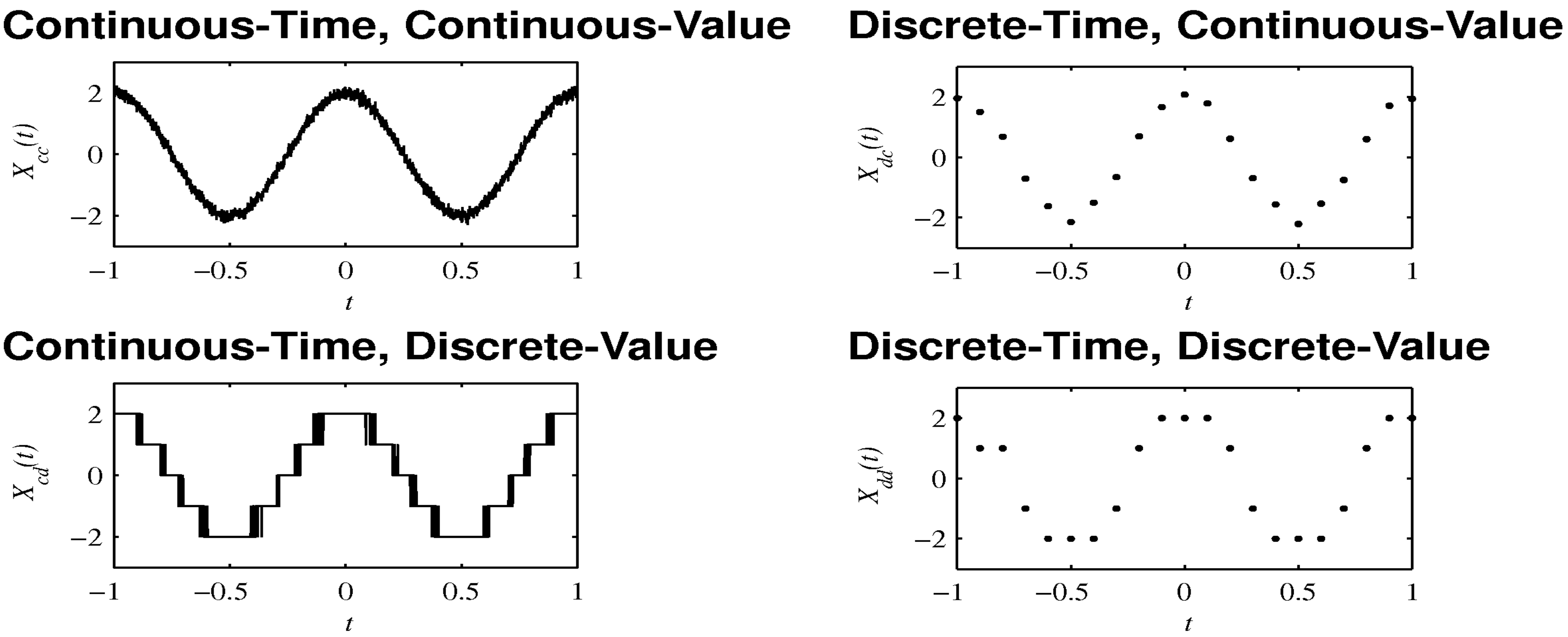
Discrete Random Process: In this random process, the random variable having only discrete values while t is continuous as shown in figure.

Continuous Random Sequence: A random process for which X is continuous but time has only discrete values is called continuous random sequence. Such a sequence is obtained by periodically sampling the ensemble members of figure.

Random sequence is also called a discrete time (DT) random process since continuous random sequence is defined at only discrete time values and usually denoted by X(n) which are the important in the analysis of various DSP systems.

Discrete Random Sequence: If both time t and random variable X are discrete in nature, then the random process is called discrete random sequence as shown in figure.

Classification of Random process



Sample functions of four kinds of stochastic processes. $X_{cc}(t)$ is a continuous-time, continuous-value process. $X_{dc}(t)$ is discrete-time, continuous-value process obtained by sampling $X_{cc})$ every 0.1 seconds. Rounding $X_{cc}(t)$ to the nearest integer yields $X_{cd}(t)$, a continuous-time, discrete-value process. Lastly, $X_{dd}(t)$, a discrete-time, discrete-value process, can be obtained either by sampling $X_{cd}(t)$ or by rounding $X_{dc}(t)$.

Classification of Random process

We can classify random process in another way also. It can be classified as

- **Deterministic Random Process:** if the future values of any sample function can be predicted from a knowledge of the past values, then the random process is called deterministic random process.

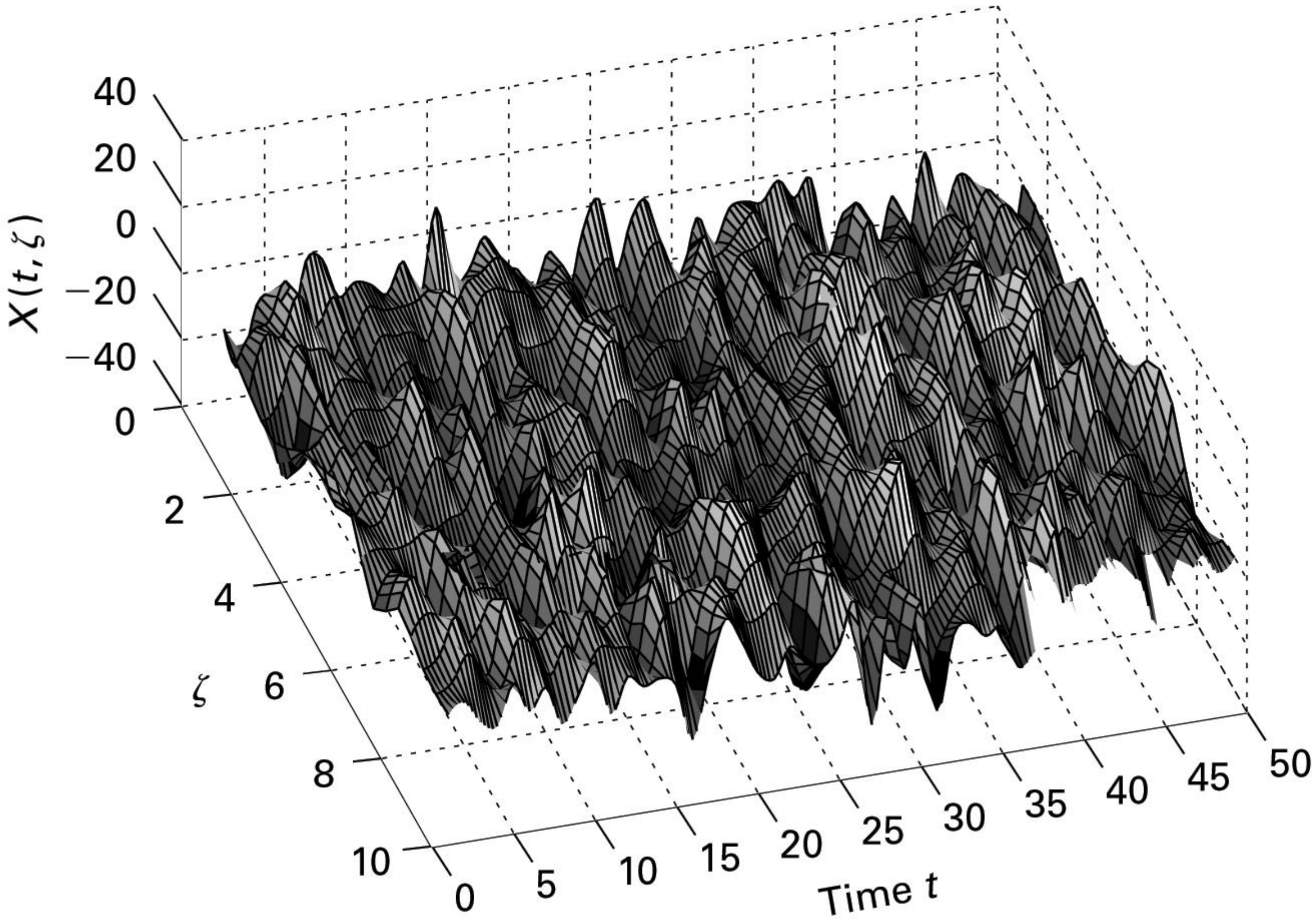
Example: Consider a random process $X(t) = A\cos(\omega t + \theta)$. This consists of a family of pure sine waves and it is completely specified in terms of the random variables A and θ. Hence, it is a deterministic random process.

- **Non-deterministic Random Process:** If the future values of a sample function cannot be predicted from the knowledge of the past values, the random process is called non-deterministic random process.

Example: A sine wave random process, $X(t) = A\sin(\omega t) + \theta$

In the case of dissolving of sugar crystals in coffee, it consists of a family of functions that cannot be described in terms of finite number of parameters. The future sample function cannot be determined from the past sample functions and so it is a non-deterministic random process.

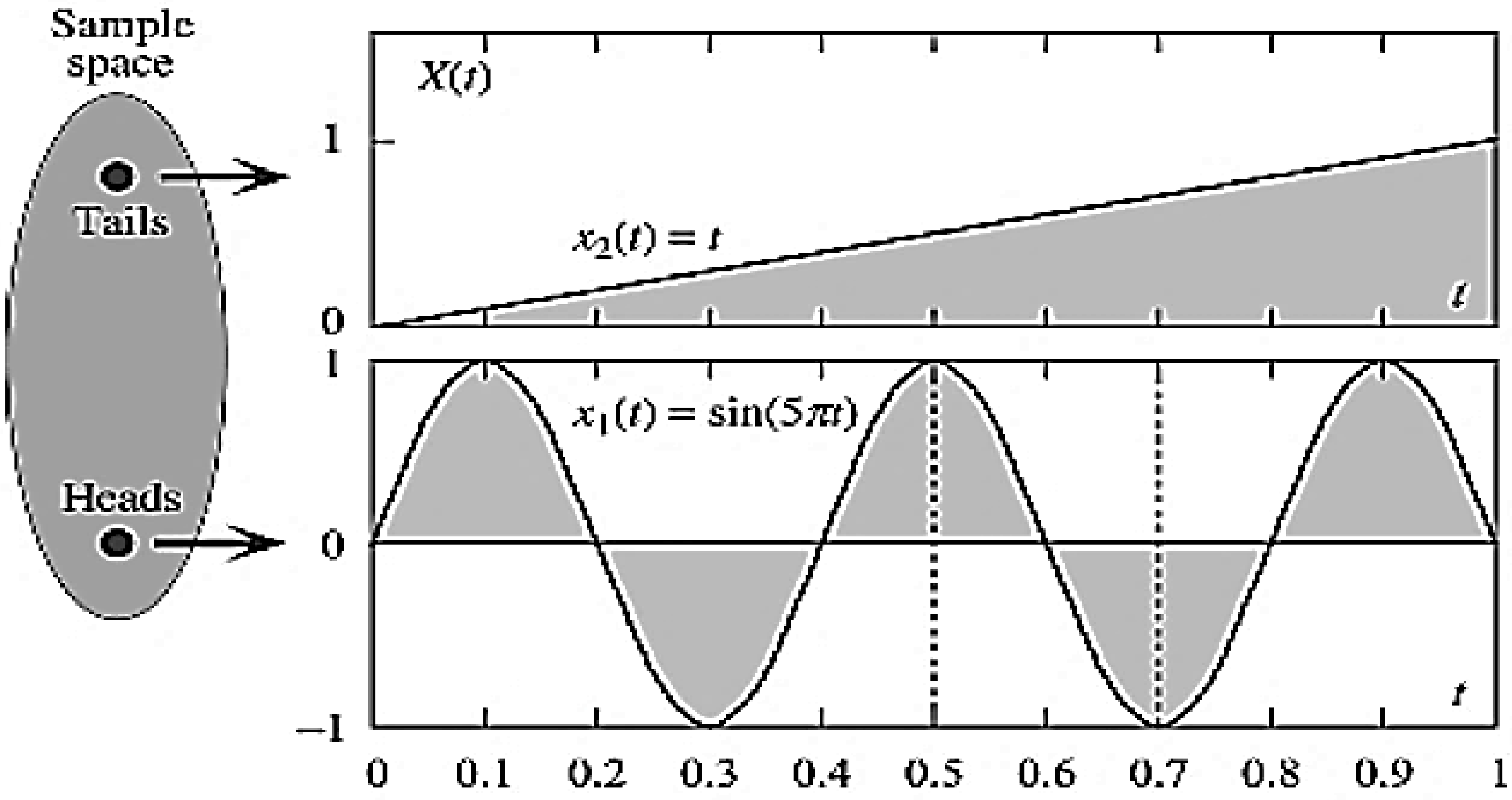
Random Process for a Continuous Sample Space



A random process for a continuous sample space $\Omega = [0, 10]$.

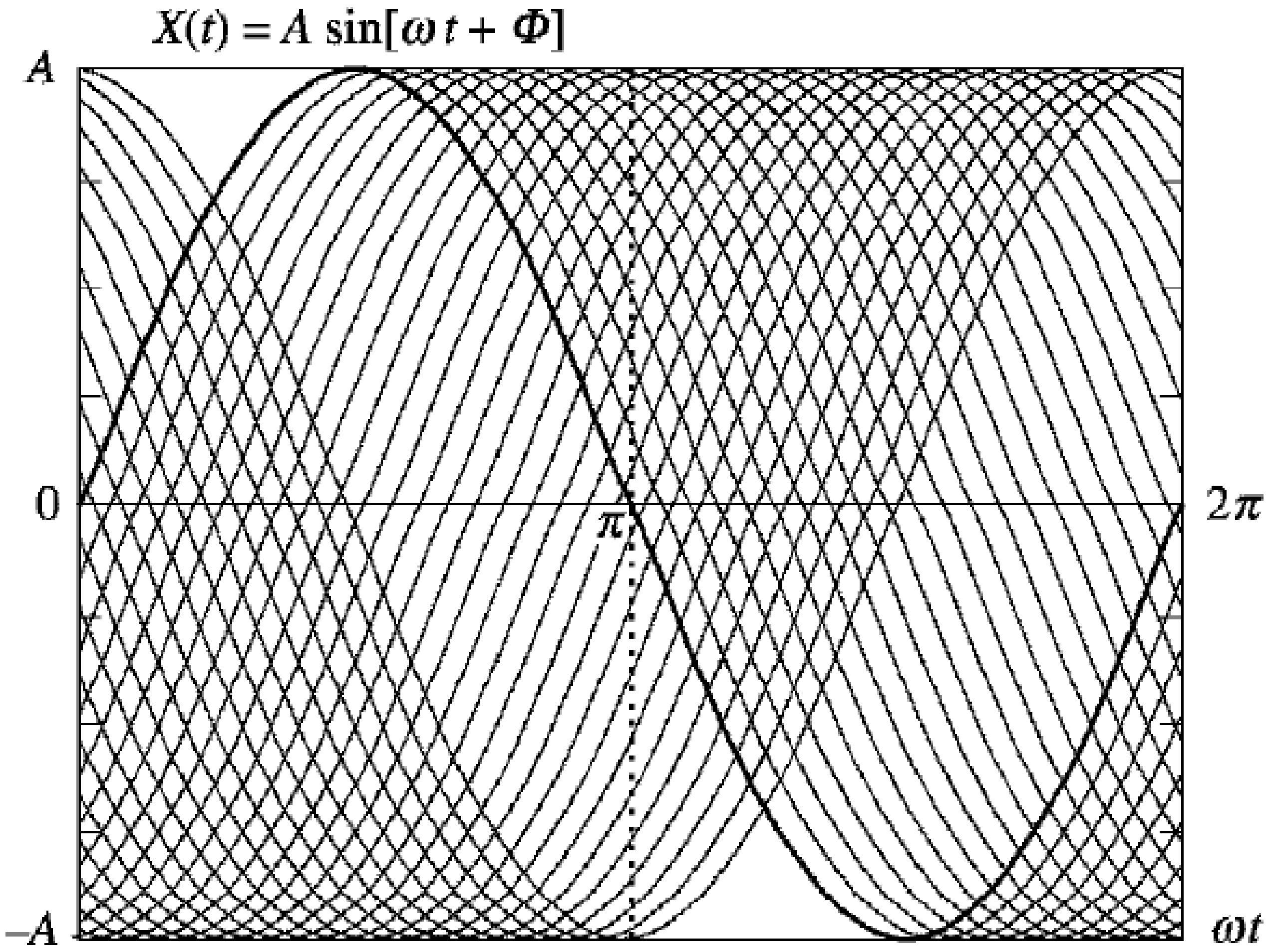
Problem 1.

A fair coin is tossed. If heads come up, a sine wave $x_1(t) = \sin(5\pi t)$ is sent. If tails come up, then a ramp $x_2(t) = t$ is sent. The resulting random process X(t) is an ensemble of two realizations, a sine wave and a ramp, and is shown in Fig. The sample space S is discrete.



Problem 2.

In this example a sine wave is in the form $X(t) = A \sin(\omega t + \varphi)$, where φ is a random variable uniformly distributed in the interval $(0, 2\pi)$. Here the sample space is continuous, and the sequence of sine functions is shown in Fig.



Note

- A random variable can be obtained from a random process at a particular instant of time t. the random variable has all the statistical properties (such as mean, moment, variance etc.) that are related to its probability density function.
- If two random variables are obtained from the random process at two different times, they will have all the statistical properties related to their joint probability density function.
- Consider a random process $X(t)$. For a single random variable at time t_1, $X_1 = X(t_1)$, the cumulative distribution function is defined as $F_X(x_1, t_1) = P\{X(t_1) \leq x_1\}$
- For two random variables at time instants t_1 and t_2, $X(t_1) = X_1$ and $X(t_2) = X_2$, the joint joint distribution is called the second order joint distribution function of the random process $X(t)$. It is given by $F_X(x_1, x_2;\ t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$

- Where x_1 and x_2 are real numbers.

- In general, for N random variables at N time instants, $X(t_i) = X_i; i = 1, 2, 3, \ldots\ldots, N$, the N^{th} order joint distribution function of $X(t)$ is defined as

$$F_X(x_1, x_2, \ldots\ldots, x_N;\ t_1, t_2, \ldots\ldots, t_N)$$
$$= P\{X(t_1) \le x_1, X(t_2) \le x_2, \ldots \ldots, X(t_N) \le x_N\}$$

Joint Density Function of A Random Processes

- **Joint density function of a random process can be obtained from the derivatives of the distribution function.**

- The first order density function of a random process X(t) is

$$f_X(x_1, t_1) = \frac{dF_X(x_1, t_1)}{dx_1}$$

- The second order joint density function of the random process is second derivative of the joint distribution function of the random process

$$\text{i.e.,} f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \, \partial x_2}$$

- The N^{th} order joint density function is given by

$$f_X(x_1, x_2, \ldots\ldots, x_N; t_1, t_2, \ldots\ldots, t_N) = \frac{\partial^N F_X(x_1, x_2, \ldots\ldots, x_N; t_1, t_2, \ldots\ldots, t_N)}{\partial x_1 \, \partial x_2 \ldots\ldots \partial x_N}$$

Joint Density Function of A Random Processes

- If X(t) is a discrete time random process, then the Nth order probability mass function is

$$P_X(x_1, x_2, \ldots\ldots, x_N;\ t_1, t_2, \ldots\ldots, t_N)$$
$$= P\{X(t_1) = x_1, X(t_2) = x_2, \ldots\ldots, X(t_N) = x_N\}$$

- **Distribution and Density Functions:** Since a random process is a random variable for any fixed time t, we can define a probability distribution and density functions as

$$F_X(x; t) = P(\xi, t\colon X(\xi; t) \le x) \text{ for a fixed } t$$

$$f_X(x; t) = \frac{\partial}{\partial x} F_X(x; t) = \lim_{\Delta x \to 0} \frac{F_X(x + \Delta x; t) - F_X(x; t)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} P(x < X(t) \le x + \Delta x)$$

These are also called first-order distribution and density functions, and in general, they are functions of time.

Independent Random Processes

- Consider a random process $X(t)$. Let $X(t_i) = x_i$, $i = 1, 2, 3, N$ be N-random variables defined at time instants $t_1, t_2, \ldots\ldots, t_N$, with density functions

$$f_X(x_1, t_1), f_X(x_2, t_2), \ldots \ldots, f_X(x_N, t_N)$$

- If the random process $X(t)$ is statistically independent, then the N^{th} order joint density function is equal to the product of the individual joint function of $X(t)$.

$$f_X(x_1, x_2, \ldots\ldots, x_N; t_1, t_2, \ldots\ldots, t_N)$$
$$= f_X(x_1, t_1)\, f_X(x_2, t_2) \ldots\ldots f_X(x_N, t_N)$$

- Similarly, let as we consider two random processes $X(t)$ and $Y(t)$. let $X(t)$ have random variables $X(t_1), X(t_2), \dots, X(t_N)$ with joint density function $f_X(x_1, x_2, \dots, x_N;\ t_1, t_2, \dots, t_N)$ and $Y(t)$ have random variables $Y(t'_1), Y(t'_2), \dots, Y(t'_M)$ with joint density function $f_Y(y_1, y_2, \dots, y_M;\ t'_1, t'_2, \dots, t'_M)$. That is, the joint density function of the random processes $X(t)$ and $Y(t)$ is equal to the product of the individual joint density function of $X(t)$ and $Y(t)$.

$$
\begin{aligned}
&f_{XY}(x_1, x_2, \dots, x_N, y_1, y_2, \dots, y_M;\ t_1, t_2, \dots, t_N, t'_1, t'_2, \dots, t'_M) \\
&= f_X(x_1, x_2, \dots, x_N;\ t_1, t_2, \dots, t_N)\, f_Y(y_1, y_2, \dots, y_M;\ t'_1, t'_2, \dots, t'_M)
\end{aligned}
$$

STATISTICAL PROPERTIES OF RANDOM PROCESSES

Mean: The mean value of a random process X(t) is equal to the expected value of the random process X(t). It is defined as

$$\overline{X(t)} = E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x, t)\, dx$$

Where $f_x(x, t)$ is the probability density function of the random process X(t).
The mean value E[X(t)] is also called the ensemble average of X(t)

Means and Variances. Analogous to random variables, we can define the mean of a random process as

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;\, t)\, dx$$

and the variance as

$$\sigma_X^2(t) = E[X(t) - \mu_X(t)]^2 = E[X^2(t)] - \mu_X^2(t)$$

$$= \int_{-\infty}^{\infty} [x - \mu_X(t)]^2 f_X(x; t) dx$$

where

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x; t) dx$$

Since the density is a function of time, the means and variances of random pro- cesses are also functions of time.

Autocorrelation

Autocorrelation, also known as serial correlation or self-correlation, is a statistical concept used to measure the degree of similarity between a time series and a lagged version of itself. In simpler terms, it quantifies the relationship between data points within a sequence at different time intervals. Autocorrelation is a fundamental concept in time series analysis, signal processing, and various fields of statistics and econometrics.

Definition: Autocorrelation is calculated as the correlation coefficient between a time series and a lagged version of itself. The lag represents the number of time periods by which the series is shifted.

Formula: The autocorrelation coefficient at lag k, denoted as ρ(k) or AC(k), can be calculated using the following formula:

ρ(k) = (Σ(x_t - μ)(x_(t-k) - μ)) / (Σ(x_t - μ)^2)

where:

- x_t represents the value at time t in the time series.
- μ is the mean of the time series.
- t represents the current time period.
- k represents the lag.

Autocorrelation: Consider a random process $X(t)$. Let X_1 and X_2 be two random variables defined at times t_1 and t_2 respectively with joint density function $f_X(x_1, x_2;\ t_1, t_2)$.

The correlation of X_1 and X_2, $E[X_1X_2] = E[X(t_1)X(t_2)]$ is called the autocorrelation function of the random process $X(t)$. It is denoted as $R_X(t_1, t_2)$

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_X(t_1, t_2) = E[X_1X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f_X(x_1, x_2;\ t_1, t_2)\ dx_1dx_2$$

Interpretation:

- If p(k) is close to 1, it indicates a strong positive autocorrelation, suggesting that values at a given time are highly correlated with values at a lag of k time units in the past.
- If p(k) is close to -1, it indicates a strong negative autocorrelation, meaning that values at a given time are highly correlated with values at a lag of k time units in the past, but with an inverse relationship.
- If p(k) is close to 0, it suggests little to no autocorrelation, indicating that there is no significant linear relationship between the values at different time lags.

Use Cases:

- Autocorrelation is widely used in time series analysis to identify patterns, trends, and
- seasonality in data.
- It helps in determining the order of autoregressive (AR) and moving average (MA) terms in time series models like ARMA and ARIMA.
- It is also used in the analysis of financial data, signal processing, and quality control, among other fields.

Cross correlation

Cross-correlation is a statistical technique used to measure the degree of similarity or relationship between two different time series or signals as a function of a time lag between them. It is often employed in signal processing, time series analysis, and various fields, including economics, engineering, and neuroscience, to analyze how one signal relates to another when one is shifted in time.

Key points about cross-correlation:

Definition: Cross-correlation quantifies the similarity or linear relationship between two time series, often referred to as the "reference" and "target" signals, as one is shifted relative to the other.

Formula: The cross-correlation at lag k, denoted as C(k) or R(k), between two signals x(t) and y(t) can be calculated using the following formula:

$C(k) = \Sigma(x\_t - \mu\_x)(y\_(t\text{-}k) - \mu\_y)$

where:

- x_t represents the value of the reference signal at time t.
- y_t represents the value of the target signal at time t.
- μ_x and μ_y are the means of the reference and target signals, respectively.
- t represents the current time period.
- k represents the lag.

Cross Correlation: Consider two random process $X(t)$ and $Y(t)$ defined with random variables X and Y at time instants t_1 and t_2 respectively.

The joint density function is $f_{XY}(x, y;\, t_1, t_2)$. Then the correlation of X and Y, $E[XY] = E[X(t_1)Y(t_2)]$ is called the cross correlation function of the random processes $X(t)$ and $Y(t)$. It is denoted as $R_{XY}(t_1, t_2)$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = E[XY]$$

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x, y;\, t_1, t_2)\, dxdy$$

Interpretation:

- If C(k) is close to 1, it suggests a strong positive cross-correlation, indicating that the two signals are similar and tend to move together with a lag of k time units.
- If C(k) is close to -1, it suggests a strong negative cross-correlation, indicating that the
- two signals are similar but tend to move in opposite directions with a lag of k time units.
- If C(k) is close to 0, it suggests little to no cross-correlation, meaning that there is no significant linear relationship between the two signals at a lag of k time units.

Use Cases:

- Cross-correlation is widely used in signal processing to analyze the similarity between
- two signals. For example, in speech recognition, it can be used to match a recorded voice sample with reference patterns.
- In time series analysis, cross-correlation can be employed to study relationships between economic indicators, stock prices, weather data, and other time-dependent phenomena.
- Cross-correlation is used in neuroscience to study the relationship between brain activity recorded from different regions or electrodes.
- It is also applied in image processing for template matching and object recognition.

Autocovariance

Autocovariance is a statistical concept closely related to autocorrelation, and it is used to measure the covariance between two values of a time series at different time points or lags. While autocorrelation measures the linear relationship between values at different lags, autocovariance measures how the covariance between these values changes as the lag increases or decreases.

Definition: Autocovariance measures the covariance between two observations of a time series at a given lag k. It quantifies how the joint variability between these two observations changes as the lag varies.

Formula: The autocovariance at lag k, denoted as $\gamma(k)$ or $Cov(k)$, can be calculated using the following formula:

$$\gamma(k) = (1/N) * \Sigma(x_t - \mu)(x_{(t-k)} - \mu)$$

where:

- x_t represents the value at time t in the time series.
- μ is the mean of the time series.
- t represents the current time period.
- k represents the lag.
- N is the total number of observations in the time series.

Interpretation:

- If y(k) is positive, it indicates that values at a given time and values at a lag of k time units tend to move in the same direction, suggesting positive dependence or similarity.
- If y(k) is negative, it indicates that values at a given time and values at a lag of k time units tend to move in opposite directions, suggesting negative dependence or dissimilarity.
- If y(k) is close to zero, it suggests little to no covariance or dependence between values at different lags.

Use Cases:

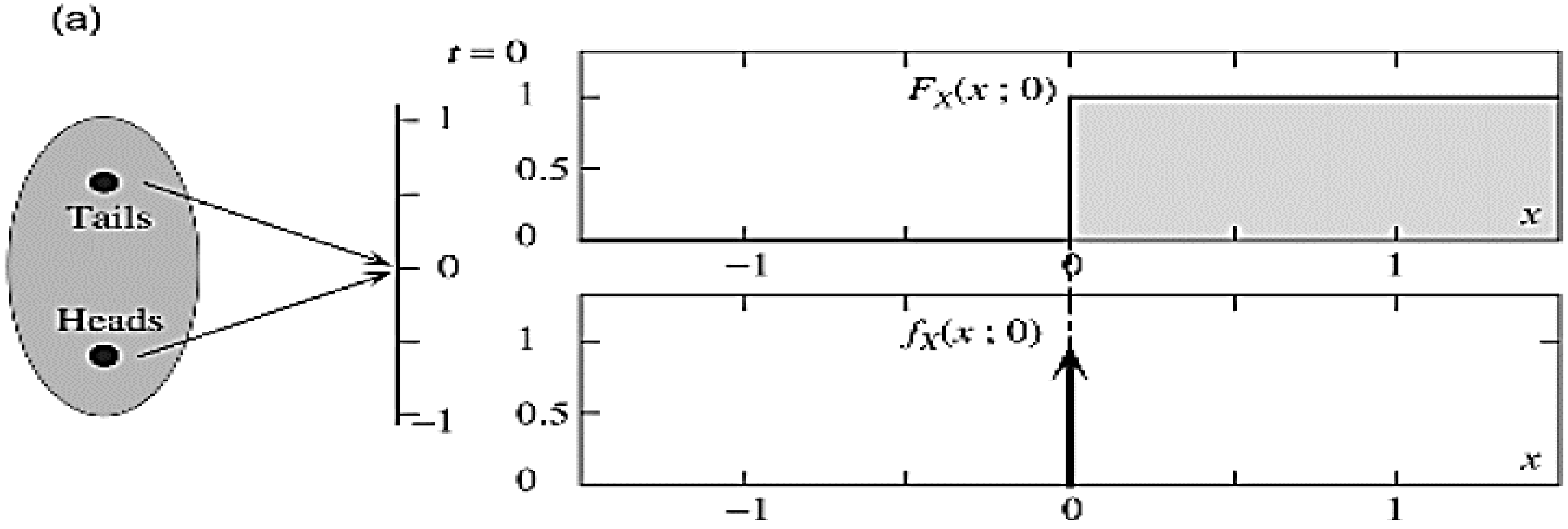
- Autocovariance is used in time series analysis and econometrics to understand the temporal dependencies and volatility clustering in data.
- It is commonly used in the analysis of financial time series to study the volatility and risk associated with financial assets.
- Autocovariance is also used in fields like signal processing, where it can help identify periodic or cyclical patterns in data.

Example 19.1.3 We shall now find the distribution and density functions along with the mean and variance for the random process of Example 19.1.1 for times $t = 0, \frac{1}{2}, \frac{7}{10}$:

$$t = 0, \quad x_1(0) = 0, \quad x_2(0) = 0$$

At $t = 0$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4a along with the corresponding distribution and density functions.

The mean value is given by $\mu_X(0) = 0; \frac{1}{2} + 0 \cdot \frac{1}{2} = 0$. The variance is given by $\sigma_X^2(0) = (0 - 0)^2 \frac{1}{2} + (0 - 0)^2 \frac{1}{2} = 0$:

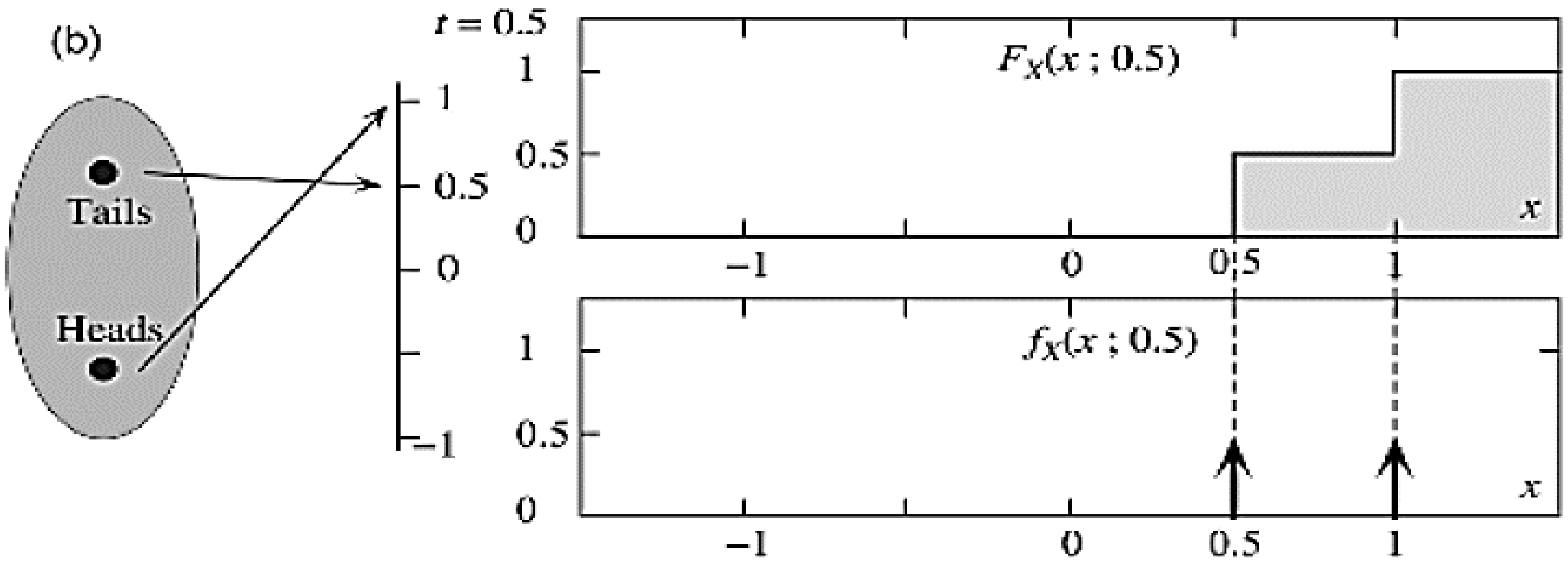


$$t=\frac{1}{2},\quad x_1\left(\frac{1}{2}\right)=1,\quad x_2\left(\frac{1}{2}\right)=\frac{1}{2}$$

At $t=\frac{1}{2}$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4b along with the corresponding distribution and density functions.

The mean value is given by $\mu_X\left(\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{1}{2}+1\cdot\frac{1}{2}=\frac{3}{4}=0.75.$

The variance is given by $\sigma_X^2\left(\frac{1}{2}\right)=\left(\frac{1}{2}-\frac{3}{4}\right)^2\frac{1}{2}+\left(1-\frac{3}{4}\right)^2\frac{1}{2}=\frac{1}{16}=0.0625$:



$$t = \frac{7}{10}, \quad x_1\left(\frac{7}{10}\right) = 1, \quad x_2\left(\frac{7}{10}\right) = \frac{7}{10}$$

At $t = \frac{7}{10}$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4c along with the corresponding distribution and density functions.

The mean value is given by $\mu_X\left(\frac{7}{10}\right) = \frac{7}{10} \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = -\frac{3}{20} = -0.15$
The variance is given by $\sigma_X^2\left(\frac{7}{10}\right) = \left(\frac{7}{10} + \frac{3}{20}\right)^2 \frac{1}{2} + \left(-1 + \frac{3}{20}\right)^2 \frac{1}{2} = \frac{289}{400} = 0.7225.$

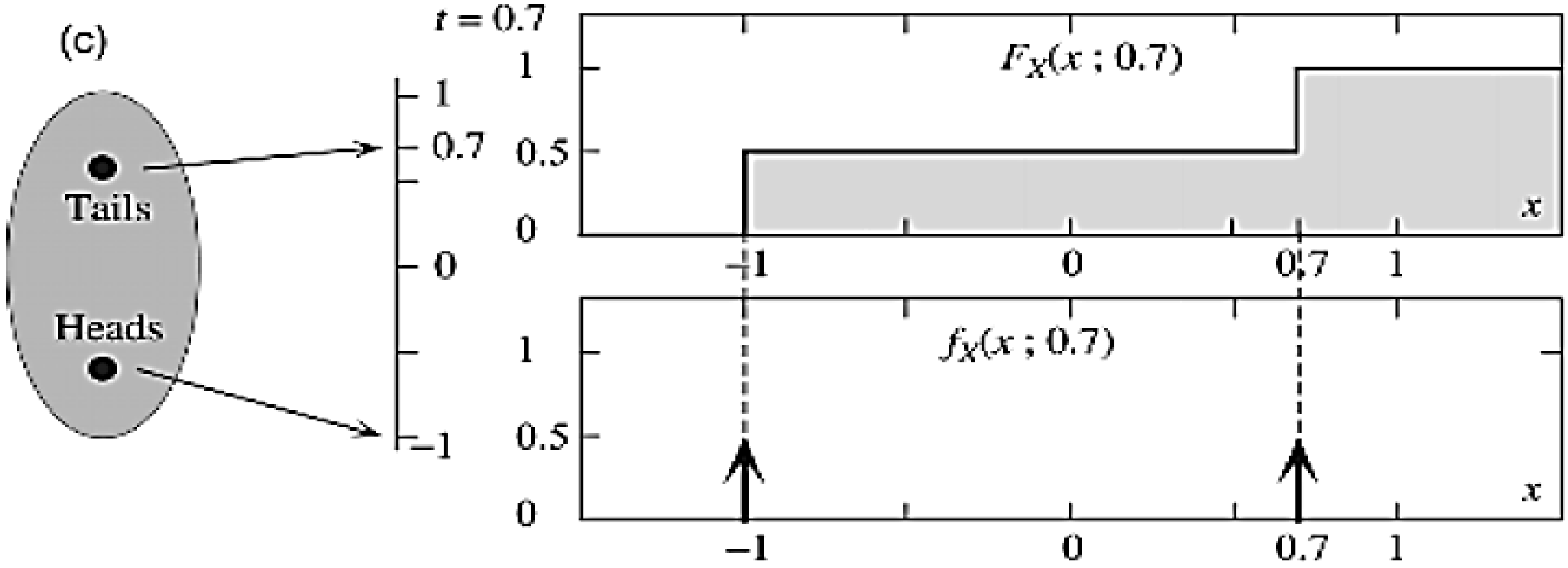


FIGURE 19.1.4

Example 19.1.4 A random process, given by $X(t) = A\sin(\omega t)$, is shown in Fig. 19.1.5, where A is a random variable uniformly distributed in the interval (0,1].

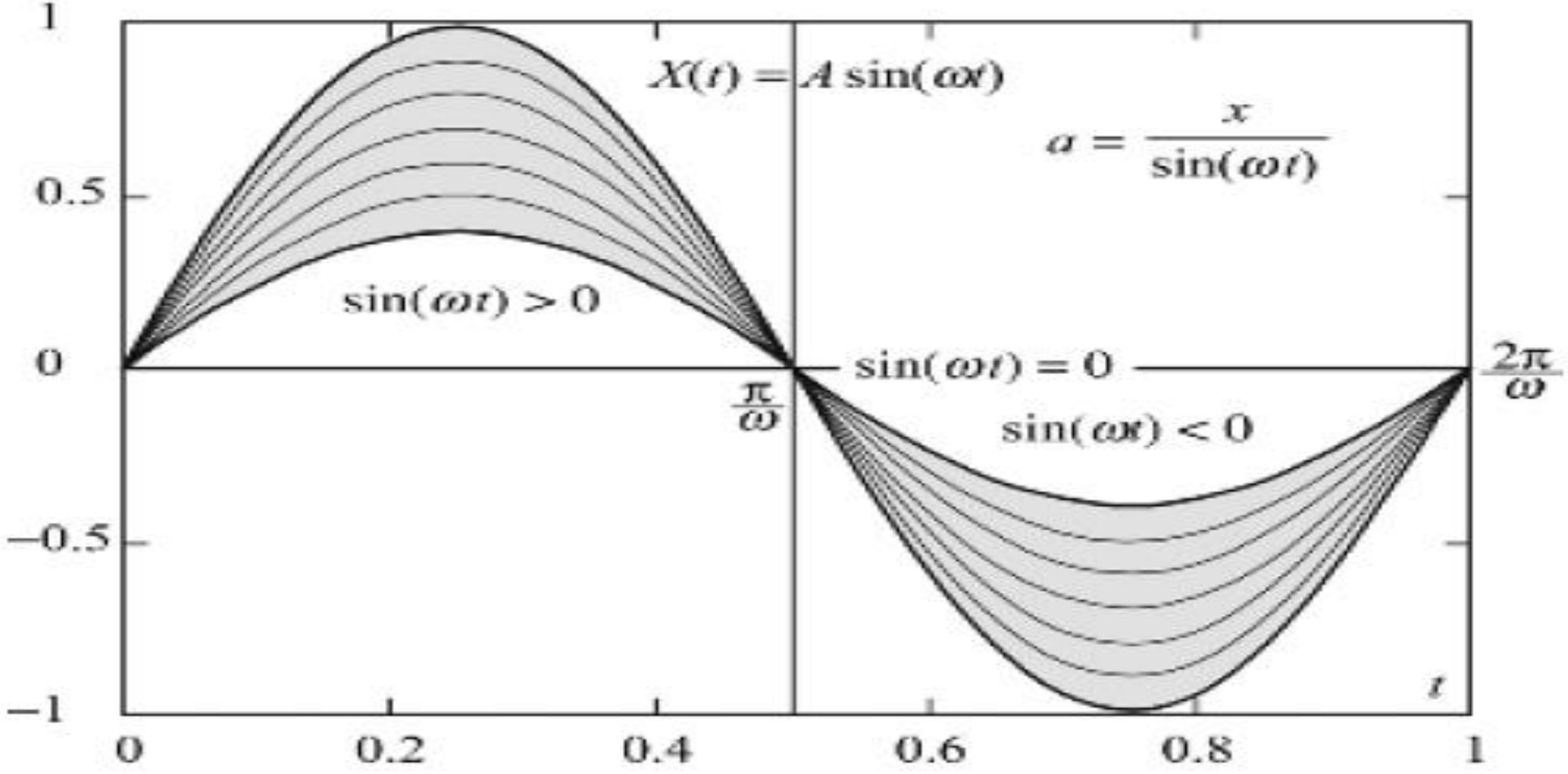
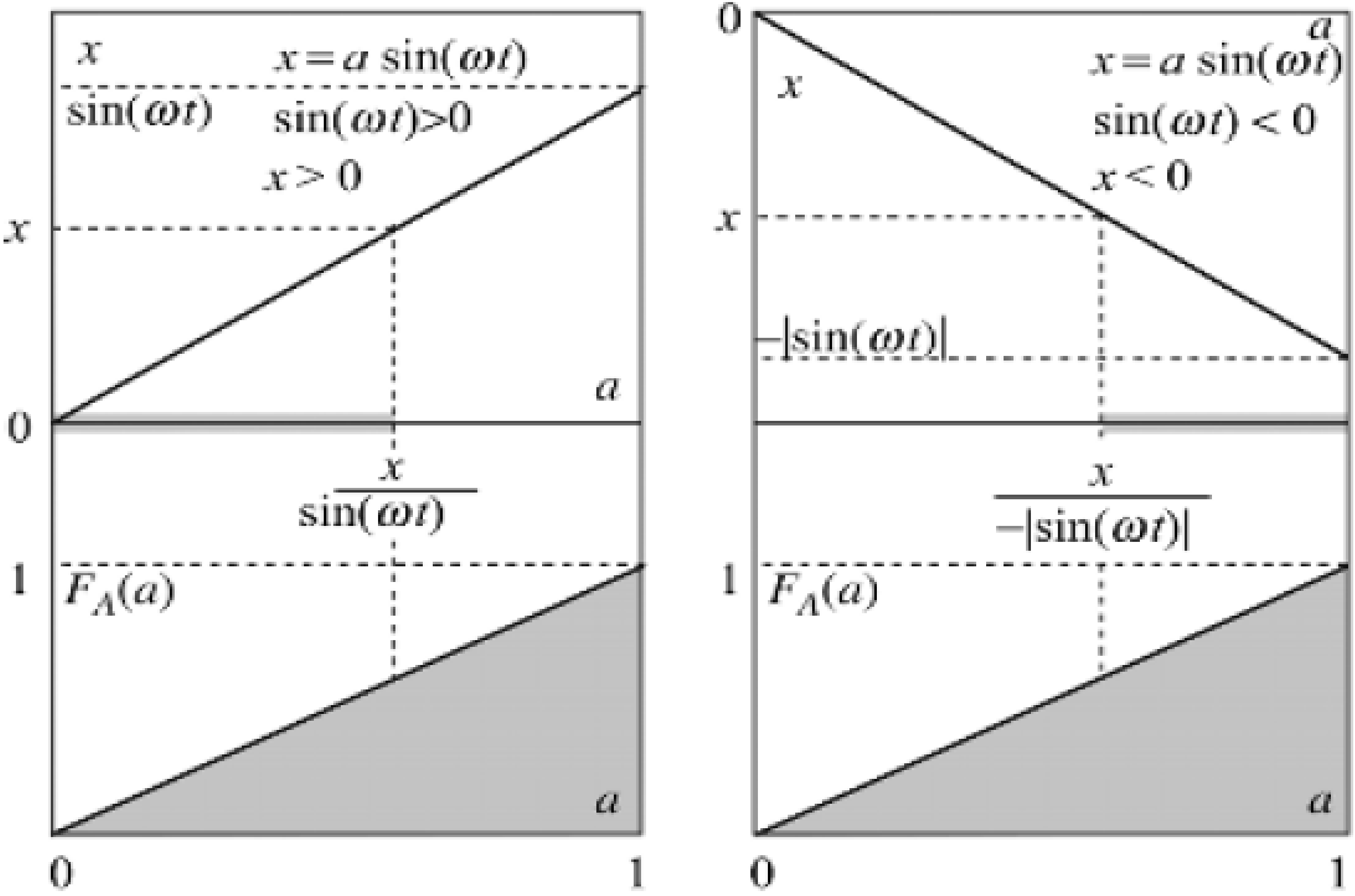


FIGURE 19.1.5

The density and distribution functions of A are

$$f_A(a) = \begin{cases} 1, & 0 < a \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad F_A(a) = \begin{cases} 0, & a \leq 0 \\ a, & 0 < a \leq 1 \\ 1, & a > 1 \end{cases}$$

We have to find the distribution function $F_X(x; t)$. For any given t, $x = a\sin(\omega t)$ is an equation to a straight line with slope $\sin(\omega t)$, and hence we can use the results of Examples 12.2.1 and 12.2.2 to solve for $F_X(x; t)$. The cases of $\sin(\omega t) > 0$ and $\sin(\omega t) < 0$ are shown in Fig. 19.1.6.



x
x = a sin(ωt)
sin(ωt)
sin(ωt)>0
x > 0
x
a
0
x
sin(ωt)
1
F_A(a)
a
0
1
0
a
x = a sin(ωt)
x
sin(ωt) < 0
x < 0
x
−|sin(ωt)|
x
−|sin(ωt)|
1
F_A(a)
a
0
1

FIGURE 19.1.6

Case 1: $\sin(\omega t) > 0$. There are no points of intersection on the a axis for $x \leq 0$, and hence $F_X(x; t) = 0$. For $0 < x \leq \sin(\omega t)$ we solve $x = a\sin(\omega t)$ and obtain $a = x/[\sin(\omega t)]$. The region I_a for which $a \sin(\omega t) \leq x$ is given by $I_a = \{0 < a \leq$

$x/[\sin(\omega t)]\}$ (Fig. 19.1.6). Thus

$$F_X(x; t) = F_A\left(\frac{x}{\sin(\omega t)}\right) - F_A(0) = F_A\left(\frac{x}{\sin(\omega t)}\right) = \frac{x}{\sin(\omega t)}$$

Finally for $x > \sin(\omega t)$, the region I_a for which $a\sin(\omega t) \leq x$, is given by $I_a = \{0 < a \leq 1\}$ and $F_X(x; t) = 1$. Thus, for $\sin(\omega t) > 0$, we have

$$F_X(x; t) = \begin{cases} 0, & x \leq 0 \\ \dfrac{x}{\sin(\omega t)}, & 0 < x \leq \sin(\omega t) \\ 1, & x > \sin(\omega t) \end{cases}$$

Case 2: $\sin(\omega t) < 0$. The region I_a for which $x > 0$ is given by $I_a = \{0 < a \leq 1\}$, and hence $F_X(x;t) = 1$. For $-|\sin(\omega t)| < x \leq 0$, we solve $x = -a|\sin(\omega t)|$ and obtain $a = [x/(-\,|\sin(\omega t)|)]$. The region I_a for which $-a\,|\sin(\omega t)| \leq x$ is given by

$$I_a = \left\{\frac{x}{-|\sin(\omega t)|} < a \leq 1\right\}$$

(Fig. 19.1.6). Thus,

$$F_X(x;\, t) = F_A(1) - F_A\left(\frac{x}{-|\sin(\omega t)|}\right) = 1 - \frac{x}{-|\sin(\omega t)|}$$

Finally, for $x \leq -|\sin(\omega t)|$, the region I_a for which $-a\,|\sin(\omega t)| \leq x$ is given by $I_a = \{1 < a \leq \infty\}$ and $F_X(x;t) = 0$. Thus, for $\sin(\omega t) < 0$, we have

$$F_X(x;\, t) = \begin{bmatrix} 0, & x \leq -|\sin(\omega t)| \\ 1 - \dfrac{x}{-|\sin(\omega t)|}, & -|\sin(\omega t)| < x \leq 0 \\ 1, & x > 0 \end{bmatrix}$$

Case 3: $\sin(\omega t) = 0$. The region I_a for which $x > 0$ is given by $I_a = \{0 < a \leq 1\}$ and $F_X(x;t) = 1$. For $x \leq 0$, $I_a = \varnothing$ and $F_X(x;t) = 0$. Thus, for $\sin(\omega t) = 0$, we have

$$F_X(x;\, t) = \begin{bmatrix} 0, & x \leq 0 \\ 1, & x > 0 \end{bmatrix}$$

Example 19.1.5 We shall now find the distribution and density functions along with the mean and variance for the random process of Example 19.1.2 and see how this process differs from the previous ones.

We are given that $X(t) = A \sin(\omega t + \Phi)$, where A is a constant and $f_\Phi(\phi) = 1/2\pi$ in the interval $(0, 2\pi)$ and we have to find $f_X(x; t)$. We will solve this problem by (1) finding the distribution $F_X(x; t)$ and differentiating it, and (2) by direct determination of the density function.

1. *Determination of Distribution Function $F_X(x; t)$.* The distribution function for Φ is given by $F_\Phi(\phi) = (\phi/2\pi), 0 < \phi \le 2\pi$. The two solutions for $x = A \sin(\omega t + \phi)$ are obtained from the two equations:

$$\sin(\omega t + \phi_1) = \frac{x}{A} \quad \text{and} \quad \sin(\pi - \omega t - \phi_2) = \frac{x}{A}$$

Hence the solutions are given by

$$\phi_1 = \sin^{-1}\left(\frac{x}{A}\right) - \omega t \quad \text{and} \quad \phi_2 = \pi - \sin^{-1}\left(\frac{x}{A}\right)$$

and are shown in Fig. 19.1.7.

For $x \le -A$, there are no points of intersection and hence $F_X(x;\, t) = 0$. For $-A < x \le A$, the set of points along the ϕ axis such that $A\sin(\omega t + \phi) \le x$ is $(0, \phi_1] \cup (\phi_2, 2\pi]$. Hence $F_X(x;\, t)$ is given by

$$\begin{aligned}
F_X(x;t) &= F_\Phi(\phi_1) - F_\Phi(0) + F_\Phi(2\pi) - F_\Phi(\phi_2) \\
&= \frac{1}{2\pi}\left\{\sin^{-1}\left(\frac{x}{A}\right) - \omega t - 0 + 2\pi - \left[\pi - \sin^{-1}\left(\frac{x}{A}\right) - \omega t\right]\right\} \\
&= \frac{1}{2\pi}\left\{2\sin^{-1}\left(\frac{x}{A}\right) + \pi\right\} = \frac{1}{\pi}\sin^{-1}\left(\frac{x}{A}\right) + \frac{1}{2}, \quad -A < x \le A
\end{aligned}$$

Finally, for $x > A$, the entire curve $A\sin(\omega t + \phi)$ is below x, and $F_X(x;t) = 1$.

2. *Determination of Density Function $f_X(x;\, t)$*

(a) The two solutions to $x = A\sin(\omega t + \phi)$ have been found earlier.

(b) The absolute derivatives $|\partial x/\partial\phi||_{\phi_1}$ and $|\partial x/\partial\phi||_{\phi_2}$ are given by

$$\left|\frac{\partial x}{\partial \phi}\right|\Bigg|_{\phi_1} = A\cos(\omega t + \phi_1) = A\cos\left[\omega t + \sin^{-1}\left(\frac{x}{A}\right) - \omega t\right]$$

$$= A\cos\left[\sin^{-1}\left(\frac{x}{A}\right)\right] = A\,\frac{\sqrt{A^2 - x^2}}{A} = \sqrt{A^2 - x^2}$$

and

$$\left|\frac{\partial x}{\partial \phi}\right|\Bigg|_{\phi_2} = A\cos(\omega t + \phi_2) = \left|A\cos\left[\omega t + \pi - \sin^{-1}\left(\frac{x}{A}\right) - \omega t\right]\right|$$

$$= \left|A\cos\left[\pi - \sin^{-1}\left(\frac{x}{A}\right)\right]\right| = A\,\frac{\sqrt{A^2 - x^2}}{A} = \sqrt{A^2 - x^2}$$

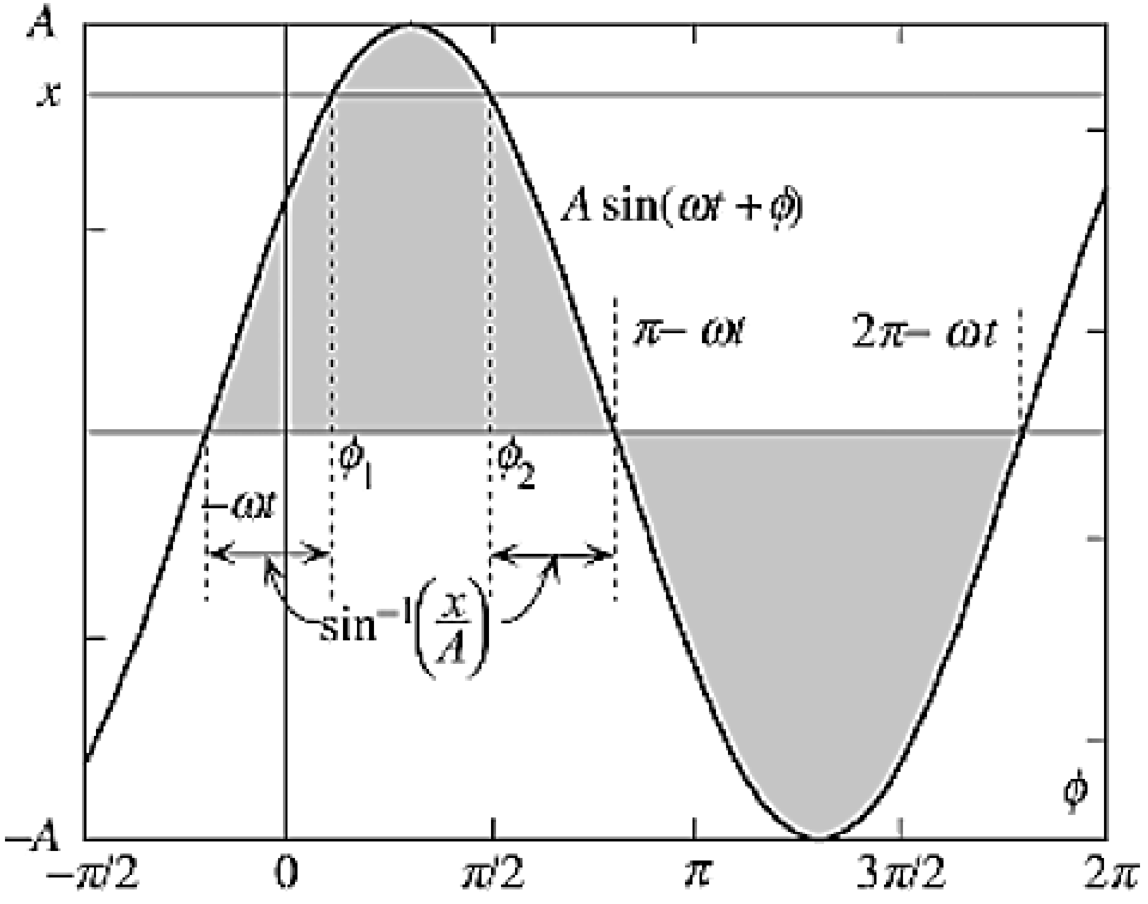


FIGURE 19.1.7

(c) With the two solutions for x, the density function $f_X(x;t)$ is given by Eq. (12.3.6):

$$f_X(x;\, t) = \frac{1}{\sqrt{A^2 - x^2}}\left(\frac{1}{2\pi} + \frac{1}{2\pi}\right) = \frac{1}{\pi\sqrt{A^2 - x^2}}, \qquad -A < x \le A$$

Integration of $f_X(x;t)$ gives the distribution function $F_X(x;t)$

$$F_X(x;\, t) = \begin{bmatrix} 0, & x \le -A \\ \dfrac{1}{\pi}\sin^{-1}\left(\dfrac{x}{A}\right) + \dfrac{1}{2}, & -A < x \le A \\ 1, & x > A \end{bmatrix}$$

and these two solutions are exactly the same as before. For this random process, we find that the density and the distribution functions are both independent of time. For $A = 1$, they become

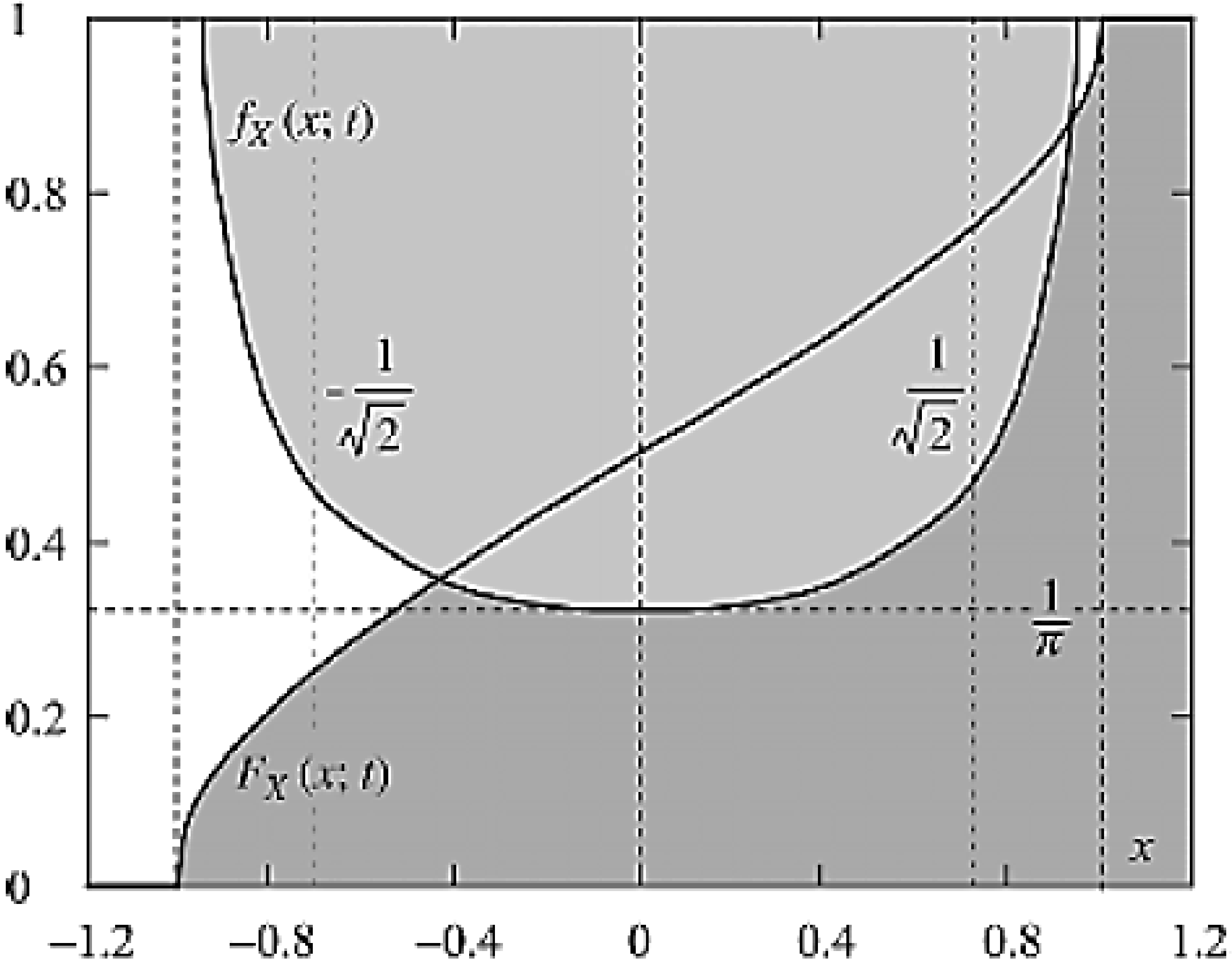
$$f_X(x;t) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad -1 < x \le 1$$

$$F_X(x;t) = \begin{bmatrix} 0, & x \le -1 \\ \dfrac{\sin^{-1}(x)}{\pi} + \dfrac{1}{2}, & -1 < x \le 1 \\ 1, & x > 1 \end{bmatrix}$$

and these functions are shown in Fig. 19.1.8. Since $f_X(x)$ has even symmetry, the mean value is 0 and the variance is obtained from

$$\sigma_X^2 = \int_{-1}^{1} \frac{x^2}{\pi\sqrt{1-x^2}} dx = \frac{1}{2}$$

The value of $\sigma = \pm 1/\sqrt{2}$ is also shown in Fig. 19.1.8.



1
0.8
0.6
0.4
0.2
0
−1.2
−0.8
−0.4
0
0.4
0.8
1.2
$f_X(x;t)$
$F_X(x;t)$
$-\frac{1}{\sqrt{2}}$
$\frac{1}{\sqrt{2}}$
$\frac{1}{\pi}$
x

FIGURE 19.1.8

Example 19.1.6 This example is slightly different from Example 19.1.4. A random process $X(t)$ is given by $X(t) = A\sin(\omega t + \phi)$ as shown in Fig. 19.1.11, where A is a uniformly distributed random variable with mean μ_A and variance σ_A^2. We will find the mean, variance, autocorrelation, autocovariance, and normalized autocovariance of $X(t)$.

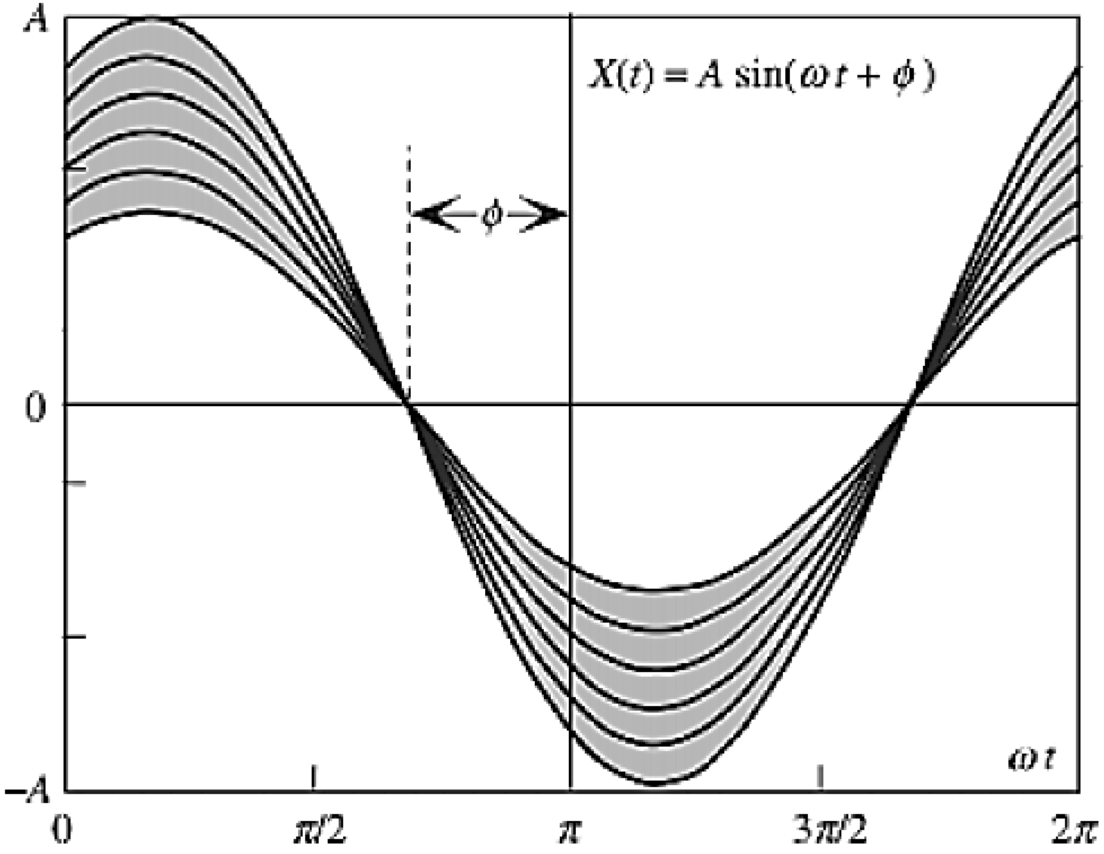


FIGURE 19.1.11

Mean:

$$E[X(t)] = \mu_X(t) = E[A \sin(\omega t + \phi)] = \mu_A \sin(\omega t + \phi)$$

Variance:

$$\begin{aligned} \mathrm{var}[X(t)] &= \sigma_X^2(t) = E[A^2 \sin^2(\omega t + \phi)] - \mu_A^2 \sin^2(\omega t + \phi) \\ &= \{E[A^2] - \mu_A^2\} \sin^2(\omega t + \phi) = \sigma_A^2 \sin^2(\omega t + \phi) \end{aligned}$$

Autocorrelation. From Eq. (19.1.14) we have

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E[A^2] \sin(\omega t_1 + \phi) \sin(\omega t_2 + \phi) \\ &= \frac{1}{2} E[A^2]\{\cos[\omega(t_1 - t_2)] - \cos[\omega(t_1 + t_2) + 2\phi]\} \end{aligned}$$

Autocovariance. From Eq. (19.1.16) we have

$$\begin{aligned}
C_X(t_1,t_2) &= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \\
&= E[A^2]\sin(\omega t_1+\phi)\sin(\omega t_2+\phi) - \mu_A^2\sin(\omega t_1+\phi)\sin(\omega t_2+\phi) \\
&= \sigma_A^2\sin(\omega t_1+\phi)\sin(\omega t_2+\phi) \\
&= \frac{1}{2}\sigma_A^2\{\cos[\omega(t_1-t_2)] - \cos[\omega(t_1+t_2)+2\phi]\}
\end{aligned}$$

Normalized Autocovariance. From Eq. (19.1.18) we have

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \frac{\sigma_A^2\sin(\omega t_1+\phi)\sin(\omega t_2+\phi)}{\sigma_A^2\sin(\omega t_1+\phi)\sin(\omega t_2+\phi)} = 1$$

Example 19.1.7 A random process $X(t)$ with k changes in a time interval t, and its probability mass function is given by $p(k; \lambda) = e^{-\lambda t}[(\lambda t)^k / k!]$. It is also known that the joint probability $P\{k_1 \text{ changes in } t_1, k_2 \text{ changes in } t_2\}$ is given by

$$
\begin{aligned}
&P\{k_1 \text{ changes in } t_1, k_2 \text{ changes in } t_2\} \\
&\quad = P\{k_1 \text{ changes in } t_1, (k_2 - k_1) \text{ changes in } (t_2 - t_1)\} \\
&\quad = P\{k_1 \text{ changes in } t_1\} P\{(k_2 - k_1) \text{ changes in } (t_2 - t_1)\} \\
&\quad = e^{-\lambda t_1} \frac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda(t_2 - t_1)} \frac{[\lambda(t_2 - t_1)]^{(k_2 - k_1)}}{(k_2 - k_1)!}
\end{aligned}
$$

We have to find the mean, variance, autocorrelation, autocovariance, and normalized autocovariance of $X(t)$:

Mean:

$$E\,[X(t)] = \mu_X(t) = \sum_{k=0}^{\infty} k e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \lambda t$$

Variance:

$$\mathrm{var}[X(t)] = \sigma_X^2(t) = \sum_{k=0}^{\infty} k^2 e^{-\lambda t} \frac{(\lambda t)^k}{k!} - (\lambda t)^2 = \lambda t$$

Autocorrelation. From Eq. (19.1.18) we have

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E\{X(t_1)[X(t_2) - X(t_1) + X(t_1)]\} \\ &= E\,[X^2(t_1)] + E\,[X(t_1)]E[X(t_2) - X(t_1)] \quad \text{(from condition given)} \\ &= (\lambda t_1)^2 + \lambda t_1 + \lambda t_1 \lambda(t_2 - t_1) \\ &= \lambda^2 t_1 t_2 + \lambda t_1 \text{ if } t_2 > t_1 \end{aligned}$$

and we have a similar result if $t_1 > t_2$:

$$R_X(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda t_2 \quad \text{if} \quad t_1 > t_2$$

Combining these two results, we have

$$R_X(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Autocovariance. From Eq. (19.1.19) we have

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \\ &= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2) - \lambda^2 t_1 t_2 = \lambda \min(t_1, t_2) \end{aligned}$$

Normalized Autocovariance. From Eq. (19.1.18) we have

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \frac{\min(t_1, t_2)}{\lambda\sqrt{t_1 t_2}} = \frac{1}{\lambda}\min\left(\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_1}}\right)$$

Stationary Processes

If a random process is stationary to all order then the random process is said to be strict sense stationary process.

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau)$$

and the corresponding density function may be written as

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = f_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau)$$

Random processes with the property of above Eq. are called nth-order stationary processes. A strict-sense or strongly stationary process is a random process that satisfies above Equation for all n. Analogously, we can also define lower orders of stationarity.

Note: 1. In general we consider upto second order density function or second order characteristics to verify whether a process to be stationary or not in the respective order.

2. If a random process fails to be atleast first order stationary then the random process is not a stationary process.

Jointly stationary in the strict sense:

Two real-valued random processes {X(t)} and {Y(t)} are said to be jointly stationary in the strict sense, if the joint distribution of X(t) and Y(t) of all order are invariant under translation of time.

First order stationary process

A random process is called stationary to order one, if its first-order density function does not change with a shift in time origin.

In otherwards,

A random process is *first-order stationary* if

$$F_X(x;\, t) = F_X(x;\, t+\tau) = F_X(x)$$

$$f_X(x;\, t) = f_X(x;\, t+\tau) = f_X(x)$$

and the distribution and density functions are *independent* of time.

- A first order stationary random process has a constant mean. (OR) The first order stationary random process X(t) has independent of t.
- A first order stationary random process has a constant variance.

Second-Order Stationary Process

A process is said to be second order stationary, if the 2nd order density function satisfies.

A random process is *second-order stationary* if

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_1 + \tau, t_2 + \tau) = F_X(x_1, x_2; \tau)$$
$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \tau, t_2 + \tau) = f_X(x_1, x_2; \tau)$$

The distribution and density functions are dependent not on two time instants t_1 and t_2 but on the time difference $\tau = t_1 - t_2$ only. Second-order stationary processes are also called *wide-sense stationary* or *weakly stationary*. Hereafter, *stationary* means wide-sense stationary, and strict-sense stationary will be specifically mentioned.

Wide-Sense and Jointly Wide-sense Stationary Processes

A random process X (t) is said to be wide-sense stationary if it satisfies the conditions

(i) $E[X(t)] = \text{constant}$

(ii) $R(t_1, t_2) = E[X(t_1) X(t_2)] = R(t_1 - t_2)$

Note: All SSS process is a WSS process but the converse is not true. i.e., Every WSS process need not be a SSS process.

Two random processes X (t) and Y (t) are called jointly wide-sense stationary if

(i) X(t) is a WSS process

(ii) Y(t) is a WSS process

(iii) $R(t_1, t_2) = E[X(t_1) Y(t_2)] = R(t_1 - t_2)$

Joint Stationary Processes

In a similar manner, two processes *X(t)* and *Y(t)* are jointly stationary if for all *n*

$$F_{XY}(x_1,\ldots,x_n;y_1,\ldots,y_n;t_1,\ldots,t_n)$$
$$=F_{XY}(x_1,\ldots,x_n;y_1,\ldots,y_n;t_1+\tau,\ldots,t_n+\tau)$$

or

$$f_{XY}(x_1,\ldots,x_n;y_1,\ldots,y_n;t_1,\ldots,t_n)$$
$$=f_{XY}(x_1,\ldots,x_n;\ y_1,\ldots,y_n;t_1+\tau,\ldots,t_n+\tau)$$

and they are jointly wide-sense stationary if

$$F_{XY}(x_1,y_2;t_1,t_2)$$
$$=F_{XY}(x_1,y_2;t_1+\tau,t_2+\tau)=F_{XY}(x_1,y_2;\tau)$$

and the joint density function

$$f_{XY}(x_1, y_2; t_1, t_2) = f_{XY}(x_1, y_2; t_1 + \tau, t_2 + \tau) = f_{XY}(x_1, y_2; \tau)$$

nth-order stationarity implies lower-order stationarities. Strict-sense stationarity implies wide-sense stationarity.

Similar to Eqs. Single Stationary processes, we can enumerate the following properties for stationary random processes X(t) and Y(t). Two random processes X(t) and Y(t) are independent if for all x and y

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

They are *uncorrelated* if for all τ

$$C_{XY}(\tau) = R_{XY}(\tau) - \mu_X\mu_Y = 0$$

or

$$R_{XY}(\tau) = \mu_X\mu_Y$$

They are *orthogonal* if for all τ

$$R_{XY}(\tau) = 0$$

Moments of Continuous-Time Stationary Processes

We can now define the various moments for a stationary random process X(t).

Mean:

$$E[X(t)] = \mu_X = \int_{-\infty}^{\infty} x f(x) dx$$

Variance:

$$E[X(t) - \mu_X]^2 = \sigma_X^2 = \int_{-\infty}^{\infty} (x - m_X)^2 f(x) dx = E[X^2(t)] - \mu_X^2$$

Autocorrelation:

$$R_X(\tau) = E[X(t)X(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau) dx_1 \ dx_2$$

Autocovariance:

$$\begin{aligned} C_X(\tau) &= E\{[X(t) - \mu_X][X(t+\tau) - \mu_X]\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X)(x_2 - \mu_X) f(x_1, x_2; \tau) dx_1 \ dx_2 \\ &= R_X(\tau) - \mu_X^2 \end{aligned}$$

Normalized Autocovariance (NACF):

$$\rho_X(\tau) = \frac{C_X(\tau)}{\sigma_X^2}$$

STATIONARY PROCESS

Integration w.r.to θ	$(0, 2\pi)$	$(-\pi, \pi)$
$\cos n\theta$	0	0
$\sin n\theta$	0	0
$\cos(\omega t + n\theta)$	0	0
$\sin(\omega t + n\theta)$	0	0
Here, n is an integer, $n \neq 0$		

1. $\int_{-\pi}^{\pi} d\theta = 2\pi$

2. $\int_{0}^{2\pi} d\theta = 2\pi$

3. $\int_{0}^{\pi} \cos[\omega t + \theta]\, d\theta = -2\sin\omega t$

Comparison of SSS and WSS processes

SSS	WSS
Strict Sense Stationary Process (or) Strictly stationary process (or) Stationary process	Wide-Sense Stationary Process Weak-Sense Stationary process (or) Covariance stationary process
Def. : A random process $X(t)$ is said to be SSS, if its statistical characteristics do not change with time i.e. (i) $E[X(t)] = \text{constant}$ (ii) $\text{Var}[X(t)] = \text{constant}$	Def. : A random process $X(t)$ is said to be WSS, if it satisfies (i) $E[X(t)] = \text{constant}$ (ii) $R(t_1, t_2) = $ function of time difference. i.e. a function of $(t_1 - t_2)$ Note : $\tau = t_1 - t_2$
Note : Every WSS process need not be a SSS process of order 2.	Note : A SSS process of order two is a WSS process but the converse is not true.
Example for SSS : 1. Bernoulli's process is a SSS. 2. Strong sense white noise. 3. Weak sense white noise	Example for WSS : 1. A random telegraph signal process is a WSS. 2. Random binary transmission process is a WSS which is not mean-ergodic. 3. Sinusoid with random phase.
Example for not SSS 1. Semi random telegraph signal process 2. Poisson process is not a stationary process.	Example for not WSS 1. Poisson process is not a WSS. 2. Random walk is not a WSS.

Problem:

Show that the random process X (t) = A sin (ωt + φ) where A and ω are constants, φ is a random variable uniformly distributed in (0, 2π) is first order stationary.

Solution:

Given: X (t) = A sin (ωt + φ)

where 'φ' is uniformly distributed in (0, 2π)

$$\Rightarrow f(\phi) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}, \quad 0 < \phi < 2\pi$$

[∵ From the definition of uniform distribution]

To prove : $X(t)$ is first order stationary.
i.e, To prove : $E[X(t)] = \text{constant}$

Proof

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\phi)\, d\phi$$

$$= \int_0^{2\pi} A \sin(\omega t + \phi) \frac{1}{2\pi}\, d\phi$$

$$= \frac{A}{2\pi} \int_0^{2\pi} \sin(\omega t + \phi)\, d\phi = \frac{A}{2\pi}[0] = 0 = \text{a constant}$$

$$\left[\because \int_0^{2\pi} \sin[\omega t + n\theta]\, d\theta = 0, \quad n \text{ is an integer, } n \neq 0\right]$$

Hence, X (t) is a first order stationary process.

Problem:

Consider the random process X(t) = cos (ω_0t + θ), where θ is uniformly distributed in the interval -л to л. Check whether X(t) is stationary or not? Find the first and second moments of the process.

Solution:

Given: X (t) = cos (ω_0t + θ), where 'θ' is uniformly distributed in (−л, л)

$$\Rightarrow \quad f(\theta) = \frac{1}{\pi - (-\pi)} = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$

To prove :	$X(t)$ is a SSS process.
i.e., To prove :	(i) $E[X(t)]$ = constant,
	(ii) Var. $[X(t)]$ = constant.

Proof:

(i) $$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta)\, d\theta = \frac{1}{2\pi}[0]$$

$= 0$ [First moment] = constant

$$\left[\because \int_{-\pi}^{\pi} \cos[\omega t + n\theta]\, d\theta = 0, \quad n \text{ is an integer}, n \neq 0\right]$$

$$E[X^2(t)] = E[\cos^2(\omega_0 t + \theta)]$$

$$= E\left[\frac{1 + \cos[2(\omega_0 t + \theta)]}{2}\right] \quad \left[\text{Formula : } \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{2} E[1 + \cos(2\omega_0 t + 2\theta)]$$

$$E[X^2(t)] = \frac{1}{2} E[1] + \frac{1}{2} E[\cos(2\omega_0 t + 2\theta)] \quad \ldots (1)$$

$$= \frac{1}{2} + \frac{1}{2} E[\cos(2\omega_0 t + 2\theta)]$$

$$[\because E\ (\text{constant}) = \text{constant}]$$

$$E[\cos(2\omega_0 t + 2\theta)] = \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta) \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta)\, d\theta = \frac{1}{2\pi}[0] = 0$$

$$\left[\because \int_{-\pi}^{\pi} \cos[\omega t + n\theta]\, d\theta = 0,\ n \text{ is an integer, } n \neq 0\right]$$

$$\therefore (1) \Rightarrow E[X^2(t)] = \frac{1}{2}(1) + 0 = \frac{1}{2} \quad [\text{Second moment}]$$

$$\boxed{\mathrm{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2}$$

$$= \frac{1}{2} - (0)^2 = \frac{1}{2} = \text{constant}$$

Hence, X (t) is a SSS process.

Problem:

Consider the random process X (t) = cos (t + φ), where is a φ random variable with density function f(φ) = 1/π, -π/2 < φ < π/2, check whether the process is stationary or not.

Solution: Given : $X(t) = \cos(t+\phi), \quad f(\phi) = \frac{1}{\pi}, \; -\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t)\, f(\phi)\, d\phi = \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, \frac{1}{\pi}\, d\phi$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, d\phi$$

$$= \frac{1}{\pi} \left[\sin(t+\phi)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi} \left[\sin\left(t+\frac{\pi}{2}\right) - \sin\left(t-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi} \left[\sin\left(\frac{\pi}{2}+t\right) + \sin\left(\frac{\pi}{2}-t\right)\right]$$

$$= \frac{1}{\pi} \left[\cos t + \cos t\right]$$

$$= \frac{2}{\pi} \cos t \neq \text{constant.}$$

Hence, X(t) is not a SSS process.

Problem:

A random process is described by X(t) = A sin t + B cos t where A and B are independent random variables with zero mean and equal variances (or equal S.D). Show that the process is stationary of second order.

Solution:

Given: X (t) = A sin t + B cos t ....(1)

E[A] = 0, E[B] = 0 ....(2)

$$E[AB] = E[A]\,E[B] \quad [\because \text{A and B are independent random variables}]$$

$$= (0)\,(0)$$

i.e., $E[AB] = 0$ ... (3)

$$E[A^2] = \sigma^2, \quad E[B^2] = \sigma^2 \quad \text{... (4)}$$

To prove:	$X(t)$ is a stationary of second order
i.e., To prove:	(i) $E[X(t)] =$ constant
	(ii) $E[X^2(t)] =$ constant.

(i) $E[X(t)] = E[A \sin t + B \cos t]$

$= \sin t\, E[A] + \cos t\, E[B]$

$= \sin t\,(0) + \cos t\,(0)$ ... by (2)

$= 0 =$ a constant.

(ii) $E[X^2(t)] = E\left[(A \sin t + B \cos t)^2\right]$

$= E[A^2 \sin^2 t + B^2 \cos^2 t + 2AB \sin t \cos t]$

$= \sin^2 t\; E[A^2] + \cos^2 t\, E[B^2] + 2 \sin t \cos t\, E[AB]$

$= \sin^2 t\,(\sigma^2) + \cos^2 t\,(\sigma^2) + 0$ ... by (3) & (4)

$= \sigma^2 [\sin^2 t + \cos^2 t]$

$= \sigma^2 (1) = \sigma^2 =$ a constant

Hence, the process X (t) is stationary of second order.

Problem

Show that the random process X (t) = A cos (ωt + θ) is wide sense stationary it A & ω are constant and 'θ' is uniformly distributed random variable in (0, 2л).

Solution :

Given X(t) = A cos (ωt + θ),

where 'θ' is uniformly distributed in (0, 2л).

$$\Rightarrow \quad f(\theta) = \begin{cases} \dfrac{1}{2\pi - 0}, & 0 < \theta < 2\pi \\ 0 & , \ otherwise \end{cases} \quad \Rightarrow f(\theta) = \frac{1}{2\pi}$$

To prove :	$X(t)$ is WSS.
i.e., To prove :	(i) $E[X(t)] =$ constant
	(ii) $R(t_1, t_2) =$ a function of $(t_1 - t_2)$

$$\text{(i)} \qquad E[X(t)] = \int_{-\infty}^{\infty} X(t)\, f(\theta)\, d\theta$$

$$\Rightarrow \quad E\left[A\cos(\omega t + \theta)\right] = \int_{-\infty}^{\infty} A\cos(\omega t + \theta) f(\theta)\, d\theta$$

$$= \int_0^{2\pi} A\cos(\omega t + \theta)\frac{1}{2\pi}\, d\theta = \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega t + \theta)\, d\theta$$

$$= 0 = \text{constant} \quad \left[\because \int_0^{2\pi} \cos[\omega t + n\theta]\, d\theta = 0,\ n \text{ is an integer, } n \neq 0\right]$$

$$\text{(ii)} \quad R(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= E[A\cos(\omega t_1 + \theta) A\cos(\omega t_2 + \theta)]$$

$$= E[A^2 \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)]$$

$$= E\left[\frac{A^2}{2}\left(\cos[\omega(t_1+t_2)+2\theta]+\cos[\omega(t_1-t_2)]\right)\right]$$

$$\boxed{\because \cos A \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]}$$

$$= E\left[\frac{A^2}{2}\cos[\omega(t_1+t_2)+2\theta\right] + E\left[\frac{A^2}{2}\cos[\omega(t_1-t_2)]\right]$$

$$= \frac{A^2}{2}E\left[\cos[\omega(t_1+t_2)+2\theta]\right] + \frac{A^2}{2}\cos[\omega(t_1-t_2)] \quad \ldots (1)$$

$$[\because E[\text{constant}] = \text{constant}]$$

$$\boxed{\text{Take}}\ E\left[\cos[2\theta+\omega(t_1+t_2)]\right] = \int_0^{2\pi}\cos[2\theta+\omega(t_1+t_2)]\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos[2\theta+\omega(t_1+t_2)]\,d\theta = \frac{1}{2\pi}[0] = 0$$

$$[\because \int_0^{2\pi}\cos[\omega t+n\theta]\,d\theta = 0,\ n \text{ is an integer, } n \neq 0]$$

$$(1) \Rightarrow R(t_1,t_2) = \frac{A^2}{2}(0) + \frac{A^2}{2}\cos[\omega(t_1-t_2)] = \frac{A^2}{2}\cos[\omega(t_1-t_2)]$$

$$= \text{a function of } (t_1-t_2)$$

Hence, X (t) is a WSS process.

TIME AVERAGES OF A RANDOM PROCESS

Time Average Function: Consider a random process X(t). Let x(t) be a sample function which exists for all time at a fixed value in the given sample space S.

The average value of X(t) taken over all times is called the time average of X(t). It is also called mean value of X(t). The time averaged mean is defined as

$$\overline{X(t)} = < m_x > = A[X(t)] = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\ dt$$

Time Auto correlation Function: Consider a random process X(t). The time average of the product of X(t) and X(t+τ) is called time auto correlation function of X(t) and it is denoted as

$$< R_{XX}(\tau) >= \overline{R_{XX}(\tau)} = A[X(t)\, X(t+\tau)]$$

- The time averaging auto correlation function is given by

$$< R_{XX}(\tau) >= \overline{R_{XX}(\tau)} = A[X(t)\, X(t+\tau)] = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x(t+\tau)\, dt$$

Where τ is a time difference or shift in time t.

Time Mean Square Function: If $\tau = 0$, the time average of $X^2(t)$ is called time mean square value of $X(t)$. It is denoted as

$$A[X^2(t)] = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\, dt$$

The mean-square value of a function is the mean value of the square of the function over a given interval.

Time Cross Correlation Function: Let X(t) and Y(t) be two random processes with sample functions x(t) and y(t) respectively.

The time average of the product of x(t) and y(t+τ) is called time cross correlation function of X(t) and Y(t). It is denoted as

$$< R_{XY}(\tau) >= \overline{R_{XY}(\tau)} = A[X(t)\,Y(t+\tau)]$$

$$< R_{XY}(\tau) >= \overline{R_{XY}(\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,y(t+\tau)\;dt$$

ERGODIC PROCESSES

Ergodic processes are processes for which time and ensemble (statistical) averages are interchangeable. The concept of ergodicity deals with the equality of time and statistical averages.

Definition: Ensemble Average:

The ensemble average of a random process {X(t)} is the expected value of the random variable X at time t.

i.e., Ensemble average = E[X(t)]

Definition: Ergodic process :

A random process X (t) is said to be ergodic, if its ensemble averages are equal to appropriate time averages.

i.e., $E[X(t)] = \overline{X}_T$ where $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$

Note: In general, ensemble averages and time averages are not equal except for a special class of random processes called ergodic process.

- **Jointly Ergodic Processes:** Let X(t) and Y(t) be two random processes with sample function x(t) and y(t) respectively.

The two random process are said to jointly ergodic if they are individually ergodic and their time cross correlation functions are equal to their respective ensemble cross correlation functions.

i. e., $E[X(t)] = < X(t) >$ and $E[Y(t)] = < Y(t) >$

$$R_{XX}(\tau) = < R_{XX}(\tau) >; \; R_{YY}(\tau) = < R_{YY}(\tau) >$$

and $R_{XY}(\tau) = < R_{XY}(\tau) >$

- **Mean Ergodic Processes:** A random process X(t) is said to be mean ergodic or ergodic in the mean if the time average of any sample function x(t) is equal to its statistical average $\overline{X}$ which is constant and the probability of all other sample functions is equal to one. That is

$$E[X(t)] = \overline{X} = A[X(t)] = \bar{x}$$

With probability one for all x(t).

- **Autocorrelation Ergodic Processes:** A stationary random process X(t) is said to be autocorrelation ergodic or ergodic in the autocorrelation if and only if the time auto correlation function of any sample function x(t) is equal to the statistical autocorrelation function of X(t), that is,

$$A[X(t)X(t+\tau)] = E[X(t)X(t+\tau)] \text{ or } < R_{XX}(\tau) >= R_{XX}(\tau)$$

- **Cross correlation Ergodic Processes:** Two stationary random processes X(t) and Y(t) is said to be cross correlation ergodic or ergodic in the cross correlation if and only if the time cross correlation function of any sample function x(t) and y(t) is equal to the statistical cross correlation function of X(t) and Y(t), that is,

$$A[X(t)Y(t+\tau)] = E[X(t)Y(t+\tau)] \text{ or } < R_{XY}(\tau) >= R_{XY}(\tau)$$

Problem

Show that the random process X(t) = cos (t + φ), where φ is a random variable uniformly distributed in (0, 2π) is (i) First order stationary (ii) Stationary in the wide-sense (iii) Ergodic (based on first order or second order averages).

Solution :

Given: X(t) = cos (t + φ)

where φ is uniformly distributed in (0, 2π)

$$\Rightarrow f(\phi) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}, \quad 0 < \phi < 2\pi$$

[∵ from the definition of uniform distribution]

(i)	To prove	:	$X(t)$ is First order stationary
	i.e., To prove :		$E[X(t)] = \text{constant.}$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\phi)\, d\phi$$

$$E[\cos(t+\phi)] = \int_{0}^{2\pi} \cos(t+\phi) \frac{1}{2\pi}\, d\phi$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \cos(t+\phi)\, d\phi$$

$$= \frac{1}{2\pi}[0] = 0 = \text{ a constant.}$$

$$\left[\because \int_{0}^{2\pi} \cos[\omega t + n\theta]\, d\theta = 0,\ n \text{ is an integer, } n \neq 0\right]$$

Hence, $X(t)$ is a first order stationary.

(ii) To prove : $X(t)$ is WSS

i.e., To prove : (a) $E[X(t)] = \text{constant}$

(b) $R(t_1, t_2) = \text{a function of } (t_1 - t_2)$

Proof : (a) $E[X(t)] = \text{constant by (i)}$

(b) $R(t_1, t_2) = E[X(t_1)\, X(t_2)]$

$$= E[\cos(t_1 + \phi)\cos(t_2 + \phi)]$$

$$= E\left[\frac{1}{2}\,[\cos(t_1 + t_2 + 2\phi) + \cos(t_1 - t_2)]\right]$$

$$\boxed{\because \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]}$$

$$= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2\phi)\right] + E\left[\frac{1}{2}\cos(t_1 - t_2)\right]$$

$$= \frac{1}{2}E\left[\cos(t_1 + t_2 + 2\phi)\right] + \frac{1}{2}\cos(t_1 - t_2) \quad \ldots (1)$$

[$\because$ E[constant] = constant]

Take $E\left[\cos(t_1 + t_2 + 2\phi)\right] = \int_0^{2\pi} \cos(t_1 + t_2 + 2\phi)\,\frac{1}{2\pi}\,d\phi$

$$= 0$$

[$\because \int_0^{2\pi} \cos[\omega t + n\theta]\,d\theta = 0$, n is an integer, $n \neq 0$]

$\therefore$ (1) $\Rightarrow$ $R(t_1, t_2) = \frac{1}{2}(0) + \frac{1}{2}\cos(t_1 - t_2) = \frac{1}{2}\cos(t_1 - t_2)$

$$= \text{a function of } (t_1 - t_2)$$

(iii) **To prove** : $X(t)$ is mean-ergodic.

i.e., To prove : $\lim\limits_{T\to\infty} \overline{X}_T = E[X(t)]$

Here, $E[X(t)] = 0$ by (i)

$$\text{L.H.S} = \lim_{T\to\infty} \overline{X}_T$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos(t+\phi)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\cos t \cos\phi - \sin t \sin\phi\right] dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[\cos\phi \int_{-T}^{T} \cos t\, dt - \sin\phi \int_{-T}^{T} \sin t\, dt\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[2\cos\phi \int_{0}^{T} \cos t\, dt - 0\right]$$

[∵ $\cos t$ is an even function

$\sin t$ is an odd function]

$$= \lim_{T\to\infty} \frac{1}{T} \cos\phi \left[\sin t\right]_0^T = \lim_{T\to\infty} \frac{1}{T} \cos\phi \sin T$$

$$= \lim_{T\to\infty} \cos\phi \frac{\sin T}{T} = \cos\phi \lim_{T\to\infty} \frac{\sin T}{T}$$

$$= (\cos\phi)\,(0) \qquad \left[\because \lim_{T\to\infty} \frac{\sin T}{T} = 0\right]$$

$$= 0$$

$$\text{L.H.S} = \text{R.H.S}$$

∴ The process $X(t)$ is mean-ergodic.

GAUSSIAN RANDOM PROCESSES

- Consider a continuous random processes $X(t)$. Let N random variables $X_1 = X(t_1), X_2 = X(t_2), \dots, X_N = X(t_N)$ be defined at time instants $t_1, t_2, t_3, \dots \dots, t_N$ respectively.
- If these random variables are jointly Gaussian for any N = 1, 2, 3, …… and at any time instants $t_1, t_2, t_3, \dots \dots, t_N$, then the random process $X(t)$ is called a Gaussian random process.
- The joint density function for a Gaussian random variable is given as

$$f_X(x_1, x_2, \dots, x_N;\ t_1, t_2, \dots, t_N) = \frac{1}{\sqrt{(2\pi)^N |[C_{XX}]|}} \exp\left\{-\frac{1}{2}[x - \overline{X}]^t [C_{XX}]^{-1} [x - \overline{X}]\right\}$$

$$f_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N)$$

$$= \frac{1}{\sqrt{(2\pi)^N |[C_{XX}]|}} \exp\left\{-\frac{1}{2}[x - \bar{X}]^t [C_{XX}]^{-1} [x - \bar{X}]\right\}$$

Where, $\bar{X} = E[X_i] = E[X(t)]$; $[C_{XX}]$ is the covariance matrix and its elements are

$$C_{ik} = E[(X_i - \bar{X_i})(X_k - \overline{X_k})]$$

C_{ik} is the auto covariance of $X(t_i)$ and $X(t_k)$. Also by expanding the above equation, we can get,

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

Where, $R_{XX}(t_i, t_k)$ is the auto correlation function of X.

- If the process is wide sense stationary, then it should satisfy the following conditions
    1. Mean value will be constant, i.e., $\bar{X}_i = E[X(t)] = \bar{X}$ is a constant
    2. The mean auto correlation and auto covariance functions will dependent only on time differences and not on absolute time.

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i) \text{ and } R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Poisson Random Process

- The Poisson process X(t) is a discrete random process which represents the number of times that some event has occurred as a function of time.
- X(t) has integer valued, non-decreasing sample functions, such as check in registers, arrival of a customer, arrival of vehicles at a particular point etc.
- In these functions, a single event occurs at a random time. Counting the number of occurrences with time is a Poisson process. It is, therefore, also called a counting process. Figure shows the sample function of a Poisson counting process.
- The conditions for a Poisson process X(t) are
    1. X(0) = 0
    2. Only one event occurs in any instant of time, i.c., in an infinitesimal time interval.

Definition :

If $X(t)$ represents the number of occurrences of a certain event in $(0, t)$, then the discrete random process $\{X(t)\}$ is called the Poisson process, provided that the following postulates are satisfied.

I. $P[1 \text{ occurrence in } (t, t+\Delta t)] = \lambda \Delta t + 0 (\Delta t)$
II. $P [0 \text{ occurrence in } (t,t+t\Delta)] = 1 - \lambda \Delta t + 0 (\Delta t)$
III. $P [2 \text{ or more occurrences in } (t, t+\Delta t)] = 0 (\Delta t)$
IV. $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.
V. The probability that the event occurs a specified number of times in (t_0, t_0+t) depends only on t, but not on to.

Poisson process is not a stationary process, as its statistical properties (mean, autocorrelation, ...) are time dependent.

Mean and Variance of the Poisson Process:

$$P\{X(t)=n\} = \frac{e^{-\lambda t}(\lambda t)^n}{\lfloor n}, \quad n = 0, 1, 2, \ldots \infty$$

$$E[X(t)] = \sum_{x=0}^{\infty} x \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= e^{-\lambda t} \sum_{x=1}^{\infty} \frac{\lambda t (\lambda t)^{x-1}}{\lfloor x-1} = \lambda t\, e^{-\lambda t} e^{\lambda t} = \lambda t$$

$$\text{Mean} = \lambda t$$

$$E[x^2(t)] = \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} [x(x-1)] \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x} + \sum_{x=0}^{\infty} \frac{x\, e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= e^{-\lambda t}(\lambda t)^2 \sum_{x=2}^{\infty} \frac{(\lambda t)^{x-2}}{\lfloor x-2} + e^{-\lambda t} \lambda t \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{\lfloor x-1}$$

$$= e^{-\lambda t}(\lambda t)^2 . e^{\lambda t} + e^{-\lambda t} \lambda t . e^{\lambda t} = (\lambda t)^2 + \lambda t$$

$$\text{Var}[X(t)] = E[X^2(t)] - [E(X(t))]^2$$

$$= (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$\therefore \text{Var}[X(t)] = \lambda t.$$

Problem

If patients arrive at a clinic according to Poisson process with mean rate of 2 per minute. Find the probability that during a 1-minute interval, no patient arrives.

Solution :

Let X(t) be the number of patients arrive at a clinic in the time interval of t.

Given:

Mean rate, $\lambda = 2$ per minute

Time interval, $t = 1$ per minute

No. of arrivals, $n = 0$

X(t) follows a Poisson process, $P[X(t) = n] = e^{-\lambda t} (\lambda t)^{n} / n!$

$$P\begin{bmatrix}\text{no patient arrives in}\\ \text{one-minute interval}\end{bmatrix} = P[X(1) = 0] = \frac{e^{-(2)(1)}[(2)(1)]^0}{0!}$$

$$= e^{-2} = 0.135$$