# Problems and Solutions on Convolution

1. Justify that the convolution of $e^{-at}u(t) * e^{-bt}u(t)$ is $\frac{1}{(b-a)}[e^{-at}u(t) - e^{-bt}u(t)]$.

**Solution:** Given that $x(t) = e^{-at}u(t)$ and $h(t) = e^{-bt}u(t)$

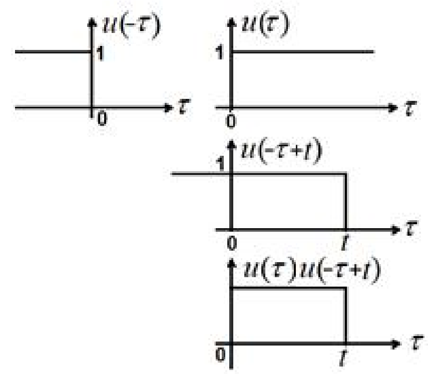
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau = \int_0^t e^{-a\tau}\, e^{-b(t-\tau)} d\tau$$

$$= e^{-bt} \int_0^t e^{-a\tau} e^{b\tau}\, d\tau = e^{-bt} \int_0^t e^{-(a-b)\tau}\, d\tau$$

$$= e^{-bt} \left[ \frac{e^{-(a-b)\tau}}{-(a-b)} \right]_0^t = \frac{e^{-bt}}{(b-a)} \left[ e^{-(a-b)t} - 1 \right] u(t)$$

$$= \frac{1}{(b-a)} \left[ e^{-at} - e^{-bt} \right] u(t)$$




2. An excitation and impulse response of the system are given. Find the responses.

(a) $x(t) = u(t);$ $\qquad h(t) = 2u(t-1) - 2u(t-4)$

Ans:

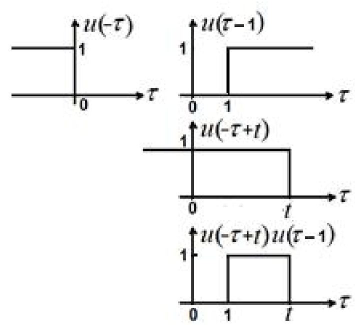
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau)\; d\tau$$

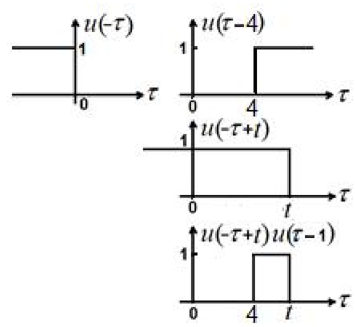
$$= \int_{-\infty}^{\infty} u(t-\tau) \left\{ 2u(\tau-1) - 2u(\tau-4) \right\} d\tau$$

$$= \int_{-\infty}^{\infty} u(t-\tau) 2u(\tau-1)\, d\tau - \int_{-\infty}^{\infty} u(t-\tau) 2u(\tau-4)\, d\tau$$

$$= 2\int_1^t 1\, d\tau + 2\int_4^t 1\, d\tau$$

$$= 2(t-1)u(t-1) - 2(t-4)u(t-4)$$




```
clear all; close all; clc;

t = -10:0.01:10;
x = double(us(t,0));
h = 2*us(t,1)-2*us(t,4);
y = 0.01*conv(x,h);
figure();
subplot(3,1,1);plot(t,x,'r','LineWidth',3); axis([-2 20
-0.1 1.2]);
subplot(3,1,2);plot(t,h,'b','LineWidth',3); axis([-2 20 -0.1 2.2]);
t1 = -20:0.01:20;
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subplot(3,1,3);plot(t1,y,'m','LineWidth',3);axis([-2 20 -0.1 6.2]);
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(b) $x(t) = 2u(t) - 2u(t-2);\qquad h(t) = 3u(t-5) - 3u(t-1)$

Ans:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\ d\tau$$

$$= \int_{-\infty}^{\infty} \{2u(t-\tau) - 2u(t-\tau-2)\}\{3u(\tau-5) - 3u(\tau-1)\}\ d\tau$$

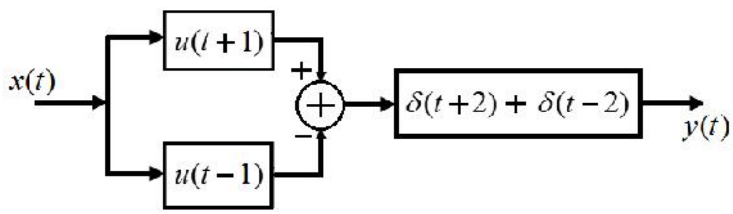
$$= 6\int_{-\infty}^{\infty} u(t-\tau)u(\tau-5)\ d\tau - \int_{-\infty}^{\infty} u(t-\tau)u(\tau-1)\ d\tau$$

$$-6\int_{-\infty}^{\infty} u(t-\tau-2)u(\tau-5)\ d\tau - \int_{-\infty}^{\infty} u(t-\tau-2)u(\tau-1)\ d\tau$$

$$= 6\left[\int_{5}^{t} 1\ d\tau - \int_{5}^{t-2} d\tau - \int_{1}^{t} d\tau + \int_{1}^{t-2} d\tau\right]$$

$$= 6\left[(t-5)u(t-5) - (t-7)u(t-7) - (t-1)u(t-1) + (t-3)u(t-3)\right]$$

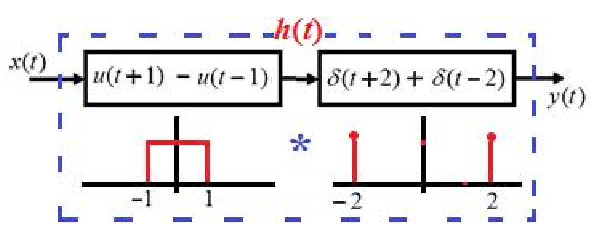
3. Three systems are interconnected as shown below

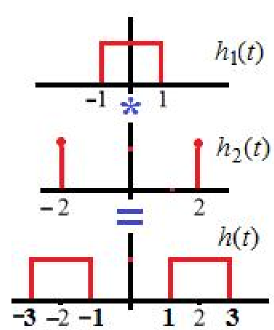



Find and plot output $y(t)$ when $x(t) = \delta(t)$

**Ans:** Let the combined impulse response in parallel structure is represented by

$h_1(t) = u(t+1) - u(t-1)$ and let $h_2(t) = \delta(t+2) - \delta(t-2)$. Then the equivalent impulse response of cascaded stage is represented by



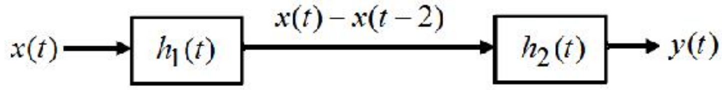

$h(t) = h_1(t) * h_2(t)$. This illustrated in the following figures.

For the given input excitation $x(t) = \delta(t)$, the response of the system is

$$y(t) = x(t) * h(t) = \delta(t) * h(t) = h(t)$$

$$= \{u(t+3) - u(t+1)\} + \{u(t-1) - u(t-3)\}$$

4. A cascaded system is shown below

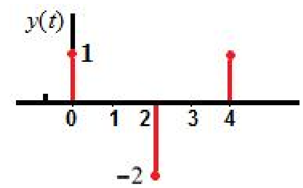



Suppose that $h_1(t)=h_2(t)$, find and plot output $y(t)$ when $x(t)=¢(t)$

**Solution:** Let the output of the system $h_1(t)$ is $z(t)=x(t)-x(t-2)$. That is input minus two units delayed input signal. This signal $z(t)$ is applied as input to the system represented by $h_2(t)$. The response of this system (in turn overall response of the system) is

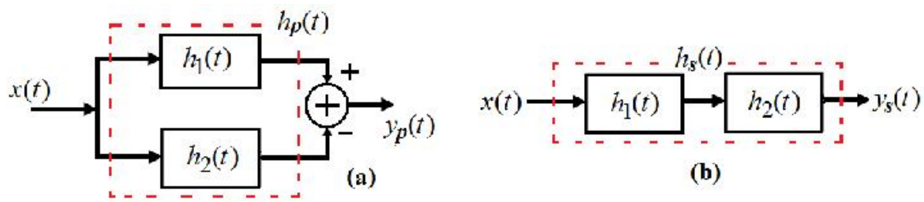
$$y(t)=z(t)-z(t-2)=\{x(t)-x(t-2)\}-\{x(t-2)-x(t-4)\}$$
$$=x(t)-2x(t-2)+x(t-4)$$

For the input excitation $x(t)=¢(t)$, the response of the system is given by $y(t)=¢(t)-2¢(t-2)+¢(t-4)$. The plot of this response is shown in Figure.




5. Two LTI systems have impulse response functions given by

$$h_1(t)=¢(t-1) \text{ and } h_2(t)=u(t-1)$$




**Ans:** When two LTI systems are connected in parallel, the overall impulse response of the system is addition of the two individual impulse responses, that is

$$h_p(t)=h_1(t)+h_2(t)=¢(t-1)+u(t-1)$$

When two LTI systems are connected in series the overall impulse response is convolution of two individual impulse responses. i.e.

$$h_c(t)=h_1(t)*h_2(t)=¢(t-1)*u(t-1)=u(t-2)$$

6. Consider a system having impulse response $h(t)=¢(t+1)+¢(t-1)$. Determine and sketch the output for the following excitations.

(a) A symmetrical rectangular pulse of unit height and unit width centered at origin

(b) A symmetrical triangular pulse of unit height and unit width centered at origin

(c) $x(t)=u(t)-u(t-1)$

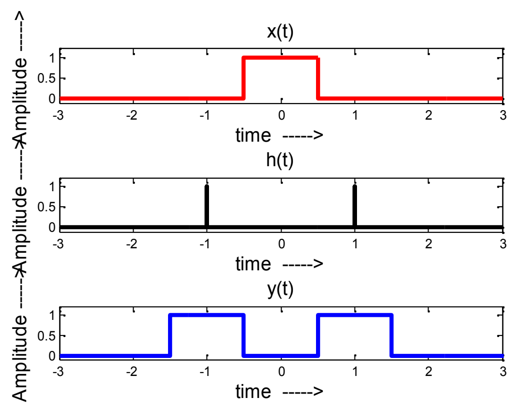
**Ans: (a)** A symmetrical rectangular pulse of unit height and unit width centered at origin is represented mathematically by $x(t) = \left(u\left(t+0.5\right) - u\left(t-0.5\right)\right)$. Given that the impulse response $h(t) = \text{¢}(t+1) + \text{¢}(t-1)$ the output of the system is computed as below.

We know that $x(t) * h(t) = y(t)$ and the property $x(t-a) * x(t-b) = y(t-(a+b))$.

Based on this property,

$$
\begin{aligned}
y(t) = x(t) * h(t) &= \left(u\left(t+0.5\right) - u\left(t-0.5\right)\right) * \left(\text{¢}(t+1) + \text{¢}(t-1)\right) \\
&= u\left(t+0.5\right) * \text{¢}(t+1) + u\left(t+0.5\right) * \text{¢}(t-1) \\
&\qquad - u\left(t-0.5\right) * \text{¢}(t+1) - u\left(t-0.5\right) * \text{¢}(t-1) \\
&= u\left(t+1.5\right) + u\left(t-0.5\right) - u\left(t+0.5\right) - u\left(t-1.5\right) \\
&= \left[u\left(t+1.5\right) - u\left(t+0.5\right)\right] + \left[u\left(t-0.5\right) - u\left(t-1.5\right)\right]
\end{aligned}
$$

The following figure illustrate the convolution operation on given signals.




**(b)** A symmetrical triangular pulse of unit height and unit width centered at origin is represented by

$$
x(t) = \begin{cases} 2t+1, & -0.5 < t < 0 \\ -2t+1, & 0 < t < 0.5 \\ 0, & Elsewhere \end{cases}
$$

In terms of unit step function $x(t) = (2t+1)u(t+0.5) - 4tu(t) + (2t-1)u(t-0.5)$

Given that $h(t) = \text{¢}(t+1) + \text{¢}(t-1)$. Then the convolution output is evaluated as below.

We know that $x(t) * h(t) = y(t)$ and the property $x(t-a) * x(t-b) = y(t-(a+b))$.

Based on this property,

$$y(t) = x(t) * h(t) = \left((2t+1)u(t+0.5) - 4tu(t) + (2t-1)u(t-0.5)\right) * \left(¢(t+1) + ¢(t-1)\right)$$

$$= \left[\{(2t+1)u(t+0.5)\} * ¢(t+1)\right] + \left[\{(2t+1)u(t+0.5)\} * ¢(t-1)\right] - \left[\{4tu(t)\} * ¢(t+1)\right]$$

$$- \left[\{4tu(t)\} * ¢(t-1)\right] + \left[\{(2t-1)u(t-0.5)\} * ¢(t+1)\right] + \left[\{(2t-1)u(t-0.5)\} * ¢(t-1)\right]$$

$$= \left[\{2tu(t+0.5) + u(t+0.5)\} * ¢(t+1)\right] + \left[\{2tu(t+0.5) + u(t+0.5)\} * ¢(t-1)\right]$$

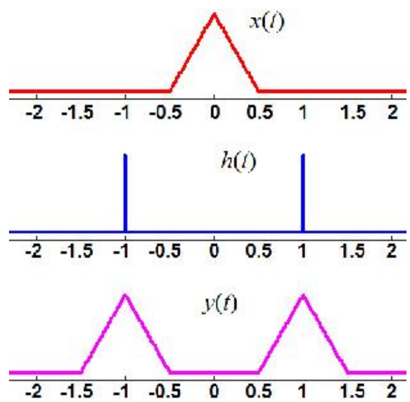
$$- \left[\{4tu(t)\} * ¢(t+1)\right] - \left[\{4tu(t)\} * ¢(t-1)\right] + \left[\{2tu(t-0.5) - u(t-0.5)\} * ¢(t+1)\right]$$

$$+ \left[\{2tu(t-0.5) - u(t-0.5)\} * ¢(t-1)\right]$$

$$= \left[\{2(t+1)u(t+1.5) + u(t+1.5)\}\right] + \left[\{2(t-1)u(t-0.5) + u(t-0.5)\}\right] - \left[\{4(t+1)u(t+1)\}\right]$$

$$- \left[\{4(t-1)u(t-1)\}\right] + \left[\{2(t+1)u(t+0.5) - u(t+0.5)\}\right] + \left[\{2(t-1)u(t-1.5) - u(t-1.5)\}\right]$$

(c) Given that $x(t) = u(t) - u(t-1)$ and $h(t) = \left[¢(t+1) + ¢(t-1)\right]$

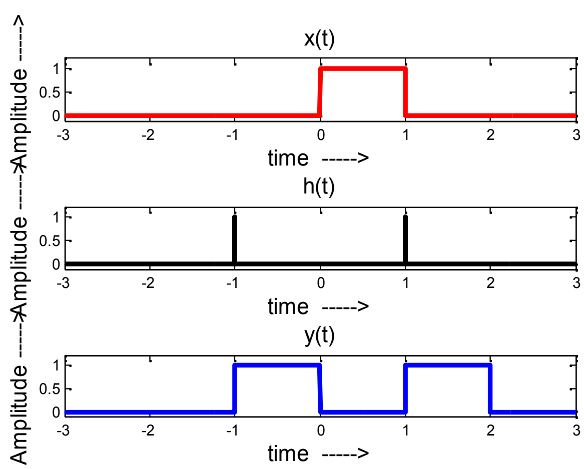



$$y(t) = x(t) * h(t) = \left[u(t) - u(t-1)\right] * \left[¢(t+1) + ¢(t-1)\right]$$

$$= u(t) * ¢(t+1) - u(t-1) * ¢(t+1) + u(t) * ¢(t-1) - u(t-1) * ¢(t-1)$$

$$= u(t+1) - u(t) + u(t-1) - u(t-2)$$

$$= \{u(t+1) - u(t)\} + \{u(t-1) - u(t-2)\}$$



7. Consider the signals shown below. Determine analytically the convolution of the following.

(i) $y(t) = x(t) * x(t)$ (ii) $z(t) = g(t) * g(t)$ (iii) $p(t) = x(t) * g(t)$.