

Vector Calculus

Scalar point function: A function  $f$  is said to be a scalar point function if it represents a Quantity without directions.

$$f = f_1 + f_2 + f_3$$

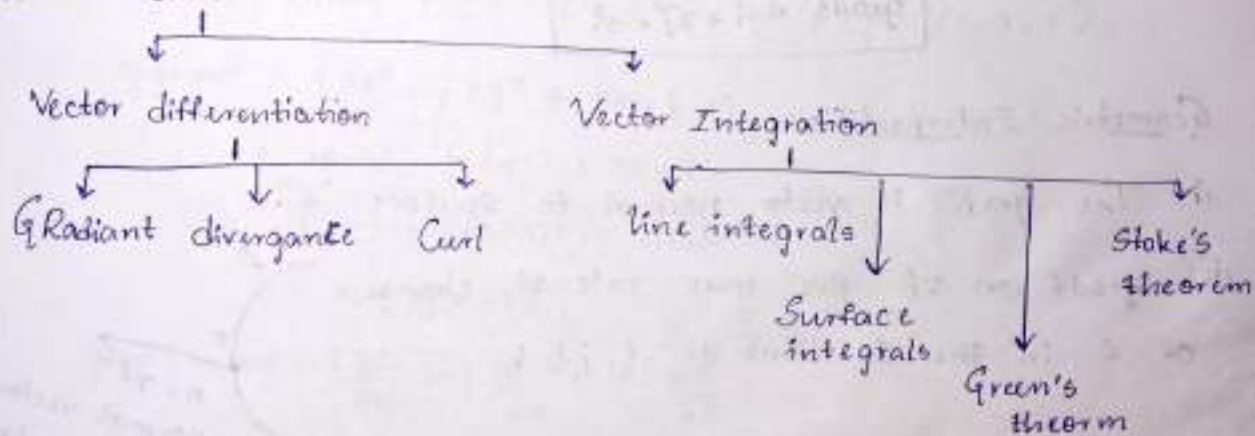
where  $f_1, f_2, f_3$  are functions in  $x, y, z$  in coordinate plane. for a point  $(x, y, z)$

Vector point function: A function  $F$  is said to be a vector point function if it represents both Quantity & Direction.

$$F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

where  $F_1, F_2, F_3$  are functions in  $x, y, z$  in coordinate plane for a point  $(x, y, z)$ .

$$\begin{array}{l|l} \hat{i} \cdot \hat{i} = 1 & \hat{i} \cdot \hat{j} = 0 \\ \hat{j} \cdot \hat{j} = 1 & \hat{j} \cdot \hat{k} = 0 \\ \hat{k} \cdot \hat{k} = 1 & \hat{k} \cdot \hat{i} = 0 \end{array}$$

Vector calculus

## Session 22 & 23

Vector differential Operator (VDO) :-

A VDO is denoted by  $\nabla$  (del)

Defined as  $\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$

Gradient of a scalar point function:

The Del applying to a scalar <sup>point</sup> function is called a gradient. Denoted by grad  $f$  (or)  $\nabla f$ .

where  $f$  is scalar point function.

$$\therefore \text{grad } f = \nabla f = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) f$$

$$\text{grad } f = \nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

Ex:- Let  $f = x + 2y + 3z$

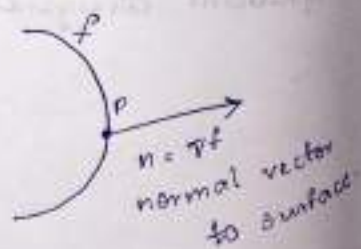
$$\text{grad } f = ?$$

$$\begin{aligned} \text{grad } f &= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} = i(1) + j(2) + k(3) \\ &= \bar{i} + 2\bar{j} + 3\bar{k} \end{aligned}$$

$$\text{grad } f = \bar{i} + 2\bar{j} + 3\bar{k}$$

Geometric Interpretation

- (i) The grad  $f$  is vector normal to surface ' $f$ '.
- (ii) grad  $f$  (or)  $\nabla f$  give max rate of change of  $f$  in the directions of  $i, j$  &  $k$ .



### Directional derivative (DD)

Let  $f(x, y, z)$  is scalar point function in the region  $R$  &  $P$  be any point in the region.

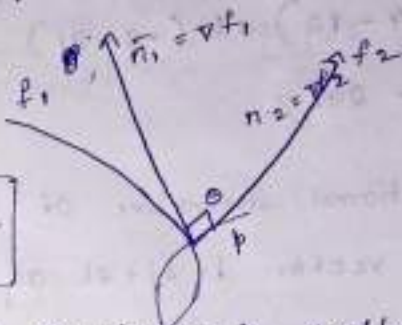
The DD of  $f$  at point  $P(x, y, z)$  in the direction of  $\vec{a}$  is

$$D.D = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

Angle b/w 2 surfaces:

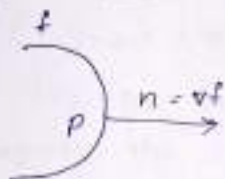
$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

$$\theta = \cos^{-1} \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$



Note: Angle between two surfaces is nothing but angle between their normals.

Unit normal vector



$$\text{unit normal vector} = \frac{\nabla f}{|\nabla f|}$$

1. Find  $\text{grad} f$ , where  $f = x^3 - y^3 + 3xyz$  at point  $(1, -1, 1)$ .

$$\begin{aligned} \nabla f &= i 3x^2 - j 3y^2 + 3xy k \\ &= i (3x^2) - j (3y^2) + 3xy k \\ &= i (3(1)) - j (3(-1)^2) + 3(1)(-1) k \\ &= 3i - 3j - 3k \end{aligned}$$

$$\text{grad} f = \nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\begin{aligned} &= i (3x^2 + 3yz) - j (3y^2 + 3xz) + 3xy k \\ &= i (3(1) + 3(-1)(1)) - j (3(-1)^2 + 3(1)(1)) + 3(1)(-1) k \\ &= -3k // \end{aligned}$$

2. Find gradient of function  $f = x^3y^2 - y^3z + 7(x^3 + x)$  at point  $(1, 2, 1)$

$$\text{grad } f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= i(3x^2y^2 - 7) + j(2x^3y - 3y^2z) + k(-y^3 - 7x^3)$$

$$= i(3(1)(2)^2 - 7) + j(2(1)(2) - 3(2)^2(1)) + k(-(2)^3 - 7(1)(1))$$

$$= 5i + j(4 - 12) + k(-8 - 7)$$

$$= 5i - 8j - 15k //$$

3. Compute directional derivative of  $f = xy^2 - y^3z - z^2x$  in the direction of vector  $i + 2j + 2k$  at the point  $(2, -1, 1)$

$$DD = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{i + 2j + 2k}{\sqrt{1 + 4 + 4}} = \frac{i + 2j + 2k}{3}$$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= i(y^2 - z^2) + j(2xy - 3y^2z) + k(-3y^3 - 2zx)$$

$$\text{at } (2, -1, 1)$$

$$= i((-1)^2 - (1)^2) + j(2(2)(-1) - 3(-1)^2(1)) + k(-3(-1)^3 - 2(1)(2))$$

$$= -7j - 5k$$

$$DD = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{(-7j - 5k)(i + 2j + 2k)}{3}$$

$$= \frac{-14 - 6}{3} = -\frac{20}{3}$$

$$DD = -\frac{20}{3}$$

4. Find DD of  $f = xyz$  at  $(1, 1, 1)$  in direction of  $i+j+k$

$$DD = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$\begin{aligned}\nabla f &= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \\ &= i(yz) + j(xz) + k(xy) \\ &= i+j+k\end{aligned}$$

$$\frac{\bar{a}}{|\bar{a}|} = \frac{i+j+k}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \frac{i+j+k}{\sqrt{3}}$$

$$DD = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= \frac{(i+j+k)(i+j+k)}{\sqrt{3}} = \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} //$$

5. The temp at point  $(x, y, z)$  in the space is given by  $T(x, y, z) = x^2 + y^2 + z$ . A mosquito located at  $(1, +1, z)$  desires to fly in such a direction that it gets cooled faster. compute the direction in which it should fly.

Sol: The given temp. function

$$T(x, y, z) = x^2 + y^2 + z^2$$

the gradient of temperature is given by  $\text{grad} T = \nabla T$

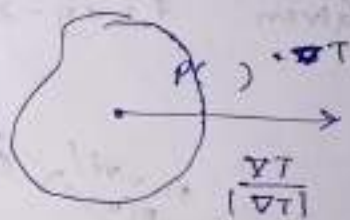
$$\nabla T = i \frac{\partial T}{\partial x} + j \frac{\partial T}{\partial y} + k \frac{\partial T}{\partial z}$$

$$\text{grad} T = i(2x) + j(2y) + k(1)$$

$$\text{grad} T = 2x\bar{i} + 2y\bar{j} + \bar{k}$$

$$\begin{aligned}\text{at } (-1, +1, z) &= -2(1)\bar{i} + 2(1)\bar{j} + \bar{k} \\ &= -2\bar{i} + 2\bar{j} + \bar{k}\end{aligned}$$

The unit normal gradient vector  $= \frac{\nabla T}{|\nabla T|} = \frac{-2\bar{i} + 2\bar{j} + \bar{k}}{3}$



To get cooled fast, it should travel in direction of  $\frac{\nabla T}{|\nabla T|}$   

$$= \frac{-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}$$

5. Find the directional derivative of  $\phi = xy - yz - zx$  at A, in the direction of AB, where  $A = (1, 2, 0)$ ,  $B = (1, 0, 3)$ .

Sol Given scalar point function

$$\phi = xy - yz - zx$$

The gradient of  $\phi \Rightarrow \nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$

$$= \nabla\phi = \mathbf{i}(y-z) + \mathbf{j}(x-z) + \mathbf{k}(-y-x)$$

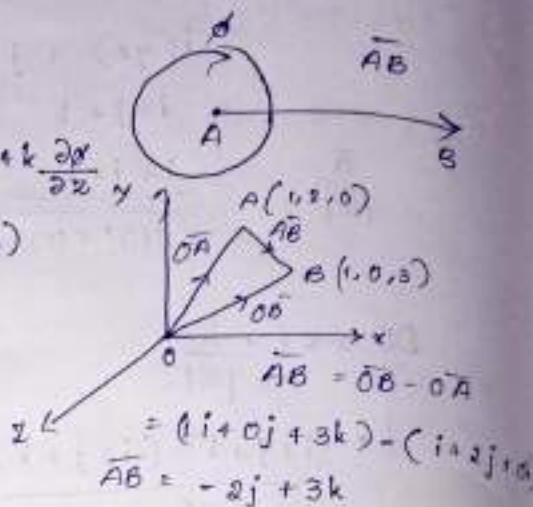
at a point  $A(1, 2, 0)$

$$\nabla\phi = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

$$D.D = \frac{\nabla\phi \cdot \vec{AB}}{|\vec{AB}|}$$

$$= \frac{(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (-2\mathbf{j} + 3\mathbf{k})}{\sqrt{4+9}}$$

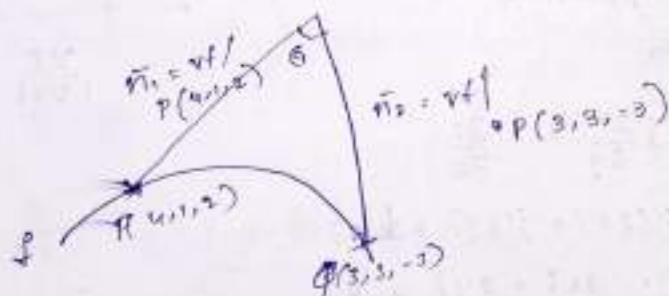
$$= \frac{-2-9}{\sqrt{13}} = -\frac{11}{\sqrt{13}}$$



$$D.D = \frac{\nabla\phi \cdot \vec{AB}}{|\vec{AB}|}$$

6. Identify the angle between the normal to the surface  $xy = z^2$  at points  $(4, 1, 2)$  &  $(3, 3, -3)$

Sol Given  $f = xy - z^2$



$$xy = z^2$$

The normal vector of surface  $f$ , at point  $P(4, 1, 2)$  is

$$\vec{n}_1 = \text{grad } f|_{\text{at } P(4, 1, 2)} = [\mathbf{i}y + \mathbf{j}x - 2z\mathbf{k}] = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

The second normal vector of  $f_2$  at point  $Q(3, 3, -3)$  is

$$\vec{n}_2 = \nabla f_2 \Big|_{\text{at } Q(3, 3, -3)} = \vec{i}y + \vec{j}x + 2\vec{k} = 3\vec{i} + 3\vec{j} + 6\vec{k}$$

Let angle between normal to surface is  $\theta$ .

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{(\vec{i} + 4\vec{j} - 4\vec{k}) \cdot (3\vec{i} + 3\vec{j} + 6\vec{k})}{\sqrt{1+16+16} \sqrt{9+9+36}}$$

$$= \frac{3+12-24}{\sqrt{33} \sqrt{54}} = \frac{-9}{\sqrt{11} \sqrt{3} \sqrt{3} \sqrt{6}}$$

$$= \frac{-1}{\sqrt{11} \sqrt{3} \sqrt{3} \sqrt{2}} = \frac{-1}{\sqrt{22}}$$

$$\theta = \cos^{-1} \left( \frac{-1}{\sqrt{22}} \right) \Rightarrow \boxed{\theta = \cos^{-1} \left( \frac{1}{\sqrt{22}} \right)}$$

7. Determine the angle between surfaces  $x^2 + y^2 + z^2 = 9$  &  $x^2 + y^2 - z = 3$  at  $(2, -1, 2)$

Given  $f_1 = x^2 + y^2 + z^2 - 9$

$f_2 = x^2 + y^2 - z - 3$

The normal vector of surface  $f_1$ , at  $(2, -1, 2)$

$$\vec{n}_1 = \nabla f_1 \Big|_{\text{at } P(2, -1, 2)}$$

$$\vec{n}_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\text{at } (2, -1, 2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\vec{n}_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

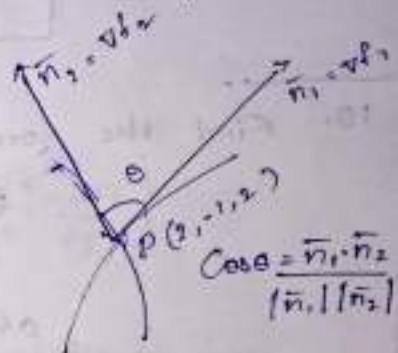
$$\text{at } (2, -1, 2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

Let  $\theta$  be angle between 2 surfaces is angle between their normals.

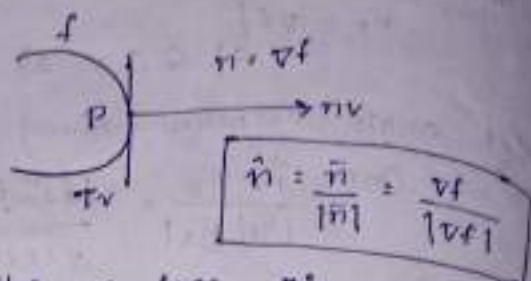
$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+16} \sqrt{16+4+1}}$$

$$= \frac{16+4-4}{\sqrt{36} \sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1} \left( \frac{8}{3\sqrt{21}} \right)}$$



## Unit normal vector to Surface



9. Find unit normal vector to the surface  $z^2 = 4(x^2 + y^2)$  at point  $(1, 0, 2)$

Sol Given surface  $f = 4x^2 + 4y^2 - z^2$

$$\nabla f = 8xi + 8yj - 2zk$$

$$\text{at } (1, 0, 2) = \nabla f = 8i - 4k$$

unit normal vector of given surface  $f$ , at point

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{8i - 4k}{\sqrt{80}}$$

10. Find the normal vector to surface  $x^2 + y^2 - z^2 = 1$  at  $(1, 3, 3)$

$$f = x^2 + y^2 - z^2 - 1$$

$$\nabla f = 2xi + 2yj - 2zk$$

$$\text{at } (1, 3, 3) = 2i + 6j - 6k$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2i + 6j - 6k}{\sqrt{4 + 36 + 36}}$$

$$\hat{n} = \frac{2i + 6j - 6k}{\sqrt{76}}$$

## Session - 24

Divergence & curl of a given vector point function.

Divergence of a vector point function:-

Let  $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$  be a vector, continuously differentiable on every point in 3-D plane, then the divergence of  $F$ , is denoted by  $\text{div } F$  (or)  $\nabla \cdot F$

Defined as  $\text{div } \vec{F} = \nabla \cdot \vec{F} = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k})$

$$\boxed{\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Solenoidal vector :-

A vector  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  is said to be solenoidal vector, if  $\boxed{\text{div } \vec{F} = 0}$  or  $\boxed{\nabla \cdot \vec{F} = 0}$

Curl of a vector point function :-

Let  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  be a vector, continuously differentiable on every point in 3-D plane, then the curl of  $\vec{F}$ , is denoted  $\text{curl } \vec{F}$  (or)  $\nabla \times \vec{F}$

$$\begin{aligned} \text{Denoted as } \text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] - \vec{j} \left[ \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right] + \vec{k} \left[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \end{aligned}$$

Irrrotational vector,

A vector  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  is said to be irrotational vector by  $\boxed{\text{Curl } \vec{F} = \vec{0}}$

$$\boxed{\vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k}}$$

1. If  $\vec{F} = x^2 y \vec{i} - 2 y^2 z \vec{j} + 3 z^2 x \vec{k}$  find  $\text{div } \vec{F}$  at  $(0, -1, 1)$

sol Given  $\vec{F} = x^2 y \vec{i} - 2 y^2 z \vec{j} + 3 z^2 x \vec{k}$

$$\text{where } \boxed{\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}}$$

$$f_1 = x^2 y, f_2 = -2 y^2 z, f_3 = 3 z^2 x$$

$$\text{div } \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 2xy + (-4yz) + 6zx$$

at  $(0, -1, 1)$   $\text{div } \vec{F} = -4(1)(1) = 4$

$$\boxed{\text{div } \vec{F} = 4}$$

2. Determine 'p' if  $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+pz)\vec{k}$  is solenoidal vector.

Given  $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+pz)\vec{k}$

where  $\vec{F} = (x+3y)\vec{i} + y\vec{j} + x\vec{k}$

$$\boxed{\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}}$$

Since,  $\vec{F}$  is solenoidal vector,  $\text{div } \vec{F} = 0$ .

$$\nabla \cdot \vec{F} = 0$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$1 + 1 + p = 0$$

$$\boxed{p = -2}$$

3. If  $\vec{F} = -x\vec{i} - y\vec{j} - z\vec{k}$ , calculate  $\text{curl } \vec{F}$ .

Given  $\vec{F} = -x\vec{i} - y\vec{j} - z\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & -z \end{vmatrix}$$

$$\text{curl } \vec{F} = \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

$$\boxed{\text{curl } \vec{F} = \vec{0}}$$

4. Find divergence & curl of  $\vec{v} = (xyz)\vec{i} - (2x^2y)\vec{j} - 3y^2z\vec{k}$  at  $(2, -1, 3)$

Sol  $\vec{v} = (xyz)\vec{i} - (2x^2y)\vec{j} - 3y^2z\vec{k}$

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

divergence of  $\vec{v}$  is  $\text{div } \vec{v}$  (or)  $\nabla \cdot \vec{v}$

$$\text{div } \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\text{div } \vec{v} = yz - 2x^2 - 3y^2$$

$$\text{at } (2, -1, 3) \quad \text{div } \vec{v} = (-1)(3) - 2(2)^2 - 3(-1)^2$$
$$= -3 - 8 - 3 = -14$$

$$\boxed{\text{div } \vec{v} = -14}$$

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -2x^2y & -3y^2z \end{vmatrix}$$

$$\text{curl } \vec{v} = \hat{i}[-6yz - 0] - \hat{j}[0 - xy] + \hat{k}[-4xy - xz]$$

$$\text{at } (2, -1, 3) \quad \vec{v} = 18\hat{i} - 2\hat{j} + 2\hat{k}$$

6. show that the fluid motion

$\vec{F} = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$  is irrotational.

$$\text{Sol} \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (x+z) & (x+y) \end{vmatrix}$$

$$\text{curl } \vec{F} = \hat{i}[1-1] - \hat{j}[1-1] + \hat{k}[1-1]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$$\boxed{\text{curl } \vec{F} = \vec{0}}$$

$\therefore \vec{F}$  is irrotational vector.

6. Determine a, b, c if  $\vec{F} = (2x+3y+az)\hat{i} + (bx+2y+3z)\hat{j} + (2x+cy+3z)\hat{k}$  is irrotational

Sol: Given vector  $\vec{F}$  is irrotational vector.

$$\text{curl } \vec{F} = \vec{0}$$

$$\nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+ax & bx+2y+3z & 2x+cy+3z \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= i[c-3] - j[2-a] + k[b-3] = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$c-3=0$$

$$c=3$$

$$\begin{cases} a=2 \\ b=3 \end{cases}$$

## Vector Integration

- (i) Line Integral
- (ii) Green's theorem for plane
- (iii) Surface integrals
- (iv) Stoke's theorem

### Session - 25

Let  $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$  be any vector point function.

Position vector  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

The line integral of  $\vec{F}$  over the curve 'C' moves from point 'A' to point 'B' is given

$$I = \int_C \vec{F} \cdot d\vec{r}$$

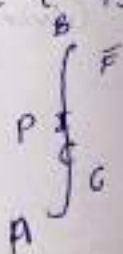
$d\vec{r}$  is displacement vector.

$$\Rightarrow I = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

Workdone by Force Function  $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$  :

Workdone by the force function  $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ , for moving a particle 'P' from point 'A' to point 'B', along curve 'C' is

$$W = \int_C \vec{F} \cdot d\vec{r}$$



2. Compute workdone by the force function  $\vec{F} = 2xy\vec{i} - 3z\vec{j} + 5x\vec{k}$ , when it moves a particle along curve  $x = t^2 + 1$ ,  $y = 2t^2$ ,  $z = t^3$  from  $t = 1$  to  $2$ .

Sol force function,

$$\vec{F} = 2xy\vec{i} - 3z\vec{j} + 5x\vec{k}$$

The work done by Force  $\vec{F}$ , to move particle from point A to point 'B', along curve 'C' is  $A \xrightarrow{P} B$

$$x = t^2 + 1$$

$$y = 2t^2$$

$$z = t^3$$

$t \rightarrow$  varies from 1 to 2

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C (2xy\vec{i} - 3z\vec{j} + 5x\vec{k}) \cdot (i dx + j dy + k dz)$$

$$= \int_C 2xy dx - 3z dy + 5x dz$$

where 'C' is curve given by

$$x = t^2 + 1 \Rightarrow dx = 2t dt$$

$$y = 2t^2 \Rightarrow dy = 4t dt$$

$$z = t^3 \Rightarrow dz = 3t^2 dt$$

$$W = \int_C 2(t^2 + 1)(2t^2) 2t dt - 3(t^3) 4t dt + 5(t^2 + 1) 3t^2 dt$$

$$= \int_C [8t^3(t^2 + 1) - 12t^4 + 15t^2(t^2 + 1)] dt$$

$$= \int_C [8t^5 + 8t^3 - 12t^4 + 15t^4 + 15t^2] dt$$

$$I = \left[ \frac{8t^6}{6} + \frac{3t^5}{5} + \frac{8t^4}{4} + \frac{15t^3}{3} \right]_1$$

$$I = \left[ \frac{8(2)^6}{6} + \frac{3(2)^5}{5} + \frac{8(2)^4}{4} + \frac{15(2)^3}{3} \right] - \left[ \frac{8(1)}{6} + \frac{3(1)}{5} + \frac{8(1)}{4} + \frac{15}{3} \right]$$

$$I = \left[ \frac{8 \times 64}{6} + \frac{3 \times 32}{5} + \frac{8 \times 16}{4} + \frac{15 \times 8}{3} \right] - \left[ \frac{8}{6} + \frac{3}{5} + \frac{8}{4} + \frac{15}{3} \right]$$

$$= 167 \frac{1}{30}$$

3. If  $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $C$  is curve  $y = 2x^2$  in the  $xy$ -plane from  $(0,0)$  to  $(1,2)$

Given  $\vec{F} = 3xy\vec{i} - y^2\vec{j}$

now,  $\int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\vec{i} - y^2\vec{j}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$

$$= \int_C 3xy dx - y^2 dy$$

$$\Rightarrow dy = 4x dx$$

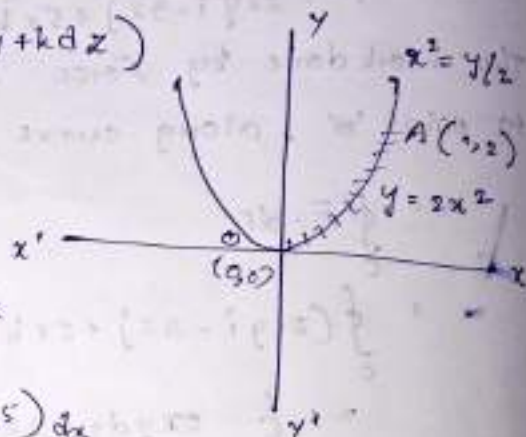
$$\Rightarrow \int 3x(2x^2) dx - (2x^2)^2 4x dx$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5) dx$$

$$= \left[ \frac{6x^4}{4} - \frac{16x^6}{6} \right]_0^1$$

$$= \frac{63}{12} - \frac{16}{6}$$

$$= \frac{9 - 16}{6} = -\frac{7}{6}$$



4. Compute the work done by the force  $\vec{F} = 2x^2\vec{i} + (4xz - y)\vec{j} + 2z\vec{k}$  when it moves from along st line from point  $(0,0,0)$  to  $(2,1,3)$

$$\vec{F} = 2x^2\vec{i} + (4xz - y)\vec{j} + 2z\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (2x^2\vec{i} + (4xz - y)\vec{j} + 2z\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C 2xz^2 + 4xz - y \, dz + 2z \, dz$$

work done by force  $\vec{F}$ , when moves a particle along straight line from  $O(0,0,0)$  to  $P(2,1,3)$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C 2(2t)^2 \cdot 3 \, dt + [4(2t)(3t) - t] \, dt + 2(3t) \cdot 3 \, dt$$

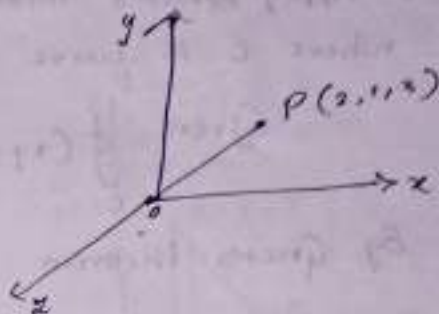
$$W = \int_0^1 (16t^2 + 24t^2 - t + 18t) \, dt$$

$$= \int_0^1 (40t^2 + 17t) \, dt$$

$$= \left[ \frac{40t^3}{3} + \frac{17t^2}{2} \right]_0^1$$

$$= \frac{40}{3} + \frac{17}{2}$$

$$= \frac{40 \times 2}{6} + \frac{17 \times 3}{6} = \frac{80 + 51}{6} = \frac{131}{6}$$



$$\text{St line} = \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$= \frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$$x = 2t \Rightarrow dx = 2 \, dt$$

$$y = t \Rightarrow dy = 1 \, dt$$

$$z = 3t \Rightarrow dz = 3 \, dt$$

$\Rightarrow$  Green's theorem for plane (XY-plane)

Let  $M(x,y)$ ,  $N(x,y)$ ,  $M_y = \left(\frac{\partial M}{\partial y}\right)$ ,  $N_x = \left(\frac{\partial N}{\partial x}\right)$ , be continuous differentiable function over the region  $E$ , in coordinate  $XY$ -plane, then

$$\oint_C M(x,y) \, dx + N(x,y) \, dy = \iint_E \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$

$\downarrow$  short form

$$\oint_C M \, dx + N \, dy = \iint_E \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$$

### Session-26

1. Apply Green's theorem to evaluate the integral  $\oint_C (xy - y^2) dx - x^2 dy$  where  $C$  is curve bounded by  $y=x$  &  $y=x^2$ .

$$\text{Given } \oint_C (xy - y^2) dx - x^2 dy$$

$$\text{By Green's theorem } \oint_C M dx + N dy = \iint_G \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

$$\text{where } M = xy - y^2, N = -x^2$$

$$\frac{\partial M}{\partial y} = x - 2y; \frac{\partial N}{\partial x} = -2x.$$

$$\oint_C (xy - y^2) dx - x^2 dy$$

### Session-26

2. Apply Green's theorem, evaluate  $\oint_C (3x - 8y^2) dx + (4y - 6xy) dy$  where  $C$  is boundary of region bounded by  $x=0, y=0, x+y=1$

$$\text{Given } \oint_C (3x - 8y^2) dx + (4y - 6xy) dy$$

By Green's theorem for plane:

$$\oint_C M dx + N dy = \iint_G \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

$$\text{where, } M = 3x - 8y^2, N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$$

~~00000000~~

$x$  limits from 0 to 1

$y$  limits from 0 to  $1-x$

$$= \oint_C \cancel{-16y dx - 6y dy}$$



Limits  $y = x^2 \rightarrow \sqrt{x}$

$x: 0 \rightarrow 1$

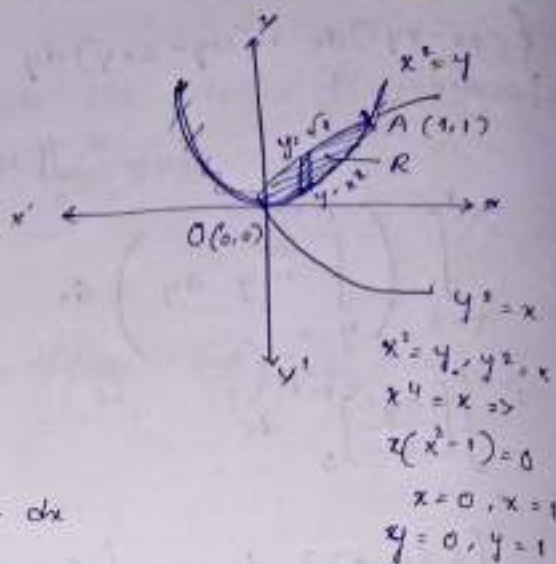
$$= \int_{x=0}^1 \left( \int_{y=x^2}^{\sqrt{x}} y \, dy \right) dx$$

$$= \int_0^1 \left[ \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[ \frac{(\sqrt{x})^2}{2} - \frac{(x^2)^2}{2} \right] dx = \int_0^1 \left[ \frac{x}{2} - \frac{x^4}{2} \right] dx$$

$$= \left[ \frac{x^2}{4} - \frac{x^5}{10} \right]_0^1 = \frac{1}{4} - \frac{1}{10} = \frac{10-4}{40} = \frac{6}{40} = \frac{3}{20} //$$

~~$= \frac{1}{2} \left( \frac{1}{2} - \frac{1}{10} \right)$~~



4. Apply Green's theorem to evaluate the integral over

$\oint_C [y - \sin x] dx + \cos x \, dy$ , where  $C$  is plane triangle enclosed by lines  $y = 0, x = \pi/2$  &  $y = \frac{2}{\pi} x$ .

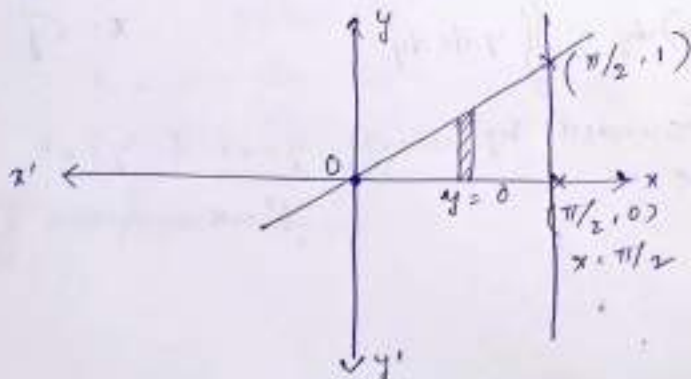
$M = y - \sin x$        $N = \cos x$

$\frac{\partial M}{\partial y} = 1$        $\frac{\partial N}{\partial x} = -\sin x$

$$\oint_C [y - \sin x] dx + \cos x \, dy = \iint_R -\sin x - \frac{1}{2} \, dx \, dy$$

Region is plane triangle enclosed by the lines

$y = 0, x = \pi/2$  &  $y = \frac{2}{\pi} x$  —  $y = mx$



$y = 0$

$x = \pi/2$

$y = \frac{2}{\pi} x$

$y = \frac{2}{\pi} x \times \frac{\pi}{2}$

$y = 1$

$$= \int_{x=0}^{\pi/2} \left( \int_{y=0}^{(2/\pi)x} (-\sin x - 1) dy \right) dx$$

$$= \int_0^{\pi/2} \left[ -y \sin x - y \right]_0^{2/\pi x} dx$$

$$= \int_0^{\pi/2} \left[ -\frac{2}{\pi} x \sin x - \frac{2}{\pi} x \right] dx = -\frac{2}{\pi} \int_0^{\pi/2} (x \sin x + x) dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x \sin x - \frac{2}{\pi} \int_0^{\pi/2} x dx$$

ILATE

$$\int f(x) g(x)$$

$$= f(x) \int g(x) dx - \left( \int f'(x) \cdot \int g(x) dx \right) dx$$

$$= -\frac{2}{\pi} \left[ \left( x(-\cos x) - \int 1 \cdot \int -\cos x dx \right) \right]_0^{\pi/2} - \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left[ -x \cos x + \sin x \right]_0^{\pi/2} - \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left[ \left( -\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (0+0) \right] - \frac{2}{\pi} \frac{(\pi/2)^2}{2}$$

$$= -\frac{2}{\pi} (1) - \frac{2}{\pi} \left( \frac{\pi^2}{8} \right)$$

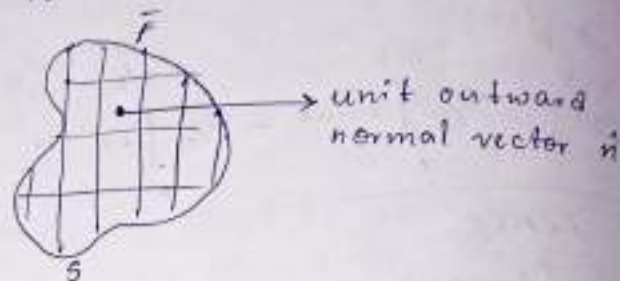
$$= - \left[ \frac{2}{\pi} + \frac{\pi}{4} \right]$$

## Session - 27

1. Surface integral
2. Stoke's theorem

1. Surface integral: Consider a continuous vector point function  $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ . If  $\hat{n}$  is a unit outward normal vector to the surface at any point, then surface integral of  $\vec{F}$ , over the given surface 'S' is

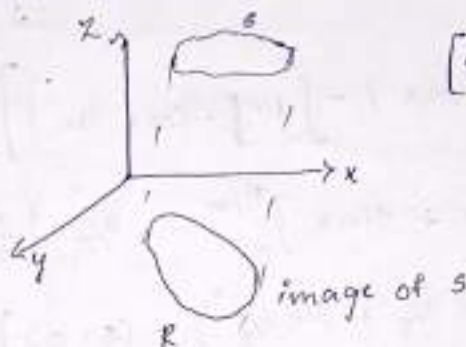
$$\int_S \vec{F} \cdot \hat{n} \, ds$$



Projects of surface:-

(i) The projection of surface on xy-plane.

$$ds = \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$$



[projection of xy-plane is  $\vec{k}$  ie  $\vec{n} = \vec{k}$ ]

(ii) Projection on yz-plane

$$ds = \frac{dydz}{|\vec{n} \cdot \vec{i}|}$$

(iii) Projection of surface of xz-plane

$$ds = \frac{dxdz}{|\vec{n} \cdot \vec{j}|}$$

2. Stoke's theorem (Relation btw Line & Surface integral)

If S be a open surface bounded by a closed curve 'C' &  $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ , be continuously differentiable over the

vector point function, then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{Curl } \vec{F} \cdot \vec{N} \, ds$$

where  $\vec{N}$  is unit external normal vector to surface  $S$

$$\vec{N} = \vec{k} ; ds = \frac{dx dy}{|\vec{k} \cdot \vec{k}|} = dx dy$$

$$\frac{dx dy}{|\vec{k} \cdot \vec{k}|} = \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

1. Apply Stoke's theorem to evaluate  $\oint_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$  &  $C$  is boundary of triangle with vertices at  $(0,0,0) \rightarrow (1,0,0)$  &  $(1,1,0)$

Given vector point function,

$$\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$$

By Stokes theorem.  $\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{Curl } \vec{F} \cdot \vec{N} \, ds$

$$\text{Now, } \text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}([0] - [0]) - \vec{j}(-1 - 0) + \vec{k}(2x - 2y) \\ = +\vec{j} + \vec{k}(2x - 2y)$$

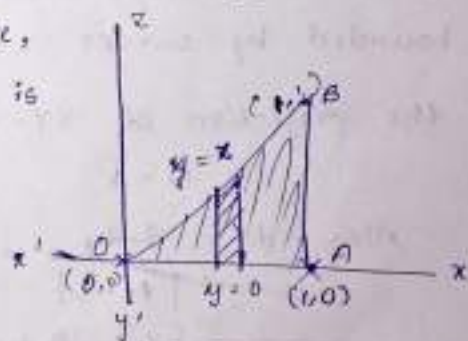
$S$  is surface of triangle  $(0,0,0), (1,0,0), (1,1,0)$ .

Since triangle region lies on  $xy$ -plane,

the projection of  $xy$ -plane is  $z$ -axis is

$$\text{i.e. } \boxed{\vec{N} = \vec{k}}$$

$$\text{Also, } ds = \frac{dx dy}{|\vec{N} \cdot \vec{k}|} = \frac{dx dy}{|\vec{k} \cdot \vec{k}|} = dx dy$$



$$\boxed{z=0 \rightarrow xy\text{-plane}}$$

$$= \oint_C \vec{F} \cdot d\vec{r} = \iint_R [\vec{j} + (2x - 2y) \vec{k}] \cdot \vec{k} \, dx dy \\ = \iint_R (2x - 2y) \, dx dy$$

$$= \int_{x=0}^1 \left( \int_{y=0}^2 (2x - 2y) dy \right) dx$$

$$= \int_0^1 \left[ 2xy - \frac{2y^2}{2} \right]_0^2 dx$$

$$= \int_0^1 \left( 2x^2 - \frac{2x^2}{2} \right) dx \quad \Rightarrow \int_0^1 x^2 dx$$

$$= \left[ \frac{x^3}{3} \right]_0^1 = 1/3 //$$

2. Apply Stokes theorem, for vector field  $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$  integrate around rectangle in plane  $z=0$  & bounded by curves  $x=0, y=0, x=a, y=b$

$$\begin{aligned} \text{Curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2) & 2xy & 0 \end{vmatrix} \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) \\ &= 4y\vec{k} \end{aligned}$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r}$$

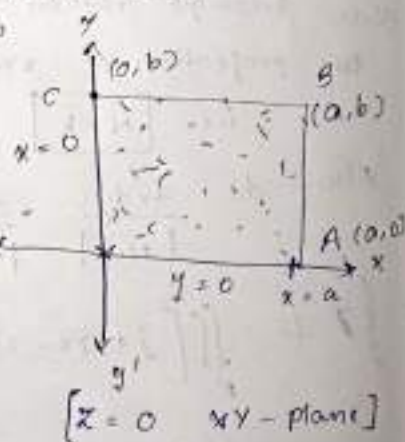
$S$  is a surface around rectangle in plane  $z=0$  bounded by curves  $x=0, y=0, x=a, y=b$

the projection of  $xy$ -plane

$$\vec{N} = \vec{k}$$

$$\text{Also } ds = \frac{dx dy}{|\vec{N} \cdot \vec{k}|} = \frac{dx dy}{|\vec{k} \cdot \vec{k}|} = dx dy$$

$$\begin{aligned} &= \oint_C \vec{F} \cdot d\vec{r} = \iint_{x=0, y=0}^{a, b} 4y\vec{k} \cdot \vec{k} dx dy \\ &= \iint_{x=0, y=0}^{a, b} (4y) dy dx \end{aligned}$$



$$= \int_0^a \left[ \frac{a^2 y^2}{2} \right]_0^b dy = \int_0^a 2b^2 dy$$

$$= [2b^2 x]_0^a = 2b^2 a \Rightarrow 2ab^2 //$$

3. Apply Stoke's theorem for vector field  $\vec{F} = x^2 \vec{i} + xy \vec{j}$  integrate around square

$$x=0, y=0, x=a, y=a$$

$$\vec{F} = x^2 \vec{i} + xy \vec{j}$$

Square in plane  $z=0$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(y-0)$$

$$= y\vec{k}$$

$$= \oint_C \vec{F} \cdot d\vec{r}$$

$$x=0, y=0, x=a, y=a$$

the projection of  $xy$ -plane

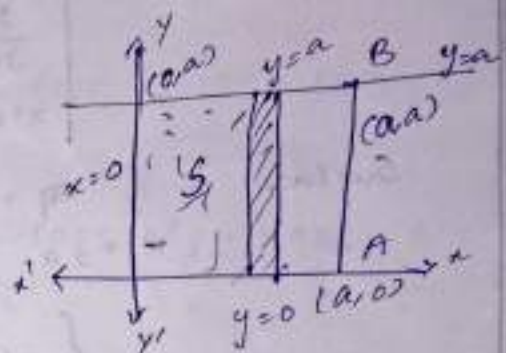
$$\vec{n} = \vec{k}$$

$$\text{Also } ds = \frac{dx dy}{|\vec{n} \cdot \vec{k}|} = \frac{dx dy}{|\vec{k} \cdot \vec{k}|} = dx dy$$

$$= \oint_C \vec{F} \cdot d\vec{r} = \int_{x=0}^a \int_{y=0}^a y \vec{k} \cdot \vec{k} dx dy$$

$$= \int_{x=0}^a \left( \int_{y=0}^a y dy \right) dx$$

$$= \int_0^a \left[ \frac{y^2}{2} \right]_0^a dx = \int_0^a \frac{a^2}{2} dx \Rightarrow \frac{a^2}{2} [x]_0^a = \frac{a^3}{2}$$



4. Apply Stokes theorem to evaluate  $\oint_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$  where  $C$  is the boundary of the rectangle bounded by lines  $x = \pm a, y = 0, y = b$ .

$$\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$$

By Stokes's theorem  $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{N} \, ds$ .

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y-2y) = -4y\vec{k}$$

Surface  $S$  is rectangle in plane  $z=0$  i.e.,  $xy$ -plane.

$$x = \pm a, y = 0, y = b.$$

$$\begin{aligned} \oint_C -4y\vec{k} \cdot \vec{k} \, ds &\Rightarrow \iint_S -4y\vec{k} \cdot \vec{k} \, dy \\ &= \int_{x=-a}^a \left( \int_{y=0}^b -4y \, dy \right) dx \\ &= \int_{x=-a}^a \left[ -\frac{4y^2}{2} \right]_0^b dx = \int_{x=-a}^a -2b^2 \, dx \end{aligned}$$

$$= \left[ -2b^2 x \right]_{-a}^a$$

$$= -2b^2 a - (-2b^2(-a))$$

$$= -4ab^2 //$$

## Tutorial-10

1. Find gradient of the function  $f = x^3y^2 - y^3z - 7(z^3 + x)$  at the point  $(1, 2, 1)$
2. Find divergence & curl of  $\vec{f} = (xyz)\vec{i} - (x^2y)\vec{j} - 3y^2z\vec{k}$  at  $(1, 1, 1)$
3. Compute directional derivative by  $f = xy^2 - 2y^3z$  in direction of vector  $\vec{i} + 2\vec{j} + 2\vec{k}$  at point  $(2, -1, 1)$
4. Identify angle between normal to surface by  $xy = 2z^2$  at points  $(1, 2, 3)$  &  $(2, 2, 2)$
5. Find unit normal vector to the surface  $x^3y - 4xz = 2$  at point  $(1, -1, 3)$

1.

$$\text{Grad } \vec{f} = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$= \frac{\partial}{\partial x} (x^3y^2 - y^3z - 7(z^3 + x)) + \frac{\partial}{\partial y} (x^3y^2 - y^3z - 7(z^3 + x)) + \frac{\partial}{\partial z} (x^3y^2 - y^3z - 7(z^3 + x))$$

$$= 3x^2y^2 - 0 - 7 + x^3y - 3y^2z - 0 + 0 - y^3 - 21z^2$$

$$= (3x^2y^2 - 7)\vec{i} + (x^3y - 3y^2z)\vec{j} + (-y^3 - 21z^2)\vec{k}$$

$$(1, 2, 1) = 3(1)^2(2)^2 - 7 + (1)^3(2) - 3(2)^2(1) - (2)^3 - 21(1)^2 - 7$$

$$= 12 - 7 + 2 - 12 - 8 - 21 - 7$$

$$\Rightarrow (12 - 7)\vec{i} + [(1)^3(2) - 3(2)^2(1)]\vec{j} + (-y^3 - 21z^2)\vec{k}$$

$$= 5\vec{i} - 8\vec{j} - 29\vec{k}$$

$$\frac{21}{21}$$

2.  $f = (xyz)\mathbf{i} - (x^2y)\mathbf{j} - 3y^2z\mathbf{k}$  at  $(1,1,1)$

$$\boxed{\begin{aligned} \text{div } f &= -2 \\ \text{curl } f &= -6\mathbf{i} + 2\mathbf{j} \end{aligned}}$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -x^2y & -3y^2z \end{vmatrix} = \mathbf{i}(-6yz - 0) - \mathbf{j}(0 - 2xz) \\ &\quad + \mathbf{k}(-2xy - 2xz) \\ &= -6yz\mathbf{i} + 2xz\mathbf{j} + \mathbf{k}(-2xy - 2xz) \\ &= -6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \\ \text{curl } f &= -6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{div } f &= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \\ &= [2yz - 2xy \cdot 0] + [2xz - x^2 - 6yz] + [2xy - 0 - 3y^2] \\ &= 2yz - x^2 - 3y^2 \\ &= 2 - 1 - 3 = -2 \end{aligned}$$

3.  $D.O = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$

$$f = xy^2 - 2y^3z$$

$$a \Rightarrow \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \text{ at } (2, -1, 1)$$

$$\bar{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$|\bar{a}| = \sqrt{1+4+4} = 3$$

$$\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

$$= y^2\mathbf{i} + (2xy - 6y^2z)\mathbf{j} + (-2y^3)\mathbf{k}$$

$$= (-1)^2\mathbf{i} + [2(2)(-1) - 6(-1)^2(1)]\mathbf{j} + [-2(-1)^3]\mathbf{k}$$

$$= (\mathbf{i} - 10\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

3

$$= \frac{1 - 20 + 4}{3} = \frac{-15}{3} = -5$$

$$1 - 20 + 4$$

.

$$4. \cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

$$xy = 2xz$$

$$f = xy - 2xz$$

$$n_1 \text{ at } (1, 2, 3) \text{ \& } n_2 \text{ at } (2, 2, 2)$$

$$\vec{n}_1 = \nabla f \text{ at } (1, 2, 3) = \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= (y)i + (x)j + (-4z)k$$

$$= 2i + j - 12k$$

$$\vec{n}_2 = \nabla f \text{ at } (2, 2, 2) = \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= (y)i + (x)j + (-4z)k$$

$$= 2i + 2j - 8k$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{4 + 2 - 96}{\sqrt{4+1+144} \sqrt{4+4+64}}$$

$$= -90$$

$$\sqrt{149} \sqrt{72}$$

$$5. x^3y - 4xz = 2 \text{ at } (1, -1, 3)$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

$$\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

$$= (3x^2y - 4z)i + (x^3)j + (-4x)k$$

$$= [3(1)^2(-1) - 4(3)]i + (1)^3j - 4k$$

$$= -15i + j - 4k$$

$$|\nabla f| = \sqrt{225 + 1 + 16}$$

$$\Rightarrow \sqrt{242}$$

$$= \sqrt{2 \times 121}$$

$$\Rightarrow 11\sqrt{2}$$

$$\hat{n} = \frac{-15i + j - 4k}{11\sqrt{2}}$$

$$\begin{array}{r} 15 \times 16 \\ \times 5 \\ \hline 115 \\ 255 \\ \hline 116 \\ 252 \end{array}$$

# Tutorial-11

1. A vector field  $\vec{F} = (2y+3)\vec{i} - 2xz\vec{j} - 3(yz-x)\vec{k}$

Evaluate  $\oint_C \vec{F} \cdot d\vec{s}$  along path C. is  $x=2t, y=t, z=t^3$   
from  $t=0$  to  $t=1$

2. If  $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ , evaluate  $\oint_C \vec{F} \cdot d\vec{s}$  where C is rectangle in xy-plane bounded by  $x=0, x=a, y=0, y=b$  - 2a<sup>2</sup>b

3. Find  $\oint_C \vec{F} \cdot d\vec{s}$  where  $\vec{F} = x^2y^2\vec{i} + y\vec{j}$  & the curve  $y^2=4x$  in xy-plane from (0,0) to (4,4).

1.  $\oint_C \vec{F} \cdot d\vec{s}$

$$= \int_C (2y+3)\vec{i} - (2xz)\vec{j} - 3(yz-x)\vec{k} \cdot (i dx + j dy + k dz)$$

$$= \int_C 2y+3 \cdot dx - 2xz \cdot dy - 3(yz-x) \cdot dz$$

$$x=2t \rightarrow dx=2dt$$

$$y=t \rightarrow dy=dt$$

$$z=t^3 \rightarrow dz=3t^2dt$$

$$= [2(t+3) \cdot 2dt - 2(2t)(t^3)dt - [3(t(t^3)-2t) \cdot 3t]dt]$$

$$= \int_0^1 [4t+6 - 4t^4 - 9t^6 + 18t^3] dt$$

$$= \left[ \frac{4t^2}{2} - 6t - \frac{4t^5}{5} - \frac{9t^7}{7} + \frac{18t^4}{4} \right]_0^1$$

$$= 2 - 6 - 4/5 - 9/7 + 9/2$$

$$= -4 - 4/5 - 9/7 + 9/2$$

$$\begin{array}{r} 35 \times 2 \\ 70 \\ -9 \times 2 + 9 \times 7 \\ 14 \end{array}$$

$$\begin{array}{r} -18 + 63 \\ 14 \end{array}$$

$$\begin{array}{r} -20 - 4 \\ 5 \\ -24 \\ 5 \end{array}$$

# CO-3 Revision problems

- Apply  $\beta$ - $\Gamma$  functions, evaluate the integral  $\int_0^{\pi/2} \sin^3 \theta \cos \theta d\theta$
- Evaluate integral  $\iint xy dx dy$  in the positive quadrant for which  $x+y \leq 1$ .
- Evaluate the integral  $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2) dy dx$  by changing into polar coordinates.

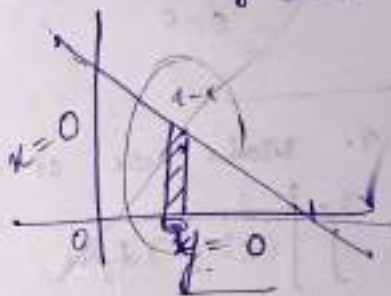
$$\begin{aligned} 2m-1 &= 3 & 2n-1 &= 7 \\ 2m &= 4 & 2n &= 8 \\ \boxed{m=2} & & \boxed{n=4} \end{aligned}$$

$$= 2 \cdot \frac{1}{2} \int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta = \beta(2, 4)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{\Gamma(2) \Gamma(4)}{\Gamma(6)} = \frac{1}{2} \cdot \frac{1! \times 3!}{5!}$$

$$= \frac{3 \times 2}{2 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{40}$$

$$\begin{aligned} x+y &= 1 \\ y &= 1-x \end{aligned}$$



$$\iint xy dx dy$$

$$x+y \leq 1$$

$$\int_{x=0}^1 \left( \int_{y=0}^{1-x} xy dy \right) dx$$

$$\int_0^1 \left[ \frac{xy^2}{2} \right]_0^{1-x} dx \Rightarrow \int_0^1 \frac{x(1-x)^2}{2} dx = \int_0^1 \frac{x - x^2}{2} dx$$

$$= \frac{1}{2} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \Rightarrow \frac{1}{2} \left( \frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{1}{2} \left( \frac{3-2}{6} \right)$$

$$= \int_0^1 \frac{x(1+x^2-2x)}{2} dx = \frac{1}{2} \int_0^1 (x + x^3 - 2x^2) dx = \frac{1}{2} \left[ \frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right]$$

$$= \frac{1}{2} \left( \frac{6+3-8}{12} \right) = \frac{1}{24}$$

$$\frac{9}{12} = \frac{3}{4}$$

$$\begin{array}{l} 1. 2, 4, 3 \\ 2. 1, 2, 3 \\ 3. 1, 1, 1 \\ 4 \times 3 = 12 \end{array}$$

$$3. \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2) dy dx$$

$x=0$   $y=0$   $x=r\cos\theta$ ,  $y=r\sin\theta$   
 $dx dy = r dr d\theta$

$$\Rightarrow \int_0^{\pi/2} \int_0^1 r^2 \cdot r dr d\theta$$

$\theta=0$   $r=0$

$$y=0 \text{ to } y=\sqrt{1-x^2}$$

$$y^2 = 1-x^2$$

$$x^2+y^2=1$$

$$x=0 \text{ to } x=1$$

$$= \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^1 d\theta \Rightarrow \frac{1}{4} [\theta]_0^{\pi/2}$$

$\theta=0$   $= \pi/8 //$

4. Make use of change the order of integration evaluate the

$$\int_0^3 \int_x^3 (x^2-y^2) dy dx$$

Given  $\int_0^3 \int_x^3 (x^2-y^2) dy dx$

$$x=0 \text{ to } x=3$$

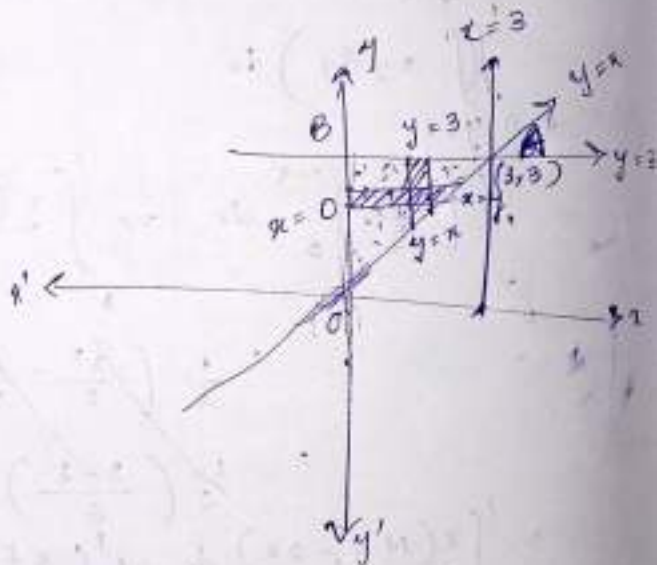
$$y=x \text{ to } y=3$$

$$x=0 \text{ to } x=3$$

By changing the order of integration

$$\int_0^3 \left( \int_0^y (x^2-y^2) dx \right) dy$$

$y=0$   $x=0$



$$\int_0^3 \left[ \frac{x^3}{3} - y^2 x \right]_0^y dy$$

$$\int_0^3 \frac{y^3}{3} - y^3 dy \Rightarrow \int_0^3 \frac{1}{3} \cdot \frac{y^4}{4} - \frac{y^4}{4}$$

$$\int_0^3 \left[ \frac{y^4}{12} - \frac{y^4}{4} \right] dy \Rightarrow \frac{(3)^4}{12} - \frac{(3)^4}{4} \times 3$$

$$\frac{(3)^4}{12} - \frac{(3)^4}{4} = \frac{81 - 243}{12} = -\frac{172}{12}$$

$$= -27/2 //$$

$$8 \times 3 \times 3 \times 3 = 27 \times 3 = 81$$

$$\frac{81 \times 3}{243} = \frac{243}{243} = 1$$

$$\frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$$

5. Evaluate integral  $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz$

$$\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz$$

$$\int_0^1 \int_0^1 \int_0^1 e^x \cdot e^y \cdot e^z dz dy dx$$

$$x=0 \quad y=0 \quad z=0$$

$$\int_0^1 \int_0^1 [e^x]_0^1 e^y \cdot e^z dy dz$$

$$= \int_0^1 \int_0^1 [e^1 - e^0] dy dz = \int_0^1 [(e-1) \cdot e^y dy] dz$$

$$(e-1) \int_0^1 \int_0^1 e^y dy \cdot e^z dz$$

$$= (e-1)^3 //$$

$$6. \int_0^a \int_0^a \int_0^a (xy + yz + zx) dx dy dz$$

$$x=0 \quad y=0 \quad z=0$$

$$\int_0^a \int_0^a \int_0^a (xy + yz + zx) dx dy dz$$

$$\int_0^a \int_0^a \left[ xyz + \frac{yz^2}{2} + \frac{z^2x}{2} \right]_0^a dy dz$$

$$x=0 \quad y=0$$

$$= \int_0^a \int_0^a \left[ xya + \frac{ya^2}{2} + \frac{a^2x}{2} \right] dy dx$$

$$x=0 \quad y=0$$

$$= \int_0^a \left[ \frac{xy^2a}{2} + \frac{y^2a^2}{4} + \frac{a^2xy}{2} \right]_0^a dx$$

$$x=0$$

$$= \int_0^a \left[ \frac{x(a)^2a}{2} + \frac{(a)^2a^2}{4} + \frac{a^2x(a)}{2} \right] dx$$

$$x=0$$

$$= \left[ \frac{x^2a^3}{2 \times 2} + \frac{a^4x}{4} + \frac{a^3x^2}{2 \times 2} \right]_0^a$$

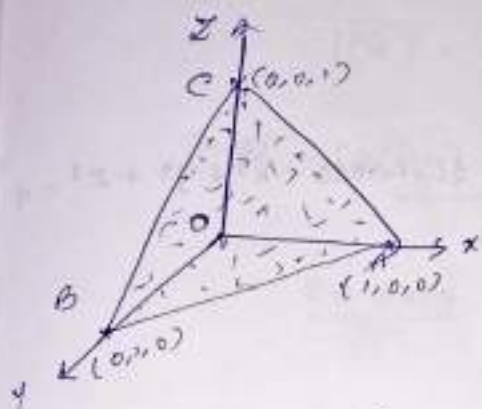
$$= \frac{a^2a^3}{4} + \frac{a^4a}{4} + \frac{a^3a^2}{4}$$

$$= \frac{a^5 + a^5 + a^5}{4} = \frac{3a^5}{4}$$

7. Evaluate volume of tetrahedron bounded by planes

$$x=0, y=0, z=0; x+y+z=1.$$

$$\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$$



OABC is tetrahedron  
Volume of tetrahedron

$$= \iiint_V dx dy dz.$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} \left( \int_{z=0}^{1-x-y} dz \right) dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} [z]_0^{1-x-y} dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} (1-x-y) dy dx$$

$$\Rightarrow \int_{x=0}^1 \left[ y - yx - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_{x=0}^1 \left( (1-x) - (1-x)x - \frac{(1-x)^2}{2} \right) dx$$

$$= \int_{x=0}^1 \left( 1-x - x + x^2 - \frac{1-x^2+2x}{2} \right) dx$$

$$= \int_{x=0}^1 \frac{2-2x-2x+2x^2-1-x^2+2x}{2} dx = \int_{x=0}^1 \frac{1-x^2-2x}{2} dx$$

$$\frac{1}{2} \left[ x - \frac{x^3}{3} - \frac{2x^2}{2} \right]_0^1 = \frac{1}{2} \left[ 1 - \frac{1}{3} - 1 \right] = -\frac{1}{6}$$

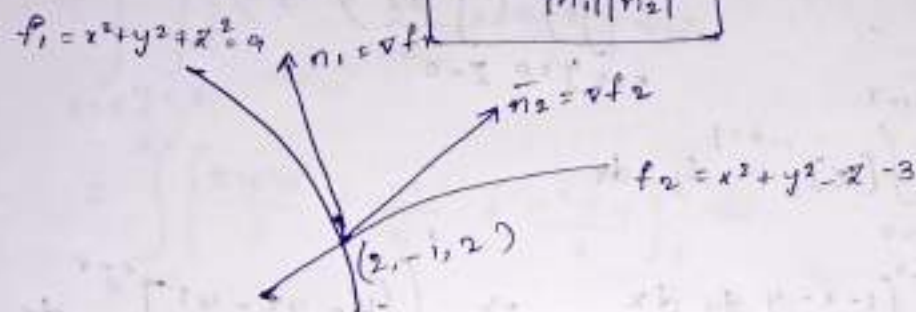
# CO-4 Revision - Problems

- Find the directional derivative of  $f = x^2 y^2 + 2z^2$  at point  $P(1, 2, 3)$  in direction of line  $PQ$  where  $Q$  is point  $(5, 0, 4)$

$$D.D = \frac{\nabla f \cdot \overrightarrow{PQ}}{|\overrightarrow{PQ}|}$$

- Identify the angle between the surfaces  $x^2 + y^2 + z^2 = 9$  &  $x^2 + y^2 - z = 3$  at point  $(2, -1, 2)$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$



- Obtain divergence & curl of  $\vec{f} = (xyz)\vec{i} - 3x^2 y\vec{j} + y^2 z\vec{k}$  at point  $(1, 2, 1)$

$$1. \quad f = x^2 - y^2 + 2xz$$

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$= (2x - y^2 + 2z^2)\vec{i} + (2xy - 2y)\vec{j} + (2x - y^2 + 4z)\vec{k}$$

$$= (2 - (2)^2(1) + 2(1)^2(1))\vec{i} + ((2)(1)^2 - 2(2) - 2(1)^2(2))\vec{j}$$

$$+ ((1)^2(1) - (2)^2(1) + 4(1))\vec{k}$$

$$= (2 - 4 + 2)\vec{i} + (2 - 4 - 4)\vec{j} + (1 - 4 + 4)\vec{k}$$

$$= -6\vec{j} + \vec{k}$$

$$af = 2x - 2y + 4z$$

$$= 2xi - 2yj + 4zk = 2i - 4j + 4k$$

$$\vec{PQ} = OQ - OP = 5i + 4k - i - 2j - 3k \\ = 4i - 2j + k$$

$$|\vec{PQ}| = \sqrt{16 + 4 + 1} = \sqrt{21}$$

$$\frac{(2i - 4j + 4k) \cdot (4i - 2j + k)}{\sqrt{21}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

$$2\sqrt{16x^2 + y^2 + z^2} = 9$$

$$3xi + 2yj + 2zk \Rightarrow 4i - 2j + 4k = 6$$

$$f_2 = x^2 + y^2 - z = 3$$

$$= 2xi + 2yj - k \Rightarrow 4i - 2j - k$$

$$\frac{(4i - 2j + 4k) \cdot (4i - 2j - k)}{\sqrt{16 + 4 + 16} \sqrt{16 + 4 + 1}}$$

$$= \frac{16 + 4 - 4}{\sqrt{36} \sqrt{21}} = \frac{16}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$