RANDOM PROCESSES -
SPECTRAL CHARACTERISTICS

Time Domain Analysis Different from Frequency Domain

- **A time domain analysis is an analysis of physical signals, mathematical functions, or time series of economic or environmental data, in reference to time.**
- **Frequency domain is an analysis of signals or mathematical functions, in reference to frequency, instead of time.**
- **a time-domain graph displays the changes in a signal over a span of time, and frequency domain displays how much of the signal exists within a given frequency band concerning a range of frequencies.**
- **In the time domain, the signal or function's value is understood for all real numbers at various separate instances in the case of discrete-time or the case of continuous-time.**
- **Ex:- an oscilloscope is a tool commonly used to see real-world signals in the time domain.**
- **a time-domain graph can show how a signal changes with time, whereas a frequency-domain graph will show how much of the signal lies within each given frequency band over a range of frequencies.**
- **a frequency-domain representation can include information on the phase shift that must be applied to each sinusoid to be able to recombine the frequency components to recover the original time signal.**
- **Ex: - a perfect example of a transform is the Fourier transform. Which converts a time function into an integral of sine-waves of various frequencies or sum, each of which symbolizes a frequency component**

RANDOM PROCESSES -SPECTRAL CHARACTERISTICS

- Both the time domain and frequency domain analysis methods exist for analyzing linear systems and deterministic wave forms.
- The spectral description of a deterministic wave form is obtained by Fourier transforming the wave from.
- The Fourier transform X(ω) of deterministic (random) signal x(t) is given by

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

- The function X(ω) sometimes called the spectrum of x(t) and it has units of volts per Hertz when x(t) is a voltage. X(ω) can be considered to be a "Voltage Density Spectrum" of x(t) and it represents both amplitude and phases of frequencies present in x(t).
- •The deterministic signal x(t) can be recovered by means of the "Inverse Fourier Transform" of X(ω).

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

Power Spectral Density or Power Density Spectrum

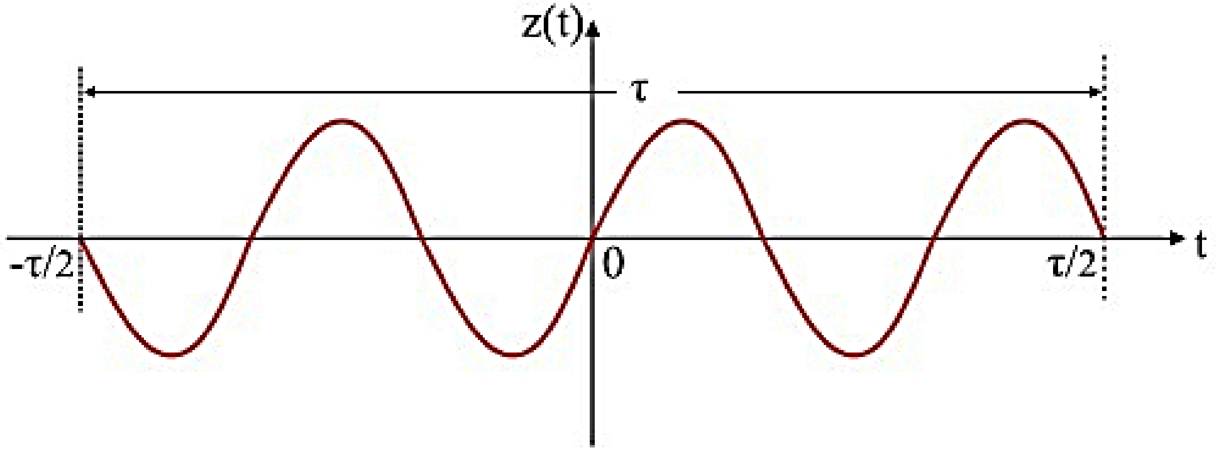
Power spectral density (PSD) is a measure of how a signal's power is distributed across different frequencies. It's also known as power spectrum.

The distribution of average power of a signal x(t) in the frequency domain is called the power spectral density (PSD) or power density (PD) or power density spectrum.

The PSD function is denoted by S(ω)

$$S(\omega) = \lim_{\tau \to \infty} \frac{|X(\omega)|^2}{\tau}$$

In order to drive the power spectral density (PSD) function, consider a power signal as a limiting case of an energy signal, i.e., the signal Z(t) is zero outside the interval |τ/2| as shown in the figure.



Power Spectral Density or Power Density Spectrum

The signal Z(t) is given by,

$$Z(t)\begin{cases} x(t) & |t| < (\frac{\tau}{2}) \\ 0 & otherwise \end{cases}$$

Where x(t) is a power signal of same magnitude extending to infinity.
As the signal Z(t) is finite duration signal of duration τ and thus, it is an energy signal having energy E, that is given by,

$$E = \int_{-\infty}^{\infty} |Z(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Z(\omega)|^2 \, d\omega$$

Where

$$Z(t) \overset{FT}{\leftrightarrow} Z(\omega)$$

Also,

$$\int_{-\infty}^{\infty} |Z(t)|^2 \, dt = \int_{-(\tau/2)}^{(\tau/2)} |x(t)|^2 \, dt$$

Power Spectral Density or Power Density Spectrum

Therefore, we have,

$$\frac{1}{\tau}\int_{-(\tau/2)}^{(\tau/2)}|x(t)|^2\,dt=\frac{1}{2\pi}\left(\frac{1}{\tau}\right)\int_{-\infty}^{\infty}|Z(\omega)|^2\,d\omega$$

Hence, when $\tau\rightarrow\infty$, then the LHS of the above equation gives the average power (P) of the signal x(t) , i.e.,

$$P=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{\tau\rightarrow\infty}\left(\frac{|Z(\omega)|^2}{\tau}\right)\,d\omega$$

If $\tau\rightarrow\infty$, then $\left(\frac{|Z(\omega)|^2}{\tau}\right)$ in above equation approaches a finite value. Assume this finite value is represented by S(ω), i.e.,

$$S(\omega)=\lim_{\tau\rightarrow\infty}\left(\frac{|Z(\omega)|^2}{\tau}\right)$$

The expression in the equation is called the power spectral density (PSD) of the signal z(t). Therefore, for the function x(t), the PSD function is given by,

Power Spectral Density or Power Density Spectrum

$$S(\omega) = \lim_{\tau \to \infty} \left(\frac{|X(\omega)|^2}{\tau} \right)$$

Hence, the average power (P) of the signal x(t) is given by,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\ d\omega = \int_{-\infty}^{\infty} S(f)\ df$$

Also, the power spectral density (PSD) of a periodic function is given by,

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(\omega - n\omega_0)$$

Properties of Power Spectral Density (PSD)

Property 1 - For a power signal, the area under the power spectral density curve is equal to the average power of that signal, i.e.,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega$$

Property 2 - If the signal x(t) is input to an LTI system with impulse response h(t), then the input and output PSD functions of the system are related as,

$$S_y(\omega) = |H(\omega)|^2\, S_x(\omega)$$

Where, |H(ω)| is the magnitude of the system transfer function.

Property 3 - The autocorrelation function R(τ) and the power spectral density function S(ω) of a power signal form a Fourier transform pair, i.e.,

$$R(\tau) \overset{FT}{\leftrightarrow} S(\omega)$$

Properties of Power Spectral Density (PSD)

Property 4 - Power Spectral Density and Expected Power

The integral of the PSD $S_X(f)$ over "all" frequencies equals the expected power in $\{X(t)\}$.
For continuous-time processes, this result can be stated as:

$$\int_{-\infty}^{\infty} S_X(f)\,df = R_X(0) = E[X(t)^2].$$

For discrete-time processes, this result can be stated as:

$$\int_{-0.5}^{0.5} S_X(f)\,df = R_X[0] = E[X[n]^2].$$

Property 5 - The zero frequency value of the PSD of a wide sense stationary random process equals the total area under the graph of the auto correlation function $S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)d\tau$

Property 6 - The power density spectrum of a WSS process is always non-negative. i. e.,

$$S_{xx}(\omega) > 0 \; for \; all \; \omega$$

WIENER-KHINCHIN RELATION [Relationship between Power Spectrum and Auto Correlation Function]

The distribution of average power of a signal in the frequency domain is called the Power Spectral Density (PSD) or Power Density (PD) or power density spectrum. The power spectral density is denoted by S(ω) and is given by,

$$S(\omega) = \lim_{\tau \to \infty} \frac{|X(\omega)|^2}{\tau}$$

The autocorrelation function gives the measure of similarity between a signal and its time-delayed version. The autocorrelation function of power (or periodic) signal x(t) with any time period T is given by,

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-(T/2)}^{T/2} x(t)\ x^*(t-\tau)\ dt$$

Where, τ is called the delayed parameter.

The power spectral density function S(ω) and the autocorrelation function R(τ) of a power signal form a Fourier transform pair, i.e.,

$$R(\tau) \overset{FT}{\leftrightarrow} S(\omega)$$

Cross Power Density Spectrum (or) Cross Spectral Density

- The PSD of a random process provides a measure of the frequency distribution of a single random process.
- Similarly, the cross power density spectrum provides a measure of the frequency interrelationship between two random processes.
- Cross power spectral density (CPSD) is the Fourier Transform of the cross-correlation function. Cross-correlation function is a function that defines the relationship between two random signals.
- Consider two random processes X(t) and Y(t) and one of their sample functions x(t) and y(t) respectively.
- The cross power spectral density, $S_{xy}(f)$ is complex-valued with real and imaginary parts given by co-spectrum ($Co_{xy}(f)$) and quadrature spectrum ($Qu_{xy}(f)$) respectively.
- Coherence function $C_{xy}(f)$ is a measure to estimate how one signal corresponds to another at each frequency and can be called normalized cross power spectral density.

- Let $x_T(t)$ and $y_T(t)$ be defined as that portion of $x(t)$ and $y(t)$ between the limits $-T$ and $+T$.

$$\text{i.e.,}\ x_T(t) = x(t);\ -T < t < T \text{ and } y_T(t) = y(t); -T < t < T$$

- We know that, $X_T(\omega)$ & $Y_T(\omega)$ are Fourier transforms of $x(t)$ and $y(t)$ respectively.

$$F[x_T(t)] = X_T(\omega) = \int_{-T}^{T} x_T(t)\ e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\ e^{-j\omega t}\, dt$$

$$F[y_T(t)] = Y_T(\omega) = \int_{-T}^{T} y_T(t)\ e^{-j\omega t}\, dt = \int_{-T}^{T} y(t)\ e^{-j\omega t}\, dt$$

- Now we define the cross power $P_{XY}(T)$ in the two processes within the interval (-T, T) is defined as

$$P_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} x_T(t)y_T(t)\,dt = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)\,dt$$

- Using Parseval's theorem,

$$P_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)Y_T(\omega)}{2T}\,d\omega$$

- By taking the expected value in above equation, we obtain the average cross power

$$P_{\overline{XY}}(T) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T} d\omega$$

$$P_{\overline{XY}}(T) = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T} d\omega \right\}$$

$$P_{\overline{XY}}(T) = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega \right\} ==> P_{\overline{XY}}(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega$$

- Where $S_{XY}(\omega)$ represents the cross density spectrum or cross spectral density of the process X(t) and Y(t) and is given by

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T}$$

- By using the same procedure as above, we can also arrive at the average cross power

$$P_{\overline{YX}}(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega$$

$$\text{Where, } S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[Y_T^*(\omega)X_T(\omega)]}{2T}$$

RELATIONSHIP BETWEEN CROSS POWER SPECTRUM AND CROSS CORRELATION FUNCTIONS

- Let X(t) and Y(t) be two jointly wide sense stationary (WSS) random process with their cross correlation functions given by $R_{XY}(\tau)$ & $R_{YX}(\tau)$. Then the cross spectral densities of this pair of random process are defined by the Fourier transforms of the cross correlation functions

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau \quad \text{and} \quad R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau} d\omega$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau} d\tau \quad \text{and} \quad R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)\, e^{j\omega\tau} d\omega$$

Fourier Transform of Different types of function

1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	

9	$\cos \omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos \omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$

LINEAR SYSTEM

What is a Linear System?

System − An entity which acts on an input signal and transforms it into an output signal is called the *system*.

Linear System − A *linear system* is defined as a system for which the principle of superposition and the principle of homogeneity are valid.

SUPERPOSITION PRINCIPLE

- The principle of superposition states that the response of the system to a weighted sum of input signals is equal to the corresponding weighted sum of the outputs of the system to each of the input signals.
- Therefore, if an input signal $x_1(t)$ produces an output signal $y_1(t)$ and another input signal $x_2(t)$ produces an output $y_2(t)$, then the system is said to be linear if,

$$T[ax_1(t) + bx_2(t)] = ay_1(t) + by_2(t)$$

Where a and b are constants.

Types of Linear Systems

Linear systems can be of the following two types –

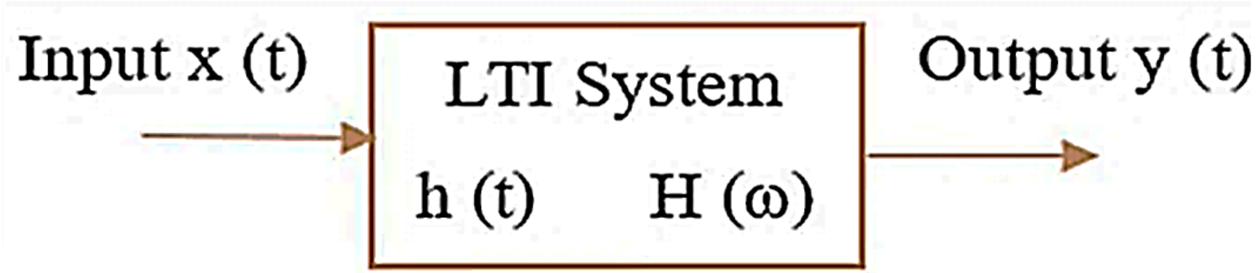
- Linear Time-Invariant [LTI] System
- Linear Time-Variant [LTV] System

A system which is both linear and time-invariant is called the *linear time-invariant system*. In other words, a system for which both the superposition principle and the homogeneity principle are valid and the input-output characteristics of the system do not change with time is called *linear time invariant (LTI) system.*

A system which is linear but time-variant is called the *linear time-variant system*. In other words, a system for which the principle of superposition and homogeneity are valid but the input-output characteristics change with time is called the *linear time-variant (LTV) system.*

Random Signal Response of Linear Systems

We describe the temporal characteristics such as mean value, mean squared value of the response, its auto correlation function and cross correlation function of a stable, linear, time invariant system as shown in figure when the input is an sample function.



Let X(t) be a wide sense stationary random processes applied to an LTI system, with impulse response h(t) and the corresponding output process be Y(t).

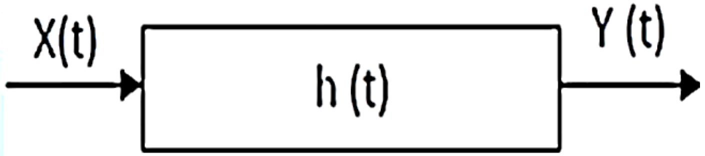
The total response of the system Y(t) can be expressed using convolution integral as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Characteristics of The System Response

When x(t) is a random signal, the LTI system response y(t) is given by convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$



Or

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

Mean Value of The System Response Y(t)

If X(t) is a random signal, the system response Y(t) is

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau$$

Taking the expected value operation on both sides, we get,

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau\right]$$

Since, expectation is applicable only for random variations

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, E[X(t-\tau)]\, d\tau$$

Since X(t) is a WSS random process its mean is a constant

$$E[X(t-\tau)] = E[X(t)] = \overline{X} = m_X$$

Therefore,

$$E[Y(t)] = \overline{X} \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

- For an LTI system, its impulse response and transfer function form a Fourier transform pair

$$\text{i.e.,}\ h(t) \overset{\text{F.T.}}{\longleftrightarrow} H(\omega)$$

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \implies H(0) = \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

- Therefore,

$$E[Y(t)] = \overline{X} \int_{-\infty}^{\infty} h(\tau)\, d\tau \implies E[Y(t)] = \overline{X}\, H(0)$$

Mean Squared Value of The System Response Y(t)

- The mean square value of Y(t) is $E[Y^2(t)] = E[Y(t)Y(t)]$

$$E[Y^2(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t-\tau_2)\,d\tau_2\right]$$

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\tau_1)X(t-\tau_2)]\,h(\tau_1)\,h(\tau_2)\,d\tau_1 d\tau_2$$

- If the input is wide sense stationary, then

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(\tau_1 - \tau_2) = E[X(t-\tau_1)X(t-\tau_2)]$$

Therefore,

$$\boxed{E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau_1 - \tau_2)\,h(\tau_1)\,h(\tau_2)\,d\tau_1 d\tau_2}$$

Auto Correlation Function of Response Y(t)

Let X(t) be wide sense stationary random process the auto correlation function of output Y(t) is

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$R_{YY}(t, t+\tau)$$
$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t+\tau-\tau_2)\,d\tau_2\right]$$

$$R_{YY}(t, t+\tau)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\tau_1)X(t+\tau-\tau_2)]\,h(\tau_1)\,h(\tau_2)\,d\tau_1 d\tau_2$$

$$R_{YY}(t, t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\,h(\tau_1)\,h(\tau_2)\,d\tau_1 d\tau_2$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$$

Cross Correlation Function of Input and Output

The cross correlation function of X(t) and Y(t) is

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

$$==> R_{XY}(t, t+\tau) = E\left[X(t)\int_{-\infty}^{\infty} h(\tau_1)X(t+\tau-\tau_1)\,d\tau_1\right]$$

$$==> R_{XY}(t, t+\tau) = \int_{-\infty}^{\infty} E[X(t)X(t+\tau-\tau_1)]\,h(\tau_1)\,d\tau_1$$

$$==> R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(t+\tau-\tau_1-t)\,h(\tau_1)\,d\tau_1$$

$$==> R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\tau_1)\,h(\tau_1)\,d\tau_1$$

$$==> R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\,R_{XX}(\tau-\tau_1)\,d\tau_1$$

$$==> R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$

Cross Correlation Function of Input and Output

Similarly, it can be shown that

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(-\tau)\, R_{XX}(\tau - \tau_1)\, d\tau_1 ==> R_{YX}(\tau)$$
$$= R_{XX}(\tau) * h(-\tau)$$

From the above equations, it can concluded that, the cross correlation function depend on t and not on absolute time 't'.

The auto correlation function $R_{yy}(t)$ and cross correlation function $R_{xy}(T)$ & $R_{yx}(t)$ are related as

$$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau) \quad \& \quad R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$

Power Density Spectrum of Response Y(t)

We prove that the power density spectrum $S_{YY}(\omega)$ of the response of a LTI system having a transfer function $H(\omega)$ is given by $S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$

Where, $S_{XX}(\omega)$ is the power spectrum of the input process X(t).

Proof: The Fourier transform of output correlation function is given by

$$S_{YY}(\omega) = F[R_{YY}(\tau)] = \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-j\omega\tau} d\tau$$

- We know that, $R_{YY}(\tau)$ for the LTI system is given by

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau + \tau_1 - \tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1 d\tau_2$$

- Therefore,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau + \tau_1 - \tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1 d\tau_2\, e^{-j\omega\tau} d\tau$$

Let, $\tau + \tau_1 - \tau_2 = t ==> d\tau = dt$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\tau_1)\ d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\ d\tau_2 \int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega(t-\tau_1+\tau_2)}\, dt$$

$$S_{YY}(\omega)$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1}\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2}\, d\tau_2 \int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega t}\, dt$$

$$S_{YY}(\omega) = H^{*}(\omega)H(\omega)S_{XX}(\omega)$$

$$==> S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

Cross Power Density Spectrum of Input & Output

- We know that, $R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

Take Fourier transform on both sides,

$$F[R_{XY}(\tau)] = F[R_{XX}(\tau) * h(\tau)]$$

$$==> S_{XY}(\omega) = F[R_{XX}(\tau)]F[h(\tau)] = S_{XX}(\omega)H(\omega)$$

- We know that, $R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$

Take Fourier transform on both sides,

$$F[R_{YX}(\tau)] = F[R_{XX}(\tau) * h(-\tau)]$$

$$==> S_{YX}(\omega) = F[R_{XX}(\tau)]F[h(-\tau)] = S_{XX}(\omega)H^*(\omega)$$

- We know that, $R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$

Take Fourier transform on both sides,

$$F[R_{YY}(\tau)] = F[R_{XY}(\tau) * h(-\tau)]$$

$$==> S_{YY}(\omega) = F[R_{XY}(\tau)]F[h(-\tau)] = S_{XY}(\omega)H^*(\omega)$$

$$==> S_{YY}(\omega) = S_{XX}(\omega)H(\omega)H^*(\omega)$$

$$==> S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

- We know that, $R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$

Take Fourier transform on both sides,

$$F[R_{YY}(\tau)] = F[R_{YX}(\tau) * h(\tau)]$$

$$==> S_{YY}(\omega) = F[R_{YX}(\tau)]F[h(\tau)] = S_{YX}(\omega)H(\omega)$$

$$==> S_{YY}(\omega) = S_{XX}(\omega)H^*(\omega)H(\omega)$$

$$==> S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$