Signals and Communication Systems

Signal Analysis in Frequency domain

Fundamentals of FT

CT Fourier Transform

The analysis and synthesis equations are given by

$$\boxed{X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt}$$

Analysis Equation

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega}$$

Synthesis Equation

Magnitude and Phase spectrum:

The FT of a signal $x(t)$ is represented by

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt \quad \text{where} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t} d\Omega$$

$X(j\Omega)$ is complex in general and need two plots for its graphical representation $X(j\Omega) = |X(j\Omega)|\, e^{j\phi(\omega)}$

where $|X(j\Omega)$ is known as magnitude spectrum, and $\phi(\omega)$ is known as phase spectrum

For real functions $x(t)$, $X^*(j\Omega) = X(-j\Omega)$,

where '*' is complex conjugate.

We know that $X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$, then $X(-j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\Omega t}\, dt$

It follows that $X^*(j\Omega) = X(-j\Omega)$.

Further $X(j\Omega) = |X(j\Omega)|\, e^{j\phi(\omega)}$, then $X(-j\Omega) = |X(j\Omega)|\, e^{-j\phi(\omega)}$.

Therefore, we conclude that

the magnitude spectrum $|X(j\Omega)$ is an even function

and $\phi(\omega)$ is an odd function.

Dirichlet conditions / Conditions to exist FT

(1) Absolutely summable $$\boxed{\int_{-\infty}^{\infty} |x(t)|\, dt < \infty}$$

Square summable i.e., it is a finite energy signal

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty}$$

(2) $x(t)$ must have a finite number of maxima and minima within any finite interval

(3) $x(t)$ must have a finite number of discontinuities, all of finite size, within any finite interval

Test the following signals for convergence of FT

$$(i)\ x(t) = e^{-2t}u(t) \qquad (ii)\ y(t) = e^{2t}u(t)$$

Condition for convergence of FT is $\boxed{\int_{-\infty}^{\infty} |x(t)|\, dt < \infty}$

Case (i) $\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} |e^{-2t}u(t)|\, dt$

$$= \int_{0}^{\infty} |e^{-2t}|\, dt = \frac{e^{-2t}}{-2}\bigg|_0^{\infty} = \frac{1}{2} < \infty$$

$e^{-2t}u(t)$ does have FT

Case (ii) $\int_{-\infty}^{\infty} |y(t)|\, dt = \int_{-\infty}^{\infty} |e^{2t}u(t)|\, dt$

$$= \int_{0}^{\infty} |e^{2t}|\, dt = \frac{e^{2t}}{2}\bigg|_0^{\infty} = \infty$$

$e^{2t}u(t)$ does not have FT

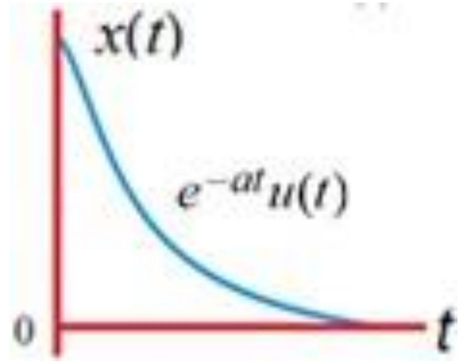
FT of standard signals

1. $x(t) = e^{-at}u(t), \; a > 0$ Right sided signal

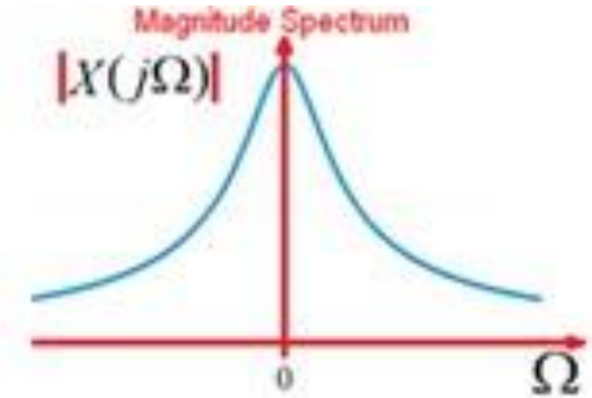
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\Omega t}\, dt$$

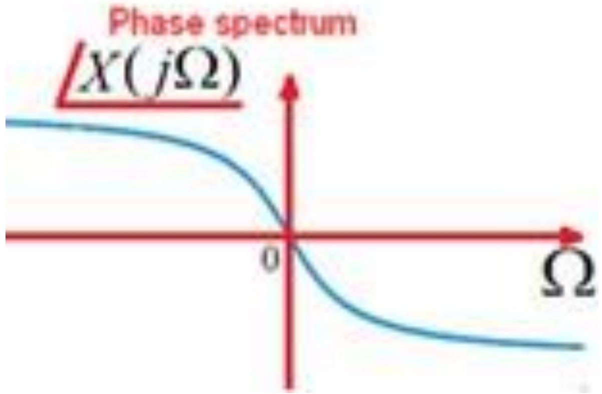
$$= \int_{0}^{\infty} e^{-(a+j\Omega)t}\, dt = \frac{1}{a + j\Omega}$$



Magnitude spectrum $|X(j\Omega)| = \dfrac{1}{\sqrt{a^2 + \Omega^2}}$



Phase spectrum $\angle X(j\Omega) = -\tan^{-1}\left(\dfrac{\Omega}{a}\right)$



$$\boxed{x(t) = e^{-at}u(t)} \xleftarrow{\text{FT}} \!\!\!\xrightarrow{} \boxed{X(j\Omega) = \frac{1}{a + j\Omega}}$$

$$x(t) = e^{-at}u(t) \xleftrightarrow{FT} X(j\Omega) = \frac{1}{a + j\Omega}$$

Find FT of a signal $x(t) = 2e^{-2t}u(t)$

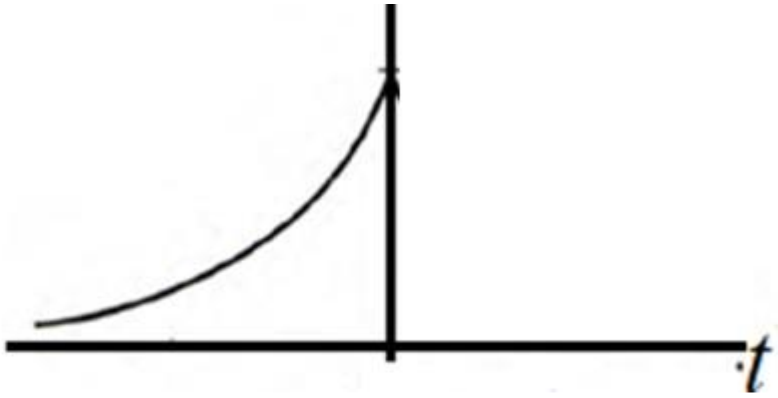
$$X(j\Omega) = \frac{2}{2 + j\Omega}$$

Find $x(t)$ if $X(j\Omega) = \dfrac{5}{4 + j\Omega}$

$$x(t) = 5e^{-4t}u(t)$$

2. Left sided signal

$$x(t) = e^{at}u(-t)$$



$$e^{at}u(-t), \; a > 0 \quad \xleftrightarrow{FT} \quad \frac{1}{a - j\Omega}$$

Find FT of a signal $x(t) = 2e^{2t}u(-t)$

$$X(j\Omega) = \frac{2}{2 - j\Omega}$$

Find $x(t)$ if $X(j\Omega) = \dfrac{5}{4 - j\Omega}$

$$x(t) = 5e^{4t}u(-t)$$

3. Two-sided signal

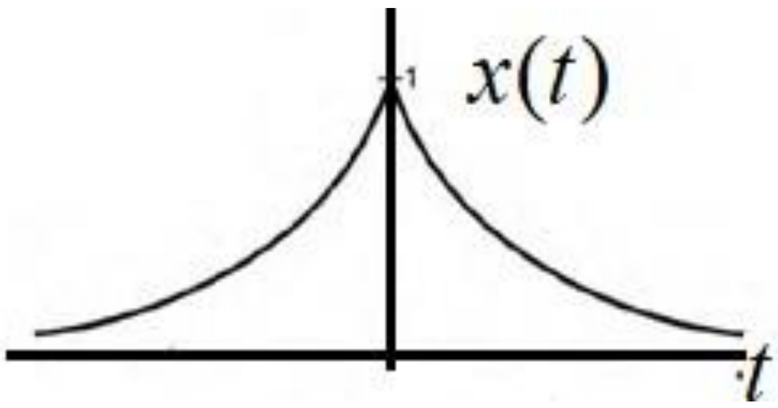
$$x(t) = e^{-a|t|},\ a > 0$$

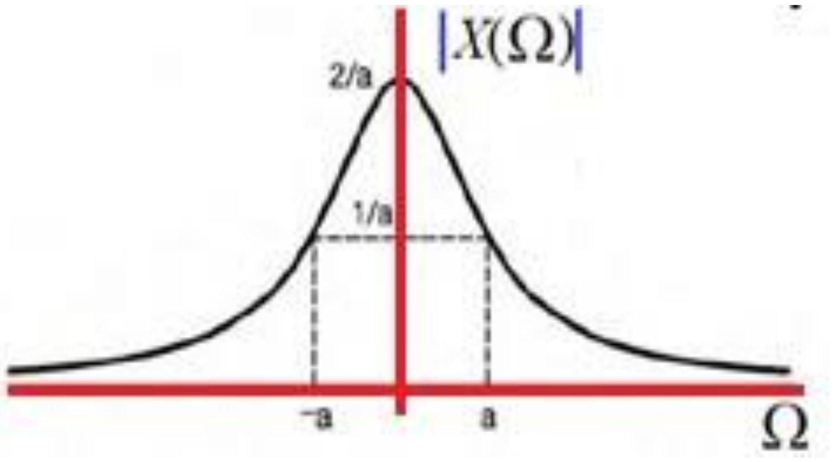
Let $x(t) = x_1(t) + x_2(t)$

$$e^{-a|t|} = e^{at}u(-t) + e^{-at}u(t)$$

$$e^{-at}u(t) \xleftrightarrow{\text{FT}} \frac{1}{a + j\Omega}$$

$$e^{at}u(-t) \xleftrightarrow{\text{FT}} \frac{1}{a - j\Omega}$$





$$e^{-a|t|} \xleftrightarrow{\text{FT}} \frac{1}{a + j\Omega} + \frac{1}{a - j\Omega} = \frac{2a}{a^2 + \Omega^2}$$

$$e^{-a|t|} \xleftrightarrow{\text{FT}} \frac{2a}{a^2 + \Omega^2}$$

Exercise questions:

$$e^{-a|t|} \xleftarrow{\ \text{FT}\ } \to \frac{2a}{a^2+\Omega^2}$$

Find FT of the following signals:

$$x(t) = e^{-2|t|}$$

$$X(j\Omega) = \frac{4}{4+\Omega^2}$$

Find Inverse FT of the following signals:

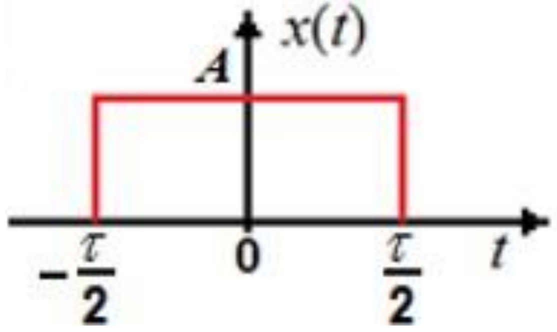
$$X(j\Omega) = \frac{3}{16+\Omega^2}$$

$$\frac{2a}{a^2+j\Omega}$$

$$x(t) = \frac{3}{8}e^{-4|t|}$$

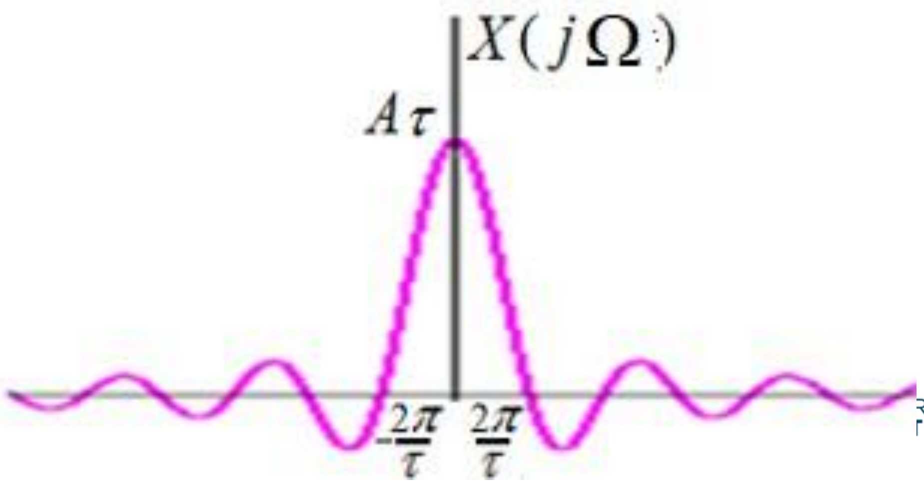
4. FT of a gate function:

$$x(t) = \begin{cases} A, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & Elsewhere \end{cases}$$

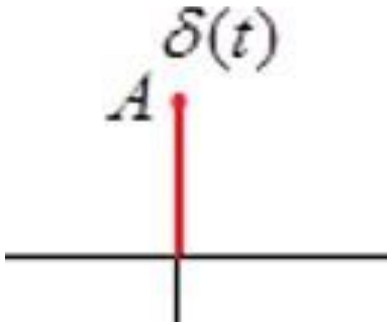


$$A\,rect\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{FT}} A\tau\left[\frac{\sin\Omega\frac{\tau}{2}}{\Omega\frac{\tau}{2}}\right] = A\tau\,\text{Sinc}\left(\frac{\Omega\tau}{2}\right)$$

$$= A\tau\,Sa\left(\frac{\Omega\tau}{2}\right)$$



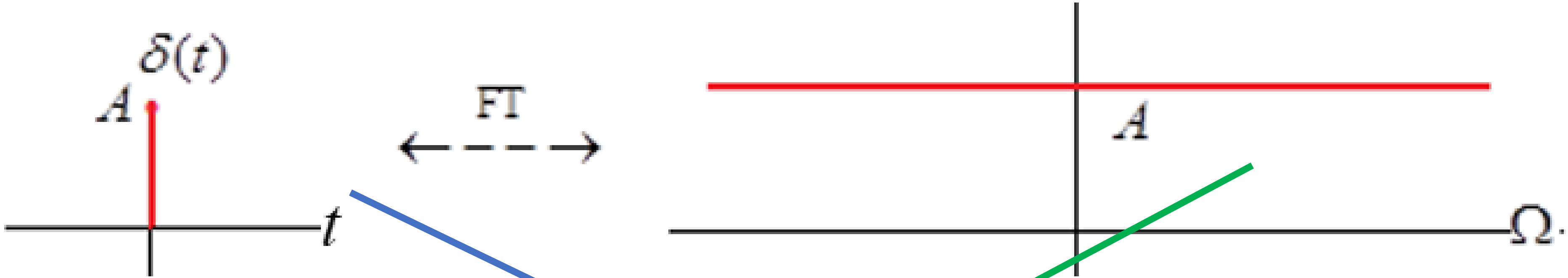
5. FT of an impulse / delta function



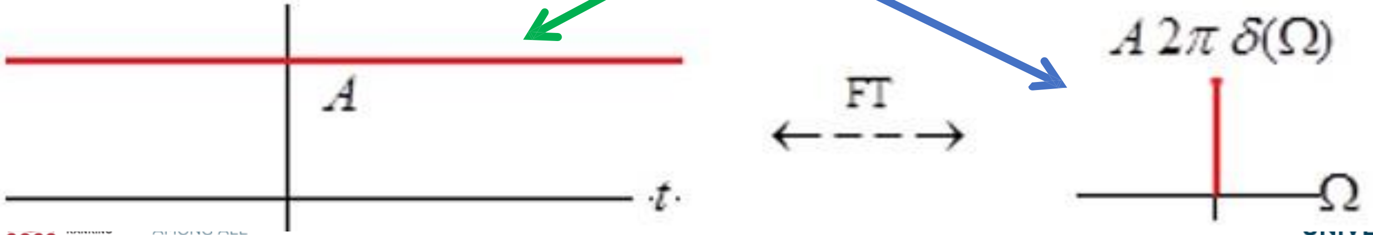
$$x(t) = A\delta(t)$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} A\delta(t)\, e^{-j\Omega t}\, dt = A, \text{ for all } \Omega$$



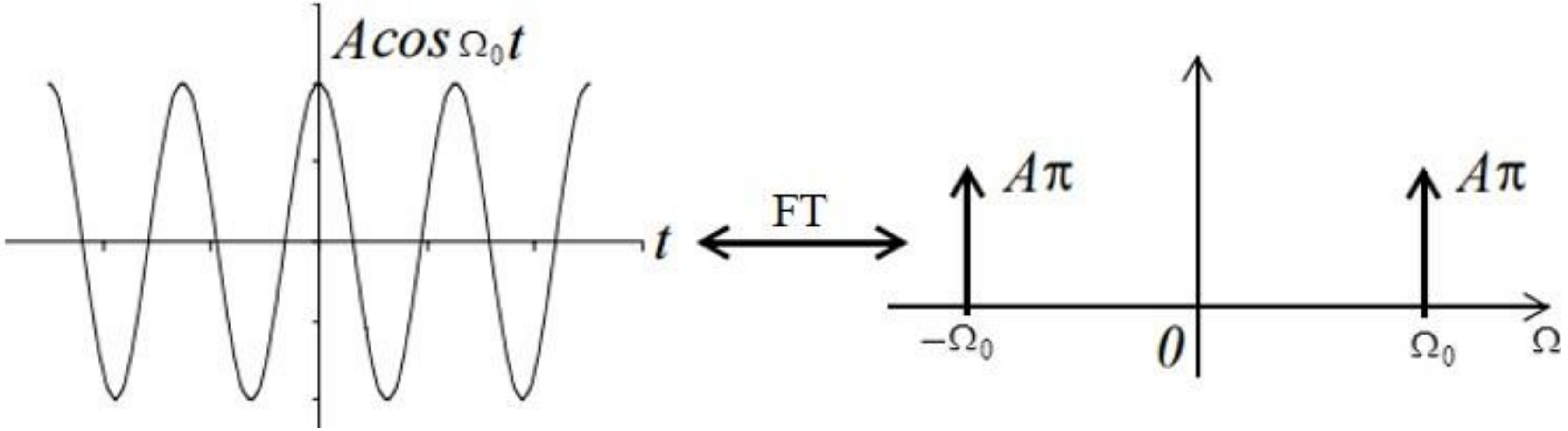
7. FT of a constant A, for all 't' $\overset{FT}{\longleftrightarrow}$ $A\, 2\pi\, \delta(\Omega)$ duality to above



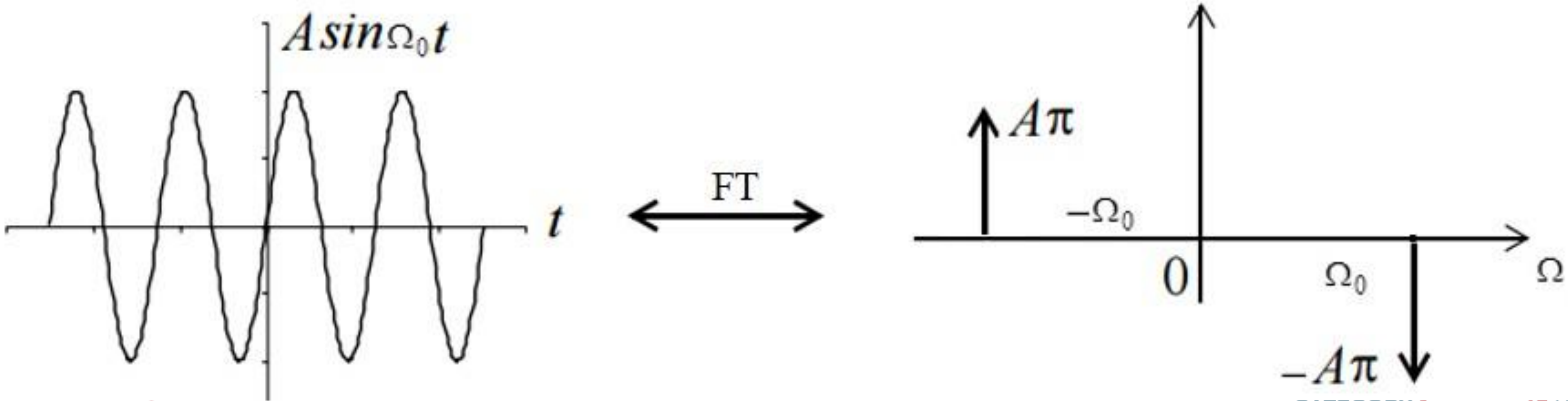
8. FT of a step function $u(t) \xleftrightarrow{\text{FT}} \pi\,\delta(\Omega) + \dfrac{1}{j\Omega}$

9. FT of a signum function $\operatorname{sgn}(t) \xleftrightarrow{\text{FT}} \dfrac{2}{j\Omega}$

10. $\cos(\Omega_0 t) \xleftrightarrow{FT} A\pi\left\{\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)\right\}$



11. $\sin(\Omega_0 t) \xleftrightarrow{FT} Aj\pi\left\{\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)\right\}$



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