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CO-3

Session-17

Double integration and it's application:-

Formulae:-

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$\int \frac{1}{x} dx = \log x$

$\int \sec^2 x \, dx = \tan x$

$\int e^x dx = e^x$

$\int \sec x \tan x \, dx = \sec x$

$\int \sin x \, dx = -\cos x$

$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \cdot \frac{1}{a}$

$\int \cos x \, dx = \sin x$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$

$\int \frac{1}{1+x^2} dx = \tan^{-1} x$

$\int \cos 2x \, dx = \frac{\sin 2x}{2}$

$\int c \, dx = c \cdot x$

$\int x+y+z \, dz = xz + yz + \frac{z^2}{2}$

Double Integration

* $I = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) \, dx \, dy$ (or)

$$I = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) \, dy \, dx.$$

* $$I = \int_{y=a}^{y=b} \int_{x=f_1(y)}^{x=f_2(y)} f(x,y) \, dx \, dy$$

* $$I = \int_{x=c}^{x=d} \int_{y=f_1(x)}^{y=f_2(x)} f(x,y) \, dy \, dx$$

* Ex:- $I = \int_{y=-1}^{y=-2}\int_{x=1}^{2}(x^2+y^2)\,dx\,dy$

$I = \int_{y=-1}^{y=-2}\left[\frac{x^3}{3}+y^2(x)\right]_1^2 dy$

$\boxed{\int x^n = \frac{x^{n+1}}{n+1}}$

$\int_{y=-1}^{y=-2}\left[\frac{(2)^3}{3}+y^2(2) - \frac{(1)^3}{3}+y^2(1)\right]dy$

$\int_{y=-1}^{y=-2}\left[\frac{8}{3}+2y^2-\frac{1}{3}+y^2\right]dy$

$\int_{y=-1}^{y=-2}\left[\frac{7}{3}+y^2\right]dy$

$\boxed{\int c\,dx = c\cdot x}$

$\left[\frac{7}{3}y+\frac{y^3}{3}\right]_1^{-2} = \left[\frac{7(-2)}{3}+\frac{(-2)^3}{3}-\left[\frac{7}{3}(-1)+\frac{(-1)^3}{3}\right]\right]$

$\left[-\frac{14}{3}+\frac{(-8)}{3}+\frac{7}{3}-\frac{(-1)}{3}\right]$

$\left[\frac{-14}{3}-\frac{8}{3}+\frac{7}{3}+\frac{1}{3}\right]$

$\left[\frac{-14-8+8}{3}\right]$

$= -14/3.$

① Evaluate the integral $\int_0^1\int_{-1}^{2} xy\,dy\cdot dx$

$I = \int_{x=0}^{x=1}\int_{y=-1}^{y=2} xy\,dy\,dx$

$= \int_{x=0}^{1} x\left(2-\frac{1}{2}\right)dx$

$$\int_{x=0} \frac{3}{2} x \, dx = \left[\frac{3}{2} \cdot \frac{x^2}{2}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - 0\right]$$

$$= \frac{3}{4}$$

(2) Evaluate the integral $\int_0^4 \int_0^{\sqrt{4-x}} (xy)\, dy\, dx$

$y = 0$ to $y = \sqrt{4-x}$

$y = 0$ to $x = 4$

$$\int_0^4 \int_0^{\sqrt{4-x}} (x) y \, dy \, dx$$

$$\int_0^4 x \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x}} dx$$

$$\int_{x=0}^4 \frac{x}{2} \left[(\sqrt{4-x})^2 - 0^2\right] dx$$

$$\int_{x=0}^4 \frac{1}{2} \left[x(4-x)\right] dx$$

$$= \frac{1}{2} \int_{x=0}^4 (4x - x^2) \, dx$$

$$= \frac{1}{2}\left[4 \cdot \frac{x^2}{2} - \frac{x^3}{4}\right]_0^4$$

$$= \frac{1}{2}\left[\cancel{4}^2 \frac{(4)^2}{\cancel{2}} - \frac{(4)^3}{3} - \frac{4(0)^3}{3}\right]$$

$$= \frac{1}{2}\left[32 - \frac{64}{3}\right]$$

$$= \frac{1}{2}\left[\frac{96-64}{3}\right]$$

$$= \frac{\cancel{32}^{16}}{\cancel{2} \times 3} = \frac{16}{3}$$

How to find limits if the region is bounded by some curves

Find the area enclosed by $y = x^2$ and $x = y^2$

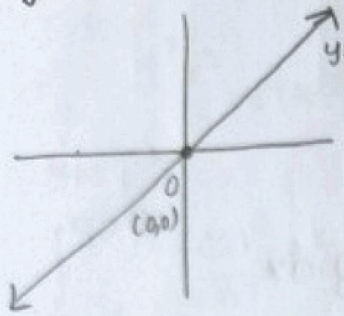
$$\boxed{\text{Area} = \iint_C dA}$$

where $dA = dx\,dy$ or $dy\,dx$

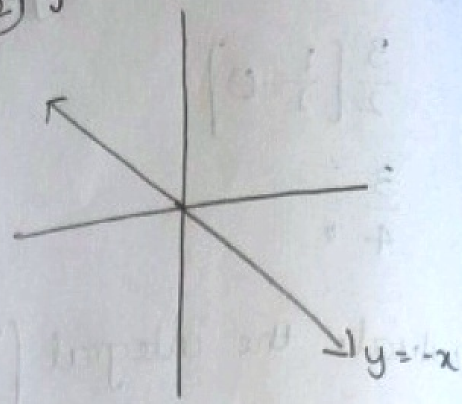
where 'C' is the region of integration.

Standard Curves:-

① $y=x$

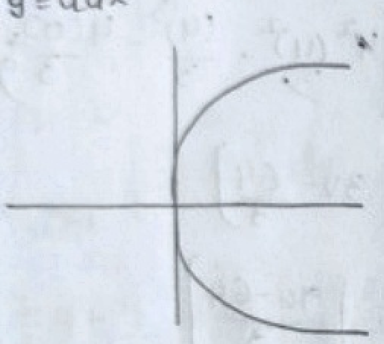


② $y=-x$

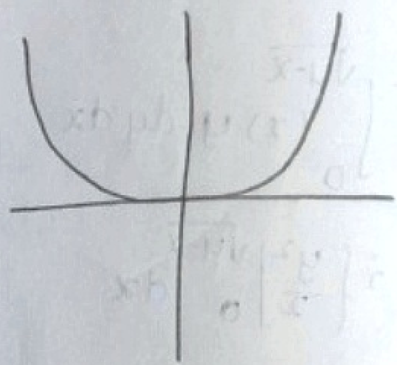


Parabola:-

③ $y^2=4ax$



④ $x^2=4ay$

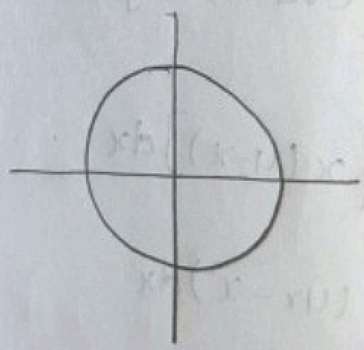


⑤ circle:-

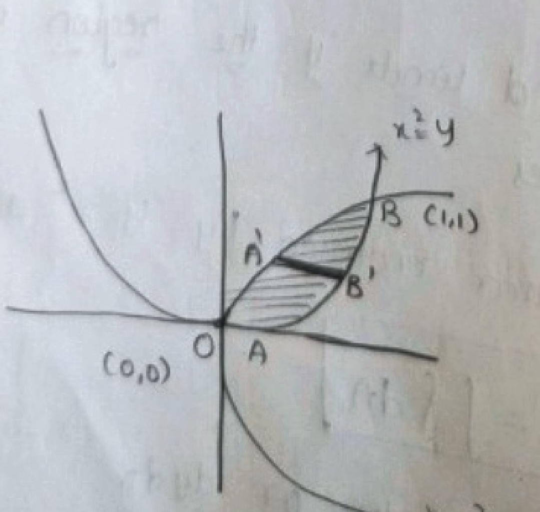
$x^2+y^2=a^2$

Center $=(0,0)$

radius $=$ 'a'



Ex:-



- $x=0$ lies on y-axis
- $y=0$ lies on x-axis
- $y=1$ lies parallel to x-axis

lower limit of x: $x=y^2$

upper limit of x: $x=$

lower limit of y: y=0

upper limit of y: y=1

(1) Area $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} dx\, dy$

$$\int_{y=0}^{1} [x]_{y^2}^{\sqrt{y}} dy = \int_{y=0}^{1} \left[\sqrt{y} - y^2\right] dy \Rightarrow \left[\frac{y^{3/2}}{3/2} - \frac{y^3}{3}\right]_0^1$$

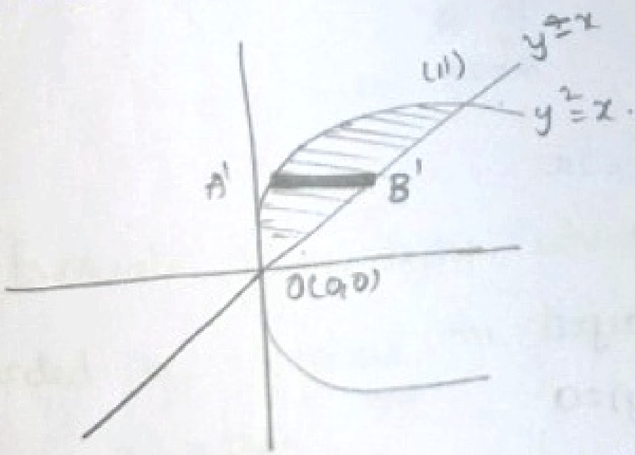
$$\left[\frac{2}{3} y^{3/2} - \frac{y^3}{3}\right]_0^1 = \left[\frac{2}{3}(1) - \frac{1}{3} - 0 + 0\right]$$

$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ units.

(4) Find the area bounded by parabola $x=y^2$ and the line $y=x$.

Area $= \iint dA$

where $dA = dx\,dy$ or $dy\,dx$



limits of x: lower limit x: $x=y^2$

upper limit x: $x=y$

limits of y: lower limit y = $y=0$

upper limit, y = $y=1$

Area $= \int_0^1 \int_{y^2}^{y} dx\, dy$

$$\int_0^1 [x]_{y^2}^{y} dy = \int_0^1 [y-y^2] dy$$

$$\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{3-2}{6} = 1/6$$

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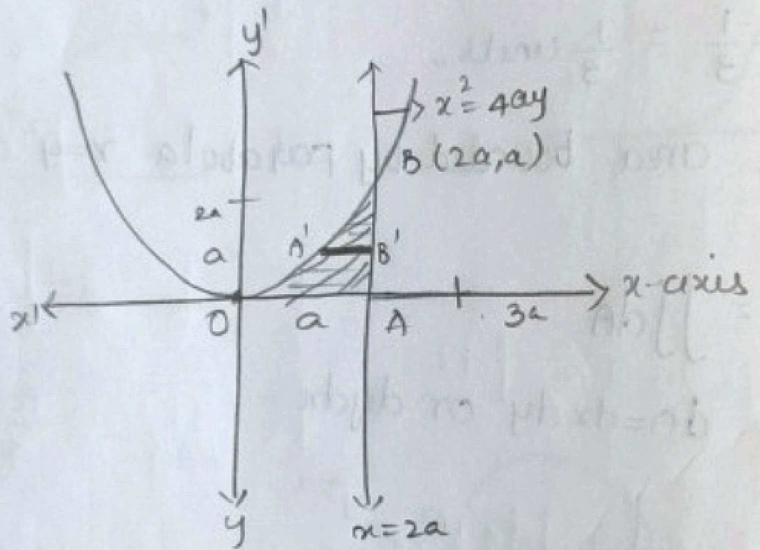
(5) Evaluate $\iint_C xy \, dx \, dy$, where C is the domain bounded by x-axis, ordinate $x=2a$, and the curve $x^2=$



$x^2 = 4ay$ & $x = 2a$

$(2a)^2 = 4ay$

$4a^2 - 4ay = 0$

$4a(a-y) = 0$

$y = a$

The region OAB, is the region of integration

$x^2 = 4ay \Rightarrow x = \sqrt{4ay}$

lower limit of $x: \sqrt{4ay}$

upper limit of $x = 2a$

lower limit $y = 0$

upper limit $y = a$

$$\iint_c xy\,dx\,dy = \int_0^a \int_{\sqrt{4ay}}^{2a} xy\,dx\,dy$$

$$\int_0^a y\left[\frac{x^2}{2}\right]_{\sqrt{4ay}}^{2a} dy$$

$$\int_0^a y\left[\frac{(2a)^2}{2} - \frac{1}{2}(\sqrt{4ay})^2\right] dy$$

$$\int_0^a y\left[2a^2 - 2ay\right] dy$$

$$\int_0^a y\left[2a^2y - 2ay^2\right] dy$$

$$\left[2a^2\left[\frac{y^2}{2}\right] - 2a\left[\frac{y^3}{3}\right]\right]_0^{2a}$$

$$\left[\frac{2a^2}{2}(2a)^2 - \frac{2a}{3}(2a)^3\right]$$

$$\frac{\cancel{4}^{2}\,a^4}{\cancel{2}} - \frac{4a^4}{3} = \frac{6a^4 - 4a^4}{6}$$

$$= \frac{2a^4}{6}$$

⑥ Evaluate $\iint_c xy\,dx\,dy$, where c is the domain bounded by y-axis the line $y=2a$ and parabola $x^2 = 4ay$

$x^2 = 4ay$

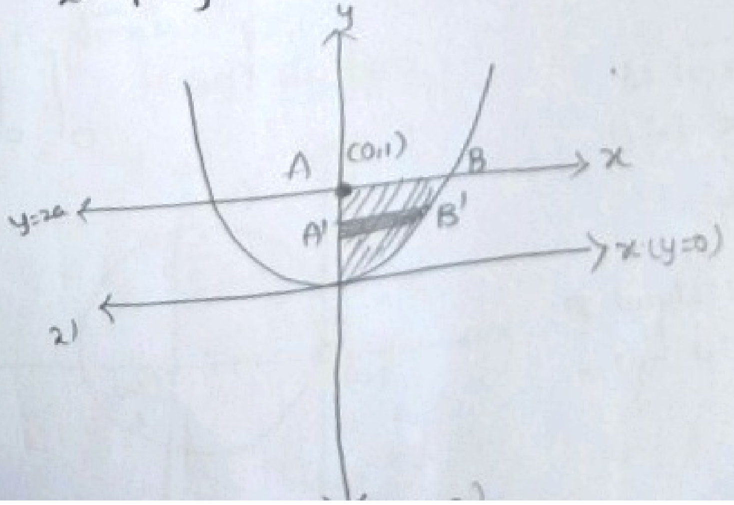
$y = 2a$

$x^2 = 8a^2$

$x = 2\sqrt{2}a$

$y = 2a$

$x^2 = 4ay$



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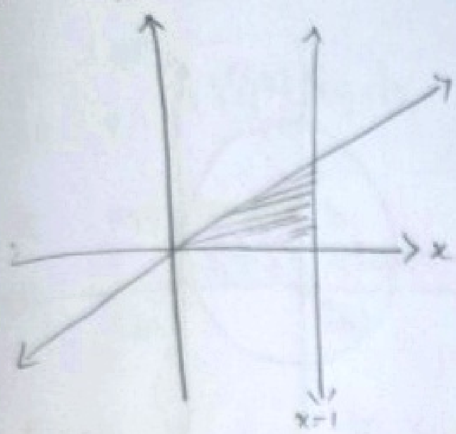
① Sketch region of integration for $\int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x,y)\,dy\,dx$



y limits $y=0$ to $y=x$

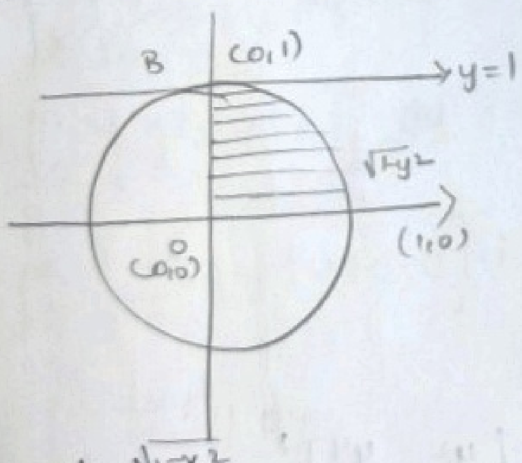
x limits $x=0$ to $x=1$

② $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dy\,dx$

$I = \int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dy\,dx$

x limits : $x=0$ to $x=\sqrt{1-y^2}$

y limits : $y=0$ to $y=1$

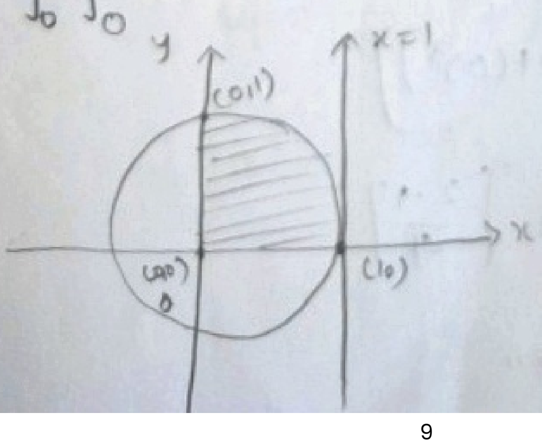


① $I = \int_0^1\int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

$y=\sqrt{1-x^2}$

$y^2=1-x^2$

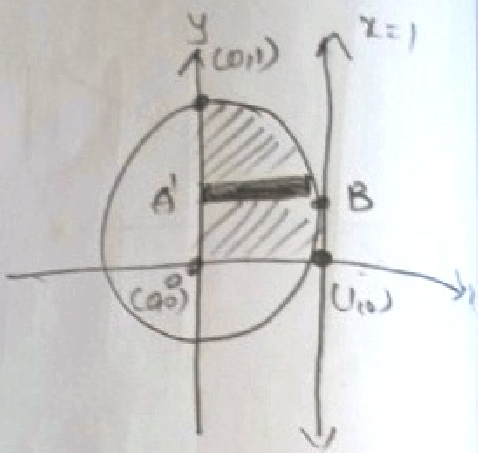
$x^2+y^2=1$



y limits: $y=0$ to $y=\sqrt{1-x^2}$

x limits $x=0$ to $x=1$

③ Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$, by changing order of integration



limits x: $x=0$ to $x=1$

limits y: $y=0$ to $y=\sqrt{1-x^2}$

new order $= dx\,dy$

limits of x: $x=0$ to $x=\sqrt{1-y^2}$

limits of y: $y=0$ to $y=1$

8) $$\int_0^1 \int_0^{\sqrt{1-x^2}} 2y \; dy\, dx$$

$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-x^2}} 2y \; dx\, dy$$

$$\int_0^1 y \left[\left[\frac{x^2}{2}\right] \right]_0^{\sqrt{1-y^2}} dy$$

$$\int_0^1 \frac{y}{2} (\sqrt{1-y^2})^2 - 0^2] \, dy$$

$$\int_0^1 \frac{y}{2} [+1-y^2] \, dy$$

$$\frac{1}{2} \int_0^1 [y-y^3]_0^1 \, dy = \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$\frac{1}{2}\left[\frac{1}{2} - \frac{(1)}{4} - [0+(0)^3]\right]$$

$$\frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] \Rightarrow \frac{1}{2}\left[\frac{2-1}{4}\right]$$

$$\frac{1}{2}\left[\frac{1}{4}\right] = \frac{1}{8}$$

(4) change the order of integration and evaluate $\int_0^a$

$\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$

$$I = \int_0^a \int_x^a (x^2+y^2)\,dy\,dx$$

limits x: $x=0$ to $x=a$

limits y: $y=x$ to $y=a$

OBC

x limits $x=0$ to $x=y$

y limits $y=0$ to $y=a$

$$\int_0^y \int_y^a (x^2+y^2)\, dy\, dx$$

$$\int_0^a \int_0^y (x^2+y^2)\, dy\, dx$$

$$\int_0^a \left[\frac{x^3}{3} + y^2 \cdot x\right]_0^y dy$$

$$\int_0^a \left[\frac{y^3}{3} + y^2 \cdot y + 0 - 0\right] dy$$

$$\int_0^a \left[\frac{4y^3}{3}\right] dy$$

$$\frac{4}{3}\int_0^a \frac{y^4}{4}\, dy = \frac{\cancel{4}}{3\times\cancel{4}}\left[y^4\right]_0^a$$

$$\frac{1}{3}\left[a^4 - 0\right] = a^4/3$$

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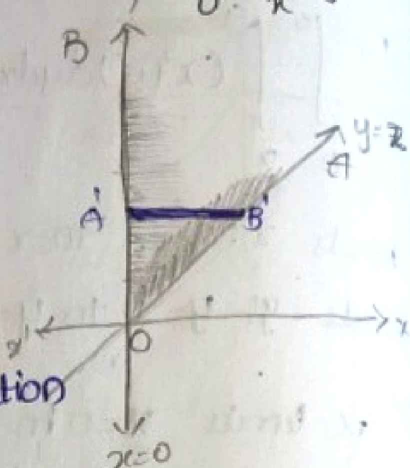
(5) change the order of integration and hence evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$

Given,

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$$



limits of y: $y = x$ to $y \to \infty$

limits of x: $x = 0$ to $x \to \infty$

OAB is an open region of integration

the new order of integration is dxdy

As we need to write drawing an horizontal strip in the region of integration

x: limits $x = 0$ to $x = y$

y: limits $y = 0$ to $y \to \infty$

$$G = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy \Rightarrow \int_0^\infty \frac{e^{-y}}{y} \left[x \right]_0^y dy = \int_0^\infty \frac{e^{-y}}{y} [y-0$$

$$\int_0^\infty \frac{e^{-y}}{\cancel{y}} \cancel{y} dy = \int_0^\infty e^{-y} dy \Rightarrow \left[-e^{-y} \right]_0^\infty = \left[-e^{-\infty} + e^0 \right]$$

Session-19

Double Integration in polar Co-ordinates:-

To convert Catecian System (i.e x-y-z plane) to Polar Co-ordinates system (r, θ) we need to use $x = r\cos$

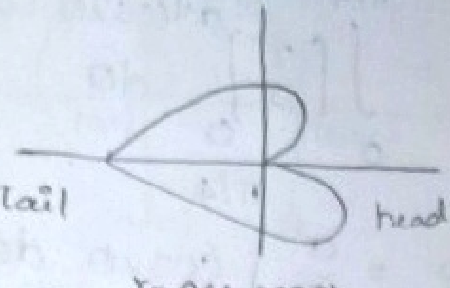
$y = r\sin\theta$

$dydx$ or $dxdy = r \cdot dr d\theta$

Some standard polar curves

i) cardioid :-

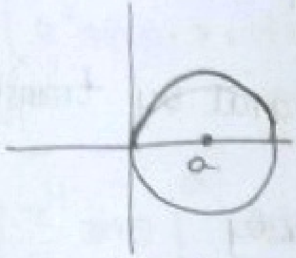
$r = a(1-\cos\theta)$



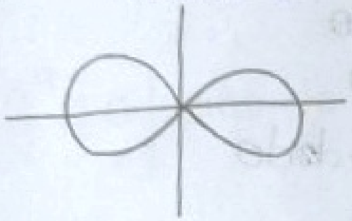
$r = a(1-\cos\theta)$

ii) circle :- $r = 2a\cos\theta$

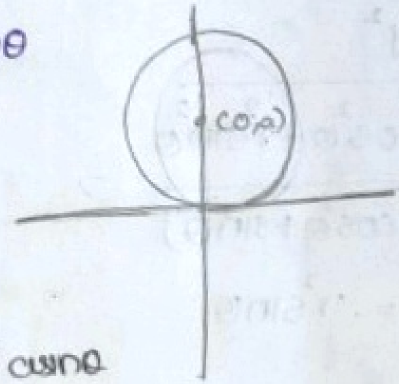
center $(a,0)$ radius 'a'



iii) lemniscate :- $r^2 = a^2\cos 2\theta$



iv) $r = 2a\sin\theta$



① Evaluate $\displaystyle\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

$$I = \int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta \Rightarrow \int_0^{\pi}\frac{1}{2}\left[a^2\sin^2\theta - 0\right]d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta \Rightarrow \frac{a^2}{4}\int_0^{\pi}\left[1-\cos 2\theta\right]d\theta \Rightarrow \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$\Rightarrow \frac{a^2}{4}\left[\pi - \frac{\sin 2\times\pi}{\pi} - 0 - \frac{\sin 0}{2}\right] = \frac{\pi a^2}{4}$$

(3) Evaluate the integral $\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$

$$I = \int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/4}\left[a^2\cos 2\theta - 0\right] d\theta = \frac{a^2}{2}\int_0^{\pi/4}\cos 2\theta \, d\theta \Rightarrow \frac{a^2}{2}\left[\frac{\sin 2\theta}{2}\right.$$

$$\frac{a^2}{4}\left[\sin 2\left[\frac{\pi}{4}\right] - 0\right] = \frac{a^2}{4}\sin \pi/2 \Rightarrow \frac{a^2}{4}(1) = a^2/4$$

(4) Evaluate the following integral by transforming into polar Co-ordinates $\int_0^a\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$.

let us, $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r.dr\,d\theta$

$$f(x,y) = y.\sqrt{x^2+y^2}$$

$$= r\sin\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$r\sin\theta\sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= r\sin\theta(r) = r^2\sin\theta$$

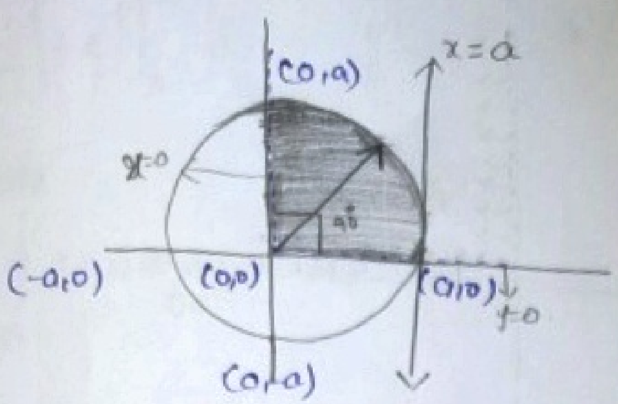
$$\int_0^a\int_0^{\sqrt{a^2-x^2}} r^2\sin\theta \, r.\,dr\,d\theta$$

Given, y limits $y=0$ to $y=\sqrt{a^2-x^2}$

$y^2 = a^2 - x^2$

$y^2 + x^2 = a^2$

x limits $x=0$ to $x=a$



θ start's from y=0, on x-axis so θ=90°

limits of r: r=0 to r=a

limits of θ: θ=0 to θ=π/2

$$I = \int_0^{\pi/2}\int_0^{a} r^2 \sin\theta \cdot r \cdot dr\, d\theta$$

$$\int_0^{\pi/2}\left[\frac{r^4}{4}\sin\theta\right]_0^a d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin\theta\, d\theta$$

$$\Rightarrow \int_0^{\pi/2}\left[\frac{a^4}{4} - 0\right]\sin\theta\, d\theta \Rightarrow \frac{a^4}{4}\int_0^{\pi/2}[\sin\theta]\, d\theta.$$

$$\frac{a^4}{4}[-\cos\theta]_0^{\pi/2} \Rightarrow \frac{a^4}{4}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$\Rightarrow \frac{a^4}{4}(-0+1) = \frac{a^4}{4}$$

(5) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar Coordinates

$x = r\cos\theta$

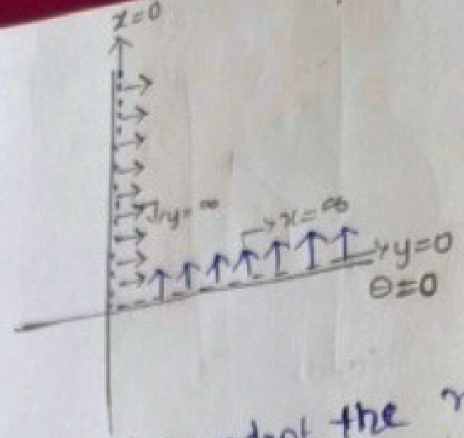
$y = r\sin\theta$

$dx\,dy = r \cdot dr\, d\theta$

$F(x) = e^{-(x^2+y^2)}$

limits x: $x=0$ to $x=\infty$

limits y: $y=0$ to $y=\infty$



in this sum the 1st quardant the regeon of integration

So, ~~$x \neq r\cos\theta$~~

$$f(x,y) = e^{-(x^2+y^2)}$$

$$= e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}$$

$$= e^{-r^2}$$

limits of r : $r=0$ to $r \to \infty$

limits of θ : $\theta = 0$ to $\theta = \pi/2$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy$$

$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr\, d\theta$$

$e^{-r^2} \cdot r$

$\frac{r^2}{2}$

Subtitute $r^2 = t$

$$2r\,dr = dt$$

$$r\,dr = \frac{dt}{2}$$

$$r = 0 \to t = 0$$

$$r \to \infty \Rightarrow t \to \infty$$

$$\int_0^{\pi/2} \int_0^\infty e^{-t} \cdot \frac{1}{2} \, dt \cdot d\theta$$

$$\int_0^{\pi/2} \frac{1}{2}\left[-e^{-t}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} \left[e^{-\infty} + e^{-0}\right] d\theta$$

$$\boxed{\therefore e^{-\infty} + e^{-0} = 0}$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} 1 \cdot d\theta = \frac{1}{2}(\theta)_0^{\pi/2} \Rightarrow \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

Session - 20 & 21

Triple integrals and its applications:-

$$\int \int \int f(x, y, z)\, dx\, dy\, dz$$

(or)

$$\int \int \int f(x, y, z)\, \cancel{dz}\, dy\, dx$$

dz

① Evaluate $\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} x^2 y z \, dx\, dy\, dz$

$$\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} x^2 y z \, dz\, dy\, dx \Rightarrow$$

$$\int_{x=0}^{1} (x^2 dx) \cdot \int_{y=0}^{1} (y \cdot dy) \cdot \int_{z=0}^{1} (z \cdot dz) \Rightarrow \left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1$$

$$\left[\frac{1}{3} - 0\right]\left[\frac{1}{2} - 0\right]\left[\frac{1}{2} - 0\right] \Rightarrow \left[\frac{1}{3}\right]\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]$$

$$= \frac{1}{12}$$

② Evaluate $\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(x+y+z)\,dx\,dy\,dz$

$$I=\int_0^1\int_0^1\int_0^1 (x+y+z)\,dz\,dy\,dx$$

$$\int (x+y+z)\,dz = \left[xz+yz+\frac{z^2}{2}\right]$$

$$\int_0^1\int_0^1 \left[(x+y)z+\frac{z^2}{2}\right]_0^1 \Rightarrow \int_0^1\int_0^1\left[(x+y)(1)+\frac{1}{2}-(x+y)(0)+\frac{0}{2}\right]dy$$

$$\int_0^1 \left[xy+\frac{y^2}{2}+\frac{1}{2}y\right]_0^1 dx \Rightarrow \int_0^1\left[x(1)+\frac{1}{2}+\frac{1}{2}-x(0)+\frac{0}{2}\right]dx$$

$$\int_0^1 [(x+1)]\,dx = \left[\frac{x^2}{2}+x\right]_0^1 \Rightarrow \left[\frac{1}{2}+1-\frac{0}{2}+0\right]$$

$$=\frac{3}{2}$$

③ Evaluate $\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} dx\,dy\,dz$

$$\left(\int_{x=0}^{a} dx\right)\left(\int_{y=0}^{b} dy\right)\left(\int_{z=0}^{c} dz\right) \Rightarrow [x]_0^a\,[y]_0^b\,[z]_0^c$$

$$(a-0)(b-0)(c-0) = abc$$

④ Evaluate $\int_0^a\int_0^x\int_1^{x+y} e^{x+y+z}\,dx\,dy\,dz$

$$I=\int_0^a\int_0^x\int_1^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$$\int_0^a\int_0^x \left[e^{x+y+z}\right]_0^{x+y} dy\,dx$$

$$\int_0^a \int_0^x \left[e^{(x+y+x+y)} - e^{(x+y)+0} \right] dy\,dx$$

$$\int_0^a \int_0^x \left[e^{2x+2y} - e^{x+y} \right] dy\,dx$$

25/03/2…

Session 15 & 16

Beta and Gamma relationships

① Gamma function:- Gamma function is denoted by "Γ" and can be defined by the improper

→ integral which is dependent on parameter 'n',

$$\boxed{\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt}, \quad n>0$$

→ Gamma function also known as Euler's integral of second kind

$$\Gamma(n+1) = \int_0^\infty e^{-t} \cdot t^{n+1-1}\, dt$$

$$\Gamma(n+1) = \int_0^\infty e^{-t} \cdot t^{n}\, dt \qquad \therefore \int uv\,dx = u\int v\,dx + v\int u\,dx$$

$$\boxed{\int u \cdot v\, dx = u\int v\, dx - \int \left[\left(\frac{d}{dx}u\right) \int v\, dx\right] dx}$$

$u = t^n \quad v = e^{-t}$

$$\int_0^\infty t^n \cdot e^{-t}\, dt = \left[\frac{t^n e^{-t}}{-1}\right]_0^\infty - \int_0^\infty n t^{n-1} \int -e^{-t} \,]\, dt$$

$$\left[e^{-\infty} + 0\right] + n\int_0^\infty e^{-t} \cdot t^{n-1}\, dt$$

$$\boxed{\Gamma(n+1) = n\Gamma n}$$

Note - I :- If 'n' is a positive integer, then by repeated application of a formula $\Gamma n+1 = n\Gamma n$, we can get

$\Gamma n+1 = n\Gamma n$

$\Gamma n-1 = n(n-1)\Gamma(n-1)$

$\Gamma n-2 = n(n-1)(n-2)\Gamma(n-2)$

$\sqrt{n+1} = n(n-1)(n-2)\ldots 3.2.1$

$\sqrt{(n+1)} = \angle n$ or $n!$

Note - II If n is a positive fraction then by repeated application of above formula $\sqrt{(n+1)} = n!$, we can get

$$\sqrt{n} = (n-1)(n-2)\ldots$$

these series of factors continued so long as the factors remains positive, multipled by

$\sqrt{n+1} = n\sqrt{n}$

$n+1 = 11/4$

Example:- $\Gamma(11/4)$

Given, $n = \frac{11}{4}$

$n+1 = \frac{11}{4}$

$n = \frac{11}{4} - 1 = 7/4$ ✓ $\boxed{\therefore n = 7/4}$

$\sqrt{(n+1)} = n\sqrt{n}$

$= \frac{7}{4}\sqrt{\frac{7}{4}}$

$n+1 = \frac{7}{4}$

$n = \frac{7}{4} - 1$

$\sqrt{(n+1)} = n\sqrt{n}$

$= \frac{7}{4}\,\frac{3}{4}\sqrt{\frac{3}{4}}$,,

$n = \frac{3}{4}$ ✓

$\sqrt{(n+1)} = n\sqrt{n}$

$n+1 = \frac{3}{4}$

$n = \frac{3}{4} - 1$

$n = 3-4/4 = -1/4$ ✗

→ gamma value is always +ve

Note - III :- If n is negative, then $\boxed{\sqrt{n} = \frac{\sqrt{n+1}}{n}}$

Note - IV :-

$\sqrt{\tfrac{1}{2}} = \sqrt{\pi}$

$= \Gamma\frac{9}{2}$

$\Gamma n+1 = n\Gamma n$ $\quad n+1 = \frac{9}{2}$

$\Gamma\frac{9}{2} = \frac{7}{2}\Gamma\frac{7}{2}$ $\quad n = \frac{9}{2} - 1 = \frac{7}{2}$

$\Gamma\frac{7}{2} = \frac{7}{2}\frac{5}{2}\Gamma\frac{5}{2}$ $\quad \frac{7}{2} - 1 = \frac{5}{2}$

$\frac{5}{2} - 1 = \frac{3}{2}$

$\Gamma\frac{5}{2} = \frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma\frac{3}{2}$ $\quad \frac{3}{2} - 1 = \frac{1}{2}$

$\Gamma\frac{3}{2} = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2} \times \pi$

b) $\Gamma(-3.5)$ $\quad \Gamma n = \frac{\Gamma n+1}{n}$

$n+1 = -3.5+1$ $\quad n+1 = -2.5$ $\quad n+1 = -1.5$

$= -2.5$ $\quad 1+n = -2.5+1$ $\quad 1+n = -1.5+1$

$1+n = -1.5$ $\quad n+1 = -0.5$

$\Gamma(-3.5) = \frac{\Gamma -2.5}{(-3.5)} = \frac{\Gamma(-1.5)}{(-3.5)(-2.5)} = \frac{\Gamma 0.5}{(-3.5)(-2.5)(-1.5)(-0.5)}$

c) $\Gamma(6)$

$\Gamma(n+1) = n!$

$\Gamma 6 = 5! = 120.$

(2) Evaluate $\int_0^\infty x^5 e^{-x} dx$

$n-1 = 5$

$\Gamma n = \int_0^\infty e^{-t} t^{n-1} dx$ $\quad n = 6.$

$\Gamma n = \Gamma(n+1) = n!$

$\int_0^\infty e^{-x} x^{6-1} dx = \Gamma 6$

$= 5!$

③ $\int_0^\infty x^6 e^{-3x}\, dx$

Substitute $3x = t$

$3dx = dt$

$dx = \frac{1}{3} dt$

$x = \frac{1}{3} t$

if $x \to 0,\ t \to 0$

$x \to \infty,\ t \to \infty$

$$\int_0^\infty \left(\frac{1}{3}t\right)^6 \cdot e^{-t} \left(\frac{1}{3}dt\right)$$

$$\frac{1}{3^7}\int_0^\infty e^{-t} t^6\, dt$$

$$\frac{1}{3^7}\int_0^\infty e^{-t} t^{7-1}\, dt \Rightarrow \frac{1}{3^7}\Gamma 7$$

$n = \Gamma 7 = 6!$

$\frac{1}{3^7} 6!$

Beta functions

Beta function can be defined as $\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx}$

where, $m > 0$, $n > 0$

This function is also known as Euler's integral of 1st kind

Note - $\beta(m,n) = \beta(n,m)$

Beta function in trignometric form

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

another form of Beta function

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p x \cdot \cos^q x \, dx.$$

Relationship between Beta and Gamma functions:-

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$$

(6) $\int_0^1 x^4 (1-x)^3 \, dx$

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$m-1=4 \quad n-1=3$

$m=5 \quad n=4$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \frac{\Gamma 5 \, \Gamma 4}{\Gamma(5+4)} = \frac{\cancel{4!}\, 3!}{8\times\cancel{7}\times 6\times 5\times\cancel{4!}}$$

$$= \frac{\cancel{3}\times\cancel{2}\times 1}{8\times 7\times\cancel{6}\times 5} = \frac{1}{35\times 8} = \frac{1}{280}$$

(7) $\int_0^3 \frac{1}{\sqrt{9-x^2}} dx$

$$I = \int_0^3 \frac{1}{\sqrt{9}\left(\sqrt{1-\frac{x^2}{9}}\right)} dx = \frac{1}{3}\int_0^3 \left(1-\left(\frac{x}{3}\right)^2\right)^{-1/2} dx$$

$t = \left(\frac{x}{3}\right)^2 \Rightarrow x/3 = \sqrt{t} \Rightarrow dt = \frac{1}{9} 2x \, dx$

$\Rightarrow x = 3\sqrt{t}$

$dx = \frac{9}{2x} dt \quad \frac{9}{2} \cdot \frac{1}{3\sqrt{t}} dt$

If $x=0 \quad t \to 0$

$x=3 \quad t \to 1$

$$\int_0^1 (1-t)^{-1/2} \cdot \frac{\cancel{9}^{\,3}}{2} \cdot \frac{1}{\cancel{3}\sqrt{t}} \, dt$$

$$\frac{3}{2}\int_0^1 t^{-1/2} \cdot (1-t)^{-1/2} \, dt$$

compare with $\boxed{\int_0^1 x^{m-1} (1-x)^{n-1} dx}$

$m-1 = \frac{-1}{2}$ $\quad n-1 = -1/2$

$m = \frac{-1}{2}+1$ $\quad \boxed{n = 1/2}$

$\boxed{m = 1/2}$

$$\frac{3}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2}\,dt = \frac{3}{2}\,\beta\left(\frac{1}{2},\frac{1}{2}\right)$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$m = 1/2$ $\quad n = 1/2$

$$\frac{3}{2}\cdot\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}}$$

$$= \frac{3}{2}\sqrt{\pi}\cdot\sqrt{\pi}$$

$$= \frac{3}{2}\pi$$

(8) $$\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta\,d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$2m-1 = 5$ $\quad 2n-1 = \frac{7}{2}$

$2m = 6$ $\quad 2n = \frac{7}{2}+1$

$m = 3$ $\quad 2n = \frac{9}{2}$

$n = \frac{9}{4}$ $= \cancel{\frac{9-4}{2} = \frac{5}{2}}$

4) $\frac{9}{8}$ 12.2

275 ($\frac{10}{8}$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{(\Gamma m+n)} = \frac{\Gamma 3\,\Gamma\frac{9}{4}}{\Gamma 3+\frac{5}{2}} = \cancel{\frac{\Gamma 3\,\Gamma 2.2}{\Gamma\frac{11}{2}}}$$

$\cancel{\Gamma 2.2} = \cancel{\boxed{n\Gamma n = \Gamma n+1}}$ $\quad \Gamma\frac{9}{4} = $ $\quad n+1 = \frac{9}{4}$

$\cancel{n+1} = \cancel{2.2}$ $\quad n = \frac{5}{4}$

$\cancel{n} = 1.2$ $\quad \Gamma\frac{9}{4} = \frac{5}{4}\Gamma\frac{5}{4}$

$$\frac{\Gamma 3\,\Gamma\frac{9}{4}}{\Gamma\frac{11}{2}}$$

$$\boxed{n\Gamma n = \Gamma n+1}$$

$\Gamma\frac{9}{4} =$ $n+1 = \frac{9}{4}$

$n = \frac{9}{4} - 1$

$\Gamma\frac{9}{4} \to \frac{5}{4}\Gamma\frac{5}{4}$ $n = \frac{5}{4}$

$\to \frac{5}{4}\cdot\frac{1}{4}\Gamma\frac{1}{4}\pi$ $n+1 = \frac{5}{4}$

$n = 1/4$

$\Gamma\frac{11}{2} = \frac{9}{2}\Gamma\frac{9}{2}$ $n+1 = \frac{11}{2}$

$n = \frac{11}{2} - 1$

$= \frac{9}{2}\cdot\frac{7}{2}\Gamma\frac{7}{2}$ $n = \frac{9}{2}$

$= \frac{9}{2}\cdot\frac{7}{2}\,\frac{5}{2}\Gamma\frac{5}{2}$ $n+1 = \frac{9}{2}$

$= \frac{9}{2}\cdot\frac{7}{2}\,\frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$ $n = \frac{9}{2} - 1$

$n = 7/2$

$= \frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} \times \pi$ $n+1 = 7/2$

$n = \frac{7}{2} - 1 = 5/2$

$n = \frac{5}{2} - 1 = \frac{3}{2}$

$n = \frac{3}{2} - 1 = \frac{1}{2}$

26/08/24

(9) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = I$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$\int_0^{\pi/2} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta \to \int_0^{\pi/2} \cos^{1/2}\theta \cdot \sin^{-1/2}\theta\, d\theta$$

$$\boxed{\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta\, d\theta}$$

$p = -1/2$ $q = 1/2$

$\frac{p+1}{2} = \frac{-\frac{1}{2}+1}{2} = \frac{\frac{-1+2}{2}}{2} = 1/4$

$\frac{q+1}{2} = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$ $= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma\frac{1}{4}+\frac{3}{4}}$$ //

$$* \int_0^{\pi/2} \sin^n x \, dx = \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \cdot \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

(10) $I = \int_0^{\pi/2} \cos^6\theta \, d\theta$

$$\boxed{\int_0^{\pi/2} \cos^n x \, dx = \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \frac{\sqrt{\pi}}{2}}$$

$n=6$ $\quad \frac{\Gamma\frac{6+1}{2}}{\Gamma\frac{6+2}{2}} \frac{\sqrt{\pi}}{2} = \frac{\Gamma\frac{7}{2}}{\Gamma\cancel{\frac{8}{2}}4} \cdot \frac{\sqrt{\pi}}{2} = \frac{\Gamma 3.5}{\Gamma 4} \cdot \frac{\sqrt{\pi}}{2}$

$\Gamma(n+1) = n\Gamma n$

$2.5\Gamma 2.5$

$= 2.5 . 1.5\Gamma 1.5$

$= 2.5.1.5.0.5\Gamma 0.5$

$= 5/2 . 3/2 . 1/2 \sqrt{\pi}$

$n+1 = 3.5$

$n = 2.5$

$n+1 = 2.5$

$n = 1.5$

$n+1 = 1.5$

$n = 0.5$

$\frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{3 \times 2 \times 1} \times \frac{\sqrt{\pi}}{2}$

$\frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\cancel{3} \times 2 \times 1} \times \frac{1}{2}$

$\Gamma(n+1) = n\Gamma n$

$(n+1) = 4$

$n = 3$

$n+1 = 3$

$n = 2$

$n+1 = 2$

$n = 1$

$\Rightarrow 3\Gamma 3$

$3.2\Gamma 2$

$3.2.1\Gamma 1$

$$\boxed{I = \frac{5\pi}{32}}$$

④ $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$

$\therefore \Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$

sub $t = \log \frac{1}{x}$

$\frac{1}{x} = e^t$

$\boxed{\therefore \log 1 = 0}$

$x = e^{-t}$

$dx = -e^{-t} dt$

$\boxed{\therefore \left[\int_a^b f(x) dx = - \int_b^a f(x) dx \right]}$

$x \to 0 \quad t \to \infty$

$x \to 1 \quad t \to 0$

$\int_\infty^0 t^{n-1} (-e^{-t}) \, dt \Rightarrow \int_0^\infty e^{-t} \cdot t^{n-1} dt = \Gamma n$

$\sqrt{\pi}$