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Signals & Communication Systems
(24EC2105)

Lesson-1a Fundamentals of Signals

1.1 Introduction to signals and systems:

- 1.1(a) Signals
- 1.1(b) Systems

1.2 Classifications of Signals

- 1.2(a) Continuous Time (C.T), Discrete Time (D.T) and Digital signals
- 1.2(b) Deterministic versus Random Signals
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1.3 Representation of basic elementary continuous time signals

- 1.3(a) Sinusoidal signals
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1.4. Basic Operations on signals

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1.6 References

Lesson-1: Basic concepts of Signals

1.2 **Introduction to signals and systems:** The concept of signals and systems arise in an extremely wide variety of fields and the ideas and techniques associated with these concepts play an important role in such diverse areas of science and technology as communications, Aeronautics and Astronauts, Circuit design, Acoustics, Seismology, Bio-medical engineering, Energy generation and distribution systems, Chemical process control and Speech processing etc. Although the physical nature of the signals and systems that arise in these various disciplines may be drastically different, they all have two very basic features in common.

1.1(a) Signals: A signal may be defined as any physical quantity that varies with time, space or any other independent variable or variables. The signals are functions of one or more independent variables and typically contain information about the behavior or nature of some phenomenon.

Mathematically, we describe a signal as a function of one or more independent variables.

$x_1(t) = 5t$ ------ (1) varies linearly with independent variable '*t*'.

$x_2(t) = 16t^2$ ------ (2) varies quadratically with independent variable '*t*'.

$f(x, y) = 3x^2 + 2xy + 10y^2$ ------ (3) This function describes a signal of the two independent variables 'x' and 'y' that could represent the two spatial co-ordinates in a plane.

The signals described in equations (1), (2) and (3) belongs to a class of signals that are precisely defined by specifying the functional dependence on the independent variables.

Other examples of signals:

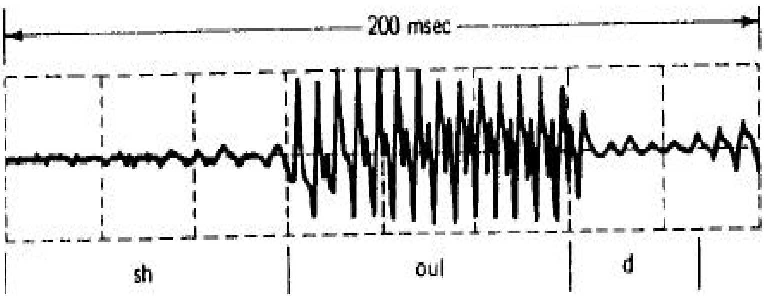


Fig. 1 (a) Speech Signal (1D signal) **(b) An image (2D signal)**

We shall often call the terms '*signals*' and '*functions*' interchangeably. A *signal* is a function of time. A function can be a multi-valued signal of variable time '*t* '.

1.1(b) Systems: A system may be defined as a physical device that performs an operation on a signal. A system can be viewed as any process that results in the transformation of signals. The system respond to the input signal and produce an output signal according to the characteristics of the system as shown in Fig.2. Thus a system has an input signal, an out put signal which is related to the input through the system transformation by $y(t) = \mathrm{T}\{x(t)\}$.

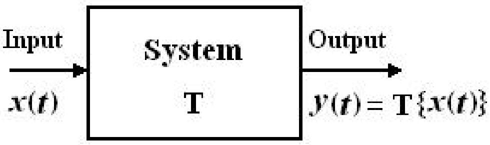


Fig.2 System

Examples: * A Digital Computer.

* A Public Address System.

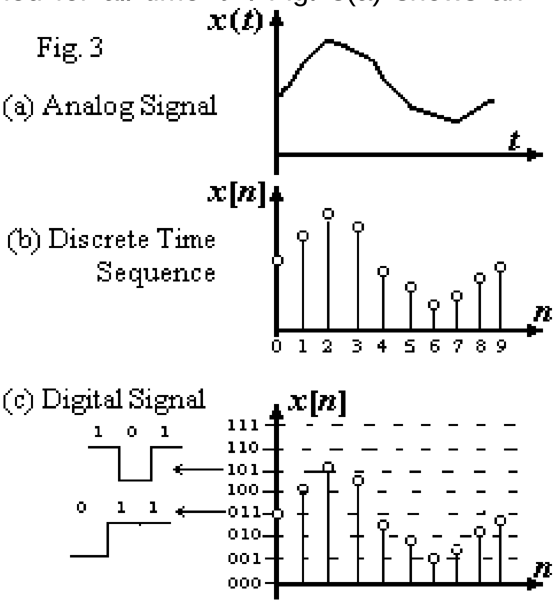
* A filter is used to reduce the noise and interference.

1.2 Classifications of Signals:

Signals are classified into different categories depending on the characteristics of time (independent) variable and the values they take.

1.2(a) Continuous Time (C.T), Discrete Time (D.T) and Digital signals: A signal is said to be ***continuous time*** or ***Analog* signal** if it is defined for all time '*t* '. Fig. 3(a) shows an example of continuous-time signal whose amplitude or value varies continuously with time. On the other hand a discrete time signal is defined only at discrete instances of time. A discrete time signal is often derived from analog signal by sampling (time axis) at a uniform rate and usually represented by the sequence numbers. Fig. 3(b) shows an example of discrete time signal derived from the analog signal. If these sampled or discrete time signals are further sampled or quantized in magnitude and specified in a particular binary format (ex 3-bit, 4-bit, 8-bit etc) is called digital signal. In digital signal each bit is either logic-1 or logic-0.

Fig. 3



1.2(b) Deterministic versus Random Signals: Any signal that can be uniquely described by an explicitly mathematical expression, a table of data or a well defined rule is called *deterministic signal*. The term deterministic is used to emphasize the fact that all past, present and future values of the signal are known precisely, without any uncertainty. $f(t) = 5t$, $x(t) = 2\cos(200\ t)$ are examples for deterministic signals. Some signals that either can not be described to any reasonable degree of accuracy by explicit mathematical formulas. Or such description is too complicated to be of any practical use. The lack of such relationship implies that such signals evolve in time in an unpredictable manner. Radio, Television, Mobile, the output of a noise generator and seismic signals are examples of *Random signals*.

1.2 (c) Real and Complex Signals: A signal $x(t)$ is a real signal if its value is a real number and a complex signal if its value is a complex number.

Example 1(a) $x(t) - 2\cos(200\pi t)$ is a real valued signal.

(b) $x(t) = 2e^{j200\pi t} = 2[\cos(200\pi t) + j\sin(200\pi t)]$ is a complex valued signal.

1.2(d) Vector Signal: Vector or Multi channel signal. $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

1.2(e) Causal and non-causal signals: A causal signal is defined as the signal having the values for $t > 0$, otherwise non causal signals. All right sided signals are referred to as causal signals. The following figure illustrates the causal and non-causal signals.

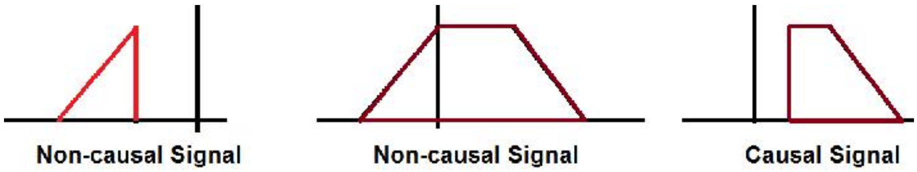


Fig.4 Causal ans non-causal signal

1.2(f) Periodic versus Non-Periodic Signals: A signal $x(t)$ is said to be periodic if the equation $x(t+nT) = x(t)$ (where *n* is an integer) for all 't ' is satisfied. Any signal that does not satisfy the above equation is known as non-periodic or aperiodic signal

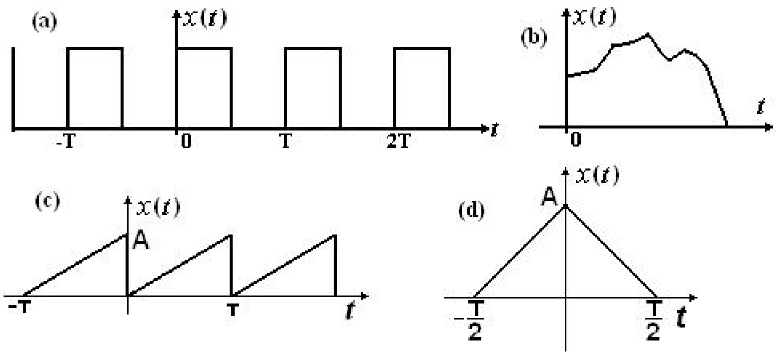


Fig.5 (a) and (c) Periodic CT Signal (b) and (d) Aperiodic CT signal

Example 2. Determine whether the signals shown in **Fig.6** is periodic or aperiodic signal. In the case of periodic signals find the fundamental periods.

Ans: CT Periodic signal, $T = 0.2$ sec.

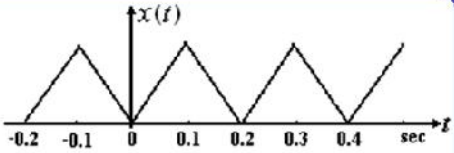


Fig.6 Periodic CT Signal

1.2(g) Energy and Power Signals: Aperiodic signals such as pulse type signals are usually referred to as *energy signals*. The signals having infinite energy such as periodic signals are called power signals.

The normalized energy content '*E*' of a signal $x(t)$ is defined as $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$.

The normalized average power 'P' of a signal is defined as $P = \underset{T\to\infty}{Lt} \frac{1}{T}\int_0^T |x(t)|^2 \, dt$.

If $0 < E < \infty$, i.e., if E is finite (so that $P = 0$) then $x(t)$ is referred to as an *Energy Signal*.

If $E = \infty$, but $0 < P < \infty$, *i.e.*, P is finite, then $x(t)$ is referred to as an *Power Signal*.

Example3: Determine whether a signal is defined by $x(t) = A\sin t$ is an energy signal or power signal. Find its value.

Ans: Since $x(t) = A\sin t$ is a sinusoidal signal and is a periodic signal, it is a power signal. By comparing with standard signal

$$x(t) = A\sin(\Omega_0\, t + \theta), \Omega_0 = \frac{2\pi}{T} = 1 \; or \; T = 2\pi .$$

Therefore $x(t + nT) = A\sin(t + n2\pi) = A\sin t = x(t)$.

Power $P = \frac{1}{T}\int_0^T |x(t)|^2 \, dt = \frac{1}{2\pi}\int_0^{2\pi} A^2 \sin^2 t \, dt = \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1 + \cos 2t) \, dt = \frac{A^2}{2} watts$.

1.2(g) Even (or Symmetric) and Odd (anti-symmetric) signals:

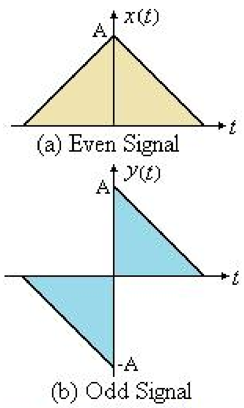
If a signal satisfies the condition $x(-t) = x(t)$, the signal is said to be an even or symmetric signal.

Similarly, a signal satisfies the relation $x(-t) = -x(t)$, the signal is called an odd or anti-symmetric signal.

Any signal can be represented by sum of an even signal $x_e(t)$ and an odd signal $x_o(t)$.

i.e., $x(t) = x_e(t) + x_o(t)$ (1)

For even signal $x_e(-t) = x_e(t)$ and for odd signal $x_o(-t) = -x_o(t)$.



Graphical representation of even and odd signals

By substituting $t = -t$ in eq(1), we get

$$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t) \qquad (2)$$

From equations (1) and (2) we get

$$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\} \text{ and } x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

From these signals, we can show that $x_e(t) + x_o(t) = \frac{1}{2}\{x(t) + x(-t)\} + \frac{1}{2}\{x(t) - x(-t)\} = x(t)$

In can be observed that, even x even = even, odd x odd = even and even x odd = odd.

Example4: Determine the even and odd signals of a signal

$$x(t) = \cos t + \sin t + \cos t \sin t .$$

Answer: $x_e(t) = \cos t$ and $x_o(t) = \sin t + \cos t \sin t$.

Even and odd components of complex valued signals: In the case of a complex valued signals, we may speak of conjugate symmetry. A complex valued signal $x(t)$ is said to be conjugate symmetric if it satisfies the equation

$$x(-t) = x^*(t)$$

where the asterisk denotes complex conjugation.

Let $x(t) = x_{\text{Re}}(t) + j\, x_{Im}(t)$, where $x_{\text{Re}}(t)$ is the real part of $x(t)$, and $x_{Im}(t)$ is the imaginary part of $x(t)$.

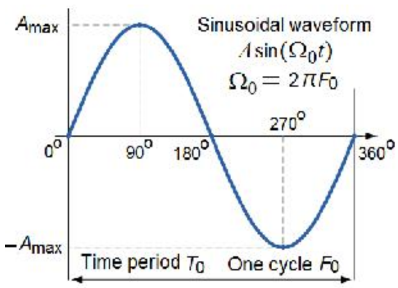
The complex valued signal is represented by $x^*(t) = x_{\text{Re}}(t) - j\, x_{Im}(t)$.

Then $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\} = \frac{1}{2}\{x(t) + x^*(t)\}$

$$= \frac{1}{2}\{(x_{\text{Re}}(t) + j\, x_{Im}(t)) + (x_{\text{Re}}(t) - j\, x_{Im}(t))\} = x_{\text{Re}}(t)$$

Therefore it follows that a complex valued signal $x(t)$ is conjugate symmetric if its real part is even and its imaginary part is odd.

1.3 Representation of basic elementary continuous time signals



1.3(a) Sinusoidal signals: The graphical representation of sinusoidal signals is shown in the following figure. A standard mathematical representation of sinusoidal signal is given by $x(t) = A\cos(\Omega_0 t + \quad)$ or $x(t) = A\sin(\Omega_0 t + \quad)$, where A is the peak or maximum value of the signal, Ω_0 is the angular $\Omega_0 = 2\ \ F_0 = \frac{2}{T_0}$), and is the phase angle measured in radians.

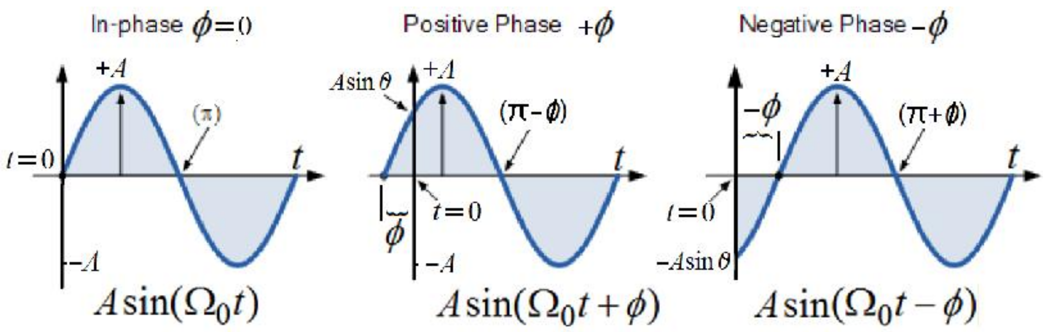


Fig Representation of sinusodal signals

$$A_{av} = \frac{2}{\pi}A_{max} = \frac{2}{\pi}A_{peak} = 0.637A_{peak}$$, and

R.M.S. value = $A_{rms} = \dfrac{A_{max}}{\sqrt{2}}$

Example 5. Characterize the sinusoidal signal $x(t) = \sqrt{2}\cos(\pi t/2 + \pi/4)$

Solution: The signal $x(t)$ is

- Deterministic, as the value of the signal can be obtained for any possible value of *t*.
- Analog, as there is a continuous variation of the time variable *t* from $-\infty$ to ∞, and of the amplitude of the signal between $-\sqrt{2}$ to $\sqrt{2}$.
- Of infinite support, as the signal does not become zero outside any finite interval.

The amplitude of the sinusoid is $\sqrt{2}$, its frequency is $\Omega_0 = 2\pi F_0 = \frac{\pi}{2}$ rad/sec (or $F_0 = \frac{1}{4}$ Hz), and its phase is $\phi = \pi/4$ rad. Because of the infinite support, this signal cannot exist in practice, but we will see that sinusoids are extremely important in the representation and processing of signals.

```
% SA1_1.m Continuous Time Co-sinusoidal signals x(t)= A*cos(wo*t+theta)
% theta = pi/4; frequemcy = 100 Hz. Amplitude = 2

t = (-pi:0.001:pi)/100;
A = 2; theta = pi/4; f = 100;
x = A*cos(2*pi*f*t + theta);
figure();plot(t,x,'m','LineWidth',1.5);grid on
set(gca,'fontsize',14)
xlabel('time  ----->','FontSize',14);ylabel('Amplitude -----
>','FontSize',14);
title('C.T. Sinusoidal signal A*cos(2*pi*100*t + pi/4)','FontSize',14);
axis([-0.03, 0.03,-2.15 2.15]);
```

1.3(b) Exponential Signals: The Euler's identity provides the relation of the sinusoids with the complex exponential

$$e^{j} = \cos + j\sin$$

Replacing with – yields

$$e^{-j} = \cos(-) + j\sin(-) = \cos - j\sin ,$$

since the cosine function is even and the sine function is odd. The sum of e^{j} and e^{-j} can be expressed as

$$\cos = \frac{e^{j} + e^{-j}}{2}$$

and the difference of e^{j} and $e^{\ j}$

$$\sin = \frac{e^{j} - e^{-j}}{2j}$$

These relations are so useful in signal and system analysis that they should be memorized. The complex exponential can also be expressed in polar form as

$$e^{j} = 1\angle$$

where the notation $R\angle$ signifies the complex function of magnitude R at the angle .

To prove consider,

$$|e^{j}| = |\cos + j\sin | = \sqrt{\cos^2 + \sin^2} = 1$$

and

$$\arg e^{j} = \tan^{-1}\left[\frac{\sin}{\cos}\right] =$$

where $\arg\{\bullet\}$ denotes the angle of $\{\bullet\}$.

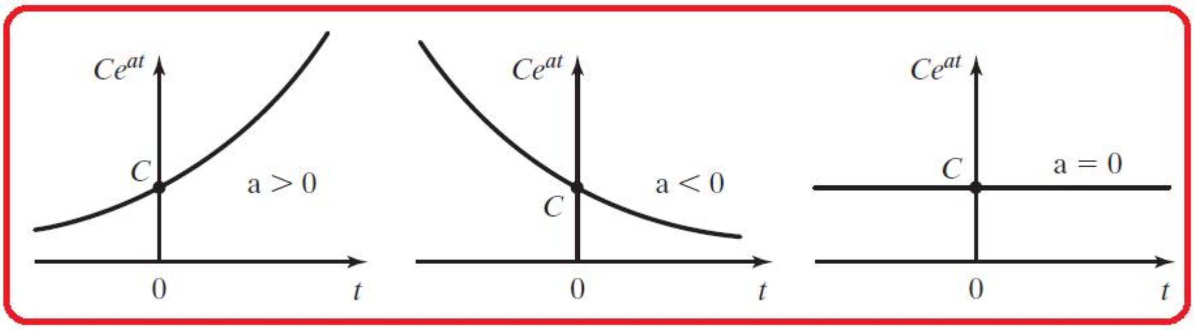
In the signal processing, it is more convenient to represent a complex exponential signals as

$$x(t) = Ce^{at}$$

where the parameters C and a are real or complex. Three cases for exponential functions will now be investigated.

Case1: C and a are real:

Consider for the first case, both C and a are real for the exponential $x(t) = Ce^{at}$. The product (at) is unit less; hence, the units of a are the reciprocal of those of t. The units of C are the same as those of $x(t)$. The signal $x(t) = Ce^{at}$ is illustrated below for $C > 0$ with $a > 0$, $a < 0$ and $a = 0$.



- For $a > 0$ the signal magnitude increases monotonically without limit with increasing time.
- For $a < 0$ the signal magnitude decreases monotonically toward zero as time increases.
- For $a = 0$, the signal is constant.

Case2: C is complex and a is imaginary:

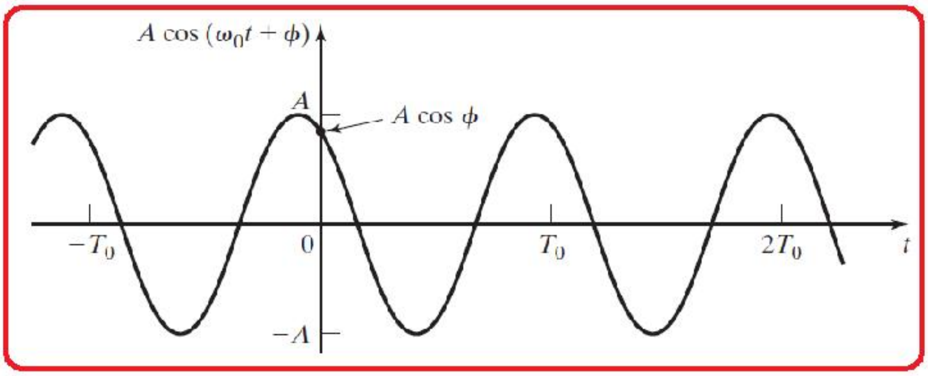
Next we consider the case that C is complex and a is imaginary-namely,

$$x(t) = Ce^{at}; \quad C = Ae^{j\ } = A\angle\ , \quad a = j\Omega_0$$

where A, and Ω_0 are real and constant. The complex exponential signal $x(t)$ can be expressed as

$$x(t) = Ce^{at} = Ae^{j\ } e^{j\Omega_0 t} = Ae^{j(\Omega_0 t +\ )}$$
$$= A\cos(\Omega_0 t +\ ) + jA\sin(\Omega_0 t +\ )$$

The sinusoids are periodic, with frequency Ω_0 and period $T_0 = 2\ /\Omega_0$. Hence, the complex exponential is also periodic. The real part of this complex exponential signal is shown below.



Case3: Both C and a are complex:

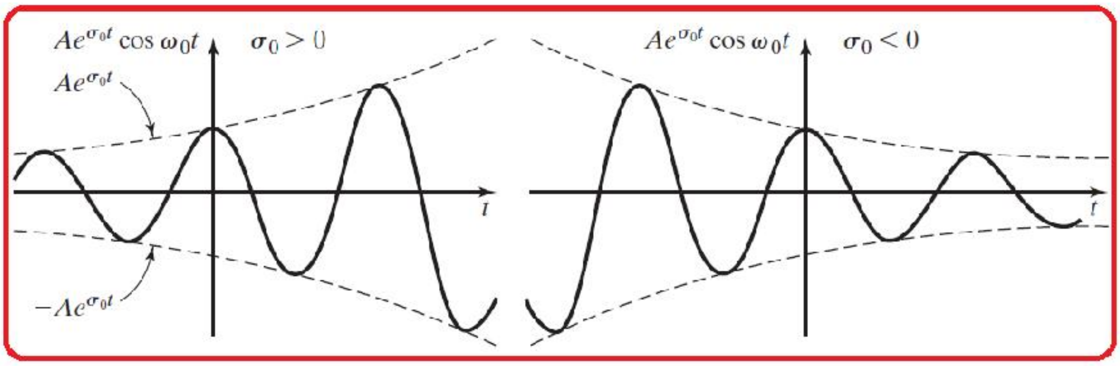
For this case $x(t) = Ce^{at}$, the complex exponential has the parameters

$$x(t) = Ce^{at}; \quad C = Ae^{j\ }, \quad a = \ _0 + j\Omega_0$$

where A , and Ω_0 are real and constant. Then the complex exponential signal $x(t)$ can be expressed as

$$
\begin{aligned}
x(t) &= Ce^{at} = Ae^{j} \; e^{(\;\; _0 + j\Omega_0 t)} = A\,e^{\;\; _0 t} \, e^{j(\Omega_0 t + \;\;)} \\
&= Ae^{\;\; _0 t} \cos(\Omega_0 t + \;\;) + jAe^{\;\; _0 t} \sin(\Omega_0 t + \;\;) \\
&= x_r(t) + j\, x_i(t)
\end{aligned}
$$

where $x_r(t)$ is the real part of $x(t)$, i.e., $x_r(t) = \mathrm{Re}\{x(t)\}$, and $x_i(t)$ is the imaginary part of $x(t)$, i.e., $x_i(t) = \mathrm{Im}\{x(t)\}$. The Real part of a complex exponential is illustrated in the



following figure.

Periodicity : By definition, a continuous-time signal $x(t)$ is periodic if

$$x(t) = x(t+T), \qquad T > 0 \tag{1}$$

for all t ,where the constant T is the period. A signal that is not satisfy the above equation is said to be *aperiodic*. In (1), we replace t with $(t+T)$ resulting in

$$x(t+T) = x(t+2T)$$

This equation is also equal to $x(t)$ from (1). By repeating this substitution, we see that a periodic function satisfies the equation

$$x(t+nT) = x(t)$$

where n is any integer. Hence, a periodic signal with period $T > 0$ is also periodic with period nT .

The minimum value of the period $T > 0$ that satisfies the definition $x(t+nT) = x(t)$ is called the *fundamental period* of the signal and is denoted as T_0 .

With T_0 in seconds, the fundamental frequency in hertz (the number of periods per second) and the fundamental frequency in rad/s are given by

$$f_0 = \frac{1}{T_0} \qquad \Omega_0 = 2 \;\; f_0 = \frac{2}{T_0} \text{ rad / sec respectively.}$$

A special case of a periodic function is that of $x(t)$ equal to a constant. A constant satisfies the definition $x(t) = x(t+T)$ for any value of T. Because there is no smallest value of T, the fundamental period of a constant signal is not defined. However, it is sometimes convenient to consider a constant signal A to be the limiting case of the sinusoid $x(t) = A\cos(\Omega_0 t)$ with Ω_0 approaching zero. For this case, the period T is unbounded.

Example 6. Test whether the following signals are periodic or not.

(a) $x(t) = e^{\sin t}$ is (b) $x(t) = te^{\sin t}$

Ans (a) $x(t+T) = e^{\sin(t+T)} = e^{\sin t} = x(t)$. Hence $x(t) = e^{\sin t}$ is periodic signal.

(b) $x(t+T) = (t+T)e^{\sin(t+T)} \neq x(t)$. Hence $x(t) = te^{\sin t}$ is aperiodic signal.

Periodicity of composite sinusoidal signals: The sum of continuous-time periodic signals is periodic if and only if the ratios of the periods of the individual signals are ratios of integers.

Let $x(t)$ and $y(t)$ are two periodic signals with fundamental periods of T_x and T_y respectively. If $z(t) = x(t) + y(t)$, the fundamental question is what is the condition for $z(t)$ to be periodic? If it is periodic, what is its fundamental period? The answers for these questions are addressed as below.

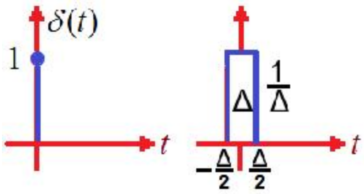
Suppose that $z(t)$ is periodic with period T_z. Since $z(t) = x(t) + y(t)$, both $x(t)$ as well as $y(t)$ must complete an integer number of their periods with in the period T_z. Suppose that $x(t)$ completes m periods and $y(t)$ completes n periods in time T_z, where m and n are integers, then $T_z = mT_x = nT_y$. Hence $\frac{T_x}{T_y} = \frac{n}{m}$ must be a rational number. Thus for $z(t)$ to be periodic. The condition to be satisfied is that $\frac{T_x}{T_y}$ must be a rational number.

Further, since $T_z = mT_x = nT_y$, where m and n are integers, when $z(t)$ is periodic with fundamental period T_z, m and n must be smallest integers satisfying the above relation. This means that $T_z = \mathrm{LCM}(T_x, T_y)$, where LCM is referred to as least common multiplier.

1.3(c) Singularity functions: In this section, we consider a class of functions called singularity functions. We define a *singularity function* as one that is related to the impulse function and associated functions unit step function and the unit ramp function.

A Unit Impulse Function: It is also called Dirac-delta function or simply delta function and is denoted as (t).

(Dirac from England was awarded the Nobel Prize for Physics in 1933.)



Unit Impulse function

Mathematically it is expressed as $(t) = \begin{cases} 1, \ for\ t = 0 \\ 0, \ for\ t \neq 0 \end{cases}$.

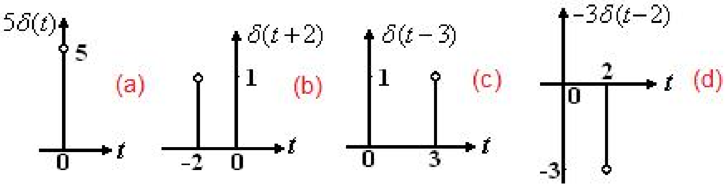
The delta function can be regarded as a pulse of width and height 1/ having unit area as shown in the figure.

Matlab code: `delta(t==0)=1;`

A generalized Matlab code for delta function:

```
function y = ud(t,k)
y = zeros(1,length(t));
y(t == k)=1;
```

Operations on delta function:

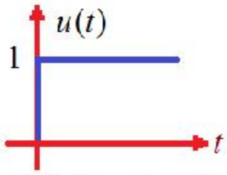


Matlab code for operations on delta function:

```
% SA1_2.m Matlab code for operations on delta function:
clear all; close all; clc;
t = -4:0.01:4;
y1 = 5*ud(t,0);  % Unit delta function
y2 = ud(t,-2);  % Left side shifted delta function
y3 = ud(t,3);  % Right side shifted delta function
y4 = -3*ud(t,2);  % Amplitude scaled right side shifted delta function
```

A Unit Step Function: The unit step function is defined by

$$u(t) = \begin{cases} 1, \ for\ t \geq 0 \\ 0, \ for\ t < 0 \end{cases}.$$



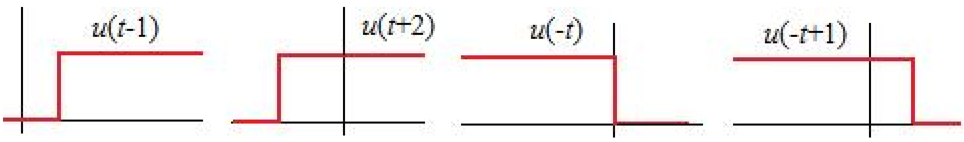
Unit Step function

Matlab code: `step(t>=0)=1;`

A generalized Matlab code for delta function:

```
function y = us(t,k)
%US Unit step
% y = us(t)
% t: time index
% y: Output step signal
y = (t >= k);
```

Operations on step function:



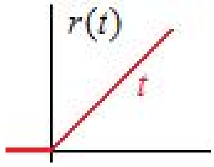
Matlab code for operations on step function:

```
% SA1_3.m Matlab code for operations on step function:
clear all; close all; clc;
t = -4:0.01:4;
y1 = us(t,1);
y2 = us(t,-2);
y3 = us(-t,0);
y4 = us(-t,-1);
```

The unit step function is also represented in terms of impulse function as $u(t)=\int_{-\infty}^{t} (\ )\, d$

A Unit Ramp function: The unit ramp function is

denoted by $r(t)$ and is defined by $r(t)=\begin{cases} t, \ for\ t \geq 0 \\ 0, \ for\ t<0 \end{cases}$

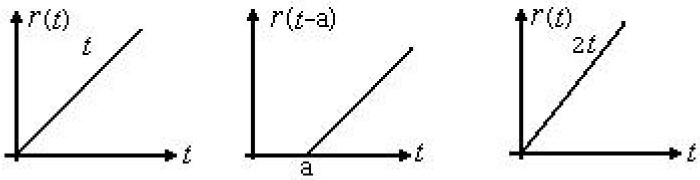


Matlab code: `r(t>=0)=t;`

A generalized Matlab code for delta function:

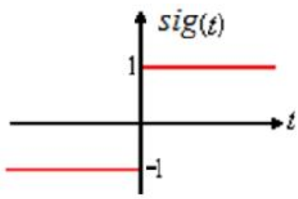
```
function y = ur(t,k)
% function y = ur(t)
%UR Unit Ramp
% y = us(t)
% t: time index
% y: output signal
y = (t-k).*us(t,k);
```

Operations on ramp function:



Signum Function: The signum function is described by the equation,

$$\operatorname{sgn}(t)=\begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases} = 2u(t)-1$$



Signum function

Therefore $u(t)=\frac{1}{2}\{1+\operatorname{sgn}(t)\}$ and $\operatorname{sgn}(t)=2u(t)-1$

```
% SA1_4.m Signum Function
% theta = pi/4; frequemcy = 100 Hz. Amplitude = 2

t = (-pi:0.001:pi)/100;
x = 2*us(t,0)-1;
z = zeros(size(t));

figure();plot(t,x,'m','LineWidth',3);grid on
title('Signum Function');
set(gca,'fontsize',14)
xlabel('time  ----->','FontSize',14);ylabel('Amplitude -----
>','FontSize',14);
title('C.T. Sinusoidal signal A*cos(2*pi*100*t + pi/4)','FontSize',14);
axis([-0.03, 0.03,-2.15 2.15]);
```

$$\frac{d}{dt}r(t)=u(t) \qquad \frac{d}{dt}u(t)= \quad (t) \ \ \text{(iii)} \ \ u(t)=\int_{-\infty}^{t} \ \ (\ \ )\,d \quad \text{(iv)} \ \ r(t)=\int_{0}^{t}u(t)\,dt$$

Properties of Singularity functions:

1. $(-t) = \quad (t)$
2. $(t) = \frac{d}{dt}u(t) \Rightarrow \quad (t-t_0) = \frac{d}{dt}u(t-t_0)$
3. $u(t-t_0) = \int_{-\infty}^{t} \quad (t-t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$
4. $\int_{-\infty}^{\infty} x(t) \quad (t-t_0) = x(t_0)$, $x(t)$, continuous at $t = t_0$
5. $\int_{-\infty}^{\infty} x(t-t_0) \quad (t) = x(-t_0)$, $x(t)$, continuous at $t = -t_0$
6. $x(t) \quad (t-t_0) = x(t_0) \quad (t-t_0)$, continuous at $t = t_0$

The impulse *signal* $\delta(t)$ is:

- Zero everywhere except at the origin where its value is not well defined (i.e., $\delta(t) = 0, t \neq 0$, and undefined at $t = 0$).
- its area is unity, i.e.,

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

The unit-step signal is
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

The $\delta(t)$ and $u(t)$ are related as follows:
$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$
$$\delta(t) = \frac{du(t)}{dt}$$

The ramp signal is defined as $r(t) = t\,u(t)$

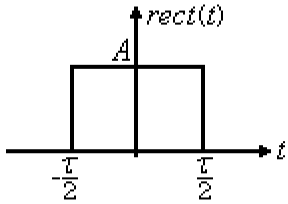
Its relation to the unit-step and the unit-impulse signals is

$$\frac{dr(t)}{dt} = \frac{dtu(t)}{dt} = u(t) + t\frac{du(t)}{dt} = u(t) + t\,\delta(t)$$
$$= u(t) + 0\,\delta(t) = u(t)$$

$$\frac{d^2r(t)}{dt^2} = \delta(t)$$

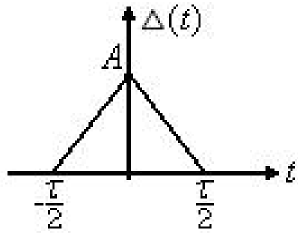
1.3(d) Other useful functions:

Gate function: A gate function is denoted by $rect(t)$ or $\Pi(t)$ of height A and width ,
centered at the origin. The mathematical expression is defined as



$$\mathrm{rect}\left(\frac{t}{\ }\right)=\begin{cases}A,\ for\ -\frac{\ }{2}<t<\frac{\ }{2}\\ 0,\qquad elsewhere\end{cases}=\begin{cases}A,\ for\ \ |t|\le\frac{\ }{2}\\ 0,\ for\ \ |t|>\frac{\ }{2}\end{cases}$$

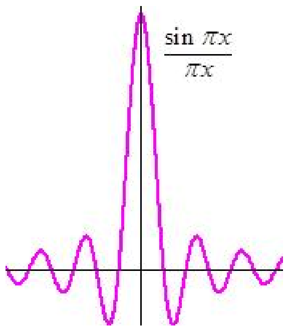
Triangular function: A triangular function is denoted by $\Delta(t)$, of height A and width ,
centered at the origin. The mathematical expression is defined as



$$\Delta(t)=\begin{cases}\frac{2A}{\ }t+A,\ for\ -\frac{\ }{2}<t<0\\ -\frac{2A}{\ }t+A,\ for\ 0<t<\frac{\ }{2}\end{cases}$$

Interpolation function: sinc(*x*) or Sa(x). The function $\frac{\sin x}{x}$ is the
'sine over argument' function denoted by sinc(*x*). It is also known as filtering or interpolating function.

- ❖ sinc(*x*) is an even function.
- ❖ Using L'Hospital' rule, sinc(0) = 1.
- ❖ sinc(*x*) is also denoted as Sa(x) in the literature. Some authors define as $\frac{\sin\ \ x}{x}$.

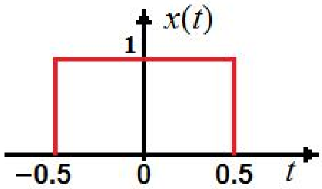



```
% SA1_5: Sinc Function
clear all; close all; clc;
t = (-pi:0.001:pi)/100;j = sqrt(-1);
A = 1;  f = 100;
x = A*sin(2*pi*f*t)./(2*pi*f*t);
figure();
plot(t,x,'m','LineWidth',3);%grid on
xlabel('time  ----->');ylabel('Amplitude ----->');
title('Sinc(2*pi*f*t)'); grid on;
axis([-0.03, 0.03,-0.4 1.15]);
```

1.3(e) Representation of signals in terms of singularity functions: Development of Matlab codes:

In this section we represent various signals in terms of functional form and in terms singularity functions. Further we discuss how to Matlab code for both methods.

Example 1: A rectangular pulse having unit width, unit height centered at origin. A graphical representation of this signal is shown in Figure.



The functional representation: $x(t)=\begin{cases}1,\ -\frac{1}{2}\le t\le\frac{1}{2}\\ 0,\ elsewhere\end{cases}$

Matlab code for functional representation:

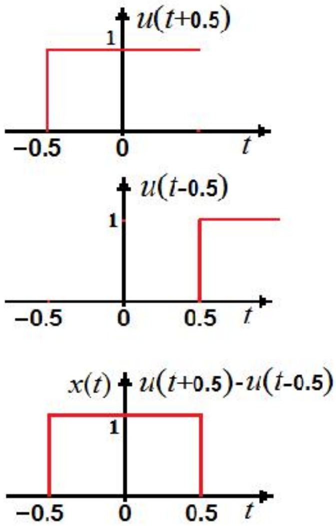
```
% SA1_6:  Rectangular function
clear all; close all; clc;
t = -3:0.01:3;
x = zeros(size(t));
% Defining Rectangular pulse from functional description
idx = find((t>-0.5) & (t<0.52));% defining the Rectangular
x(idx) = 1;

figure(); set(gca,'fontsize',14);
z = zeros(size(t));
plot(t, x,'r','LineWidth',3); grid on;
title('Rectangular Pulse');axis([-3 3 -0.1 1.2]);
xlabel('time  ----->');ylabel('Amplitude ----->');
```

Representation of signal in terms of step function: The rectangular signal can be represented as subtraction of two signals as shown in Figure.

$$x(t) = u(t+0.5) - u(t-0.5)$$

Matlab code:

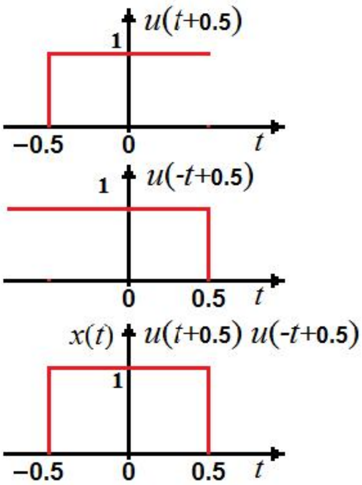



```
% SA1_7.m Rect function (Sum of two Step functions)
clear all; close all; clc;
t = -3:0.01:3;
x = zeros(size(t));
% Defining Rectangular pulse in terms of step
function
%by subtraction of two step signals
x = us(t,-0.5) - us(t,0.5);

figure(); set(gca,'fontsize',14);
z = zeros(size(t));
plot(t, z,'k','LineWidth',1.5); hold on;  % defining
x-axis as time axis
plot(z, t,'k','LineWidth',1.5); hold on; % defining
y-axis as amplitude of the signal
plot(t, x,'r','LineWidth',3); axis([-3 3 -0.1 1.2]);
title('Rectangular Function');
```

The rectangular signal can also be represented by multiplication of two step signals as illustrated in Figure.

$$x(t) = u(t+0.5)u(0.5-t)$$

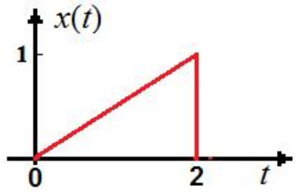



```
% SA1_8.m Rect function (Multiplication of two Step
functions)

clear all; close all; clc;
t = -3:0.01:3;
x = zeros(size(t));
% Defining Rectangular pulse in terms of step
function
% That is multiplication of two step signals
x = us(t,-0.5).* us(-t,-0.5);
 figure(); set(gca,'fontsize',14);
z = zeros(size(t));
plot(t, z,'k','LineWidth',1.5); hold on;  % defining
```

```
        x-axis as time axis
plot(z, t,'k','LineWidth',1.5); hold on; % defining y-axis as amplitude of
the signal
plot(t, x,'r','LineWidth',3);
title('Rectangular Pulse');axis([-3 3 -0.1 1.2]); grid on;
xlabel('time  ----->');ylabel('Amplitude ----->');
```

Example 2: Consider a sawtooth pulse shown in the Figure.



This signal is defined as the ramp function over the time period $0 < t < 2$.

The slope of this signal is $m = \frac{1}{2}$. Then the signal is represented by

$$x(t) = \begin{cases} \frac{1}{2}t, 0 \le t \le 2 \\ 0, elsewhere \end{cases}$$

Matlab code for functional representation:

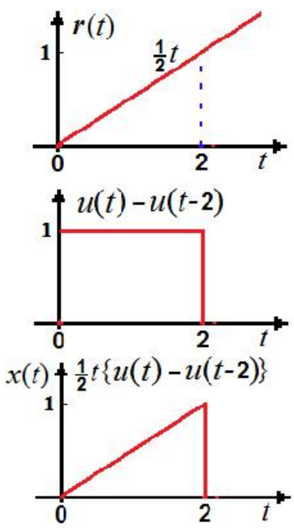
```
% SA1_9.m  Sawtooth wave
clear all; close all; clc;
t = -3:0.01:3;

x = zeros(size(t));
% Defining Sawtooth pulse from functional description
idx = find((t>0) & (t<2));% defining the Sawtooth pulse
x(idx) = 0.5*t(idx);

figure(); set(gca,'fontsize',14);
z = zeros(size(t));
plot(t, z,'k','LineWidth',1.5); hold on;  % defining x-axis as time axis
plot(z, t,'k','LineWidth',1.5); hold on; % defining y-axis as amplitude of
the signal
plot(t, x,'r','LineWidth',3);
title('Sawtooth Pulse');axis([-1 3 -0.1 1.2]); grid on;
xlabel('time  ----->');ylabel('Amplitude ----->');
```

Representation of sawtooth signal in terms of step signals: The saw tooth signal be represented in terms of step function as illustrated below and refer the figure.

$$x(t) = \frac{1}{2}t\left\{u(t) - u(t-2)\right\}$$



Matlab code:

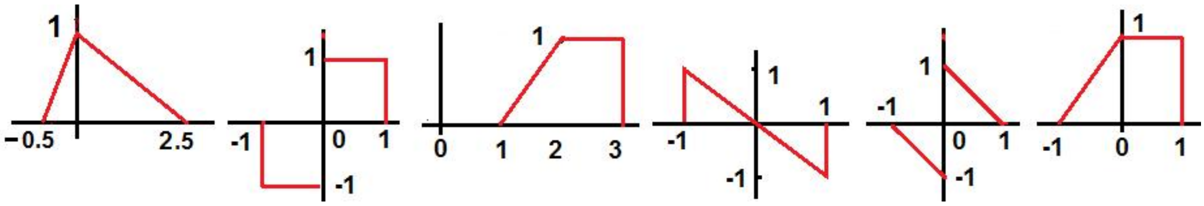
```
% SA1_10.m     Sawtooth pulse in terms of step function
clear all; close all; clc;
t = -3:0.01:3;
x = zeros(size(t));
% Defining Sawtooth pulse in terms of step function
x = 0.5*t.*(us(t,0) - us(t,2));

figure(); set(gca,'fontsize',14);
z = zeros(size(t));
plot(t, z,'k','LineWidth',1.5); hold on;  % defining x-
axis as time axis
plot(z, t,'k','LineWidth',1.5); hold on; % defining y-
axis as amplitude of the signal
```

```
        plot(t, x,'r','LineWidth',3);
title('Sawtooth Pulse');axis([-1 3 -0.1 1.2]); grid on;
xlabel('time  ----->');ylabel('Amplitude ----->');
```

Exercise:

1. Write mathematical expressions (a) functional representation, and (b) in terms of step functions for the following signals.



2. Sketch the following signals and represent in terms of step functions. Then develop Matlab codes and verify.

(i) $x(t) = \begin{cases} 5-2t, & 0 \le t \le 2 \\ 0, & elsewhere \end{cases}$ (ii) $x(t) = \begin{cases} t, & 0 < t < 1 \\ -t+2, & 1 < t < 2 \\ 0, & Otherwise \end{cases}$ (iii) $x(t) = \begin{cases} 1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \end{cases}$

1.4. Basic Operations on signals: The operations performed on signals can be broadly classified into two types: Operations on dependent variables, and Operations on independent variables.

1.4(a): Operations on dependent variables: The operations of the dependent variable can be classified into five types: amplitude scaling, addition, multiplication, integration and differentiation.

Amplitude scaling: Amplitude scaling of a signal $x(t)$ given by equation

$$y(t) = ax(t)$$

results in amplification of $x(t)$ if a >1, and attenuation if a <1.

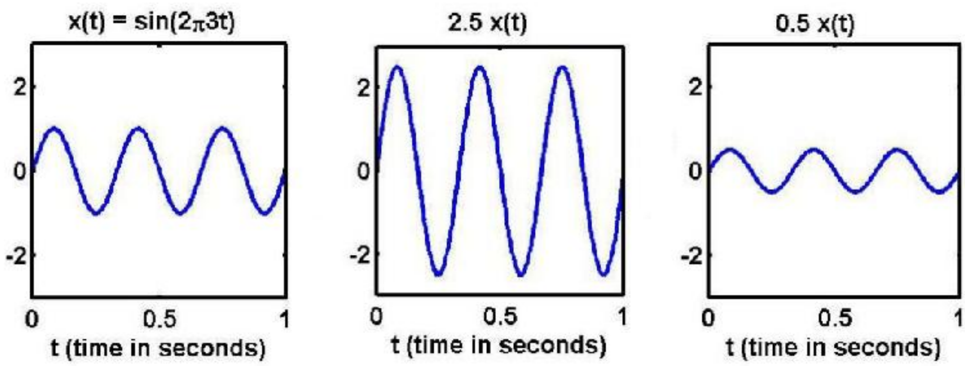


Fig. Amplitude scaling of sinusoidal signal

Signal Addition: The addition of signals is given by equation $y(t) = x_1(t) + x_2(t)$. An example of the addition of a sinusoidal signal with a signal of constant amplitude (positive constant) is illustrated below.

$$x_1(t) = \sin(2\ \ 4t)\,;\quad x_2(t) = 1.25 \Rightarrow x(t) = x_1(t) + x_2(t) = 1.25 + \sin(2\ \ 4t)$$

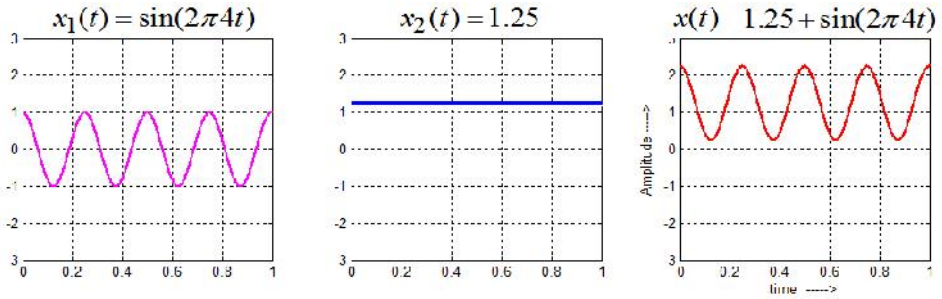


Fig. An example of signal addition

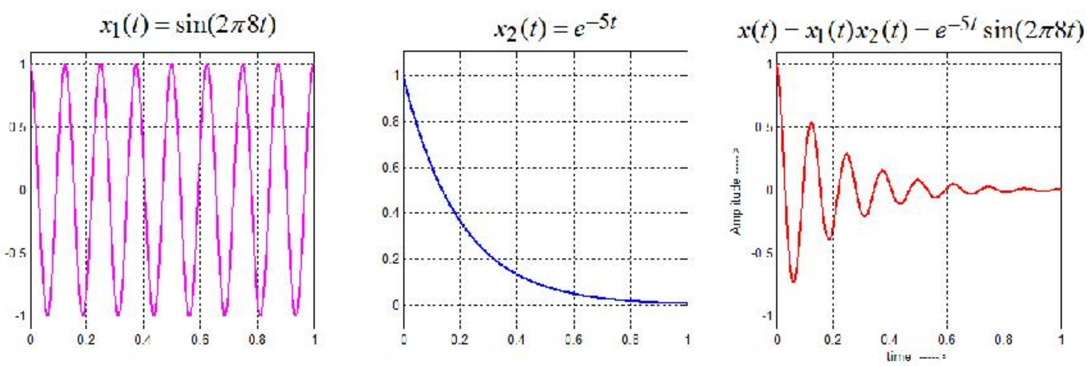
Matlab code for addition example:

```
% SA1_11:Addition of two signals
clear all; close all; clc;
t = 0:0.001:1;
x1 = cos(2*pi*4*t);
x2 = 1.25;
y = x1+x2;
figure();set(gca,'fontsize',14)
subplot(3,1,1);plot(t,x1,'m','LineWidth',3);grid on
axis([0, 1,-3 3]);
subplot(3,1,2);plot(t,x2,'b','LineWidth',3);grid on
axis([0, 1,-3 3]);
subplot(3,1,3);plot(t,y,'r','LineWidth',3);grid on
axis([0, 1,-3 3]);
xlabel('time  ----->');ylabel('Amplitude ----->');
```

Physical significance of this operation is to add two signals like in the addition of the back ground music along with the human audio. Another example is the undesired addition of noise along with the desired audio signals.

Signal Multiplication: The multiplication of signals is given by the simple equation $x(t) = x_1(t)x_2(t)$. An example is illustrated below.

$$x_1(t) = \sin(2\ \ 8t)\,;\quad x_2(t) = e^{-5t} \Rightarrow x(t) = x_1(t)x_2(t) = e^{-5t}\sin(2\ \ 8t)$$

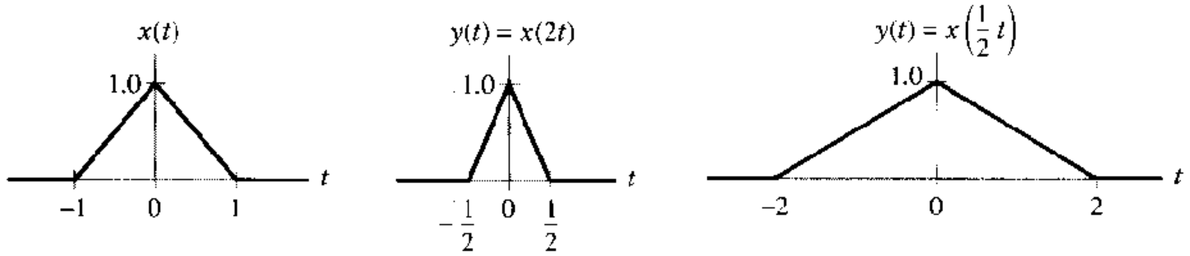


Matlab code for Multiplication example:

```
% SA1_12: Multiplication of two signals
clear all; close all; clc;
t = 0:0.001:1;
x1 = cos(2*pi*8*t);
x2 = exp(-5*t);
y = x1.*x2;
figure();
subplot(3,1,1);plot(t,x1,'m','LineWidth',3);grid on
axis([0, 1,-1.1 1.1]);
subplot(3,1,2);plot(t,x2,'b','LineWidth',3);grid on
axis([0, 1,-0.1 1.1]);
subplot(3,1,3);plot(t,y,'r','LineWidth',3);grid on
axis([0, 1,-1.1 1.1]);
xlabel('time  ----->');ylabel('Amplitude ----->');
```

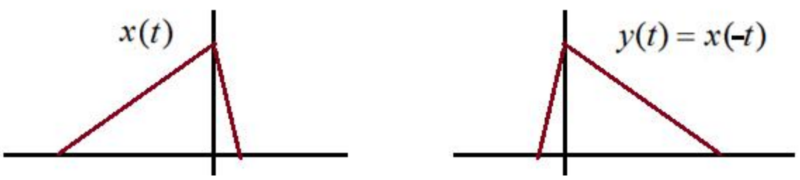
1.4(b): Operations on independent variables: There are various kind of operations on independent variables illustrated below.

Time scaling: Time scaling operation is given by equation $y(t) = x(at)$. This operation results in expansion in time for $a < 1$, and compression in time for $a > 1$. An example of time scaling is illustrated below.



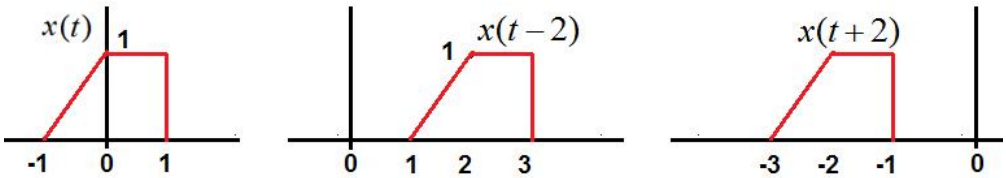
An example of this operation is the compression or expansion of the time scale that results in the 'fast-forward' or the 'slow motion' in a video, provided we have the entire video in some stored form.

Time reflection: Time reflection is given by equation $y(t) = x(-t)$. An example is illustrated below.



Time shifting: The equation representing time shifting operation is given by $y(t) = x(t - t_0)$. Here, $y(t)$ is a version of the original signal $x(t)$ that has been shifted by an amount t_0. If t_0 is positive constant, the operation is right shift. If t_0 is negative the operation is left shift.

Rule: set argument $t - t_0 = 0$ and move origin of $x(t)$ to t_0. An example shown below.



Multiple transformations / operations: In practice we usually require to do multiple operations. These multiple operations are required to follow certain procedure rules. We discuss two procedures as below.

Combined Time shifting and scaling operations:

Case1: The combined transformation of shifting and scaling is represented by an equation $y(t) = x(at - t_0)$.

Procedure rule for Time shifting and scaling: Let $y(t)$ denote a continuous time signal that is derived from another continuous time signal $x(t)$ through a combination of time shifting and time scaling, as described below.

$$y(t) = x(at - t_0)$$

where 'a' is scaling constant and t_0 is time shifting.

The relation between $y(t)$ and $x(t)$ satisfies the following conditions

At time $t = 0$, $y(0) = x(-t_0)$ and

At time $t = \frac{t_0}{a}$, $y\left(\frac{t_0}{a}\right) = x(0)$

which provide useful checks on $y(t)$ in terms of corresponding values of $x(t)$.

To correctly obtain $y(t)$ from $x(t)$, the time shifting and time scaling operations must be performed in the correct order. The proper order is based on the fact that the scaling operation always replaces 't' by 'at', while shifting operation always replaces 't' by '$t - t_0$'. Hence the time shifting operation is performed first on $x(t)$, resulting in an intermediate signal

$$v(t) = x(t - t_0)$$

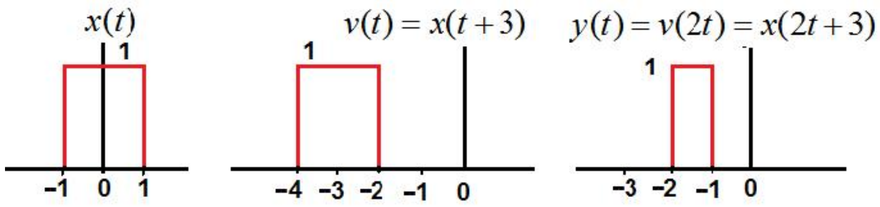
The time shift has replaced 't' in $x(t)$ by $t - t_0$. Next, the time scaling operation is performed on $v(t)$. This replaces 't' by 'at, resulting in the desired output

$$y(t) = v(at) = x(at - t_0)$$

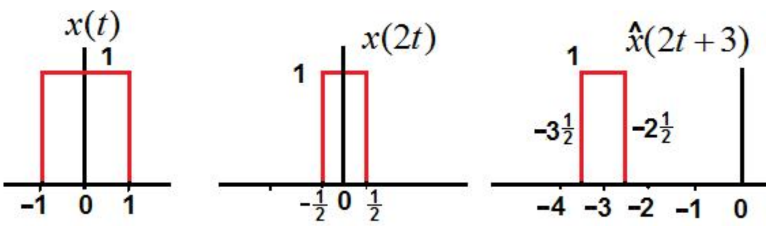
Summary: Multiple transformation for the case of $y(t) = A\,x(at - t_0)$

$$x(t) \xrightarrow{\text{Amplitude Scaling, } A} A\,x(t) \xrightarrow[t \to (t-t_0)]{\text{Time Shifting, } t_0} A\,x(t - t_0) \xrightarrow[t \to at]{\text{Time Scaling, } a} A\,x(at - t_0)$$

Example : Consider the rectangular pulse $x(t)$ of unit amplitude and width of 2 time units as depicted in figure. Find $y(t) = x(2t+3)$ and sketch.



Suppose next that we purposely do not follow the precedence rule; that is, we first apply time scaling, followed by time shifting. The results are shown below, does not agree with the above results.



Case2: The combined transformation of shifting and scaling is represented by an equation

$$y(t) = A\,x\left(\frac{t-t_0}{a}\right).$$

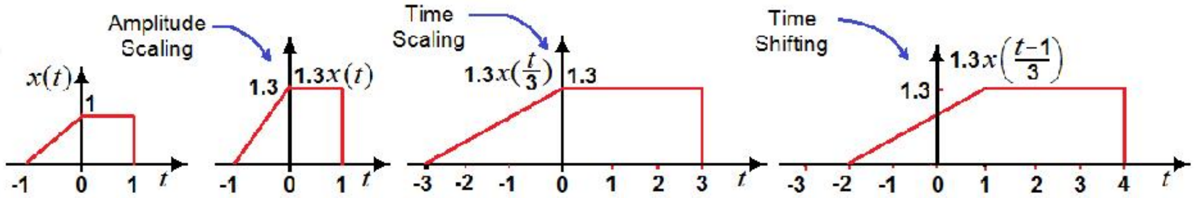
In this case the multiple operations procedure is as follows:

$$x(t) \xrightarrow{\text{Amplitude Scaling}} A\,x(t) \xrightarrow[t \,\to\, \frac{t}{a}]{\text{Time Scaling}} A\,x\left(\frac{t}{a}\right) \xrightarrow[t \,\to\, (t-t_0)]{\text{Time Shifting}} A\,x\left(\frac{t-t_0}{a}\right)$$

$$x(t) \xrightarrow{\text{Amplitude Scaling}} A\,x(t) \xrightarrow[t \,\to\, (t-t_0)]{\text{Time Shifting}} A\,x(t-t_0) \xrightarrow[t \,\to\, \frac{t}{a}]{\text{Time Scaling}} A\,x\left(\frac{t}{a}-t_0\right) \neq A\,x\left(\frac{t-t_0}{a}\right)$$

Example: Perform the multiple operation $y(t) = 1.3\,x\left(\frac{t-1}{2}\right)$ on a signal $x(t)$ shown below.

Ans:



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