UNIT-IV
STOCHASTIC PROCESSES-SPECTRAL CHARACTERISTICS

INTRODUCTION:

In majority of the communication related problems transmission power and bandwidth plays a significant role. Describing the characteristics of random process is very much needed in the design of communication experiments.

Autocorrelation function, Cross-Correlation function, and Covariance are used to describe the statistical properties of random processes in time domain.

Similarly frequency domain can also be used to characterize the random processes. Power spectral density is the fundamental tool used to calculate the average power of random processes.

Fourier transform is the fundamental tool used to get the frequency information of a signal (or) a process. However for any process to have Fourier transform must satisfy the "Dirichlet's conditions". One's Fourier transform is applied then the average power is area under power spectral density curve.

POWER SPECTRAL DENSITY

Let $x(t)$ be the one sample function of a random process $X(t)$. Further consider $x_T(t)$ represent the portion of $x(t)$ $between -T < t < T$.

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & elsewhere \end{cases}$$

The energy contained in the waveform $x_T(t)$ is in the interval $(-T, T)$ is

$$E(T) = \int_{-T}^{T} x_T{}^2(t)\mathrm{dt} = \int_{-T}^{T} x^2(t)\, \mathrm{dt}$$

By dividing the above expression by 2T, we obtain the power $P(T)$ of the truncated (small) portion is

$$P(T) = \frac{1}{2T} \int\limits_{-T}^{T} x_T{}^2(t) \, \mathrm{dt} = \frac{1}{2T} \int\limits_{-T}^{T} x^2(t) \, \mathrm{dt}$$

From the above equation it is observed that

- ✓ It does not represent the power in an entire sample function.
- ✓ The expression is only the power in one sample function and does not represent entire random process.

Hence to obtain power density spectrum for the random process

- ✓ Let making T arbitrarily large $i.e; \ T \rightarrow \infty$ and
- ✓ $P(T)$ is actually a random variable with respect to the random process. By taking the expected value for $P(T)$, we can obtain an average power P_{XX} for the random process.

$$P_{XX} = E[P(T)]$$

$$P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int\limits_{-T}^{T} E[x^2(t)] \, \mathrm{dt}$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T} \, d\omega$$

$\lim\limits_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T} \ is \ called \ power \ spectral \ density. (PSD)$

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$where \ \ X_T(\omega) = F.T[x_T(t)]$$

$$= \int\limits_{-T}^{T} x(t) \, e^{-j\omega t} dt$$

Properties of power spectral density

1. Power spectral density is always a non-negative quantity.

$$S_{XX}(\omega) \geq 0$$

Proof:

It is known that Power spectral density of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

It is known that the term $|X_T(\omega)|^2$ is always positive and expected value of appositive quantity is always positive. Thus

$$S_{XX}(\omega) \geq 0$$

2. Power spectral density is an even function

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

Proof:

It is known that Power spectral density of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

It is known that the term $|X_T(-\omega)|^2$ is always equal to $|X_T(\omega)|^2$

$$S_{XX}(-\omega) = \lim_{T \to \infty} \frac{E[|X_T(-\omega)|^2]}{2T}$$

$$= \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

3. Power spectral density is always real valued function.

i.e; Imaginary part of $S_{XX}(\omega) = 0$

Proof:

It is known that Power spectral density of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

It is known that the term $|X_T(\omega)|^2$ is always positive and real quantity. Thus Imaginary part of $S_{XX}(\omega) = 0$

4. Power spectral density of derivative of random process is equal to ω^2 times power spectral density of random process.

$$\boldsymbol{S_{\dot{X}\dot{X}}(\omega) = \omega^2 \, S_{XX}(\omega)}$$

Proof:

It is known that Power spectral density of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$X_T(\omega) = F.T[x_T(t)] = \int_{-T}^{T} x(t) \, e^{-j\omega t} dt$$

$$\dot{X}_T(\omega) = F.T\left[\frac{d}{dt} x_T(t)\right] = \int_{-T}^{T} \frac{d}{dt} x(t) \, e^{-j\omega t} dt$$

$$\dot{X}_T(\omega) = (j\omega) \, X_T(\omega)$$

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T \to \infty} \frac{E\left[|\dot{X}_T(\omega)|^2\right]}{2T}$$

$$= \lim_{T \to \infty} \frac{E[|(j\omega) \, X_T(\omega)|^2]}{2T}$$

$$= \omega^2 \, \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$\boldsymbol{S_{\dot{X}\dot{X}}(\omega) = \omega^2 \, S_{XX}(\omega)}$$

5. The power spectral density and time average of auto correlation function form a Fourier transform pair.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \, e^{j\omega\tau} \, d\omega = A[R_{XX}(t, t+\tau)]$$

Proof:

It is known that Power spectral density of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$|X_T(\omega)|^2 = X_T(\omega) \, X_T^*(\omega)$$

$$X_T(\omega) = \int_{-T}^{T} x(t_1) \, e^{-j\omega t_1} \, dt_1$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t_2) \, e^{j\omega t_2} \, dt_2$$

Where t_1, t_2 are dummy variables such that $-T < (t_1, t_2) < T$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1) \, e^{-j\omega t_1} \, dt_1 \int_{-T}^{T} x(t_2) \, e^{j\omega t_2} \, dt_2 \right]$$

Rearranging order of integration and expectation

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} E[x(t_1) \, x(t_2)] \, e^{-j\omega t_1} \, e^{j\omega t_2} \, dt_1 \, dt_2$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) \, e^{-j\omega(t_1-t_2)} \, dt_1 \, dt_2$$

Apply inverse Fourier transform

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \, e^{j\omega\tau} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) \, e^{-j\omega(t_1-t_2)} \, e^{j\omega\tau} \, dt_1 \, dt_2 \, d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) \ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau+t_2-t_1)} \ d\omega \ dt_1 \ dt_2$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega x} \ d\omega$$

$$\delta(t) \leftrightarrow 2\pi \ \delta(\omega)$$

$$\frac{1}{2\pi} \ \delta(t) \leftrightarrow \delta(\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) \ \delta(\tau + t_2 - t_1) \ dt_1 \ dt_2$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t_1, t_2) \left(\int_{-T}^{T} \delta(\tau + t_2 - t_1) \ dt_1 \right) dt_2$$

$$\int_{-\infty}^{\infty} \delta(\tau + t_2 - t_1) \ dt_1 = 1$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t_1, t_2) \ dt_2$$

Let $t_1 = t, and \ t_2 = t + \tau$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t, t + \tau) \ dt$$

$$A[R_{XX}(t, t + \tau)] \ = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t, t + \tau) \ dt$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \ e^{j\omega\tau} \ d\omega = A[R_{XX}(t, t + \tau)]$$

Hence, the inverse Fourier transform of the power density spectrum is the time average of the process autocorrelation function.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t, t + \tau)] \; e^{-j\omega\tau} \; d\tau$$

Which shows that $S_{XX}(\omega)$ and $A[R_{XX}(t, t + \tau)]$ form a Fourier transform pair.

$$A[R_{XX}(t, t + \tau)] \;\leftrightarrow\; S_{XX}(\omega)$$

Note:

If X(t) is at least wide sense stationary,

$$A[R_{XX}(t, t + \tau)] = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \; e^{-j\omega\tau} \; d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \; e^{j\omega\tau} \; d\omega$$

$$R_{XX}(\tau) \;\leftrightarrow\; S_{XX}(\omega)$$

The above equations are called Wiener-Khintchine relations.

CROSS POWER SPECTRAL DENSITY

Let $x(t) \; and \; y(t)$ denote sample function of a random process . Further consider $x_T(t) \; and \; y_T(t)$ represent the portion of $x(t) \; and \; y(t)$. such that

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & elsewhere \end{cases}$$

$$y_T(t) = \begin{cases} y(t) & -T < t < T \\ 0 & elsewhere \end{cases}$$

In such a case, the energy (cross energy) of portion of sample function is given as

$$E(T) = \int_{-T}^{T} x_T(t) \; y_T(t) \; \mathrm{dt} \; = \; \int_{-T}^{T} x(t) \; y(t) \; \mathrm{dt}$$

By dividing the above expression by 2T, we obtain the power $P(T)$ of the truncated (small) portion is

$$P(T) = \frac{1}{2T} \int_{-T}^{T} x_T(t) \, y_T(t) \, dt = \frac{1}{2T} \int_{-T}^{T} x(t) \, y(t) \, dt$$

From the above equation it is observed that

✓ It does not represent the power in an entire sample function.

✓ The expression is only the power in sample function and does not represent entire random process.

Hence to obtain power density spectrum for the random process

✓ Let making T arbitrarily large $i.e; \; T \to \infty$ and

✓ $P(T)$ is actually a random variable with respect to the random process. By taking the expected value for $P(T)$, we can obtain an average power P_{XY} for the random process.

$$P_{XY} = E[P(T)]$$

$$P_{XY} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x(t) \, y(t)] \, dt$$

By using Parseval's theorem

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{E[X^*(\omega)Y(\omega)]}{2T} \, d\omega$$

$\lim\limits_{T \to \infty} \dfrac{E[X^*(\omega)Y(\omega)]}{2T}$ *is called cross power spectral density.*

$$S_{XY}(\omega) = \lim_{T \to \infty} \frac{E[X^*(\omega)Y(\omega)]}{2T}$$

$$where \;\; Y_T(\omega) = F.T[y_T(t)] = \int_{-T}^{T} y(t) \, e^{-j\omega t} dt$$

$$X_T{}^*(\omega) = F.T[x_T(t)] = \int_{-T}^{T} x(t) \, e^{j\omega t} dt$$

Properties of power spectral density

1. The cross power spectral density and time average of cross correlation function form a Fourier transform pair.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XY}(\omega)\ e^{j\omega\tau}\ d\omega = A[R_{XY}(t,t+\tau)]$$

Proof:

It is known that cross power spectral density between two random processes $X(t)\ and\ Y(t)$ is given as

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{E[X^*(\omega)Y(\omega)]}{2T}$$

$$X_T{}^*(\omega) = \int_{-T}^{T}x(t_1)\ e^{j\omega t_1}\ dt_1$$

$$Y_T(\omega) = \int_{-T}^{T}y(t_2)\ e^{-j\omega t_2}\ dt_2$$

Where t_1, t_2 are dummy variables such that $-T < (t_1, t_2) < T$

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{-T}^{T}x(t_1)\ e^{j\omega t_1}\ dt_1 \int_{-T}^{T}y(t_2)\ e^{-j\omega t_2}\ dt_2\right]$$

Rearranging order of integration and expectation

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}E[x(t_1)\ y(t_2)]\ e^{j\omega t_1}\ e^{-j\omega t_2}\ dt_1\ dt_2$$

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XY}(t_1,t_2)\ e^{j\omega(t_1-t_2)}\ dt_1\ dt_2$$

Apply inverse Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XY}(\omega)\ e^{j\omega\tau}\ d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XY}(t_1,t_2)\ e^{j\omega(t_1-t_2)}e^{j\omega\tau}\ dt_1\ dt_2\ d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XY}(t_1, t_2) \ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau+t_1-t_2)} \ d\omega \ dt_1 \ dt_2$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega x} \ d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XY}(t_1, t_2) \ \delta(\tau + t_1 - t_2) \ dt_1 \ dt_2$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t_1, t_2) \left(\int_{-T}^{T} \delta(\tau + t_1 - t_2) \ dt_1 \right) dt_2$$

$$\int_{-\infty}^{\infty} \delta(\tau + t_1 - t_2) \ dt_1 = 1$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t_1, t_2) \ dt_2$$

Let $t_1 = t, and \ t_2 = t + \tau$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \ e^{j\omega\tau} \ d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t + \tau) \ dt$$

$$A[R_{XY}(t, t + \tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t + \tau) \ dt$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \ e^{j\omega\tau} \ d\omega = A[R_{XY}(t, t + \tau)]$$

Hence, the inverse Fourier transform of the cross power density spectrum is the time average of the process of cross correlation function.

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t, t + \tau)] \ e^{-j\omega\tau} \ d\tau$$

Which shows that $S_{XY}(\omega)$ and $A[R_{XY}(t, t + \tau)]$ form a Fourier transform pair.

$$A[R_{XY}(t, t + \tau)] \ \leftrightarrow \ S_{XY}(\omega)$$

Note:

If X(t) is at least wide sense stationary,

$$A[R_{XY}(t, t+\tau)] = R_{XY}(\tau)$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \, e^{-j\omega\tau} \, d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \, e^{j\omega\tau} \, d\omega$$

Thus for wide sense stationary random processes cross power spectral density and cross correlation form a Fourier transform pair.

$$S_{XY}(\omega) \quad \leftrightarrow \quad R_{XY}(\tau)$$

$$S_{YX}(\omega) \quad \leftrightarrow \quad R_{YX}(\tau)$$

2. The real part of cross power spectral density is an even function

$$Re \, \{S_{XY}(-\omega)\} = Re \, \{S_{XY}(\omega)\}$$

$$Re \, \{S_{YX}(-\omega)\} = Re \, \{S_{YX}(\omega)\}$$

Proof:

It is known that for wide sense stationary random process

$$S_{XY}(\omega) \quad \leftrightarrow \quad R_{XY}(\tau)$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \, e^{-j\omega\tau} \, d\tau$$

$$e^{-j\omega\tau} = cos\omega\tau - j \, sin\omega\tau$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \, [cos\omega\tau - j \, sin\omega\tau] \, d\tau$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \, cos\omega\tau \, d\tau \, - j \int_{-\infty}^{\infty} R_{XY}(\tau) \, sin\omega\tau \, d\tau$$

$$Re\ \{S_{XY}(\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\ cos\omega\tau\ d\tau$$

$$Re\ \{S_{XY}(-\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\ cos(-\omega\tau)\ d\tau$$

$$Re\ \{S_{XY}(-\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\ cos\omega\tau\ d\tau$$

$$Re\ \{S_{XY}(-\omega)\} = Re\ \{S_{XY}(\omega)\}$$

3. Imaginary part of cross power spectral density is an odd function

$$Img\ \{S_{XY}(\omega)\} = -Img\ \{S_{XY}(\omega)\}$$

$$Img\ \{S_{YX}(\omega)\} = -Img\ \{S_{YX}(-\omega)\}$$

Proof:

It is known that for wide sense stationary random process

$$S_{XY}(\omega)\ \leftrightarrow\ R_{XY}(\tau)$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\ e^{-j\omega\tau}\ d\tau$$

$$e^{-j\omega\tau} = cos\omega\tau - j\ sin\omega\tau$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\ [cos\omega\tau - j\ sin\omega\tau]\ d\tau$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\ cos\omega\tau\ d\tau\ - j \int_{-\infty}^{\infty} R_{XY}(\tau)\ sin\omega\tau$$

$$S_{XY}(-\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\ cos\omega\tau\ d\tau + j \int_{-\infty}^{\infty} R_{XY}(\tau)\ sin\omega\tau$$

$$Img\ \{S_{XY}(\omega)\} = -Img\ \{S_{XY}(\omega)\}$$

$$Img\ \{S_{YX}(\omega)\} = -Img\ \{S_{YX}(-\omega)\}$$

4. When $X(t)$ and $Y(t)$ are two uncorrelated constant mean random processes then their cross power spectral density will be equal to

$$S_{XY}(\omega) = 2\pi \, \bar{X} \, \bar{Y} \, \delta(\omega)$$

Proof:

It is known that for wide sense stationary random process

$$S_{XY}(\omega) \quad \leftrightarrow \quad R_{XY}(\tau)$$

$$S_{XY}(\omega) \; = F.T \, [R_{XY}(\tau)]$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \, e^{-j\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} E[X(t) \, Y(t+\tau)] \, e^{-j\omega\tau} \, d\tau$$

Given $X(t)$ and $Y(t)$ are two uncorrelated constant mean random processes

$$E[X(t) \, Y(t+\tau)] = E[X(t)]E[Y(t+\tau)] = \bar{X} \, \bar{Y}$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} \bar{X} \, \bar{Y} \, e^{-j\omega\tau} \, d\tau$$

$$S_{XY}(\omega) = \bar{X} \, \bar{Y} \int_{-\infty}^{\infty} e^{-j\omega\tau} \, d\tau$$

$$S_{XY}(\omega) = 2\pi \, \bar{X} \, \bar{Y} \, \delta(\omega)$$

5. When two random processes $X(t)$ and $Y(t)$ are orthogonal then their cross power spectral density is zero.

Proof:

It is known that for wide sense stationary random process

$$S_{XY}(\omega) \quad \leftrightarrow \quad R_{XY}(\tau)$$

$$S_{XY}(\omega) \; = F.T \, [R_{XY}(\tau)]$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \, e^{-j\omega\tau} \, d\tau$$

$X(t) \; and \; Y(t)$ are orthogonal then $R_{XY}(\tau) = 0$, then

$$S_{XY}(\omega) = 0$$

BANDWIDTH OF RANDOM PROCESS:

Whenever it is required to describe a random variable, probability density function is used as one attribute/ parameter. In this the standard deviation is a measure of the spread from a given reference value. This reference value can be (mean) either zero (or) a non-zero value.

The standard deviation indicates how the probabilities vary for a given random variable. Similar to the standard deviation, the power spectral density also has a parameter that is a measure of spread (or) distribution of power. This is called as bandwidth.

Let X(t) be a base band process. For a base band process, the spectral components are concentrated over the origin. For such a base band process, the bandwidth can be the second moment upon normalization.

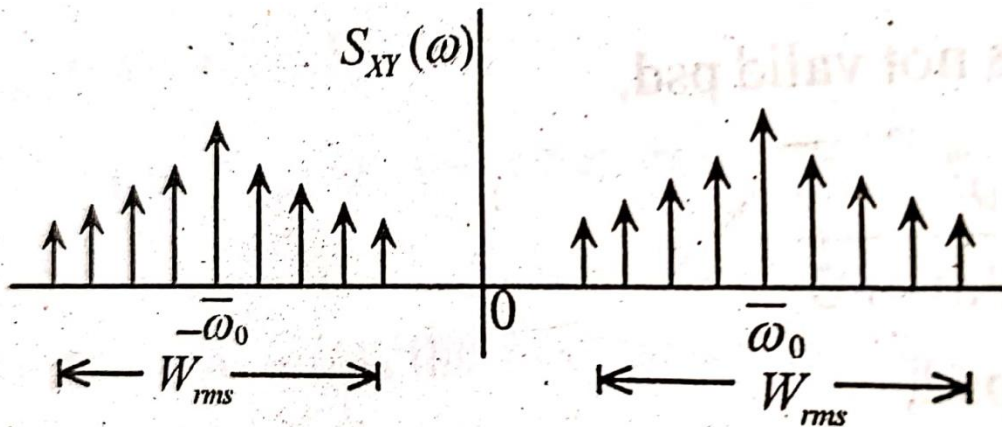The rms bandwidth of a base band process is given as

$$\omega^2_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 \, S_{XX}(\omega) \, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) \, d\omega}$$
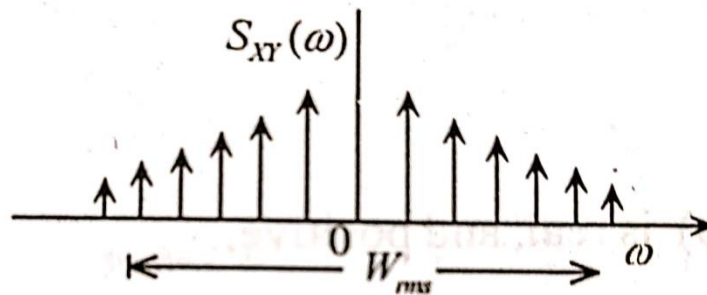
Similarly the rms bandwidth can also be defined for band pass processes. For a band pass process, the spectral/ frequency components are concentrated over some reference value (or) mean value ω_0.

The mean frequency of a band pass process is given as

$$\bar{\omega}_0 = \frac{\int_0^\infty \omega^2 \, S_{XX}(\omega) \, d\omega}{\int_0^\infty S_{XX}(\omega) \, d\omega}$$

The rms bandwidth of band pass process is

$$\omega^2{}_{rms} = \frac{\int_0^\infty (\omega - \bar{\omega}_0)^2 \, S_{XX}(\omega) \, d\omega}{\int_0^\infty S_{XX}(\omega) \, d\omega}$$

1. Describe the rms bandwidth of the random processes.
2. Interpret the properties of cross power spectral density.
3. Interpret the Wiener-Khintchine relation for auto power spectral density and autocorrelation of a random process.
4. Judge the statement that cross power spectral density and cross correlation function of random processes X(t) & Y(t) form a Fourier transform pair.
5. Choose relevant expressions to verify the properties of auto power spectral density.
6. Derive the expression for cross power between X(t) and Y(t) using cross power spectral density.

PROBLEMS

1. If X(t) is WSS process, Develop the power spectrum of $Y(t) = A_0 + B_0 X(t)$ in terms of the power spectrum of X(t), if A_0, B_0 are real constants.(assume zero mean)
2. A random process W(t) = AX(t)+BY(t), A , B are real constants and X(t), Y(t) are jointly WSS, then Determine
   (i) The power spectrum $S_{WW}(\omega)$ of W(t).
   (ii) The power spectrum $S_{WW}(\omega)$ of W(t) if X(t) & Y(t) are uncorrelated.
3. Demonstrate whether given power spectral densities are valid or not.

$(i) S_{XX}(\omega) = \dfrac{\omega^2}{\omega^6 + 3\omega^2 + 3}$ $(ii) S_{XX}(\omega) = \dfrac{\cos(3\omega)}{1 + \omega^2}$ $(iii) S_{XX}(\omega) = \dfrac{|\omega|}{1 + 2\omega + \omega^2}$

4. Calculate the rms bandwidth of a random process whose power spectral density is given as 0

$$s_{XX}(\omega) = P\cos\left(\frac{\pi\omega}{2W}\right) \ for \ |\omega| \leq W$$
$$= 0 \qquad |\omega| > W.$$

5. The cross power spectral density is given

$$S_{XY}(\omega) = a + \frac{jb\omega}{W}, \ -W \leq \omega \leq W$$

$$\mathbf{0} \quad , \quad \textbf{\textit{Otherwise}}$$

where a, b are real constants, then estimate cross correlation function.

6. Calculate the rms bandwidth of a random process whose power spectral density is given as

$$s_{XX}(\omega) = P\omega \ for \quad |\omega| \leq W$$
$$= 0 \qquad |\omega| > W.$$

7. Evaluate the rms bandwidth of a random process whose power spectral density is given as

$$S_{XX}(\omega) = \begin{cases} \left[1 - \dfrac{|\omega|}{W}\right] & for\ |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

8. Determine and plot the power density spectrum of random process whose auto correlation function $R_{XX}(\tau) = \dfrac{A_0^{\ 2}}{2} Cos(\omega_0 \tau)$.

9. The power spectral density of X(t) is given by
$$S_{XX}(\omega) = \begin{cases} 1 + \omega^2 & |\omega| < 1 \\ 0 & elsewhere \end{cases}$$
Find out the auto correlation function.

10. Determine and plot the power density spectrum of random process whose auto correlation function
$$R_{XX}(\tau) = \begin{cases} A_0 \left[1 - \dfrac{|\tau|}{T}\right] & for\ -T \le \tau \le T \\ 0 & OTHETWISE \end{cases}$$

11. Obtain the PSD of a WSS random process X(t) whose auto correlation function is
$$R_{XX}(\tau) = ae^{-b|\tau|}$$

12. Obtain auto correlation function for the power spectral density of X(t) is given by
$$S_{XX}(\omega) = \dfrac{8}{(\alpha + j\omega)^3}$$

1. If $X(t)$ is WSS process, develop the power spectrum of $Y(t) = A_0 + B_0 X(t)$ in terms of the power spectrum of $X(t)$, if A_0, B_0 are real constants. (*Assume zero mean*)

Sol: Given,

$X(t)$ is WSS process

And $Y(t) = A_0 + B_0 X(t)$

We know that,

$$S_{YY}(\omega) = FT[R_{YY}(\tau)]$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega\tau}\, d\tau$$

Now,

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\left[(A_0 + B_0 X(t))(A_0 + B_0 X(t+\tau))\right]$$

$$= E\left[A_0{}^2 + A_0 B_0 X(t+\tau) + A_0 B_0 X(t) + B_0{}^2 X(t)X(t+\tau)\right]$$

$$= E\left[A_0{}^2\right] + A_0 B_0 E[X(t+\tau)] + A_0 B_0 E[X(t)] + B_0{}^2 E[X(t)X(t+\tau)]$$

$$= A_0{}^2 + A_0 B_0 E[X(t)] + A_0 B_0 E[X(t)] + B_0{}^2 R_{XX}(\tau)[\because X(t) \text{ is } WSS]$$

$$= A_0{}^2 + A_0 B_0 (0) + A_0 B_0 (0) + B_0{}^2 R_{XX}(\tau)[Assuming\ zero\ mean]$$

$$\therefore R_{YY}(\tau) = A_0{}^2 + B_0{}^2 R_{XX}(\tau)$$

Now,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(A_0{}^2 + B_0{}^2 R_{XX}(\tau) \right) e^{-j\omega\tau} d\tau$$

$$= A_0{}^2 \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau + B_0{}^2 \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= A_0{}^2 \, \delta(\omega) + B_0{}^2 S_{XX}(\omega)$$

$$\therefore S_{YY} = A_0{}^2 \, \delta(\omega) + B_0{}^2 S_{XX}(\omega)$$

2. A random process $W(t) = AX(t) + BY(t), A, B$ are real constants and $X(t), Y(t)$ are jointly WSS then, determine
   i. The power spectrum $S_{WW}(\omega)$ of $W(t)$.
   ii. The power spectrum $S_{WW}(\omega)$ of $W(t)$ if $X(t) \,\&\, Y(t)$ are Uncorrelated.

Sol: Given,

$W(t) = AX(t) + BY(t)$ and $X(t)$ and $Y(t)$ are jointly WSS.

(i) Power spectrum of $W(t)$:

We know that,

$$S_{WW}(\omega) = FT[R_{WW}(\tau)]$$

$$S_{WW}(\omega) = \int_{-\infty}^{\infty} R_{WW}(\tau) e^{-j\omega\tau} d\tau$$

Now,

$$R_{WW}(\tau) = E[W(t)W(t+\tau)]$$

$$= E\big[\big(AX(t) + BY(t)\big)\big(AX(t+\tau) + BY(t+\tau)\big)\big]$$

$$= E[A^2 \, X(t)X(t+\tau) + AB \, X(t)Y(t+\tau) + AB \, Y(t)X(t+\tau) + B^2 \, Y(t)Y(t+\tau)]$$

$$= A^2 \, E[X(t)X(t+\tau)] + AB \, E[X(t)Y(t+\tau)] + AB \, E[Y(t)X(t+\tau)] + B^2 E[Y(t)Y(t+\tau)]$$

$$\therefore R_{WW}(\tau) = A^2 R_{XX}(\tau) + AB\ R_{XY}(\tau) + AB\ R_{YX}(\tau) + B^2 R_{YY}(\tau)$$

Now,

$$S_{WW}(\omega) = \int_{-\infty}^{\infty} R_{WW}(\tau)e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} [A^2 R_{XX}(\tau) + AB\ R_{XY}(\tau) + AB\ R_{YX}(\tau) + B^2 R_{YY}(\tau)]\,e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} A^2 R_{XX}(\tau)\,e^{-j\omega\tau}\,d\tau + \int_{-\infty}^{\infty} AB\ R_{XY}(\tau)e^{-j\omega\tau}\,d\tau$$

$$+ \int_{-\infty}^{\infty} AB\ R_{YX}(\tau)e^{-j\omega\tau}\,d\tau + \int_{-\infty}^{\infty} B^2 R_{YY}(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= A^2 \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau + AB \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\,d\tau$$

$$+ AB \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j\omega\tau}\,d\tau + B^2 \int_{-\infty}^{\infty} R_{YY}(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$\therefore \mathbf{S_{WW}(\omega) = A^2 S_{XX}(\omega) + AB\ S_{XY}(\omega) + AB\ S_{YX}(\omega) + B^2\ S_{YY}(\omega)}$$

(ii) Power spectrum of W(t)if X(t) and Y(t) are uncorrelated:

Given, X(t) & Y(t) are uncorrelated

Now,

$$R_{WW}(\tau) = E[W(t)W(t+\tau)]$$

$$= E[(AX(t) + BY(t))(AX(t+\tau) + BY(t+\tau))]$$

$$= E[A^2\ X(t)X(t+\tau) + AB\ X(t)Y(t+\tau) + AB\ Y(t)X(t+\tau) + B^2\ Y(t)Y(t+\tau)]$$

$$= A^2\, E[X(t)X(t+\tau)] + AB\, E[X(t)Y(t+\tau)] + AB\, E[Y(t)X(t+\tau)] + B^2\, E[Y(t)Y(t+\tau)]$$

$$= A^2 R_{XX}(\tau) + AB\, E[X(t)]E[Y(t+\tau)] + AB\, E[Y(t)]E[X(t+\tau)] + B^2 R_{YY}(\tau)$$

$$[\because\ X(t)\,\&\,Y(t)\ \text{are uncorrelated}]$$

$$= A^2 R_{XX}(\tau) + AB\, E[X(t)]E[Y(t)] + AB\, E[Y(t)]E[X(t)] + B^2 R_{YY}(\tau)$$

$$[\because\ X(t)\,\&\,Y(t)\ \text{are jointly WSS}]$$

$$= A^2 R_{XX}(\tau) + AB\, \overline{XY} + AB\, \overline{XY} + B^2 R_{YY}(\tau)$$

$$\therefore R_{WW}(\tau) = A^2 R_{XX}(\tau) + 2AB\, \overline{XY} + B^2 R_{YY}(\tau)$$

Now,

$$S_{WW}(\omega) = \int_{-\infty}^{\infty} R_{WW}(\tau) e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} [A^2 R_{XX}(\tau) + 2AB\, \overline{XY} + B^2 R_{YY}(\tau)] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} A^2 R_{XX}(\tau) e^{-j\omega\tau}\, d\tau + \int_{-\infty}^{\infty} B^2 R_{YY}(\tau)\, e^{-j\omega\tau}\, d\tau + \int_{-\infty}^{\infty} 2AB\, \overline{XY}\, e^{-j\omega\tau}\, d\tau$$

$$= A^2 \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau}\, d\tau + B^2 \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-j\omega\tau}\, d\tau + 2AB\, \overline{XY} \int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau$$

$$\therefore\ \mathbf{S_{WW}(\omega) = A^2 S_{XX}(\omega) + B^2\, S_{YY}(\omega) + 2AB\, \overline{XY}\, \delta(\omega)}$$

3. Demonstrate whether given spectral densities are valid or not.

$(i)\ S_{XX}(\omega) = \dfrac{\omega^2}{\omega^6 + 3\omega^2 + 3}$ $(ii)\ S_{XX}(\omega) = \dfrac{cos(3\omega)}{1 + \omega^2}$ $(iii)\ S_{XX}(\omega) = \dfrac{|\omega|}{1 + 2\omega + \omega^2}$

Sol: <u>Conditions to be satisfied to be a valid PSD</u>:

> 1. PSD is always a real valued function $\Rightarrow S_{XX}(0) =$ Some real value
> 2. PSD is an even function $\Rightarrow S_{XX}(\omega) = S_{XX}(-\omega)$

$(i)\ S_{XX}(\omega) = \dfrac{\omega^2}{\omega^6 + 3\omega^2 + 3}$

At $\omega = 0$,

$$S_{XX}(0) = \frac{0}{0 + 0 + 3} = 0 \Rightarrow Real$$

Now,

$$S_{XX}(-\omega) = \frac{(-\omega)^2}{(-\omega)^6 + 3(-\omega)^2 + 3}$$

$$= \frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$$

$$\therefore S_{XX}(-\omega) = S_{XX}(\omega) \Rightarrow Satisfies\ even\ symmetry$$

Hence, it's a valid PSD.

$(ii)\ S_{XX}(\omega) = \dfrac{cos(3\omega)}{1 + \omega^2}$

At $\omega = 0$,

$$S_{XX}(0) = \frac{1}{1 + 0} = 1 \Rightarrow Real$$

Now,

$$S_{XX}(-\omega) = \frac{cos(3(-\omega))}{1 + (-\omega)^2}$$

$$= \frac{cos(3\omega)}{1 + \omega^2}$$

$$\therefore S_{XX}(-\omega) = S_{XX}(\omega) \Rightarrow Satisfies\ even\ symmetry$$

Hence, it's a valid PSD.

$$(iii)\ S_{XX}(\omega) = \frac{|\omega|}{1 + 2\omega + \omega^2}$$

At $\omega = 0$,

$$S_{XX}(0) = \frac{0}{1 + 0 + 0} = 0 \Rightarrow Real$$

Now,

$$S_{XX}(-\omega) = \frac{|-\omega|}{1 + 2(-\omega) + (-\omega)^2}$$

$$= \frac{|\omega|}{1 - 2\omega + \omega^2}$$

$$\therefore S_{XX}(-\omega) \neq S_{XX}(\omega) \Rightarrow Doesn't\ satisfy\ even\ symmetry$$

Hence, it's an invalid PSD.

4. Calculate the rms bandwidth of a random process whose power spectral density is given as
$$S_{XX}(\omega) = \begin{cases} P\ cos\left(\frac{\pi\omega}{2W}\right) & for\ |\omega| \leq W \\ 0 & for\ |\omega| > W \end{cases}$$

Sol:Given,

$$S_{XX}(\omega) = \begin{cases} P\ cos\left(\frac{\pi\omega}{2W}\right) & for\ |\omega| \leq W \\ 0 & for\ |\omega| > W \end{cases}$$

We know that,

$$\omega^2{}_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 \, S_{XX}(\omega) \, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) \, d\omega}$$

Now,

$$\int_{-\infty}^{\infty} S_{XX}(\omega) \, d\omega = \int_{-W}^{W} P \cos\left(\frac{\pi\omega}{2W}\right) d\omega$$

$$= P \int_{-W}^{W} \cos\left(\frac{\pi\omega}{2W}\right) d\omega$$

$$= P \left[\frac{\sin\left(\frac{\pi\omega}{2W}\right)}{\frac{\pi}{2W}} \right]_{-W}^{W}$$

$$= \frac{2WP}{\pi} \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) \right]$$

$$= \frac{2PW}{\pi} \left[2\sin\left(\frac{\pi}{2}\right) \right]$$

$$= \frac{2PW}{\pi} [2(1)]$$

$$\therefore \int_{-\infty}^{\infty} S_{XX}(\omega) \, d\omega = \frac{4PW}{\pi}$$

And

$$\int_{-\infty}^{\infty} \omega^2 \, S_{XX}(\omega) \, d\omega = \int_{-W}^{W} \omega^2 . P \cos\left(\frac{\pi\omega}{2W}\right) d\omega$$

$$= P \int_{-W}^{W} \omega^2 \cos\left(\frac{\pi\omega}{2W}\right) d\omega$$

$$= P\left[\left[\frac{\omega^2 sin\left(\frac{\pi\omega}{2W}\right)}{\frac{\pi}{2W}}\right]_{-W}^{W} - \int_{-W}^{W} 2\omega\left(\frac{sin\left(\frac{\pi\omega}{2W}\right)}{\frac{\pi}{2W}}\right)\right]$$

$$= \left[\frac{2W}{\pi}\left[W^2 sin\left(\frac{\pi}{2}\right) - (-W)^2 sin\left(-\frac{\pi}{2}\right)\right] - 2\left[-\frac{\omega cos\left(\frac{\pi\omega}{2W}\right)}{\left(\frac{\pi}{2W}\right)^2}\right]_{-W}^{W} + 2\left[-\frac{sin\left(\frac{\pi\omega}{2W}\right)}{\left(\frac{\pi}{2W}\right)^3}\right]_{-W}^{W}\right]$$

$$= P\left[\frac{2W}{\pi}\left[W^2 + W^2\right] + \frac{8W^2}{\pi^2}\left[W cos\left(\frac{\pi}{2}\right) - (-W)cos\left(-\frac{\pi}{2}\right)\right]\right.$$

$$\left. - \frac{16W^3}{\pi^3}\left[sin\left(\frac{\pi}{2}\right) - sin\left(-\frac{\pi}{2}\right)\right]\right]$$

$$= P\left[\frac{4W^3}{\pi} + \frac{8W^2}{\pi^2}[0] - \frac{16W^3}{\pi^3}[1+1]\right]$$

$$= P\left[\frac{4W^3}{\pi} - \frac{32W^3}{\pi^3}\right]$$

$$= \frac{4PW^3}{\pi}\left[1 - \frac{8}{\pi^2}\right]$$

$$\therefore \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega = \frac{4PW^3}{\pi}\left[1 - \frac{8}{\pi^2}\right]$$

Finally,

$$\omega^2{}_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$$

$$= \frac{\frac{4PW^3}{\pi}\left[1 - \frac{8}{\pi^2}\right]}{\frac{4PW}{\pi}}$$

$$\therefore \omega^2{}_{rms} = W^2\left[1 - \frac{8}{\pi^2}\right]$$

5. The cross power spectral density is given

$$S_{XY}(\omega) = \begin{cases} a + \dfrac{jb\omega}{W} & , -W \le \omega \le W \\ 0 & , \quad otherwise \end{cases}$$

where a, b are real constants, then estimate cross correlation function.

Sol: Given,

$$S_{XY}(\omega) = \begin{cases} a + \dfrac{jb\omega}{W} & , -W \le \omega \le W \\ 0 & , \quad otherwise \end{cases}$$

We know that,

$$\boldsymbol{R_{XY}(\tau) = IFT[S_{XY}(\omega)]}$$

$$\boldsymbol{R_{XY}(\tau) = \frac{1}{2\pi} \int\limits_{-W}^{W} S_{XY}(\omega)e^{j\omega\tau}\,d\omega}$$

$$\Rightarrow R_{XY}(\tau) = \frac{1}{2\pi} \int\limits_{-W}^{W} \left[a + \frac{jb\omega}{W}\right] e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\left[\int\limits_{-W}^{W} a e^{j\omega\tau}d\omega + \int\limits_{-W}^{W} \frac{jb\omega}{W} e^{j\omega\tau}d\omega\right]$$

$$= \frac{1}{2\pi}\left[a \int\limits_{-W}^{W} e^{j\omega\tau}d\omega + \frac{jb}{W} \int\limits_{-W}^{W} \omega e^{j\omega\tau}d\omega\right]$$

$$= \frac{1}{2\pi}\left[a \left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-W}^{W} + \frac{jb}{W}\left[\omega\left(\frac{e^{j\omega\tau}}{j\tau}\right) - 1\left(\frac{e^{j\omega\tau}}{(j\tau)^2}\right)\right]_{-W}^{W}\right]$$

$$= \frac{1}{2\pi}\left[\frac{a}{j\tau}[e^{jW\tau} - e^{-jW\tau}] + \frac{jb}{W}\left[\frac{We^{jW\tau}}{j\tau} + \frac{e^{jW\tau}}{\tau^2} - \left(-\frac{We^{-jW\tau}}{j\tau} + \frac{e^{-jW\tau}}{\tau^2}\right)\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{a}{j\tau}(2j\ sinW\tau) + \frac{jb}{W}\left[\frac{W}{j\tau}\left(e^{jW\tau} + e^{-jW\tau}\right) + \frac{1}{\tau^2}\left(e^{jW\tau} - e^{-jW\tau}\right)\right]\right]$$

$$\left[\because\ sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2a}{\tau}(sinW\tau) + \frac{b}{\tau}(2\ cosW\tau) + \frac{jb}{W\tau^2}(2j\ sinW\tau)\right]$$

$$\left[\because\ cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{a}{\tau}(sinW\tau) + \frac{b}{\tau}(cosW\tau) - \frac{b}{W\tau^2}(sinW\tau)\right]$$

$$\therefore\ \boldsymbol{R_{XY}(\tau) = \frac{1}{\pi\tau}\left[a\ sinW\tau + b\ cosW\tau - \frac{b\ sinW\tau}{W\tau}\right]}$$

6. Evaluate the rms bandwidth of a random process whose power spectral density is given as

$$S_{XX}(\omega) = \begin{cases} \left[1 - \dfrac{|\omega|}{W}\right] & for\ \ |\omega| \leq W \\ 0 & for\ \ \ |\omega| > W \end{cases}$$

Sol:Given,

$$S_{XX}(\omega) = \begin{cases} \left[1 - \dfrac{|\omega|}{W}\right] & for\ \ |\omega| \leq W \\ 0 & for\ \ \ |\omega| > W \end{cases}$$

We know that,

$$\omega^2{}_{rms} = \frac{\int_{-\infty}^{\infty}\omega^2\ S_{XX}(\omega)\ d\omega}{\int_{-\infty}^{\infty}S_{XX}(\omega)\ d\omega}$$

Now,

$$\int\limits_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \int\limits_{-W}^{W} \left[1 - \frac{|\omega|}{W}\right] d\omega$$

$$= \int\limits_{-W}^{0} \left[1 - \frac{(-\omega)}{W}\right] d\omega + \int\limits_{0}^{W} \left[1 - \frac{\omega}{W}\right] d\omega$$

$$= \int\limits_{-W}^{0} 1.\,d\omega + \frac{1}{W}\int\limits_{-W}^{0} \omega.\,d\omega + \int\limits_{0}^{W} 1.\,d\omega - \frac{1}{W}\int\limits_{0}^{W} \omega.\,d\omega$$

$$= [\omega]_{-W}^{0} + \frac{1}{W}\left[\frac{\omega^2}{2}\right]_{-W}^{0} + [\omega]_{0}^{W} - \frac{1}{W}\left[\frac{\omega^2}{2}\right]_{0}^{W}$$

$$= [0 - (-W)] + \frac{1}{2W}[0 - (-W)^2] + [W - 0] - \frac{1}{2W}[W^2 - 0]$$

$$= W - \frac{W}{2} + W - \frac{W}{2}$$

$$= 2W - W$$

$$\therefore \int\limits_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = W$$

And

$$\int\limits_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega = \int\limits_{-W}^{W} \omega^2 \left[1 - \frac{|\omega|}{W}\right] d\omega$$

$$= \int\limits_{-W}^{0} \omega^2 \left[1 - \frac{(-\omega)}{W}\right] d\omega + \int\limits_{0}^{W} \omega^2 \left[1 - \frac{\omega}{W}\right] d\omega$$

$$= \int\limits_{-W}^{0} \left[\omega^2 + \frac{\omega^3}{W}\right] d\omega + \int\limits_{0}^{W} \left[\omega^2 - \frac{\omega^3}{W}\right] d\omega$$

$$= \int_{-W}^{0} \omega^2 . d\omega + \frac{1}{W} \int_{-W}^{0} \omega^3 . d\omega + \int_{0}^{W} \omega^2 . d\omega - \frac{1}{W} \int_{0}^{W} \omega^3 . d\omega$$

$$= \left[\frac{\omega^3}{3}\right]_{-W}^{0} + \frac{1}{W} \left[\frac{\omega^4}{4}\right]_{-W}^{0} + \left[\frac{\omega^3}{3}\right]_{0}^{W} - \frac{1}{W} \left[\frac{\omega^4}{4}\right]_{0}^{W}$$

$$= \left[0 - \frac{(-W)^3}{3}\right] + \frac{1}{4W}[0 - (-W)^4] + \left[\frac{W^3}{3} - 0\right] - \frac{1}{4W}[W^4 - 0]$$

$$= \frac{W^3}{3} - \frac{W^3}{4} + \frac{W^3}{3} - \frac{W^3}{4}$$

$$= 2\left[\frac{W^3}{3} - \frac{W^3}{4}\right]$$

$$= 2\left[\frac{W^3}{12}\right]$$

$$\therefore \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) \, d\omega = \frac{W^3}{6}$$

Finally,

$$\omega^2_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) \, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) \, d\omega}$$

$$= \frac{\left(\frac{W^3}{6}\right)}{W}$$

$$\therefore \omega^2_{rms} = \frac{W^2}{6}$$

7. Determine and plot the power density spectrum of random process whose auto correlation function is given as

$$R_{XX}(\tau) = \frac{A_0{}^2}{2} Cos(\omega_0\tau)$$

Sol: Given,

$$R_{XX}(\tau) = \frac{A_0{}^2}{2} cos(\omega_0\tau)$$

We know that,

$$S_{XX}(\omega) = FT[R_{XX}(\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau} d\tau$$

$$\Rightarrow S_{XX}(\omega) = \int_{-\infty}^{\infty} \left[\frac{A_0{}^2}{2} cos(\omega_0\tau)\right]e^{-j\omega\tau} d\tau$$

$$= \frac{A_0{}^2}{2} \int_{-\infty}^{\infty} \left[\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right]e^{-j\omega\tau}d\tau \left[\because cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}\right]$$

$$= \frac{A_0{}^2}{4}\left[\int_{-\infty}^{\infty} \left(e^{j\omega_0\tau}.e^{-j\omega\tau}\right)d\tau + \int_{-\infty}^{\infty} \left(e^{-j\omega_0\tau}.e^{-j\omega\tau}\right)d\tau\right]$$

$$= \frac{A_0{}^2}{4}\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau} d\tau + \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)\tau}d\tau\right]$$

$$\therefore S_{XX}(\omega) = \frac{A_0{}^2}{4}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]\left[\because \delta(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}d\tau\right]$$

8. Evaluate the power spectrum of $X(t)$, whose autocorrelation is

$$R_{XX}(\tau) = \begin{cases} A\left[1 - \left(\dfrac{|\tau|}{T}\right)\right] & ,-T \le t \le T \\ 0 & , \quad elsewhere \end{cases}$$

Plot the auto correlation & PSD.

Sol: Given,

$$R_{XX}(\tau) = \begin{cases} A\left[1 - \left(\dfrac{|\tau|}{T}\right)\right] & ,-T \le t \le T \\ 0 & , \quad elsewhere \end{cases}$$

We know that,

$$S_{XX}(\omega) = FT[R_{XX}(\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

$$= \int_{-T}^{T} A\left[1 - \left(\frac{|\tau|}{T}\right)\right] e^{-j\omega\tau}\, d\tau$$

$$= A \int_{-T}^{0} \left(1 - \left(\frac{-\tau}{T}\right)\right) e^{-j\omega\tau}\, d\tau + A \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) e^{-j\omega\tau}\, d\tau$$

$$= A \int_{-T}^{0} \left(1 + \frac{\tau}{T}\right) e^{-j\omega\tau}\, d\tau + A \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) e^{-j\omega\tau}\, d\tau$$

$$= A \int_{-T}^{0} e^{-j\omega\tau}\, d\tau + \frac{A}{T} \int_{-T}^{0} \tau e^{-j\omega\tau}\, d\tau + A \int_{0}^{T} e^{-j\omega\tau}\, d\tau - \frac{A}{T} \int_{0}^{T} \tau e^{-j\omega\tau}\, d\tau$$

i) $A \int\limits_{-T}^{0} e^{-j\omega\tau} \, d\tau = A \left[\dfrac{e^{-j\omega\tau}}{-j\omega} \right]_{-T}^{0} = A \left[\dfrac{1}{-j\omega} + \dfrac{e^{j\omega T}}{j\omega} \right]$

ii) $\dfrac{A}{T} \int\limits_{-T}^{0} \tau e^{-j\omega\tau} \, d\tau = \dfrac{A}{T} \left[\tau \left[\dfrac{e^{-j\omega\tau}}{-j\omega} \right]_{-T}^{0} - \left[\dfrac{e^{-j\omega\tau}}{j^2\omega^2} \right]_{-T}^{0} \right]$

$= \dfrac{A}{T} \left[0 - (-T) \left(\dfrac{e^{j\omega T}}{-j\omega} \right) - \left(\dfrac{1}{-\omega^2} - \dfrac{e^{j\omega T}}{j^2\omega^2} \right) \right]$

$= A \left[-\dfrac{e^{j\omega T}}{j\omega} + \dfrac{1}{T\omega^2} - \dfrac{e^{j\omega T}}{T\omega^2} \right]$

iii) $A \int\limits_{0}^{T} e^{-j\omega\tau} \, d\tau = A \left[\dfrac{e^{-j\omega\tau}}{-j\omega} \right]_{0}^{T} = A \left[\dfrac{e^{-j\omega T}}{-j\omega} + \dfrac{1}{j\omega} \right]$

iv) $\dfrac{A}{T} \int\limits_{0}^{T} \tau e^{-j\omega\tau} \, d\tau = \dfrac{A}{T} \left[\tau \left[\dfrac{e^{-j\omega\tau}}{-j\omega} \right]_{0}^{T} - \left[\dfrac{e^{-j\omega\tau}}{j^2\omega^2} \right]_{0}^{T} \right]$

$= \dfrac{A}{T} \left[T \left(\dfrac{e^{-j\omega T}}{-j\omega} \right) - 0 - \left(\dfrac{e^{-j\omega T}}{j^2\omega^2} - \left(\dfrac{1}{-\omega^2} \right) \right) \right]$

$= A \left[-\dfrac{e^{-j\omega T}}{j\omega} + \dfrac{e^{-j\omega T}}{T\omega^2} - \dfrac{1}{T\omega^2} \right]$

$\therefore S_{XX}(\omega) = A \left[\dfrac{1}{-j\omega} + \dfrac{e^{j\omega T}}{j\omega} - \dfrac{e^{j\omega T}}{j\omega} + \dfrac{1}{T\omega^2} - \dfrac{e^{j\omega T}}{T\omega^2} - \dfrac{e^{-j\omega T}}{j\omega} + \dfrac{1}{j\omega} \right.$

$\left. - \left(-\dfrac{e^{-j\omega T}}{j\omega} + \dfrac{e^{-j\omega T}}{T\omega^2} - \dfrac{1}{T\omega^2} \right) \right]$

$= A \left[\dfrac{1}{-j\omega} + \dfrac{e^{j\omega T}}{j\omega} - \dfrac{e^{j\omega T}}{j\omega} + \dfrac{1}{T\omega^2} - \dfrac{e^{j\omega T}}{T\omega^2} - \dfrac{e^{-j\omega T}}{j\omega} + \dfrac{1}{j\omega} + \dfrac{e^{-j\omega T}}{j\omega} - \dfrac{e^{-j\omega T}}{T\omega^2} + \dfrac{1}{T\omega^2} \right]$

$$= A \left[\frac{2}{T\omega^2} + \frac{1}{T\omega^2} \left(-e^{j\omega T} - e^{-j\omega T} \right) \right]$$

$$= A \left[\frac{2}{T\omega^2} - \frac{1}{T\omega^2} \left(e^{-j\omega T} + e^{j\omega T} \right) \right]$$

$$= A \left[\frac{2}{T\omega^2} - \frac{1}{T\omega^2} (2\cos\omega T) \right] \left[\because \cos\theta = \frac{e^{-j\theta} + e^{j\theta}}{2} \right]$$

$$= \frac{2A}{T\omega^2} [1 - \cos\omega T]$$

$$= \frac{2A}{T\omega^2} \left[2\sin^2 \left(\frac{\omega T}{2} \right) \right]$$

$$= \frac{4A}{T\omega^2} \times \frac{T}{T} \left[\sin^2 \left(\frac{\omega T}{2} \right) \right]$$

$$= AT \times \frac{4}{T^2\omega^2} \left[\sin^2 \left(\frac{\omega T}{2} \right) \right]$$

$$= AT \left[\frac{\sin^2 \left(\frac{\omega T}{2} \right)}{\left(\frac{\omega T}{2} \right)^2} \right]$$

$$= AT \left[\frac{\sin \left(\frac{\omega T}{2} \right)}{\left(\frac{\omega T}{2} \right)} \right]^2$$

$$\therefore S_{XX}(\omega) = ATSinc^2 \left(\frac{\omega T}{2} \right)$$

13.   Obtain the PSD of a WSS random process X(t) whose auto correlation function is

$$R_{XX}(\tau) = ae^{-b|\tau|}$$

Sol: We know that,

$$S_{XX}(\omega) = FT[R_{XX}(\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau$$

$$\therefore \text{we know that F.T pair } e^{-\alpha|\tau|} \quad \overset{F.T}{\Longleftrightarrow} \quad \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$S_{XX}(\omega) = \frac{2ab}{b^2 + \omega^2}$$

(OR)

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} ae^{-b|\tau|}\,e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{0} ae^{b\tau}\,e^{-j\omega\tau}\,d\tau + \int_{0}^{\infty} ae^{-b\tau}\,e^{-j\omega\tau}\,d\tau$$

$$S_{XX}(\omega) = \frac{2ab}{b^2 + \omega^2}$$

14. Obtain auto correlation function for the power spectral density of X(t) is given by

$$S_{XX}(\omega) = \frac{8}{(\alpha + j\omega)^3}$$

Sol: Given,

$$S_{XX}(\omega) = \frac{8}{(\alpha + j\omega)^3} = \frac{4 * 2}{(\alpha + j\omega)^3}$$

We know that,

$$R_{XY}(\tau) = IFT[S_{XY}(\omega)]$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-W}^{W} S_{XY}(\omega)e^{j\omega\tau} d\omega$$

$$\therefore we\ know\ that\ F.T\ pair\ \boldsymbol{u(t)\ t^2\ e^{-\alpha\tau}} \overset{\boldsymbol{F.T}}{\Leftrightarrow} \frac{2}{(\boldsymbol{\alpha + j\omega})^3}$$

$$\boldsymbol{R_{XY}(\tau) = 4\ u(t)\ t^2\ e^{-\alpha\tau}}$$