

Session 23 &amp; 24

Vector Calculus

Scalar point function: A function  $f$  is said to be a scalar point function if it represents a Quantity without directions.

$$f = f_1 + f_2 + f_3$$

where  $f_1, f_2, f_3$  are functions in  $x, y, z$  in coordinate plane for a point  $(x, y, z)$

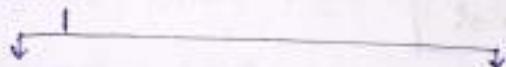
vector point function: A function  $F$  is said to be a vector point function if it represents both Quantity & Direction.

$$F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

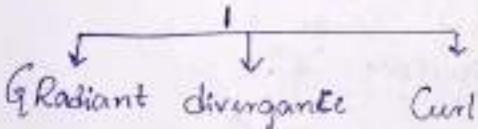
where  $F_1, F_2, F_3$  are functions in  $x, y, z$  in coordinate plane for a point  $(x, y, z)$ .

$$\begin{array}{l|l} i \cdot i = 1 & i \cdot j = 0 \\ j \cdot j = 1 & j \cdot k = 0 \\ k \cdot k = 1 & k \cdot i = 0 \end{array}$$

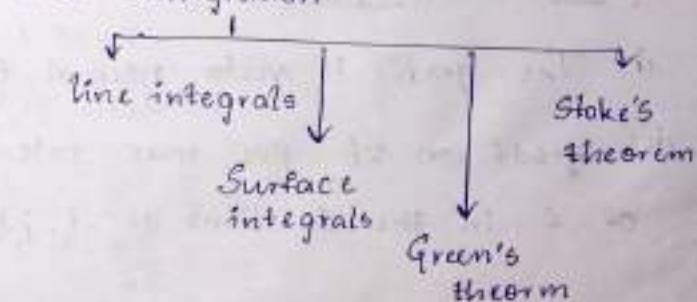
## Vector calculus



## Vector differentiation



## Vector Integration



Session 22 & 23

### Vector differential Operator (VDO) :-

A VDO is denoted by  $\nabla$  (del)

Defined as 
$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

Gradient of a scalar point function:

The Del applying to a scalar <sup>point</sup> function is called a gradient. Denoted by  $\text{grad } f$  (or)  $\nabla f$ .

where  $f$  is scalar point function.

$$\therefore \text{grad } f = \nabla f = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) f$$

$$\text{grad } f = \nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

Ex:- Let  $f = x + 2y + 3z$

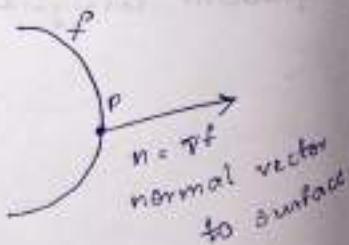
$$\text{grad } f = ?$$

$$\begin{aligned} \text{grad } f &= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} = i(1) + j(2) + k(3) \\ &= \vec{i} + 2\vec{j} + 3\vec{k}. \end{aligned}$$

$$\boxed{\text{grad } f = \vec{i} + 2\vec{j} + 3\vec{k}.}$$

### Geometric Interpretation

- (i) The grad  $f$  is vector normal to surface ' $f$ '.
- (ii) grad  $f$  (or)  $\nabla f$  give max rate of change of  $f$  in the directions of  $i, j$  &  $k$ .



## Directional derivative (DD)

Let  $f(x, y, z)$  is scalar point function in the region  $R$  &  $p$  be any point in the region.

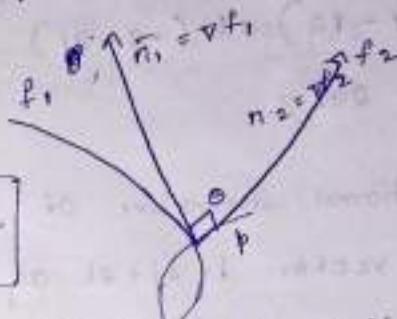
The DD of  $f$  at point  $p(x, y, z)$  in the direction of  $\vec{a}$  is

$$\boxed{D.D = \nabla f \cdot \vec{a} / |\vec{a}|}$$

Angle b/w 2 surfaces :

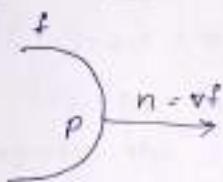
$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

$$\boxed{\theta = \cos^{-1} \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}}$$



Note: Angle between two surfaces is nothing but angle between their normals.

Unit normal vector



$$\boxed{\text{unit normal vector} = \frac{\nabla f}{|\nabla f|}}$$

1. Find grad f, where  $f = x^3 - y^3 + 3xyz$  at point  $(1, -1, 1)$

$$\begin{aligned} \nabla f &= i \partial f / \partial x + j \partial f / \partial y + k \partial f / \partial z \\ &= i(3x^2) - j(3y^2) + 3xyk \\ &= i(3(1)^2) - j(3(-1)^2) + 3(1)(-1)k \\ &= 3i - 3j - 3k \end{aligned}$$

$$\begin{aligned} \nabla f &= \nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \\ &= i(3x^2 + 3yz) + j(3y^2 + 3xz) + 3xyk \\ &= i(3(1)^2 + 3(-1)(1)) + j(3(-1)^2 + 3(1)(1)) + 3(1)(-1)k \\ &= -3k \end{aligned}$$

2. Find gradient of function  $f = x^3y^3 - y^3z + 7(x^3 + x)$  at point  $(1, 2, 1)$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= i(3x^2y^2 - 7) + j(3x^3y - 3y^2z) + k(-y^3 - 7xz^2)$$

$$= i(3(1)(2)^2 - 7) + j(3(1)(2) - 3(2)^2(1)) + k(-(2)^3 - 7(1))$$

$$= 5i + j(4 - 12) + k(-8 - 21)$$

$$= 5i + 8j - 29k //.$$

3. Compute directional derivative of  $f = xy^2 - y^3z - z^2x$  in the direction of vector  $i + 2j + 2k$  at the point  $(2, -1, 1)$

$$DD = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= \frac{i + 2j + 2k}{\sqrt{1 + 4 + 4}} = \frac{i + 2j + 2k}{3}$$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= i(y^2 - z^2) + j(2xy - 3y^2z) + k(-3y^3 - 2zx)$$

$$\text{at } (2, -1, 1)$$

$$= i((-1)^2 - (1)^2) + j(2(2)(-1) - 3(-1)^2(1)) + k(-3(-1)^3 - 2(1)(2))$$

$$= -2i - 8j - 3k$$

$$DD = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|} = \frac{(-2i - 8j - 3k)(i + 2j + 2k)}{3}$$

$$= \frac{-14 - 6}{3} = -\frac{20}{3}$$

$$DD = -\frac{20}{3}$$

4. Find DO of  $i + xyz$  at  $(1, 1, 1)$  in direction of  $\vec{r} + \vec{k}$

$$DO = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= i(yz) + j(zx) + k(xy)$$

$$= i + j + k.$$

$$\frac{\vec{a}}{|\vec{a}|} = \frac{i + j + k}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \frac{i + j + k}{\sqrt{3}}$$

$$DO = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\frac{(i + j + k)(i + j + k)}{\sqrt{3}}, \quad \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} //$$

5. The temp at point  $(x, y, z)$  in the space is given by

$T(x, y, z) = x^2 + y^2 + z^2$ . A mosquito located at  $(1, +1, 2)$  desires to fly in such a direction that it gets cooled faster compute the direction in which it should fly.

Sol The given temp. function

$$T(x, y, z) = x^2 + y^2 + z^2$$

the gradient of temperature is given by  $\text{grad } T = \nabla T$

$$\nabla T = i \frac{\partial T}{\partial x} + j \frac{\partial T}{\partial y} + k \frac{\partial T}{\partial z}$$

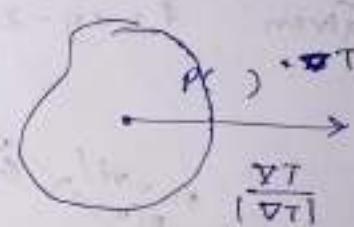
$$\text{grad } T = i(2x) + j(2y) + k(2z)$$

$$\text{grad } T = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\text{at } (1, +1, 2) = -2(1)\vec{i} + 2(+1)\vec{j} + \vec{k}$$

$$= -2\vec{i} + 2\vec{j} + \vec{k}$$

The unit normal gradient vector  $\frac{\nabla T}{|\nabla T|} = \frac{-2\vec{i} + 2\vec{j} + \vec{k}}{3}$



To get cooled fast, it should travel in direction of  $\frac{\nabla \phi}{|\nabla \phi|}$   
 $= -\frac{2i + 2j + 2k}{3}$

5. Find the directional derivative of  $\phi = xy - yz - zx$  at  
 in the direction of  $\vec{AB}$ , where  $A = (1, 2, 0)$ ,  $B = (1, 0, 3)$ .

Sol Given scalar point function

$$\phi = xy - yz - zx$$

$$\text{The gradient of } \phi \Rightarrow \nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$= \nabla \phi = i(y-z) + j(x-z) + k(-y-x)$$

at a point  $A(1, 2, 0)$

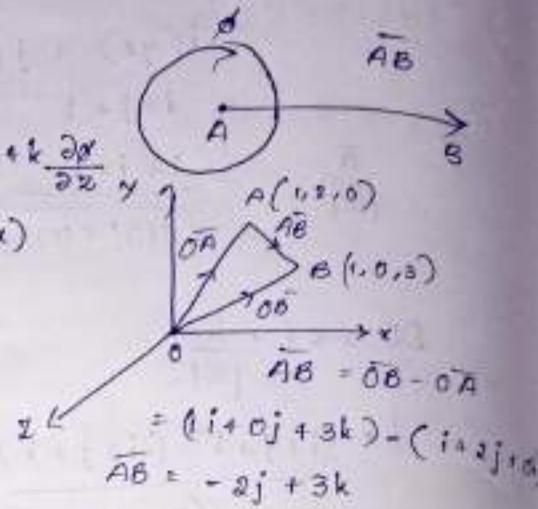
$$\nabla \phi = 2i + j - 3k$$

$$D \cdot D = \nabla \phi \cdot \frac{\vec{AB}}{|\vec{AB}|}$$

$$= (2i + j - 3k) \cdot (-2j + 3k)$$

$$\sqrt{4+9}$$

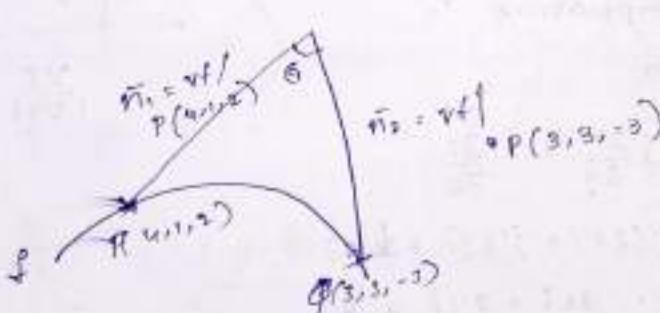
$$= \frac{-2-9}{\sqrt{13}} = -\frac{11}{\sqrt{13}}$$



$$\boxed{D.D = \nabla \phi \cdot \frac{\vec{AB}}{|\vec{AB}|}}$$

6. Identify the angle between the normal to the surface  $xy = z^2$  at points  $(4, 1, 2)$  &  $(3, 3, -3)$

Sol Given  $f = xy - z^2$



The normal vector of surface  $f$ , at point  $P(4, 1, 2)$  is

$$\vec{n}_1 = \nabla f \Big|_{\text{at } P(4, 1, 2)} = [iy + jx - 2zk] = i + 4j - 4k$$

The second normal vector of  $f_2$  at point Q (3,3,-3) is

$$\bar{n}_2 = \nabla f_2 \Big|_{\text{at } Q(3,3,-3)} = \hat{i}y + \hat{j}x - 2z\hat{k} = 3\hat{i} + 3\hat{j} + 6\hat{k}$$

Let angle between normal to surface 'G' 'Q'

$$\begin{aligned}\text{Cos} \theta &= \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|} = \frac{(i+4j-4k)(3i+3j+6k)}{\sqrt{1+16+16} \sqrt{9+9+36}} \\ &= \frac{3+12-24}{\sqrt{33} \sqrt{54}} = \frac{-9}{\sqrt{11} \sqrt{3} \sqrt{9} \sqrt{6}} \\ &= \frac{-1}{\sqrt{11} \sqrt{3} \cdot 3 \sqrt{2}} = \frac{-1}{\sqrt{66}}\end{aligned}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{66}}\right) \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{-1}{\sqrt{66}}\right)}$$

7. Determine the angle between surfaces  $x^2+y^2+z^2=9$  &  
 $x^2+y^2-z=3$  at (2, -1, 2)

$$\text{Given } f_1 = x^2+y^2+z^2-9$$

$$f_2 = x^2+y^2-z-3$$

The normal vector of surface  $f_1$ , at (2, -1, 2)

$$\bar{n}_1 = \nabla f_1 \Big|_{\text{at } P(2, -1, 2)}$$

$$\bar{n}_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{at } (2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

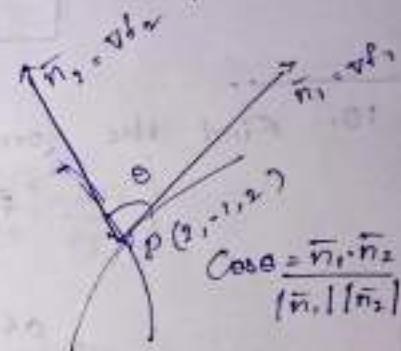
$$\bar{n}_2 = 2x\hat{i} + 2y\hat{j} - z\hat{k}$$

$$\text{at } (2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}$$

Let  $\theta$  be angle between 2 surfaces is angle between  
 their normals.

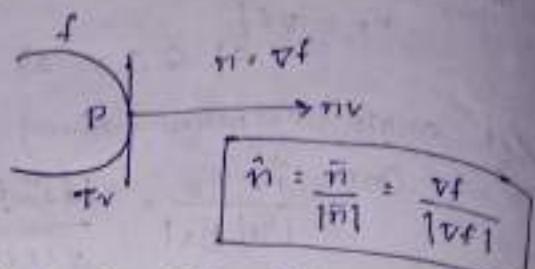
$$\begin{aligned}\cos \theta &= \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{16+4+16} \sqrt{16+4+1}} \\ &= \frac{16+4-4}{\sqrt{36} \sqrt{21}} = \frac{16}{6\sqrt{21}}\end{aligned}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)}$$



$$\text{Cos} \theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|}$$

## Unit normal vector to Surface



9. Find unit normal vector to the surface  $Z^2 = 4(x^2 + y^2)$  at point  $(1, 0, 2)$

Sol Given Surface  $f = 4x^2 + 4y^2 - Z^2$

$$\nabla f = 8xi + 8yj - 2Zk$$

$$\text{at } (1, 0, 2) \Rightarrow \nabla f = 8i - 4k$$

unit normal vector of given surface  $f$ , at point

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{8i - 4k}{\sqrt{80}}$$

10. Find the normal vector to surface  $x^2 + y^2 - Z^2 = 1$  at  $(1, 3, 3)$

$$f = x^2 + y^2 - Z^2 - 1$$

$$\nabla f = 2xi + 2yj - 2Zk$$

$$\text{at } (1, 3, 3) = 2i + 6j - 6k$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2i + 6j - 6k}{\sqrt{4 + 36 + 36}}$$

$$\hat{n} = \frac{2i + 6j - 6k}{\sqrt{76}}$$

## Session - 24

Divergence & curl. of a given vector point function.

Divergence of a vector point function:-

Let  $F = F_1 i + F_2 j + F_3 k$  be a vector, continuously differentiable on every point in 3-D plane, then the divergence of  $F$ , is denoted by  $\operatorname{div} F$  (or)  $\nabla \cdot F$

Defined as  $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$

$$\boxed{\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Solenoidal vector :-

A vector  $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$  is said to be solenoidal vector, if  $\boxed{\operatorname{div} \vec{F} = 0}$  or  $\boxed{\nabla \cdot \vec{F} = 0}$

Curl of a vector point function:

Let  $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$  be a vector, continuously differentiable on every point in 3-D plane, then the curl of  $\vec{F}$ , is denoted  $\operatorname{curl} \vec{F}$  (or)  $\nabla \times \vec{F}$

Denoted as  $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \hat{i} \left[ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] - \hat{j} \left[ \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right] + \hat{k} \left[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right]$$

Irrational vector:

A vector  $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$  is said to be irrational vector by  $\boxed{\operatorname{curl} \vec{F} = \vec{0}}$

$$\vec{0} = 0 \hat{i} + 0 \hat{j} + 0 \hat{k}$$

1. If  $\vec{F} = x^2 y \hat{i} - 2yz \hat{j} + 3z^2 x \hat{k}$  find  $\operatorname{div} \vec{F}$  at  $(0, -1, 1)$

sol Given  $\vec{F} = x^2 y \hat{i} - 2yz \hat{j} + 3z^2 x \hat{k}$

where  $\boxed{\vec{F} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}}$

$$f_1 = x^2 y, f_2 = -2yz, f_3 = 3z^2 x$$

$$\operatorname{div} \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 2xy + (-4yz) + 6zx$$

at  $(0, -1, 1)$   $\operatorname{div} \vec{F} = -4(1)(1) = 4$

$$\boxed{\operatorname{div} \vec{F} = 4}$$

2. Determine 'p' if  $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+pz)\vec{k}$  is solenoidal vector.

Given  $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+pz)\vec{k}$

where  $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\boxed{\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}}$$

Since,  $F$  is solenoidal vector,  $\operatorname{div} \vec{F} = 0$ .

$$\nabla \cdot \vec{F} = 0$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$1 + 1 + p = 0$$

$$\boxed{p = -2}$$

3. If  $\vec{F} = -x\vec{i} - y\vec{j} - z\vec{k}$ , calculate  $\operatorname{curl} \vec{F}$ .

Given  $\vec{F} = -x\vec{i} - y\vec{j} - z\vec{k}$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & -z \end{vmatrix}$$

$$\operatorname{curl} \vec{F} = \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0]$$

$$+ 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

$$\boxed{\operatorname{curl} \vec{F} = \vec{0}}$$

4. Find divergence & curl of  $\vec{v} = (xyz)\vec{i} - (2x^2y)\vec{j} - 3y^2z\vec{k}$   
at  $(2, -1, 3)$

501  $\vec{v} = (xyz)\vec{i} - (2x^2y)\vec{j} - 3y^2z\vec{k}$

$$\boxed{\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}}$$

divergence of  $\vec{v}$  is  $\operatorname{div} \vec{v}$  (or)  $\vec{v} \cdot \vec{v}$

$$\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\operatorname{div} \vec{v} = yz - 2x^2 - 3y^2$$

$$\text{at } (2, -1, 3) \quad \operatorname{div} \vec{v} = (-1)(3) + 2(2)^2 - 3(-1)^2 \\ = -3 - 8 - 3 = -14$$

$$\boxed{\operatorname{div} \vec{v} = -14}$$

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -2x^2y & -3y^2z \end{vmatrix}$$

$$\operatorname{curl} \vec{v} = i[-6yz - 0] - j[0 - xy] + k[-4xy - xz]$$

$$\text{at } (2, -1, 3) \quad \vec{v} = 18i - 2j + 2k.$$

5. Show that the fluid motion

$$\vec{f} = (y+z)i + (x+z)j + (x+y)k \text{ is irrotational.}$$

$$\text{Sol} \quad \operatorname{curl} \vec{f} = \nabla \times \vec{f} =$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (x+z) & (x+y) \end{vmatrix}$$

$$\operatorname{curl} \vec{f} = i[1 - 1] - j[1 - 1] + k[1 - 1] : \quad \boxed{\operatorname{curl} \vec{f} = 0}$$

$$= 0i + 0j + 0k = \vec{0}$$

$\therefore \vec{f}$  is irrotational vector.

6. Determine a, b, c if  $\vec{f} = (ax+by+cz)i + (bx+cy+az)j + (cx+ay+bz)k$  is irrotational

Sol: Given vector  $\vec{f}$  is irrotational vector,

$$\operatorname{curl} \vec{f} = 0$$

$$\nabla \times \vec{f} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax+by+cz & bx+cy+dz & az+bx+cy+dz \end{vmatrix} = 0i + 0j + 0k$$

$$\therefore i[c-0] - j[0-a] + k[b-0] = 0i + 0j + 0k$$

$$c-0=0$$

$$c=0$$

$$\begin{cases} a=2 \\ b=3 \end{cases}$$

$$a=2$$

$$b=3$$

## Vector Integration

- (i) Line Integral
- (ii) Green's theorem for plane
- (iii) Surface integrals
- (iv) Stoke's theorem.

### Session - 25

Let  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  be any vector point function.

Position vector  $\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$ .

The line integral of  $\vec{F}$  over the curve 'C' moves from point 'A' to point 'B' is given

$$\boxed{I = \oint_C \vec{F} \cdot d\vec{r}}$$

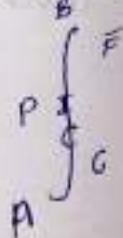
$d\vec{r}$  is displacement vector.

$$\Rightarrow \boxed{I = \int_C (F_1 dx + F_2 dy + F_3 dz)}$$

Workdone by Force Function  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ :

Workdone by the force function  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ , for moving a particle 'P' from point 'A' to point 'B', along circle 'C' is

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}}$$



2. Compute workdone by the force function  $\vec{F} = 2xy\vec{i} - 3z\vec{j} + 5x\vec{k}$ , when it moves a particle along curve  $x = t^2 + 1$ ,  $y = 2t^2$ ,  $z = t^3$  from  $t = 1$  to  $2$ .

Sol force function,

$$\vec{F} = 2xy\vec{i} - 3z\vec{j} + 5x\vec{k}$$

The work done by Force  $\vec{F}$ , to move particle from point A to point 'B', along curve 'c' is  $A \int_C^B$

$$w = \int_C \vec{F} \cdot d\vec{r}$$

$$\begin{aligned} x &= t^2 + 1 \\ y &= 2t^2 \\ z &= t^3 \end{aligned}$$

$t \rightarrow$  varies from 1 to 2

$$= \int_C (2xy\vec{i} - 3z\vec{j} + 5x\vec{k}) \cdot (i dx + j dy + k dz)$$

$$= \int_C 2xy dx - 3z dy + 5x dz$$

where 'c' is curve given by

$$x = t^2 + 1 \Rightarrow dx = 2t dt$$

$$y = 2t^2 \Rightarrow dy = 4t dt$$

$$z = t^3 \Rightarrow dz = 3t^2 dt$$

$$w = \int_C 2(t^2+1)(2t^2)2t dt - 3(t^3)4t dt + 5(t^2+1)3t^2 dt$$

$$= \int_C [8t^3(t^2+1) - 12t^4 + 15t^2(t^2+1)] dt$$

$$= \int_C [8t^5 + 8t^3 - 12t^4 + 15t^4 + 15t^2] dt$$

$$t = \left[ \frac{8t^6}{6} + \frac{3t^5}{5} + \frac{8t^4}{4} + \frac{15t^3}{3} \right]^2,$$

$$t = \left[ \frac{8(2)^6}{6} + \frac{3(2)^5}{5} + \frac{8(2)^4}{4} + \frac{15(2)^3}{3} \right]^2 - \left[ \frac{8(1)^6}{6} + \frac{3(1)^5}{5} + \frac{8(1)^4}{4} + \frac{15(1)^3}{3} \right]^2$$

$$t = \left[ \frac{8 \times 64}{6} + \frac{3 \times 32}{5} + \frac{8 \times 16}{4} + \frac{15 \times 8}{3} \right]^2 - \left[ \frac{8}{6} + \frac{3}{5} + \frac{8}{4} + \frac{15}{3} \right]^2$$

$$= 16711.$$

3. If  $\vec{F} = 3xyi - y^2j$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $C$  is curve  $y = 2x^2$  in the XY-plane from  $(0,0)$  to  $(1,2)$

Given  $\vec{F} = 3xyi - y^2j$

$$\text{now, } \int_C \vec{F} \cdot d\vec{r} = \int_C (3xyi - y^2j) \cdot (idx + jdy + kdz)$$

$$= \int_C 3xy dx - y^2 dy$$

$$\Rightarrow dy = 4x dx$$

$$\Rightarrow \int_C 3x(2x^2) dx - (2x^2) \cdot 4x dx$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C (6x^3 - 16x^3) dx$$

$$= \left[ \frac{6x^4}{4} - \frac{16x^4}{6} \right]_0^1$$

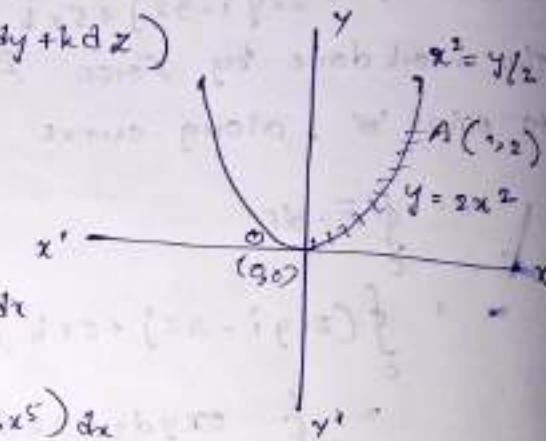
$$= \frac{63}{4} - \frac{16}{6}$$

$$= \frac{9 - 16}{6} = -\frac{7}{6}.$$

4. Compute the work done by the force  $\vec{F} = 3x^2i + (4xz - y)j + 2z\vec{k}$  when it moves along the line from point  $(0,0,0)$  to  $(2,1,2)$

$$\vec{F} = 3x^2i + (4xz - y)j + 2z\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (3x^2i + (4xz - y)j + 2z\vec{k}) \cdot (idx + jdy + kdz)$$



$$\int_C 3x^2 dy + 4xz - y dz \cdot 2z dz$$

work done by force  $\vec{F}$ , when moves a particle along straight line from  $O(0,0,0)$  to  $P(2,1,3)$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C 3(2t)^2 2 \cdot dt + [4(2t)(3t) - t] dt + 2(3t) 3 dt$$

$$W = \int_C (16t^2 + 24t^2 - t + 18t) dt$$

$$= \int_0^1 (40t^2 + 17t) dt$$

$$= \cancel{40t^3} \frac{40t^3}{3} + \frac{17t^2}{2}$$

$$= \frac{40}{3} + \frac{17}{2}$$

$$= \frac{40 \times 2}{6} + \frac{17 \times 3}{6} = \frac{80 + 51}{6} = \frac{131}{6}$$

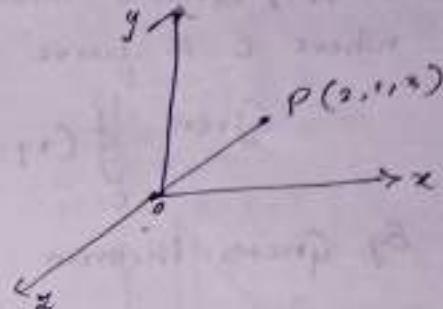
$\Rightarrow$  Green's theorem for plane ( $XY$ -plane)

Let  $M(x,y)$ ,  $N(x,y)$ ,  $M_y = \left(\frac{\partial M}{\partial y}\right)$ ,  $N_x = \left(\frac{\partial N}{\partial x}\right)$ , be continuous differentiable function over the region  $E$ , in coordinate  $XY$ -plane, then

$$\boxed{\int_C M(x,y) dx + N(x,y) dy = \iint_E \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.}$$

$\downarrow$  short form

$$\boxed{\int_C M dx + N dy = \iint_E \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy}$$



$$\text{St line} = \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$= \frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$$= x = 2t \Rightarrow dx = 2 dt$$

$$y = t \Rightarrow dy = 1 dt$$

$$z = 3t \Rightarrow dz = 3 dt$$

### Session - 26

1. Apply Green's theorem to evaluate the integral  $\oint_{C} (xy - y^2) dx + x^2 dy$ , where C is curve bounded by  $y=x$  &  $y=x^2$ .

Given  $\oint_C (xy - y^2) dx + x^2 dy$

By Green's theorem  $\oint_C M dx + N dy = \iint_G \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$ .

where  $M = xy - y^2$ ,  $N = x^2$

$$\frac{\partial M}{\partial y} = x - 2y ; \quad \frac{\partial N}{\partial x} = 2x.$$

$\iint_G (x - 2y - 2x) dxdy$

### Session - 26

2. Apply Green's theorem, evaluate  $\oint_C (3x - 8y^2) dx + (4y - 6xy) dy$ , where C is boundary of region bounded by  $x=0, y=0, x+y=1$ .

Given  $\oint_C (3x - 8y^2) dx + (4y - 6xy) dy$

By Green's theorem for plane:

$\oint_C M dx + N dy = \iint_G \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$ .

where,  $M = 3x - 8y^2$ ,  $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$$

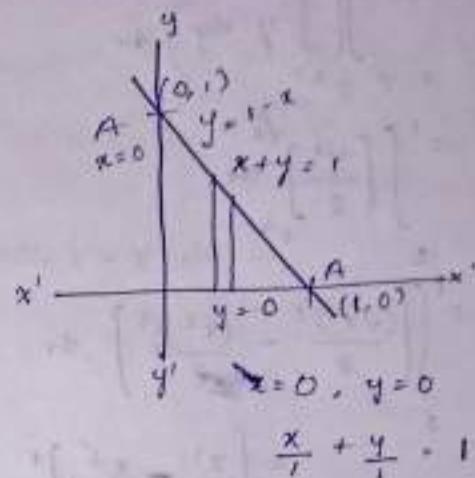
~~upper half~~

x limits from 0 to ~~1~~ 1 ~~1~~

y limits from 0 to ~~1-x~~

=  ~~$\oint$~~  ~~-16y dx - 6y dy~~

$$\begin{aligned}
 & \oint_C (3x^2 - 2y^2) dx + (4y - 3xy) dy = \iint_S -6y + 16y \, dxdy \\
 &= \iint_S 10y \, dxdy \\
 &= \int_{x=0}^1 \left( \int_{y=0}^{1-x} 10y \, dy \right) dx \\
 &= \int_0^1 \left[ \frac{10y^2}{2} \right]_0^{1-x} dx \\
 &= \int_0^1 5(1-x)^2 \, dx = \int_0^1 5(1+x^2 - 2x) \, dx \\
 &= \left[ 5\left(x + \frac{x^3}{3} - \frac{2x^2}{2}\right) \right]_0^1 = \left[ 5x + \frac{5x^3}{3} - \frac{10x^2}{2} \right]_0^1 \\
 &= 5 + \frac{5}{3} - 10 = 5/3 \quad \text{Ans}
 \end{aligned}$$



3. Apply Green's theorem.  
where C is bounded by

$$M = 3x^2 - 2y^2$$

$$\frac{\partial M}{\partial y} = -4y$$

$$\oint_C (3x^2 - 2y^2) dx + (4y - 3xy) dy \quad \text{curves } y = \sqrt{x}, y = x^2.$$

$$N = 4y - 3xy$$

$$\frac{\partial N}{\partial x} = 0 - 3y$$

$$\begin{aligned}
 & \oint_C (3x^2 - 2y^2) dx + (4y - 3xy) dy = \iint_S -8y + 4y \, dxdy \\
 &= \iint_S 4y \, dxdy
 \end{aligned}$$

$$\begin{aligned}
 & \iint_S 4y \, dxdy = \int_0^1 \int_{y=x^2}^{\sqrt{x}} 4y \, dy \, dx \\
 &= \int_0^1 4y^2 \Big|_{y=x^2}^{\sqrt{x}} \, dx = \int_0^1 4(\sqrt{x})^2 - 4(x^2)^2 \, dx = \int_0^1 4x - 4x^5 \, dx
 \end{aligned}$$

$$\begin{aligned}
 & \oint_C (3x^2 - 2y^2) dx + (4y - 3xy) dy = \iint_S 4y \, dxdy \\
 & \text{where } S \text{ is region bounded by curves } y = \sqrt{x} \text{ & } y = x^2
 \end{aligned}$$

$$y^2 = x$$

Limits  $y = x^2 \rightarrow \sqrt{x}$   
 $x = 0 \rightarrow 1$

$$= \int_{x=0}^1 \left( \int_{y=x^2}^{\sqrt{x}} y \, dy \right) dx$$

$$= \int_0^1 \left[ \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[ \left( \frac{\sqrt{x}}{2} \right)^2 - \left( \frac{x^2}{2} \right)^2 \right] dx = \int_0^1 \frac{x}{2} - \frac{x^4}{2} dx$$

$$= \left[ \frac{x^2}{4} - \frac{x^5}{10} \right]_0^1 = \frac{1}{4} - \frac{1}{10} + \frac{10 \cdot 4}{40} = \frac{8^3}{40} = \frac{3}{20} //.$$



$$\begin{aligned} & y^2 = x \\ & x^2 = y, y^2 = x \\ & x^4 = x \Rightarrow \\ & x(x^3 - 1) = 0 \\ & x = 0, x = 1 \\ & y = 0, y = 1 \end{aligned}$$

4. Apply Green's theorem to evaluate the integral over  
 $\oint_C [y - \sin x] dx + \cos x dy$ , where C is plane triangle enclosed by  
 lines  $y = 0$ ,  $x = \pi/2$  &  $y = \frac{2}{\pi}x$ .

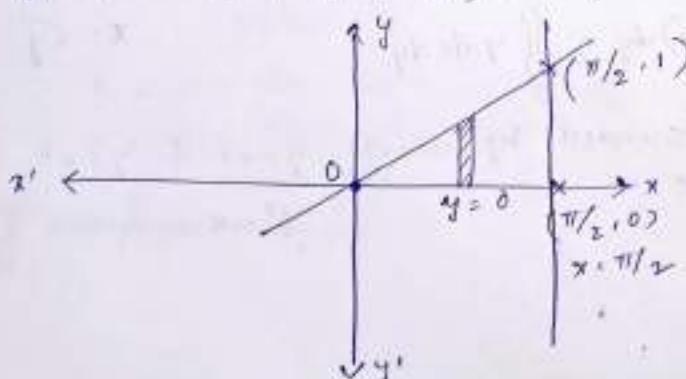
$$M = y - \sin x \quad N = \cos x$$

$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = -\sin x.$$

$$\oint_C [y - \sin x] dx + \cos x dy = \iint_R -\sin x - \frac{1}{\pi} dy dx$$

Region is plane triangle enclosed by the lines

$$y = 0, x = \pi/2 \text{ & } y = (2/\pi)x - y = mx$$



$$\begin{aligned} & y = 0 \\ & x = \pi/2 \\ & y = \frac{2}{\pi}x \\ & y = 2/\pi \times \pi/2 \\ & y = 1 \end{aligned}$$

$$= \int_{x=0}^{\pi/2} \left( \int_{y=0}^{(1/x)x} (-\sin x - 1) dy \right) dx$$

$$= \int_{x=0}^{\pi/2} [-\sin x - x]_{0}^{\pi/2} dx$$

$$\begin{aligned} &= \int_0^{\pi/2} -\frac{2}{\pi} x \sin x - \frac{2}{\pi} x dx = -\frac{2}{\pi} \int_0^{\pi/2} x \sin x + x dx \\ &= -\frac{2}{\pi} \int_0^{\pi/2} x \sin x - \frac{2}{\pi} \int_0^{\pi/2} x dx \end{aligned}$$

INTEGRATE

$$\int f(x) g(x)$$

$$= f(x) \int g(x) dx - \int \left( f'(x) \cdot \int g(x) dx \right) dx$$

$$= -\frac{2}{\pi} \left[ \left[ x(-\cos x) - \int 1 \cdot \int -\cos x dx \right] \right]_0^{\pi/2} - \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left[ -x \cos x + \sin x \right]_0^{\pi/2} - \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left[ \left( -\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (0+0) \right] - \frac{2}{\pi} \frac{(\pi/2)^2}{2}$$

$$= -\frac{2}{\pi} (1) - \frac{2}{\pi} \left( \frac{\pi^2}{8} \right)$$

$$= -\left[ \frac{2}{\pi} + \frac{\pi^2}{4} \right] \text{Ans}$$

## Session - 27

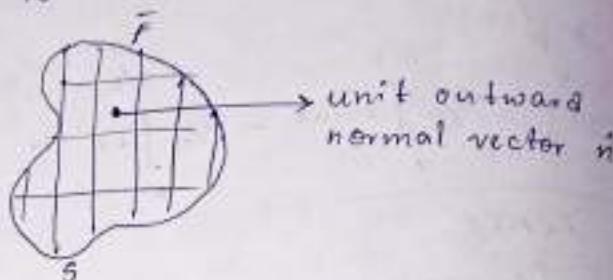
### 1. Surface integral

### 2. Stoke's theorem

1. Surface integral: Consider a continuous vector point function

$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ . If  $\hat{n}$  is a unit outward normal vector to the surface at any point, then surface integral of  $\vec{F}$ , over the given surface 'S' is

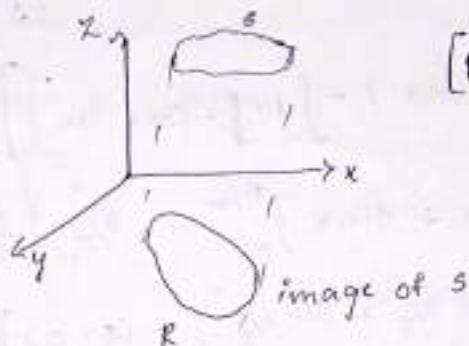
$$\int_S \vec{F} \cdot \hat{n} \, ds$$



Projects of surface:-

(i) The projection of surface on XY-plane.

\*  $ds = \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$



[projection of XY-plane  
is Z i.e.  $\hat{n} \cdot \hat{k}$ ]

(ii) Projection on YZ-plane

$$ds = \frac{dy dz}{|\hat{n} \cdot \hat{i}|}$$

(iii) Projection of surface on ZX-plane

$$ds = \frac{dx dz}{|\hat{n} \cdot \hat{j}|}$$

2. Stoke's theorem (Relation btw Line & Surface integral)

If S be a open surface bounded by a closed curve 'C'

&  $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ , be continuously differentiable over the

vector point function, then

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl } \bar{F} \cdot \bar{N} \, ds$$

where  $\bar{N}$  is unit external normal vector to surface  $S'$

$$N = \bar{k}; \, ds = \frac{dx dy}{|\bar{k} \cdot \bar{k}|} = dx dy$$

$$\frac{dx dy}{|\bar{k} \cdot \bar{k}|} = \frac{dx dy}{|n \cdot k|}$$

1. Apply Stokes theorem to evaluate  $\oint_C \bar{F} \cdot d\bar{r}$ , where  $\bar{F} = y^2 \hat{i} + x^2 \hat{j} - (x+z) \hat{k}$  &  $C$  is boundary of triangle with vertices at  $(0,0,0)$ ,  $(1,0,0)$  &  $(1,1,0)$

Given vector point function,

$$\bar{F} = y^2 \hat{i} + x^2 \hat{j} - (x+z) \hat{k}$$

By Stokes theorem.

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl } \bar{F} \cdot \bar{N} \, ds$$

Now,  $\text{curl } \bar{F} = \nabla \times \bar{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$\begin{aligned} & \Rightarrow \hat{i}([0] - [0]) - \hat{j}(-1 - 0) + \hat{k}(2x - 2y) \\ & \quad = +\hat{j} + \hat{k}(2x - 2y) \end{aligned}$$

$S$  is surface of triangle  $(0,0,0)$ ,  $(1,0,0)$ ,  $(1,1,0)$ .

Since triangle region lies on  $xy$ -plane,

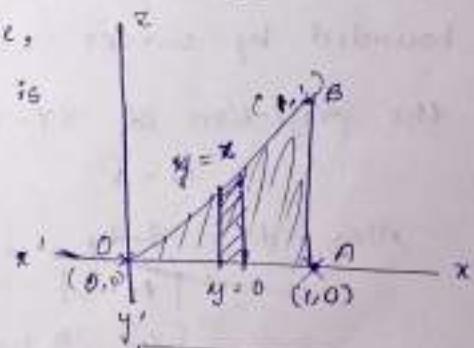
the projection of  $xy$ -plane is  $z$ -axis is

i.e.  $N = \bar{k}$

Also,  $ds = \frac{dx dy}{|\bar{k} \cdot \bar{k}|} = \frac{dx dy}{|\bar{k} \cdot \bar{k}|} = dx dy$

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R [j + (2x - 2y) \hat{k}] \cdot \bar{k} \cdot dx dy$$

$$\Rightarrow \iint_R (2x - 2y) dx dy$$



$$z=0 \Rightarrow xy\text{-plane}$$

$$= \int_{x=0}^1 \left( \int_{y=0}^2 (2x - 2y) dy \right) dx$$

$$= \int_0^1 \left[ xy - \frac{y^2}{2} \right]_0^2 dx$$

$$= \int_0^1 2x^2 - \frac{8x^2}{2} dx \Rightarrow \int_0^1 x^2 dx$$

$$= \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3} //$$

2. Apply Stokes theorem, for vector field  $\bar{F} = (x^2 - y^2)i + 2xyj$   
integrate around rectangle in plane  $z=0$  & bounded by  
curves  $x=0, y=0, x=a, y=b$

$$\begin{aligned} \text{Curl } \bar{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2) & 2xy & 0 \end{vmatrix} \\ &= i(0-0) - j(0-0) + k(2y+2y) \\ &= 4yk \end{aligned}$$

$$\rightarrow \oint_C \bar{F} \cdot d\bar{r}$$

$S$  is a surface around rectangle in plane  $z=0$   
bounded by curves  $x=0, y=0, x=a, y=b$

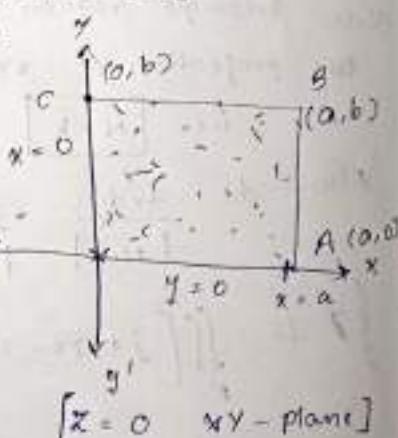
the projection of XY-plane

$$N = \bar{k}$$

$$\text{Also } ds = \frac{dx dy}{|N \cdot \bar{k}|} = \frac{dx dy}{|\bar{i} \cdot \bar{k}|} = dx dy$$

$$= \oint_C \bar{F} \cdot d\bar{r} = \iint_{x=0, y=0}^{a, b} 4yk \cdot \bar{k} dx dy$$

$$= \iint_{x=0, y=0}^{a, b} 4y ) dy ) dx$$



$$= \int_0^a \left[ \frac{xy^2}{2} \right]_0^b dx = \int_0^a ab^2 dx$$

$$= [ab^2 x]_0^a = ab^2 a = ab^3 //.$$

3. Apply Stoke's theorem for vector field  $\bar{F} = x^2 i + xy j$  integrate around square

$$x=0, y=0, x=a, y=a$$

$$\bar{F} = x^2 i + xy j$$

Square in plane  $z=0$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(y-0)$$

$$= yk$$

$$= \oint_C F \cdot d\bar{r}$$

$$x=0, y=0, x=a, y=a$$

the projection of XY-plane

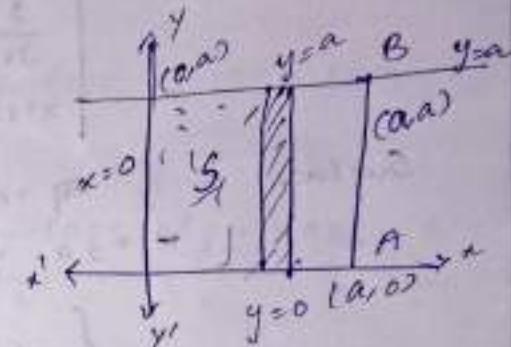
$$N = \bar{k}$$

$$\text{Also } ds = \frac{dx dy}{|N \cdot \bar{k}|} = \frac{dx dy}{|\bar{k} \cdot \bar{k}|} = dx dy$$

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \int_{x=0}^a \int_{y=0}^a y \bar{k} \cdot \bar{k} dx dy$$

$$= \int_{x=0}^a \left( \int_{y=0}^a y dy \right) dx$$

$$= \int_0^a \left[ \frac{y^2}{2} \right]_0^a dx = \int_0^a \frac{a^2}{2} dx \Rightarrow a^2/2 [x]_0^a \\ \Rightarrow a^3/2.$$



4. Apply Stokes theorem to evaluate  $\oint_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = (x^2 + y^2)i - xyj$ , where  $C$  is the boundary of the rectangle bounded by lines  $x=a$ ,  $y=0$ ,  $y=b$ .

$$\vec{F} = (x^2 + y^2)i - xyj$$

By Stoke's theorem  $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$ .

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -xy & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y - 0) = -4y\vec{k}$$

Surface  $S$  is a rectangle in plane  $z=0$  i.e.,  $xy$ -plane.

$$x = \pm a, y = 0, y = b.$$

$$\begin{aligned} \iint_S -4y\vec{k} \cdot \vec{k} dS &\Rightarrow \iint_S -4y dxdy \\ &= \int_{-a}^a \left( \int_{y=0}^b -4y dy \right) dx \\ &= \int_{x=-a}^a \left[ -\frac{4y^2}{2} \right]_0^b dx = \int_{x=-a}^a -2b^2 dx \\ &= \left[ -2b^2 x \right]_{-a}^a \\ &= -2b^2 a - (-2b^2(-a)) \\ &= -4ab^2 //. \end{aligned}$$

### Tutorial-10

1. Find gradient of the function  $f = x^3y^2 - y^3z - 7(z^3+x)$  at the point  $(1, 2, 1)$
  2. Find divergence & curl of  $\vec{f} = (xyz)\vec{i} + (x^2y)\vec{j} - 3yz\vec{k}$  at  $(1, 1, 1)$
  3. Compute directional derivative by  $f = xy^2 - 2y^3z$  in direction of vector  $\vec{i} + 2\vec{j} + 2\vec{k}$  at point  $(2, -1, 1)$
  4. Identify angle between normal to surface by  $xy = 8z^2$  at points  $(1, 2, 3)$  &  $(2, 2, 2)$
  5. Find unit normal vector to the surface  $x^3y - 4xz = 2$  at point  $(1, -1, 3)$
- 

$$1. \text{ Grad } \vec{f} = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$= \frac{\partial}{\partial x} (x^3y^2 - y^3z - 7(z^3+x)) + \frac{\partial}{\partial y} (x^3y^2 - y^3z - 7(z^3+x)) \\ + \frac{\partial}{\partial z} (x^3y^2 - y^3z - 7(z^3+x))$$

$$= 3x^2y^2 - 0 - 7 + x^3y - 3y^2z - 0 + 0 - y^3 - 21z^2$$

~~$= (3x^2y^2 + x^3y - 3y^2z) \vec{i} + y^3 \vec{j} - 21z^2 \vec{k}$~~

~~$(1, 2, 1) = 3(1)^2(2)^2(7) + (1)^3(2) - 3(2)^2(1) - (2)^3 - 21(1)^2(-7)$~~

~~$= 56 \vec{i} + 2 \vec{j} + 12 \vec{k}$~~

$\Rightarrow (12 - 7)\vec{i} + [(1)^3(2) - 3(2)^2(1)]\vec{j} + (-4^3 - 21(1)^2(-7))\vec{k}$

$= 5\vec{i} - 10\vec{j} - 29\vec{k}$

$\frac{5}{21}$

$$2. \quad f = (2xyz) i - (x^2y) j - 3y^2z k \text{ at } (1, 1, 1)$$

$$\boxed{\begin{aligned} \operatorname{div} f &= -2 \\ \operatorname{curl} f &= -6i + 2j \end{aligned}}$$

$$\begin{aligned} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & -x^2y & -3y^2z \end{vmatrix} = i(-6yz - 0) + j(0 - 2xz) \\ &\quad + k(-2xy - 2xz) = -6yz i + 2xz j + k(-2xy - 2xz) \\ &= -6\bar{i} + 2\bar{j} - 4\bar{k} \\ \operatorname{curl} f &= -6\bar{i} + 2\bar{j} - 4\bar{k} \end{aligned}$$

$$\begin{aligned} \operatorname{div} f &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= \cancel{[2yz - 2xy]}_0 + \cancel{[2xz - x^2 - 6yz]} + \cancel{[2xy - 0 - 3y^2]} \\ &= \cancel{-2} - \cancel{x} + \cancel{x} - 1 - 6 + \cancel{2} = 3 \\ &= 2yz - x^2 - 3y^2 \\ &= 2 - 1 - 3 = -2 // \end{aligned}$$

$$3. \quad D \cdot D = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\phi = xy^2 - 2y^3z$$

$$a \Rightarrow i + 2\bar{j} + 2\bar{k} \text{ at } (2, -1, 1)$$

$$\vec{a} = i + 2\bar{j} + 2\bar{k}$$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

$$= y^2 i + (2xy - 6y^2z) j + (-2y^3) k$$

$$= (-1)^2 i + [2(2)(-1) - 6(-1)^2(-1)] j + [-2(-1)^3] k$$

$$= (i - 10j + 2k) \cdot (i + 2j + 2k) = \frac{1 - 80 + 4}{3} = -\frac{75}{3} = -25$$

$$4. \quad \cos \theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|}$$

$$xy = 2xz$$

$$f = xy - 2xz$$

$\bar{n}_1$  at  $(1, 2, 1)$  &  $\bar{n}_2$  at  $(1, 2, 2)$

$$\begin{aligned}\bar{n}_1 = \nabla f \text{ at } (1, 2, 1) &= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \\ &= (y)i + (x)j + (-4z)k \\ &= 2i + j - 4z k\end{aligned}$$

$$\begin{aligned}\bar{n}_2 = \nabla f \text{ at } (2, 2, 2) &= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \\ &= (y)i + (x)j + (-4z)k \\ &= 2i + 2j - 8k\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|} = \frac{4+2-96}{\sqrt{4+1+16} \sqrt{4+4+64}} \\ &= \frac{-90}{\sqrt{149} \sqrt{72}}\end{aligned}$$

5.  $x^3y - 4xz = 2$  at  $(1, -1, 3)$

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \\ &= (3x^2y - 4z)i + (x^3)j + (-4x)k \\ &= [3(1)^2(-1) - 4(3)]i + (1)^3j - 4k \\ &= -15i + j - 4k\end{aligned}$$

$$|\nabla f| = \sqrt{225 + 1 + 16} \Rightarrow \sqrt{242} = \sqrt{2 \times 121} \Rightarrow 11\sqrt{2}$$

$$\hat{n} = \frac{-15i + j - 4k}{11\sqrt{2}}$$

$\frac{15 \times 16}{121}$   
 $\sqrt{242}$   
 $\sqrt{2}$   
 $\sqrt{121}$   
 $\sqrt{2}$

### Tutorial - 11

1. A vector field  $\vec{F} = (2y+3)\hat{i} - 2xz\hat{j} - 3(yz-x)\hat{k}$   
 Evaluate  $\oint_C \vec{F} \cdot d\vec{r}$  along path C. if  $x=2t$ ,  $y=t$ ,  $z=t^3$   
 from  $t=0$  to  $t=1$
2. If  $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ , evaluate  $\oint_C \vec{F} \cdot d\vec{r}$  where C is rectangle in XY-plane bounded by  $x=0+a$ ,  $y=0$ ,  $y=b$ ,  $x=a+b$
3. Find  $\oint_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = x^2y^2\hat{i} + \hat{y}\hat{j}$  & the curve  $y^2 = 4x$  in XY-plane from  $(0,0)$  to  $(4,4)$ .

$$1. \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C (2y+3)\hat{i} - (2xz)\hat{j} - 3(yz-x)\hat{k} \cdot (i dx + j dy + k dz)$$

$$= \oint_C 2y+3 \cdot dx - 2xz \cdot dy - 3(yz-x) \cdot dz$$

$$x = 2t \rightarrow dx = 2dt$$

$$y = t \rightarrow dy = dt$$

$$z = t^3 \rightarrow dz = 3t^2 dt$$

$$= [2(4t)+3] \cdot 2dt - 2(2t)(t^3)dt - [3(t(t^3)-2t) \cdot 3t^2]dt$$

$$= \int_0^1 [4t+6 - 4t^4 - 9t^6 + 18t^3] dt$$

$$= \left[ \frac{4t^2}{2} - \frac{4t^5}{5} - \frac{4t^8}{8} - \frac{9t^7}{7} + \frac{18t^4}{4} \right]_0^1$$

$$= 2 - 4/5 - 9/7 + 9/2$$

$$= -4 - 4/5 - 9/7 + 9/2$$

$$=$$

$$\begin{aligned}
 & -4 - \frac{4}{5} - \frac{9}{7} + \frac{9}{2} \\
 & = -\frac{20}{5} - \frac{4}{5} - \frac{70}{7} + \frac{81}{14} \\
 & = -\frac{24}{5} - \frac{70}{7} + \frac{81}{14} \\
 & = -\frac{336}{35} - \frac{350}{35} + \frac{315}{35} \\
 & = -\frac{671}{35}
 \end{aligned}$$

CO-3 Revision problems

(1) Apply Beta functions, evaluate the integral  $\int_0^{\pi/2} \sin^a \theta \cos^b \theta d\theta$

(2) Evaluate integral  $\iint xy dx dy$  in the positive quadrant for which  $x+y \leq 1$ .

(3) Evaluate the integral  $\int_0^{\sqrt{r-x^2}} \int_0^{(x^2+y^2)} dy dx$  by changing into polar coordinates.

$$2m+1 = 3$$

$$2n+1 = 7$$

$$2m = 4$$

$$2n = 8$$

$$\boxed{m=2}$$

$$\boxed{n=4}$$

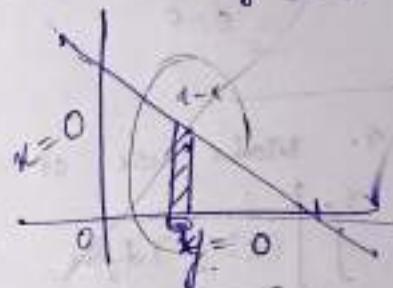
$$= 2 \cdot \frac{1}{2} \int_0^{\pi/2} \sin^{2(2)-1} \theta \cos^{2(4)-1} \theta d\theta = \beta(3,4)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{\Gamma(3) \Gamma(4)}{\Gamma(6)} = \frac{1}{2} \cdot \frac{2! \times 3!}{5!}$$

$$\Rightarrow \frac{3 \times 2}{2 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{40}$$

$$x+y=1$$

$$y = 1-x$$



$$\iint xy dx dy$$

$$\text{for } x+y \leq 1.$$

$$\int_{x=0}^{1-x} \left( \int_{y=0}^{1-x} xy dy \right) dx$$

$$x=0, y=0$$

$$\int_0^1 \left[ \frac{xy^2}{2} \right]_0^{1-x} dx \Rightarrow \int_0^1 \frac{x(1-x)^2}{2} dx = \int_0^1 \frac{x-x^3}{2} dx$$

$$= \frac{1}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \Rightarrow \frac{1}{2} \left( \frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{1}{2} \left( \frac{3-2}{6} \right)$$

$$\begin{aligned} &= \int_0^1 \frac{x(4x^2-2x)}{2} dx = \frac{1}{2} \int_0^1 x + x^3 - 2x^2 dx = \frac{1}{2} \left[ \frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3} \right] \\ &\Rightarrow \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right] \end{aligned}$$

$$= \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right]$$

$$= \frac{1}{2} \left( \frac{6+3-8}{12} \right) = \frac{1}{24}$$

$$\frac{1}{2} \times 12 \\ 2 \quad 8 \\ 12$$

$$\begin{array}{c} 1, 4, 3 \\ 2, 3, 3 \\ 3, 1, 1 \\ 1, 1, 1 \\ 12 \end{array}$$

$1 \times 3 = 12$

3.  $\int \int (x^2+y^2) dy dx$

$$x=0, y=0 \quad x=r\cos\theta, y=r\sin\theta \\ dy dx = r dr d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \cdot r dr d\theta$$

$$y=0 \text{ to } y=\sqrt{1-x^2}$$

$$y^2 = 1 - x^2$$

$$x^2 + y^2 = 1$$

$$x=0 \text{ to } x=1$$

$$= \int_{\theta=0}^{\pi/2} \left[ \frac{r^4}{4} \right]_0^1 dr d\theta \Rightarrow \frac{1}{4} [0]_0^{\pi/2} \\ = \frac{\pi}{8} //$$

4. Make use of change the order of integration evaluate the

$$\int_0^3 \int_x^3 (x^2 - y^2) dy dx$$

Given  $\int \int (x^2 - y^2) dy dx$ .

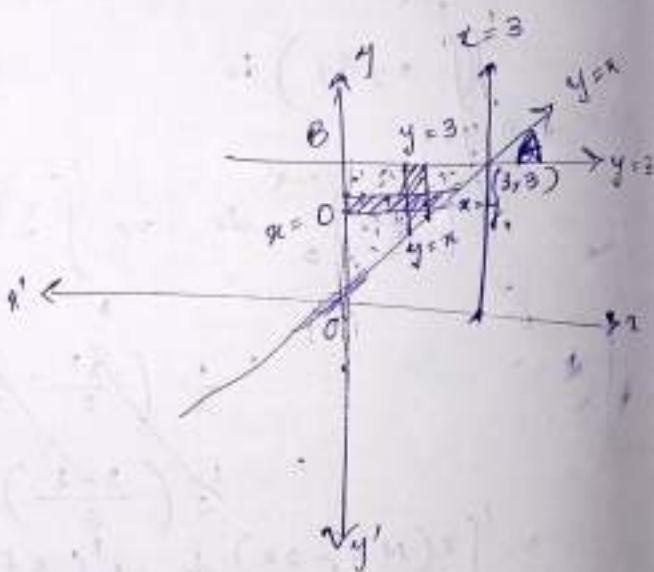
$$x=0 \text{ to } x$$

$$y=x \text{ to } y=3$$

$$x=0 \text{ to } x=3$$

By changing the order of integration

$$\int_{y=0}^3 \left( \int_{x=0}^y (x^2 - y^2) dx \right) dy$$



$$\int_0^3 \left[ \frac{x^3}{3} - y^2 x \right]_0^y dy$$

$$\int_0^3 \frac{y^3}{3} - y^5 dy \Rightarrow \int_0^3 \frac{1}{3} \cdot \frac{y^4}{4} - \frac{y^4}{4}$$

$$\int_0^3 \left[ \frac{y^4}{12} - \frac{y^4}{4} \right] = \frac{(3)^4}{12} - \frac{(3)^4 \times 3}{4}$$

$$\frac{(3)^4}{12} - \frac{(3)^4}{4} = \frac{81 - 243}{12} = -\frac{172}{12} = -\frac{86}{6} = -\frac{43}{3}$$

5. Evaluate integral  $\iiint e^{x+y+z} dx dy dz$

$$\iiint e^{x+y+z} dx dy dz = e^{x+y+z} \Big|_0^1 = \frac{e^{x+y+z}}{2}$$

$$\iiint e^x \cdot e^y \cdot e^z dz dy dx$$

$$x=0, y=0, z=0$$

$$\int_0^1 \int [e^x]_0^1 e^y \cdot e^z dy dz$$

$$= \int_0^1 \int [e^y - e^0] dy dz = \int_0^1 \left[ (e-1) \cdot e^y \right] dy dz$$

$$(e-1) \int_0^1 \int e^y dy \cdot e^z dz$$

$$= (e-1)^3$$

$$6. \int_0^a \int_0^a \int_0^a (xy + yz + zx) dx dy dz$$

$$\int_{x=0}^a \int_{y=0}^a \int_{z=0}^a (xy + yz + zx) dz dy dx$$

$$\int_{x=0}^a \int_{y=0}^a \left[ xyx + \frac{yzx^2}{2} + \frac{zx^2y}{2} \right]_0^a dy dx$$

$$= \int_{x=0}^a \int_{y=0}^a \left[ xy^2 + \frac{y^2a^2}{2} + \frac{a^2x^2y}{2} \right] dy dx$$

$$= \int_{x=0}^a \left[ \frac{xy^2a}{2} + \frac{y^2a^2}{4} + \frac{a^2xy}{2} \right]_0^a dx$$

$$= \int_{x=0}^a \left[ \frac{x(a)^2a}{2} + \frac{(a)^2a^2}{4} + \frac{a^2x(a)}{2} \right] dx$$

$$= \left[ \frac{x^2a^3}{2 \times 2} + \frac{a^4a}{4} + \frac{a^3x^2}{2 \times 2} \right]_0^a$$

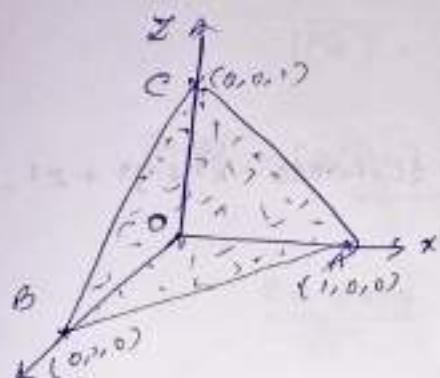
$$= \frac{a^2a^3}{4} + \frac{a^4a}{4} + \frac{a^3a^2}{4}$$

$$= \frac{a^5 + a^5 + a^5}{4} = \frac{3a^5}{4} //$$

7. Evaluate volume of tetrahedron bounded by planes

$$x=0, y=0, z=0; x+y+z=1.$$

$$\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$$



OABC is tetrahedron

Volume of tetrahedron

$$= \iiint dxdydz.$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} \left( \int_{z=0}^{1-x-y} dz \right) dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} [z]_0^{1-x-y} dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} 1-x-y dy dx$$

$$\Rightarrow \int_{x=0}^1 \left[ y - yx - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_{x=0}^1 (1-x) - (1-x)x - \frac{(1-x)^2}{2} dx$$

$$= \int_{x=0}^1 1-x - x + x^2 - \frac{1-x^2+2x}{2} dx$$

$$= \int_{x=0}^1 \frac{2-2x-2x^2+2x^3}{2} dx$$

$$= \int_{x=0}^1 \frac{1-x^2-2x}{2} dx$$

$$\frac{1}{2} \left[ x - \frac{x^3}{3} - \frac{2x^2}{2} \right]_0^1$$

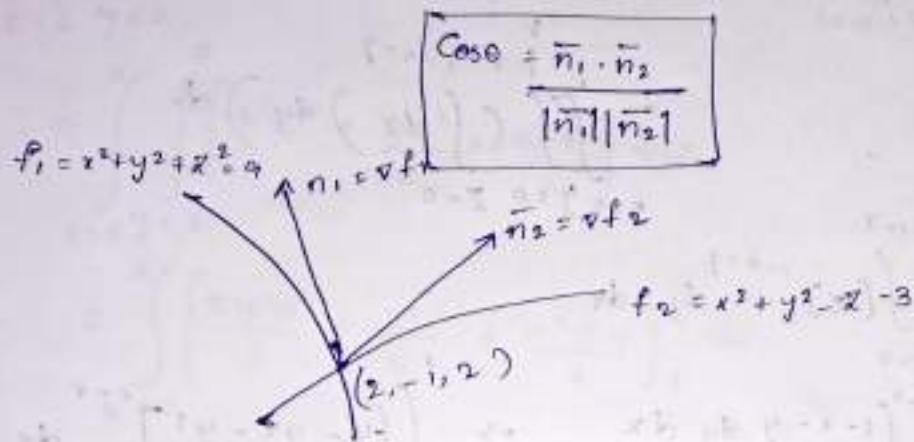
$$\Rightarrow \frac{1}{2} \left( 1 - \frac{1}{3} \right) = -\frac{1}{6}$$

CO-4 Revision - problems

1. Find the directional derivative of  $f = x^2 + y^2 + z^2$  at point  $P(1, 2, 3)$  in direction of line  $PQ$  where  $Q$  is point  $(5, 0, 4)$

$$\boxed{D \cdot D = \nabla f \cdot \frac{\vec{PQ}}{|\vec{PQ}|}}$$

2. Identify the angle between the surfaces  $x^2 + y^2 + z^2 = 9$  &  $x^2 + y^2 - z = 3$  at point  $(2, -1, 2)$



3. Obtain divergence & curl of  $\vec{f} = (xyz)\vec{i} - 3x^2y\vec{j} + y^2z\vec{k}$  at point  $(1, 2, 1)$

1.  $f = x^2 - y^2 + z^2$

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$= (2x - 0) \hat{i} + (0 - 2y) \hat{j} + (2z) \hat{k}$$

$$= (2 - (2)^2(1) + 2(1)^2(1)) \hat{i} + ((2)(1)^2 - 2(2) - 2(1)^2(1)) \hat{j}$$

$$+ ((1)^2(1) - (2)^2(1) + 4(1)) \hat{k}$$

$$= (2 - 4 + 2) \hat{i} + (2 - 4 - 4) \hat{j} + (1 - 4 + 4) \hat{k}$$

$$= -6 \hat{j} + \hat{k}$$

$$rf = 2x - 2y + 4z$$

$$= 2xi - 2yj + 4zk \quad = 2i - 4j + 4k$$

$$\vec{PQ} = OQ - OP = 5i + 4k - i - 2j - 3k \\ = 4i - 2j + k$$

$$|\vec{PQ}| = \sqrt{16+4+1} = \sqrt{21}$$

$$\Theta = \frac{(2i - 4j + 4k) \cdot (4i - 2j + k)}{\sqrt{21}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

$$2 + \sqrt{12} = x^2 + y^2 + z^2 = 9$$

$$2xi + 2yj + 2zk \Rightarrow 4i - 2j + 4k = 6$$

$$f_1 = x^2 + y^2 - z = 3$$

$$= 2xi + 2yj - * \Rightarrow 4i - 2j - k.$$

$$= \frac{(4i - 2j + 4k) \cdot (4i - 2j - k)}{\sqrt{16+4+16} \sqrt{16+4+1}}$$

$$= \frac{16 + 4 - 4}{\sqrt{36} \sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\Theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$