Signals and Communication Systems

Laplace Transform

Laplace Transform

(Generalization of Fourier Transform)

Laplace Transform

Laplace Transform is referred to as generalization of FT.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$ **Analysis equation**

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\Omega}^{\sigma+j\Omega} X(s)\, e^{st} d\Omega$$ **Synthesis equation**

- For $s = \sigma + j\Omega$, $X(\sigma + j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}dt$

$$= \int_{-\infty}^{\infty} \left\{e^{-\sigma t}x(t)\right\}e^{-j\Omega t}dt\ .$$

That is LT can be interpreted as FT of $\left\{e^{-\sigma t}x(t)\right\}$.

- If $s = j\Omega$, then above equations becomes FT.

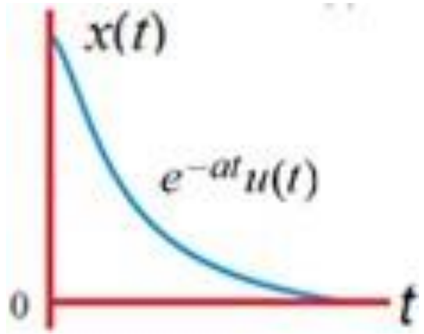
- We note, in particular, that just as the Fourier transform does not converge for all signals, the Laplace transform may converge for some values of $\mathrm{Re}\{s\}$ and not for others.

- ROC: In general, the range of values of '*s*' for which the integral converges is referred to as the *region of convergence* (ROC) of the Laplace transform.

- The Laplace transform depends on a complex variable $s = \sigma + j\Omega$, composed of damping σ and frequency Ω, while the Fourier transform considers only frequency Ω.

LT of standard signals:

Right sided Signals: $x(t) = e^{-at}u(t)$

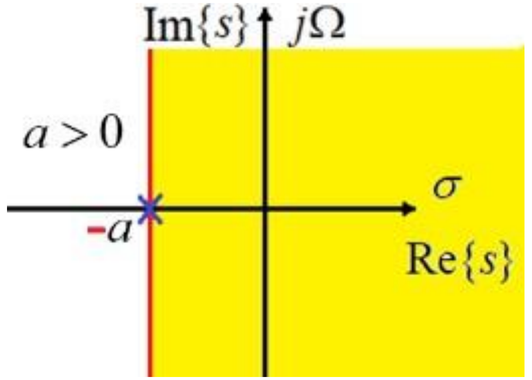


$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^{\infty}$$

$$= \frac{1}{s+a}, \quad \mathrm{Re}(s+a) > 0 \;\; or \;\; \mathrm{Re}(s) > -a$$

$$X(s) = \frac{1}{s+a}, \;\; \mathrm{Re}\{s\} > -a$$

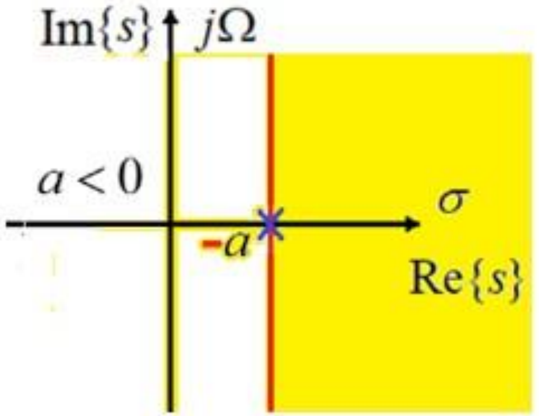


FT exists.

$$e^{-at}u(t) \xleftrightarrow{\text{L.T}} \frac{1}{s+a}, \;\; \mathrm{Re}\{s\} > -a$$

We observe that: Zeros - Nil

Poles at $s = -a$



FT don't exists

Ex1: Find the LT of a signal $x(t) = e^{-2t}u(t)$.

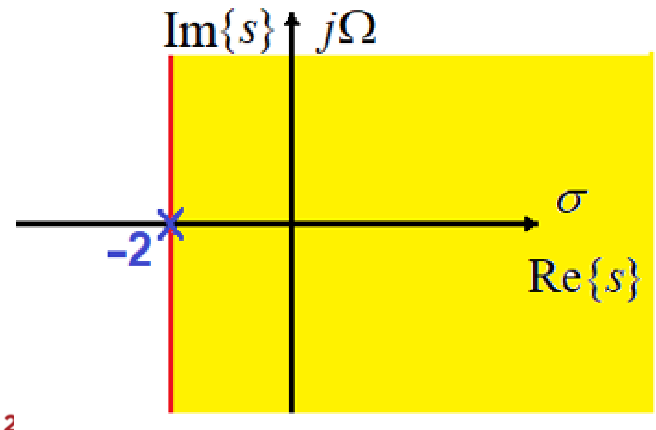
Sketch the pole-zero diagram and indicate the ROC.

Ans:

We know that $e^{-at}u(t) \xleftrightarrow{\text{L.T}} \dfrac{1}{s+a}, \quad \text{Re}\{s\} > -a$

Then $e^{-2t}u(t) \xleftrightarrow{\text{L.T}} \dfrac{1}{s+2}, \quad \text{Re}\{s\} > -2$

A pole at $s = -2$. The pole zero diagram with ROC is shown in figure.



Ex2: Find the LT of a signal $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$.

Sketch the pole-zero diagram and indicate the ROC.

Ans: We know that $e^{-2t}u(t) \xleftrightarrow{L.T} \dfrac{1}{s+2}, \; \mathrm{Re}\{s\} > -2$,

and $e^{-3t}u(t) \xleftrightarrow{L.T} \dfrac{1}{s+3}, \; \mathrm{Re}\{s\} > -3$

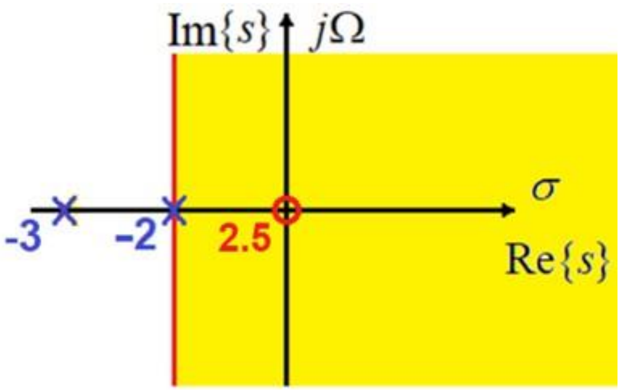
Combinedly, $X(s) = \dfrac{1}{s+2} + \dfrac{1}{s+3} = \dfrac{2s+5}{s^2+5s+6}, \; \mathrm{Re}\{s\} > -2$

ROC $\mathrm{Re}\{s\} > -2$ is chosen as the common area of $\mathrm{Re}\{s\} > -2$ and $\mathrm{Re}\{s\} > -3$

We observe that Zeros: A zero at s = 2.5, and

Poles: Two poles: at s= -2 and s=-3.

The pole zero diagram with ROC is shown in figure.



Ex3. Find the signal $x(t)$ if $X(s) = \frac{2s+5}{s^2+5s+6}, \quad \mathrm{Re}\{s\} > -2$

Ans: By partial fraction expansion,

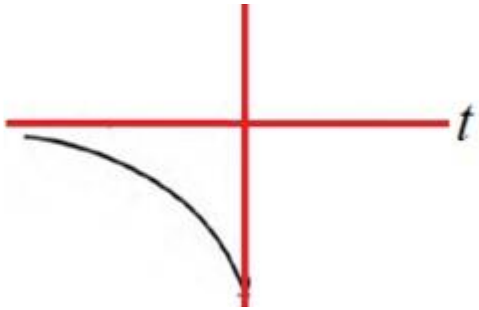
$$X(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} = \frac{1}{(s+2)} + \frac{1}{(s+3)}$$

Since the ROC $\mathrm{Re}\{s\} > -2$, the signal is right sided signal.

With a standard format $e^{-at}u(t) \xleftrightarrow{\text{L.T}} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$

We found that $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$

Left sided Signals: $x(t) = -e^{-at}u(-t)$

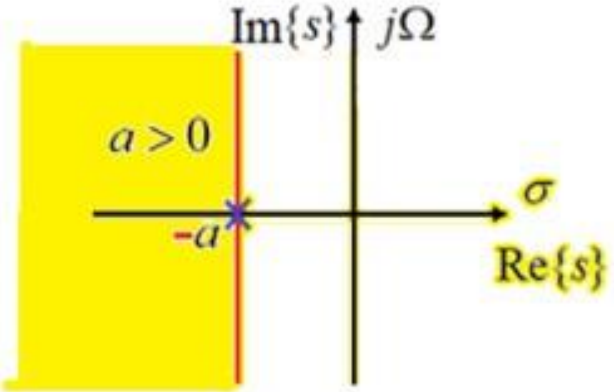


$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{-\infty}^{\infty} -e^{-at} u(-t)\, e^{-st} dt$$

$$= \int_{-\infty}^{0} -e^{-(s+a)t} dt = \frac{1}{s+a}\left[e^{-(s+a)t}\right]_{-\infty}^{0}$$

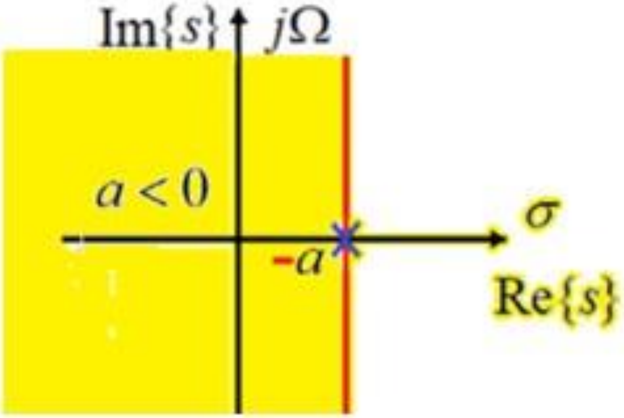
$$= \frac{1}{s+a}, \quad \mathrm{Re}(s) < -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\text{L.T}} \frac{1}{s+a}, \ \mathrm{Re}\{s\} < -a$$



We observe that: Zeros - Nil

Poles at $s = -a$



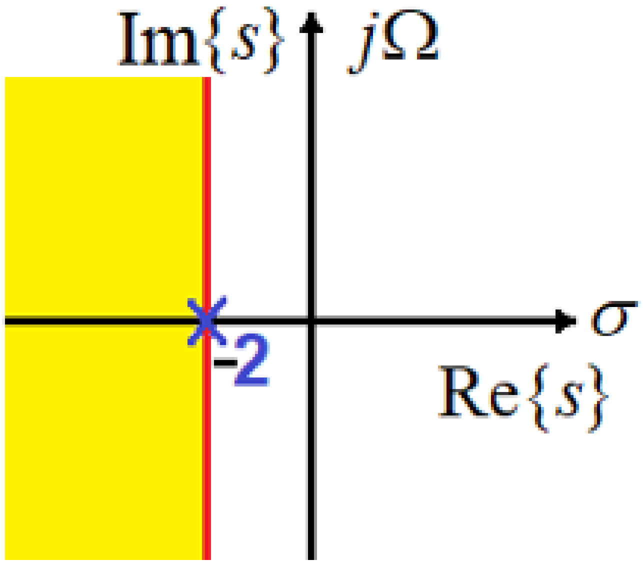
Ex4: Find the LT of a signal $x(t) = -e^{-2t}u(-t)$.

Sketch the pole-zero diagram and indicate the ROC.

Ans: We know that $-e^{-at}u(-t) \xleftrightarrow{\text{L.T}} \dfrac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$

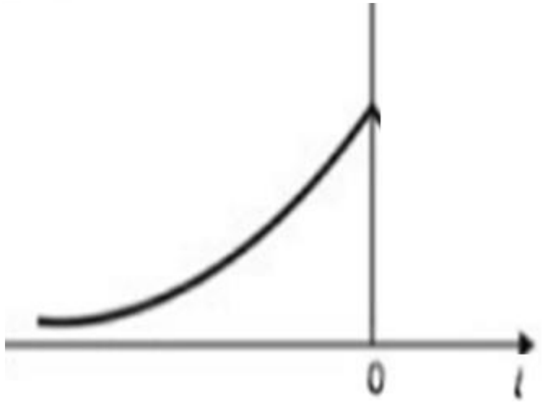
Then $-e^{-2t}u(-t) \xleftrightarrow{\text{L.T}} \dfrac{1}{s+2}, \quad \mathrm{Re}\{s\} < -2$

A pole at $s = -2$. The pole zero diagram with ROC is shown in figure.



Ex5: Find the LT of a signal $x(t) = e^{at}u(-t)$.

Sketch the pole-zero diagram and indicate the ROC.

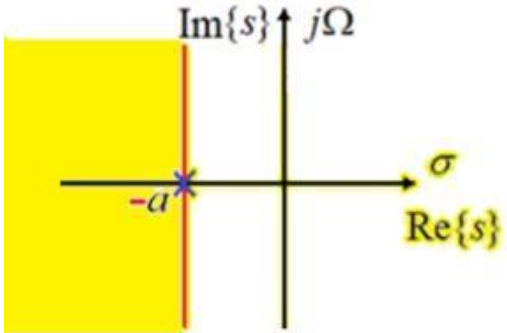


Ans: Using direct LT integral formula

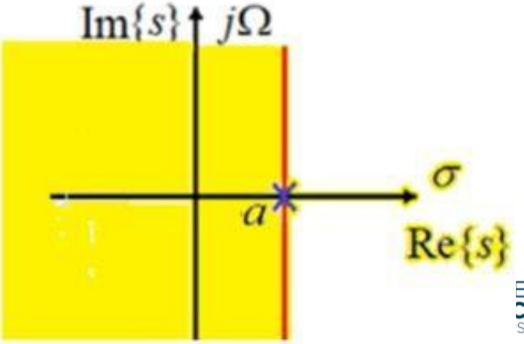
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{-\infty}^{\infty} e^{at} u(-t)\, e^{-st} dt$$

$$= \int_{-\infty}^{0} e^{-(s-a)t} dt = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_{-\infty}^{0}$$

$$= -\frac{1}{s-a}, \quad \mathrm{Re}(s) < a$$

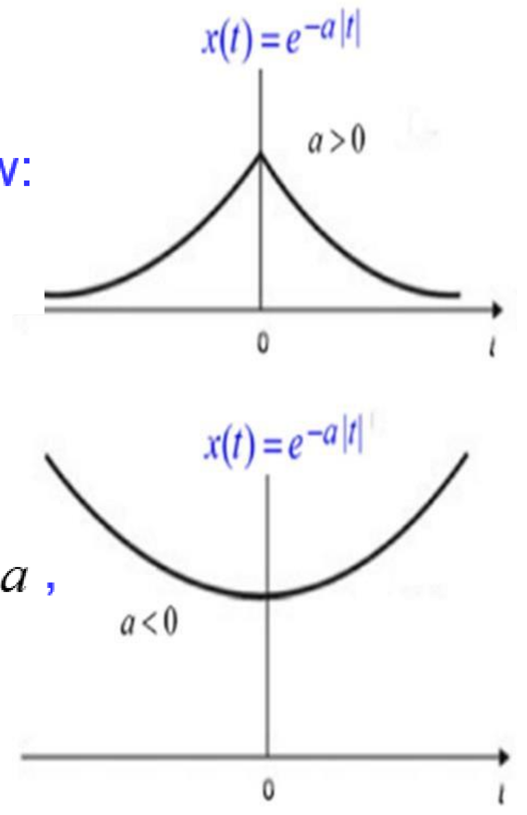


$$e^{at}u(-t) \xleftrightarrow{\text{L.T}} -\frac{1}{s-a}, \quad \mathrm{Re}\{s\} < a$$



Two sided Signals: $x(t) = e^{-a|t|}$

The time domain representation of signal is shown below:



The given signal can be represented as

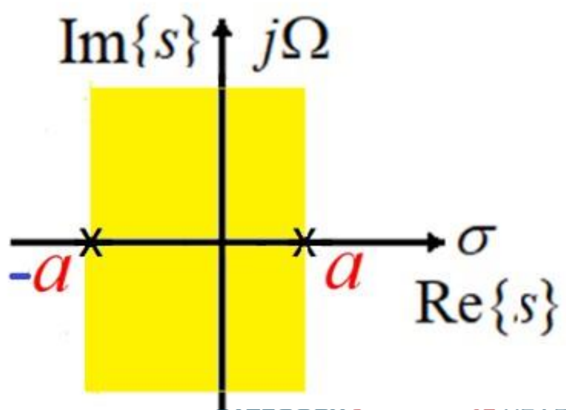
$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

We know that $e^{-at}u(t) \xleftrightarrow{\text{L.T}} \dfrac{1}{s+a}, \ \mathrm{Re}\{s\} > -a$,

and $e^{at}u(-t) \xleftrightarrow{\text{L.T}} -\dfrac{1}{s-a}, \ \mathrm{Re}\{s\} < a$

Then $e^{-a|t|} \xleftrightarrow{\text{L.T}}$

$$\frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2 - a^2}, \ -a < \mathrm{Re}\{s\} < a$$



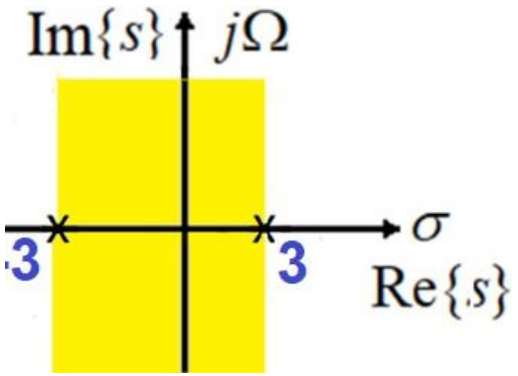
Ex6: Sketch the pole-zero diagram and find the signal $x(t)$,

$$\text{if } X(s) = \frac{2}{s^2 - 9}, \quad -3 < \text{Re}\{s\} < 3 .$$

Ans: The pole-zero diagram with ROC is shown in figure:

Given that $X(s) = \dfrac{2}{s^2 - 9}, \quad -3 < \text{Re}\{s\} < 3$ which is a LT of two-sided signal.

We know that $e^{-a|t|} \xleftrightarrow{\text{L.T}} \dfrac{2a}{s^2 - a^2}, \quad -a < \text{Re}\{s\} < a$



From the given equation, we observe that $a = 3$,

and the given equation can be represented as

$$X(s) = \frac{2a}{s^2 - a^2} = \frac{2}{s^2 - 9} = 2\frac{1}{6}\left(\frac{6}{s^2 - 3^2}\right) = \frac{1}{3}\left(\frac{6}{s^2 - 3^2}\right)$$

Therefore $X(s) = \dfrac{1}{3}\left(\dfrac{6}{s^2 - 3^2}\right), \quad -3 < \text{Re}\{s\} < 3 \quad \xleftrightarrow{\text{L.T}} \quad x(t) = \dfrac{1}{3} e^{-3|t|}$

The required signal is $x(t) = \frac{1}{3} e^{-3|t|}$

Ex7: Find the LT and sketch the pole zero diagrams with ROC for the following signals.

(a) $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$ **(b)** $x(t) = e^{2t}u(-t) + e^{-3t}u(t)$ **(c)** $x(t) = e^{2t}u(t) + e^{-3t}u(-t)$

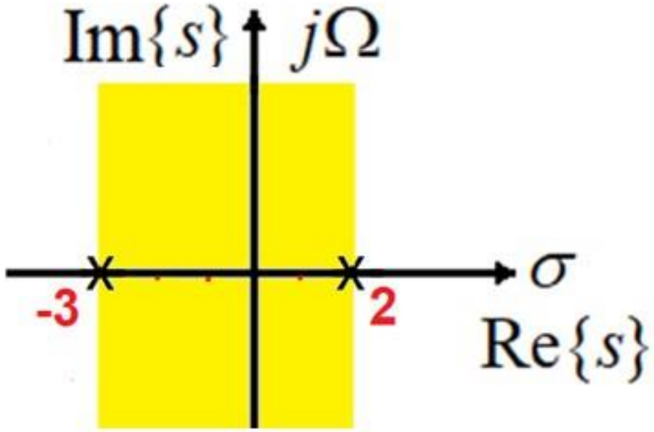
Ans: (a) Refer Ex2.

(b) Given that $x(t) = e^{2t}u(-t) + e^{-3t}u(t)$

We know that $e^{2t}u(-t) \xleftrightarrow{L.T} -\dfrac{1}{s-2}$, $\mathrm{Re}\{s\} < 2$, and

$$e^{-3t}u(t) \xleftrightarrow{L.T} \frac{1}{s+3}, \quad \mathrm{Re}\{s\} > -3$$

Combinedly, $X(s) = -\dfrac{1}{s-2} + \dfrac{1}{s+3} = \dfrac{-5}{s^2+s-6}, \quad -3 < \mathrm{Re}\{s\} < 2$



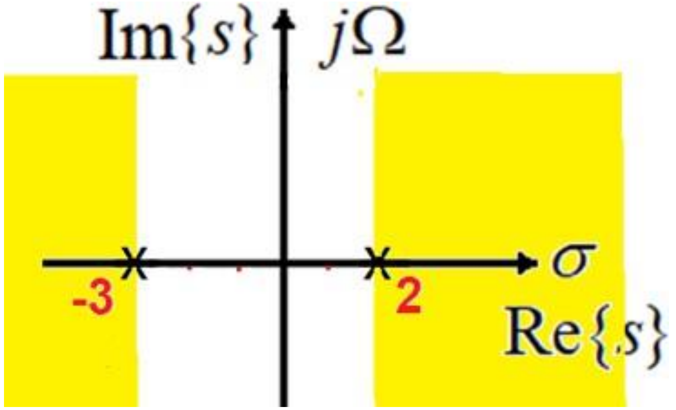
(c) Given that $x(t) = e^{2t}u(t) + e^{-3t}u(-t)$

We know that $e^{2t}u(t) \xleftarrow{\text{L.T}}\!\!\xrightarrow{} \dfrac{1}{s-2}, \quad \mathrm{Re}\{s\} > 2$,

and $e^{-3t}u(-t) \xleftarrow{\text{L.T}}\!\!\xrightarrow{} -\dfrac{1}{s+3}, \quad \mathrm{Re}\{s\} < -3$

Combinedly,

$$X(s) = -\frac{1}{s-2} - \frac{1}{s+3} = \frac{1}{s^2+s-6}, \quad -3 > \mathrm{Re}\{s\} > 2$$



Ex8: Consider a LT of signal is given by $X(s)=\dfrac{2s+5}{s^2+5s+6}$.

Find the corresponding right sided, left sided and two-sided signals.

Ans: By partial fraction expansion,

$$X(s)=\frac{2s+5}{(s+2)(s+3)}=\frac{A}{(s+2)}+\frac{B}{(s+3)}=\frac{1}{(s+2)}+\frac{1}{(s+3)}$$

From the given equation, we observe that there two poles at $s=-2$ and $s=-3$.

Case(i): Right sided signal: For right-sided signal, the ROC must be $\mathrm{Re}\{s\}>-2$

Therefore, the required signal is $x(t)=e^{-2t}u(t)+e^{-3t}u(t)$

Case(ii): Left sided signal: For left-sided signal, the ROC must be $\mathrm{Re}\{s\}<-3$.

Therefore, the required signal is $x(t)=-e^{-2t}u(-t)-e^{-3t}u(-t)$

Case(iii): Two sided signal: For two-sided signal, the ROC must be $-3<\mathrm{Re}\{s\}<-2$.

Therefore, the required signal is $x(t)=-e^{-2t}u(-t)+e^{-3t}u(t)$

Exercise1: Sketch the pole zero diagrams and indicate the ROC and find the inverse LT of the following signals.

(a) $X(s) = \dfrac{1}{s^2 + 5s + 6}, \quad \mathrm{Re}\{s\} > -2$

(b) $X(s) = \dfrac{2s + 4}{s^2 + 4s + 3}, \quad \mathrm{Re}\{s\} > -1$

(c) $X(s) = \dfrac{-5}{s^2 + s - 6}, \quad -3 < \mathrm{Re}\{s\} < 2$

Comparison between FT and LT

Signal	FT	LT
$\delta(t)$	1, for all Ω	1, ROC is the entire '*s*-plane'
$u(t)$	$\pi\,\delta(\Omega)+\frac{1}{j\Omega}$	$\frac{1}{s}$, $\mathrm{Re}\{s\}>0$
$e^{-at}u(t)$	$\frac{1}{a+j\Omega}$	$\frac{1}{s+a}$, $\mathrm{Re}\{s\}>-a$
$-e^{-at}u(-t)$	$-\frac{1}{a+j\Omega}$	$\frac{1}{s+a}$, $\mathrm{Re}\{s\}<-a$
$e^{at}u(-t)$	$\frac{1}{a-j\Omega}$	$-\frac{1}{s-a}$, $\mathrm{Re}\{s\}<a$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\Omega^2}$	$\frac{2a}{s^2-a^2}$, $-a<\mathrm{Re}\{s\}<a$

Comparison between FT and LT

Signal	FT	LT
Time shifting Property. $x(t-t_0)$	$e^{-j\Omega t_0}X(j\Omega)$	$e^{-st_0}X(s)$
Frequency shifting property	$e^{j\Omega_0 t}x(t)\overset{\text{F.T}}{\longleftrightarrow}X(j(\Omega-\Omega_0))$	$e^{s_0 t}x(t)\overset{\text{L.T}}{\longleftrightarrow}X(s-s_0)$
Time differentiation property $\frac{d}{dt}x(t)$	$j\Omega\, X(j\Omega)$	$s\, X(s)$
Time Convolution property $x(t)*h(t)$	$X(j\Omega)\, H(j\Omega)$	$X(s)H(s)$

End