CO-2

Multi differential caluclus :-

1. **derivative** : If $y = f(x)$ is a function of x then it's deeivative is written as $\frac{dy}{dx}$, defined as derivative with respect to independent variable (x) to dependent variable (y)

partial derivative: If $y = f(x)$ is a function of x the partial derivative of y with respect to x is denoted by $\frac{\partial y}{\partial x}$, Here we treated

x as variable remaining all as constants

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\log x = \frac{1}{x}$$

$$\frac{d}{dx}a^x = a^x \log x$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y\frac{dx}{dx} = x\frac{dy}{dx} + y$$

$$\frac{\partial}{\partial x}(xy) = y\frac{\partial x}{\partial x} = y$$

order: The order of the differential equation is the order of the higest derivative involved in the d.e

Degree: The Degree of the D.E is the degree of the higest order derivative

$$\left(\frac{d^2y}{dx}\right)+2\frac{dy}{dx}+5y=0$$

order → 2
degree → 1

$$\left(\frac{d^2y}{dx^2}\right)^3+5x\left(\frac{dy}{dx}\right)^4+e^x y=0$$

order → 2
degree → 3

$$\frac{d^2y}{dx^2}+\left(\frac{d^3y}{dx^3}\right)^2+\left(\frac{dy}{dx}\right)^5+e^x y=0$$

order = 3
degree = 2

First order partial derivatives:

$$f(x,y)$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

Second order partial derivatives:

$$f(x,y)$$

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial x\,\partial y}$$

$\boxed{\text{NOTE:}}$ $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Session-9:

1. find all the first and second order partial derivatives of the function

$f(x.y) = x^2+y^2-2axy, \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Sol: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2-2axy)$

$= \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial x}(2axy)$

$= 2.x^{2-1} + 0 - 2ay\frac{\partial x}{\partial x}$

$\frac{\partial f}{\partial x} = 2x - 2ay$

$\frac{\partial}{\partial y}f = \frac{\partial}{\partial y}(x^2+y^2-2axy)$

$= \frac{\partial}{\partial y}x^2 + \frac{\partial}{\partial y}y^2 - \frac{\partial}{\partial y}2axy$

$= 0 + 2y^{2-1} - 2ax\frac{\partial y}{\partial x}$

$\frac{\partial f}{\partial y} = 2y - 2ax$

Second: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x - 2y)$

$= \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial x}(2ay)$

$= 2 - 0$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y-2ax)$$

$$= \frac{\partial}{\partial y}(2y) - \frac{\partial}{\partial y}(2ax)$$

$$= 2-0$$

$$= 2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2y-2ax)$$

$$= \frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial x}(2ax)$$

$$= 0 - 2a\frac{\partial x}{\partial x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2a.$$

Veeification

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(2x-2ay)$$

$$= \frac{\partial}{\partial y}(2x) - \frac{\partial}{\partial y}(2ay)$$

$$= 0 - 2a\frac{\partial}{\partial y}(y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2a$$

2. complate all the first and second order partial derivatives of the function

$$f(x,y) = e^{xy} + x^2 - y^3$$

Sol: Given $f(x,y) = e^{xy} + x^2 - y^3$

First

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{xy} + x^2 - y^3\right)$$

$$= \frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial x} y^3$$

$$= e^{xy} \frac{\partial}{\partial x}(xy) + 2x + 0$$

$$= e^{xy} y + 2x$$

$$\frac{\partial f}{\partial x} = ye^{xy} + 2x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{xy} + x^2 - y^3\right)$$

$$= \frac{\partial}{\partial y} e^{xy} + \frac{\partial}{\partial y} x^2 - \frac{\partial}{\partial y} y^3$$

$$= e^{xy} \frac{\partial}{\partial xy}(xy) + 0 - 3y^2$$

$$= e^{xy} x - 3y^2$$

$$\frac{\partial}{\partial y} = xe^{xy} - 3y^2$$

Second order: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial x \partial y}$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial x}\left(ye^{xy}+2x\right)$$

$$= \frac{\partial f}{\partial x} ye^{xy} + \frac{\partial}{\partial x}(2x)$$

$$= y\frac{\partial}{\partial x}e^{xy} + 2\frac{\partial}{\partial x}$$

$$= y\cdot e^{xy}\frac{\partial}{\partial x}(xy) + 2$$

$$= ye^{xy}(y)+2$$

$$= y^2e^{xy}+2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial xy}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(xe^{xy} - 3y^2\right)$$

$$= \frac{\partial}{\partial}\left(xe^{xy}\right) - \frac{\partial}{\partial y}3y^2$$

$$= x\frac{\partial}{\partial xy}e^{xy} - \frac{\partial}{\partial y}3y^2 .$$

$$= e^{xy}\frac{\partial}{\partial y}(xy) - 6y$$

$$= xe^{xy}(x)+6y$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 e^{xy} - 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(xe^{xy} - 3y^2\right)$$

$$= \frac{\partial}{\partial x}\left(xe^{xy}\right) - \frac{\partial}{\partial x}\left(3y^2\right)$$

$$= \left(x\frac{\partial}{\partial x}e^{xy} + e^{xy}\frac{\partial}{\partial x}x\right) - 0$$

$$= x \cdot e^{xy}\frac{\partial}{\partial x}(xy) + e^{xy}(1)$$

$$= x e^{xy}(y) + e^{xy}$$

$$= xye^{xy} + e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy}\left(1 + xy\right)$$

3. $f(x, y) = 7\sin(2x+y) + 6\cos(x-y)$

<u>sol:</u> $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(7\sin(2x+y) + 6\cos(x-y)\right)$$

$$= 7\cos(2x+y)\frac{\partial}{\partial x}(2x+y)$$

$$-6\sin(x-y)\frac{\partial}{\partial x}(x-y)$$

$$= 7\cos(2x+y)(2) - 6\sin(x-y)$$
$$(1)$$

$$\frac{\partial f}{\partial x} = 14\cos(2x+y) - 6\sin(x-y)$$

$$\frac{\partial f}{\partial y} = (7 \sin(2x+y) + 6 \cos(x-y) \quad)$$

$$\frac{\partial f}{\partial y} = 7 \cos(2x+y) \frac{\partial}{\partial y}(2x+y) + 6 \sin(x-y) \frac{\partial}{\partial y}(x-y)$$

$$= 7 \cos(2x+y).(1) - 6 \sin(x-y) (-1)$$

$$\frac{\partial f}{\partial y} = 7 \cos(2x+y) + 6 \sin(x-y$$

<u>second derivative</u>

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(14 \cos(2x+4) - 6 \sin(x-y))$$

$$= \frac{\partial}{\partial x}$$

$$= 14 - \sin(2x+4) \frac{\partial}{\partial x}(2x+y) - 6 \cos(x-y) \frac{\partial}{\partial x}(x-y)$$

$$= -14 \sin(2x+4)(2) - 6 \cos(x-4)(1)$$

$$\frac{\partial^2 f}{\partial x 2} = -28 \sin(2x+4) - 6 \cos(x-y)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \{7\cos(2x+y) + 6\sin(x-y)\}$$

$$= -7\sin(2x+y)\frac{\partial}{\partial y}(2x+y) + 6\cos(x-y)\frac{\partial}{\partial y}(x-y)$$

$$= -7\sin(2x+y)(1) + 6\cos(x-y)(-1)$$

$$\frac{\partial^2 f}{\partial y^2} = -7\sin(2x+y) - 6\cos(x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(7\cos(2x+y) + 6\sin(x-y)\right)$$

$$= -7\overset{\sin}{\cancel{\cos}}(2x+y)\frac{\partial}{\partial x}(2x+y) + 6\cos(x-y)\frac{\partial}{\partial x}(x-y)$$

$$= -7\overset{\sin}{\cancel{\cos}}(2x+y)(2) + 6\cos(x-y)(1)$$

$$= -14\sin(2x+y) + 6\cos(x-y)$$

→ Verify $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ when $v = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

$v = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{y^2}{y^2+x^2}\frac{1}{y}\left(\frac{\partial x}{\partial x}\right)$$

$$= \frac{y^2}{y(x^2+y^2)}$$

$$\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= \frac{y^2}{x^2+y^2}\, x\, \frac{\partial}{\partial y}\left(\frac{1}{y}\right)$$

$\frac{d}{dx}\left(\frac{1}{x}\right)' = -1/x^2$

$$= \frac{y^2\, x}{x^2+y^2}\left(-1/y^2\right)$$

$\frac{d}{dx}x^{-1} = (-1)x^{-1-1}$

$= -1/x^2$

$$\frac{\partial v}{\partial y} = \frac{-x}{x^2+y^2}$$

L.H.S $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= -\left[\frac{(x^2+y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)}\right]$$

$$= -\left[\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)}\right]$$

$$= -\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{y^2-x^2}{(x^2-y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2}$$

R.H.S $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$$= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$=$ [~~$-(x^2+y^2)\frac{\partial}{\partial x}(y) - y\frac{\partial}{\partial x}($~~

$$= \left[\frac{x^2+y^2\left(\frac{\partial y}{\partial y}\right) - \frac{\partial}{\partial y}(x^2+y^2)y}{(x^2+y^2)^2}\right]$$

$$= \left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore$ L.H.S = R.H.S.

4. $f(x\cdot y) = \cos(2x) - x^2e^{5y} + 3y^2$

Sol: $\frac{\partial f}{\partial x} = -\sin(2x) - e^{5y}(2x) + 0$

$$\frac{\partial f}{\partial x} = 0 - x^2e^{5y}(5) + 6y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(-2\sin(2x) - 2xe^{5y}\right)$$

$$= (-2)\cos(2x)(2) - 2e^{5y}(1)$$

$$\frac{\partial^2 f}{\partial x^2} = -4\cos(2x) - 2e^{5y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(-5x^2 e^{5y} + 6y\right)$$

$$\frac{\partial^2 f}{\partial y^2} = -5x^2 e^{5y}(5) + 6$$

$$\frac{\partial^2 f}{\partial y^2} = -25x^2 e^{5y} + 6$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-5x^2 e^{5y} + 6y\right)$$

$$= -5e^{5y}(2x) + 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = -10xe^{5y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(-2\sin 2x - e^{5y}(2x)\right)$$

$$= 0 - 2x e^{5y}(5)$$

$$= -10xe^{5y}$$

Session-10:

Total Derivative

$u = u(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \eta(t)$

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt} + \frac{du}{dz} \cdot \frac{dz}{dt}$$

$u = \cos\left(\frac{x}{y}\right)$ $\quad x = e^t$, $y = t^2$ find $\frac{du}{dt}$

Sol: $u = \cos(x/y)$

$x = e^t$

$y = t^2$

Total derivative:

$$\frac{du}{dt} = \frac{du}{dt} + \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \longrightarrow ①$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\cos(x/y)\right) = -\sin(x/y) \frac{\partial}{\partial x}\left(\frac{x}{y}\right) =$$

$$-\sin(x/y) \frac{1}{y} \frac{\partial x}{\partial x} = -\frac{1}{y} \sin\left(\frac{x}{y}\right)$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(e^t\right) = e^t \frac{dt}{dt} = e^t$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\cos\left(\frac{x}{y}\right)\right) = -\sin\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= -\sin\left(\frac{x}{y}\right) x \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -x\sin(x/y)$$

$$(-1/y^2) = \frac{x}{y^2}\sin(x/y)$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^2) = 2t.$$

Now eq ①

$$\frac{du}{dt} = -\frac{1}{y}\sin(x/y)\, e^t + \frac{x}{y^2}\sin\left(\frac{x}{y}\right) 2t$$

$$\frac{du}{dt} = \frac{1}{y}\sin\left(\frac{x}{y}\right)\left[-e^t + \frac{2xt}{y}\right]$$

$$\frac{du}{dt} = \frac{1}{t^2}\sin\left(\frac{e^t}{t^2}\right)\left(-e^t + \frac{2e^t t}{t^2}\right)$$

2. $$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$u = x^2+y^2+z^2$, $x = e^{2t}$, $y = \cos 3t$, $z = \sin 3t$

sol:

$$\frac{\partial u}{dx} = \frac{\partial}{\partial x}(x^2+y^2+z^2) = 2x$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^{2t}) = e^{2t}\frac{d}{dt}(2t) = 2e^{2t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+z^2) = 2y$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 3t) = -\sin 3t\frac{d}{dt}(3t)$$

$$= -3\sin 3t$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right) = 2z$$

$$\frac{dz}{dt} = \frac{d}{dt}\left(\sin 3t\right) = \cos 3t \frac{d}{dt}(3t) = 3\cos 3t$$

$$\frac{du}{dt} = 2x\left(2e^{2t}\right) - 6y \sin 3t + 6z \cos 3t$$

$$\frac{du}{dt} = 4e^{2t}\cdot e^{2t} - 6\cancel{\cos 3t \sin 3t + 6\sin 3t \cos 3t}$$

$$\frac{du}{dt} = 4e^{4t}$$

Sol: Given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$

by Total variables we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \longrightarrow ①$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(y^2 - 4ax\right) = -4a$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(at^2\right) = 2at$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(y^2 - 4ax\right) = 2y$$

$$\frac{dy}{dt} = \frac{d}{dt}(2at) = 2a$$

$$\frac{\partial u}{\partial t} = (-4a)(2at) + (2y)(2a)$$

$$= -8a^2t + 4ay$$

$$\frac{du}{dt} = -8a^2t + 4a(2at)$$

$$= -8a^2t + 8a^2t$$

$$= 0$$

Jacobian:

if $u = u(x,y)$. $v = v(x,y)$ then the (u,v) with respect to x and y is denoted determined by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left(\frac{u,v}{x,y}\right)$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

NOTE: Always $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

for 3 variables:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

4. $\frac{\partial(u,v)}{\partial(x,y)}$ for the functions $u = x - 2y$
$V = 2x + y$

sol: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x}(x-2y) & \frac{\partial}{\partial y}(x-2y) \\ \frac{\partial}{\partial x}(2x+y) & \frac{\partial}{\partial y}(2x+y) \end{vmatrix}$$

$$\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1+4=5$$

8. If $u = x^2-2y$, $v = x+y+z$, $w = x-2y+3z$

$\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,0,0)$

sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2)+2(3-1)+0(\ )$$

$$= 10x+4$$

$$\frac{\partial(v,u,w)}{\partial(x,y,z)} = 10x+4$$

$$\frac{\partial(v,u,w)}{\partial(x,y,z)}_{1,0,0} = 14$$

3. $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

<u>sol:</u> Given $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$\frac{-yz}{x^2}\left(\frac{x^2y^2}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(\frac{-xyz}{yz^2} - \frac{xyz}{yz^2}\right) + \frac{y}{x}\left(\frac{xz}{yz} + \frac{xyz}{y^2z}\right.$$

$$= \frac{-yz}{x^2}\left(\cancel{\frac{x^2yz}{y^2z^2}} - \cancel{\frac{x^2yz}{y^2z^2}}\right) - \frac{z}{x}\left(\frac{-xyz}{yz^2} - \frac{xyz}{yz^2}\right) +$$

$$\frac{y}{x}\left(\frac{xyz}{y^2z} + \frac{xyz}{y^2z}\right)$$

$$= 0 - \frac{z}{x}\left(\frac{-2xyz}{yz^2}\right) + \frac{y}{x}\left(\frac{2xyz}{y^2z}\right)$$

$$= + 2\left(\cancel{\frac{xyz^2}{xyz^2}}\right) + \left(\cancel{\frac{2xy^2z}{xy^2z}}\right) \Rightarrow 2+2=4$$

Tayler's series

A function $f(x,y)$ expanding in powers of $(x-a)$ and $(y-b)$ then the Tayler series is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots$$

$\boxed{\text{NOTE}}$: If $a=0$ and $b=0$ then the Tayler series becomes

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy(0,0) + y^2 f_{yy}(0,0)\right] + \ldots\ldots$$

1. $f(x,y) = x^2y + 6y + 5x + 2$ $\quad (x-1)(y+2)$

Sol. Given $f(x,y) = x^2y + 6y + 5x + 2$

By Tayler series we have

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) \overset{\frac{\partial f}{\partial x}}{f_x}(a,b) + (y-b) \overset{\frac{\partial f}{\partial y}}{f_y}(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots\ldots$$

$$f(x,y) = x^2y + 6y + 5x - 2 \Rightarrow f(1,-2) = (1)^2(-2) + 6(-2)$$

$$+5(1) - 2$$

$$= -2 - 12 + 5 - 2$$

$$f(1,-2) = -11$$

$$f_x = \frac{\partial f}{\partial x} = 2xy + 5 \Rightarrow f_x(1,-2) = 2(1)(-2) + 5 = 1$$

$$f_y = \frac{\partial f}{\partial y} = x^2 + 6 \Rightarrow f_y(1,-2) = 7$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy + 5\right) = 2y$$

$$f_{xx}(1,-2) = -4$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 + 6\right) = 2x$$

$$\Rightarrow f_{xy}(1,-2) = 2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 + 6\right) = 0$$

$$\Rightarrow f_{yy}(1,-2) = 0$$

$$f(x,y) = f(1,-2) + \frac{1}{1!}\left[(x-1) \quad + (y+2)7\right] + \frac{1}{2!}\Big[$$

$$x^2 \; -4 + 2(2) + y^2(0)\Big] + \cdots$$

2. Expand $f(x,y) = \sin x \cdot \cos y$ in power of x and y up to the second degree form.

Sol: Given $f(x,y) = \sin x \cdot \cos y$

By Maclaurin's form

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x\, f_x(0,0) + y\, f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \ldots$$

$$f(x,y) = \sin x \cdot \cos y \Rightarrow f(0,0) = 0$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sin x \cdot \cos y) = \cos x \cdot \cos y$$

$$\Rightarrow f_x(0,0) = 1$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\sin x \cdot \cos y) = -\sin x \cdot \sin y$$

$$\longrightarrow f_y(0,0) = 0$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos x \cdot \cos y) = -\sin x \cdot \cos y$$

$$\Rightarrow f_{xx}(0,0) = 0$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-\sin x \cdot \sin y) = -\cos x \cdot \sin y$$

$$f_{xy}(0,0) = 0$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-\sin x \sin y) = -\sin x \cdot \cos y.$$

$$f_{yy}(0,0) = 0$$

$$f(x,y) = 0 + \frac{1}{1!}\left[x(1) + y(0)\right] + \frac{1}{2!}\left[x^2(0) + 2xy(0\right.$$

$$\left. + y^2(0)\right] + \cdots\cdots$$

$$f(x,y) = 0 + \frac{1}{1!}[x]$$

$$= x$$

3. Expand $f(x,y) = \cos xy$ in the neighbourhood of $(1, -\pi/2)$

4. $f(x,y) = \cos xy \quad (1, -\pi/2)$ $\quad \cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta.$

Sol. $$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

$$f(x,y) = \cos(x,y) \Rightarrow f(1,-\pi/2) = \cos(1(-\pi/2)) =$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\cos(x,y)) = -\sin(xy)\frac{\partial}{\partial x}(xy)$$

$$= -y \sin xy$$

$$\Rightarrow f_x(1,-\pi/2) = -(-\pi/2)\sin(1(-\pi/2))$$

$$= \frac{\pi}{2}(-\sin \pi/2) = -\pi/2.$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\cos xy) = -\sin xy\,(x) = -x\sin(xy)$$

$$f_y(1,-\pi/2) = -(1)\sin(1(-\pi/2)) = 1$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(-y\sin(xy)\right)$$

$$= (-y)\cos xy\,(y)$$

$$f_{xx} = -y^2 \cos(x,y)$$

$$f_{xx}(1, -\pi/2) = -\left(-\frac{\pi}{2}\right)^2 \cos\left((1)\left(-\frac{\pi}{2}\right)\right)$$

$$= -\frac{\pi^2}{4} \cos\frac{\pi}{2} = 0$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-x \sin(xy)\right)$$

$$= -\left(x \frac{\partial}{\partial x}(\sin xy) + \sin(xy)\frac{\partial}{\partial x} x\right)$$

$$= -\left(x \cdot \cos(xy) \cdot y + \sin xy\right)$$

$$f_{xy}(1, -\pi/2) = -\left(0 + \sin\left(1\left(-\frac{\pi}{2}\right)\right)\right) = -(-1) = 1$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-x \sin(xy)\right)$$

$$= -x \cos(xy) \cdot (x)$$

$$f_{yy} = -x^2 \cos(xy)$$

$$f_{yy} = (1, -\pi/2) = 0$$

$$[(x-1)(-\pi/2) + (y+\pi/2)(1)]$$

$$f(x,y) = f(1, \pi/2) + \frac{1}{1!}\{(x-1)(-\pi/2) + (y+\pi/2)(1)]$$

$$+ \frac{1}{2!}\left[(x-1)^2(0) + 2(x-1)\left(y + \frac{\pi}{2}\right)(1) + (y+\pi/2)^2 \underset{(0)}{}\right]$$

$$= f(1, -\pi/2 + \frac{1}{1}[(x-1)(-\pi/2) + (y+\pi/2)(1) + \frac{1}{2}$$

$$(2(x-1)(y+\pi/2) + 0]$$

Session-12

A function $f(x,y)$ is said to have a maxima / minima according as $f(a,b) > f(a+h, b+k)$ (or) $f(a,b) < f(a+h, b+k)$

Statinary pointes:

A point $f(a,b)$ is said to stationary points of the function $f(x,y) = 0$, if $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Extreme Values: A function which has either maxima (or) minima is called Extreme Values.

Saddle point: A function which has neither maxima nor minima is called saddle point.

working rules:

1. Consider the given function as $f(x,y)$
2. find f_x and f_y
3. Equalate $f_x = 0$ and $f_y = 0$.
4. solve the equation's we get statinary points
5. find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

i, If $rt - s^2 > 0$ and $r > 0$ then the point is minimum.

ii, If $rt - s^2 > 0$ and $r < 0$ then the point is maximum.

iii, If $rt - s^2 < 0$ then the point is saddle point

iv, If $rt - s^2 = 0$ then the point is doughtful point, further investigation is necessary.

Lagromge's method of undetermined multipler.

Let the function is $f(x,y,z)$ and the condition $\phi(x,y,z)=0$ then by Lagromge's multipler method we have,

$$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$$

where λ is the Lagromge's multipler. which can be determined as

$$\frac{\partial F}{\partial x}=0\ ,\ \frac{\partial F}{\partial y}=0 \text{ and } \frac{\partial F}{\partial z}=0$$

By solving this we can find the unknown variables (x,y,z).

session-12.

1. Determine the maxima and minima of $f(x,y) = x^2+y^2+6x+12$.

sol: Given $f(x,y) = x^2+y^2+6x+12$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+6x+12\right)$$

$$= 2x+0+6+0.$$

$$= 2x+6.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2+y^2+6x+12\right)$$

$$= 0+2y+0+0$$

$$= 2y$$

Now $f_x=0$ and $f_y=0$.

$2x+6=0$ | $2y=0$

$x=-3$ | $y=0$

$(-3,0)$ is the stationary point.

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(2x+6)$$

$$= 2.$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2y)$$

$$= 0.$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(2y)$$

$$= 2$$

$\Rightarrow rt - s^2 > 0$ and $r > 0$.

$(2)(2) - 0^2 > 0$

$4 - 0 > 0$

$4 > 0$

$\rightarrow 2 > 0.$

from $(-3,0)$ is minima point.

Minimum value $f(x,y) = x^2+y^2+6x+12$

$$f(-3,0) = 9-18+12 = 3$$

2. find the stationary points of the functions

$f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

sol: Given $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^4+y^4-2x^2+4xy-2y^2\right)$$

$$= 4x^3+0-4x+4y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^4+y^4-2x^2+4xy-2y^2\right)$$

$$= 0+4y^3-0+4x-4y.$$

$$= 4y^3+4x-4y.$$

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0.$

$4x^3-4x+4y = 0 \Rightarrow x^3-x+y$ ①

$4y^3+4x-4y = 0 \Rightarrow y^3+x-y$ ②

$$x^3 - \cancel{x} + \cancel{y} = 0$$
$$y^3 + \cancel{x} + \cancel{y} = 0$$
$$\overline{\quad x^3+y^3 = 0 \quad}$$

$x^3+y^3 = 0$

$(x+y)(x^2-xy+y^2) = 0$

$x+y = 0$ | $x^2-xy+y^2 = 0.$

$x = -y$ | implicit ×

$(-y)^3 - (-y) + y = 0$

$-y^3 + y + y = 0$

$\Rightarrow y^3 - 2y = 0$

$y(y^2 - 2) = 0$

$y = 0 \quad \Big| \quad y^2 = 2$

$\qquad\qquad y = \pm\sqrt{2}$

$y = 0 \Longrightarrow x = 0$

$y = \sqrt{2} \Rightarrow x = -\sqrt{2}$

$y = -\sqrt{2} \Rightarrow x = \sqrt{2}$

the sati stationary points $(0,0)(\sqrt{2}, -\sqrt{2})$ $(-\sqrt{2}, \sqrt{2})$

$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(4x^3 - 4x + 4y) = 12x^2 - 4 = -4$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(4y^3 + 4x - 4y) = 4$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(4y^3 + 4x - 4y) = 12y^2 - 4 = -4$

$rt - s^2 \longrightarrow (-4)(-4) - 4 \Rightarrow 16 - 4 = 12$

$\therefore$ It is maximum pt.

$(\sqrt{2}, -\sqrt{2})$

$r = 12x^2 - 4 = 12(\sqrt{2})^2 - 4) = 12(2) - 4 = 24 - 4 = 20$

$s = 4$

$t = 12y^2 - 4 = 12(-\sqrt{2})^2 = 20$

$rt - s^2 \Rightarrow (20)(20) - 4 = 400 - 4 = 396$

minimum pt.

$r = 20, \quad s = 4, \quad t = 20$

It is minimum point:

3. find the minimum value of $x^2+y^2+z^2$ given that $xyz = a^3$.

ol: $x^2+y^2+z^2, \quad xyz = a^3$

$f(x,y,z) = x^2+y^2+z^2$

$\phi(x,y,z) = xyz - a^3 = 0$

By Lagrange's method we have.

$F(x,y,z) = f(x,y,z) + \lambda \; \phi(x,y z)$

$F = x^2+y^2+z^2 + \lambda(xyz - a^3)$

where λ is the Lagrange's multiplier which can be determined as

$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial z} = 0$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda(yz) = 0$$

$$\lambda = \frac{-2x}{yz} \text{ ——— } ①$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda(xz) = 0$$

$$\lambda = \frac{-2y}{xz} \text{ ——— } ②$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda(xy) = 0$$

$$\lambda = \frac{-2z}{xy} \text{ ——— } ③$$

i.e $\frac{-2x}{yz} = \frac{-2y}{xz} = \frac{-2z}{xy}$

$$\frac{\cancel{-2}x}{y\cancel{z}} = \frac{\cancel{-2}y}{x\cancel{z}} \qquad \Bigg| \qquad \frac{\cancel{-2}y}{\cancel{x}z} = \frac{\cancel{-2}z}{\cancel{x}y}$$

$$x^2 = y^2 \qquad \qquad y^2 = z^2$$

$$x = y \qquad \qquad y = z$$

$$x = y = z$$

Given $xyz = a^3$

$$(x)(x)(x) = a^3$$

$$x^3 = a^3$$

$$x = a$$

$$y = a$$
$$z = a$$

Given (x. $f(a,a,a) = a^2+a^2+a^2$

$= 3a^2$

4. A rectangular box open at the top is to have a volume of 32 cubic ft. Find the dimensions of the box requiring least Material for it's Construction.

Sol: the surface area $S = xy+2yz+2zx$

Volume $lbh = 32$

$xyz = 32$

$\Rightarrow xyz - 32 = 0$

By Lagrange's method

$F = S(x,y,z) + \lambda V(xyz)$

$F = xy+2yz+2zx + \lambda(xyz-32)$

where λ is Lagrange's multiplier

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial x} = y + 2z + \lambda(yz) = 0 \rightarrow \lambda = \frac{-(y+2z)}{yz}$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda(xz) = 0 \Rightarrow \lambda = \frac{-(x+2z)}{xz}$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda(xy) = 0 \Rightarrow \lambda = \frac{-(2y+2x)}{xy}$$

$$\frac{-(y+2z)}{yz} = \frac{-(x+2z)}{xz} = \frac{-(2y+2x)}{xy}$$

$$\frac{\cancel{\lambda}(y+2z)}{y\cancel{z}} = \frac{\cancel{\lambda}(x+2z)}{x\cancel{z}}$$

$$x\cancel{y} + 2zx = \cancel{xy} + 2zy$$

$$2zx - 2zy = 0$$

$$x = y$$

$$\frac{\cancel{\lambda}(x+2z)}{\cancel{x}z} = \frac{\cancel{\lambda}(2y+2x)}{xy}$$

$$xy + 2z\cancel{y} = 2\cancel{y}z + 2xz$$

$$\cancel{x}y - 2z\cancel{x} = 0$$

$$y = 2z.$$

$$\boxed{x = y = 2z}$$

Given $xyz = 32$

$$(x)(x)\left(\frac{x}{2}\right) = 32$$

$$x^3 = 64$$

$$x = 4$$

$$y = 4$$

$$z = 2 \quad \left(\frac{x}{2}\right)$$

Minimum value $x^2+y^2+z^2 = 4^2+4^2+2^2$

$= 16+16+4$

$= 36$

5. Divide 36 into 3 parts such that the product of the first square of the second and cube of the third is maximum.

Sol: Given $f(x,y,z) = xy^2z^3$

$\phi(x,y,z) = x+y+z = 36.$

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

$F = xy^2z^3 + \lambda(x+y+z-36)$

$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = y^2z^3 + \lambda = 0 \qquad \lambda = -y^2z^3$

$\frac{\partial F}{\partial y} = 2xyz^3 + \lambda = 0 \qquad \lambda = -2xyz^3$

$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda = 0 \qquad \lambda = -3xy^2z^2$

$-y^2z^3 = -2xyz^3 = -3xy^2z^2$

$y^2\cancel{z^3} = 2xy\cancel{z^3}$

$y = 2x$

$2\cancel{x}\cancel{y}z^2 = 3x\cancel{y}\cancel{z}$

$2z = 3y$

$y = \frac{2z}{3}$

$2x = y = \frac{2z}{3}$

Given :

$$x+y+z=36$$

$$\frac{y}{2}+y+\frac{3y}{2}=36$$

$$\frac{y+2y+3y}{2}=36$$

$$\frac{6y}{2}=36$$

$$6y=72$$

$$\boxed{y=12}$$

$$x=\frac{y}{2}$$

$$x=\frac{\cancel{12}^{6}}{\cancel{2}}$$

$$x=6$$

$$z=\frac{3y}{2}$$

$$z=\frac{3(\cancel{12})^{6}}{\cancel{2}}$$

$$z=18$$

<u>Session -13</u>

Second order differential equations

$$y=\underbrace{C.F}_{}\xrightarrow{\text{complementary function}} + P.I \text{ — particular integral}$$

P.I: i, e^{ax}

ii, $\cos ax / \sin ax$

C.F / S.N	Roots	C.F
1.	m_1, m_2 be the real and distinct roots th	$c_1e^{m_1x}+c_2e^{m_2x}$
2.	If the roots are real and Equal $m_1=m_2$.	$(c_1+c_2x)e^{m_1x}$
3.	If m_1, m_2, m_3 be the real and distinct roots	$c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}$
4.	If roots are complex $a\pm ib$	$e^{ax}[c_1\cos bx + c_2\sin bx]$

P.I i) $f(D)y = e^{ax}$

$$y = \frac{1}{f(D)}e^{ax}$$

$$\boxed{D=a}$$

$$y_p = \frac{1}{f(a)}e^{ax}$$

ii) $f(D)y = \sin ax / \cos ax$

$$y_p = \frac{1}{f(D)}\sin ax / \cos ax$$

$$\boxed{D^2 = -(a^2)}$$

Session-13

1. solve the differential equation $\frac{d^2y}{dx^2} - 4y = 0$

Sol:

$$\frac{d^2y}{dx^2} - 4y = 0$$

The given D.E $\frac{d^2y}{dx^2} - 4y = 0$

$\frac{d}{dx} = D$

the operator form is $(D^2 - 4)y = 0$

$\frac{d^2}{dx^2} = D$

the Auxillary equation is $f(m) = 0$

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$m = \pm 2$$

$$m = 2, -2$$

$$\boxed{C.F = c_1 e^{2x} + c_1 e^{-2x}}$$

G.S $y = C.F + P.I$

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

where c_1 and c_2 are the constants.

2. solve the differential equation

$$\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$$

sol: the given D.E = $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

the operator form is $(D^3 - 3D + 2) = 0$

The Auxillary equation is $f(m) = 0$

$$m^3 - 3m + 2 = 0.$$

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & 1 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$m^2+m-2=0$

$m^2+2m-m-2=0$

$m(m+2)-1(m+2)=0$

$(m-1)(m+2)=0$

$m=1,-2.$

the roots are $m=1,1,-2.$

$C.F=(c_1+c_2x)e^{(1)x}+c_3e^{-2x}$

General solution:

$y=C.F+P.I$

$y=(c_1+c_2x)e^{x}+c_3e^{-2x}.$

where c_1, c_2 and c_3 are the constants.

5: solve the D.E $y''+y'-2y=0$ with $y(0)=1$, and $y'(0)=1$

Sol: The given D.E is $y''+y'-2y=0$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

the operator form is $(D^2+D-2)y=0$

the Auxillary equation is $f(m)=0$

$m^2+m-2=0$

$m^2+2m-m-2=0$

$m(m+2)-1(m+2)=0$

$(m-1)(m+2)=0$

$m=1,-2$

the roots are $1, -2$

$C.F = c_1e^{x} + c_2e^{-2x}$

G.S $y = C.F + P.I$

$y = c_1e^{x} + c_2e^{-2x}$ —— ①

$y' = c_1e^{x} + c_2(e^{-2x})(-2)$

$y' = c_1e^{x} - 2c_2e^{-2x} \longrightarrow$ ②

$y(0) = 4 \Rightarrow x = 0$ and $y = 4$

① $\longrightarrow$ $4 = c_1e^{0} + c_2e^{0} \Rightarrow c_1 + c_2 = 4 \longrightarrow$ ③

Given $y'(0) = 1 \Rightarrow x = 0$ & $y' = 1$

(2) $\Rightarrow$ $1 = c_1e^{0} - 2c_2e^{0} \Rightarrow c_1 - 2c_2 = 1 \rightarrow$ ④

solving (3) and ④

$c_1 = 3 \quad c_2 = 1$

finally the required solution is substitute in eq^n ①

$y = 3e^{x} + 1\cdot e^{-2x}$

4. solve the differential equation.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$$

Sol: Given the D.E is $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

The operator form is $(D^2 + 4D + 3)y = e^{2x}$

The Auxillary form is $(f)m = 0$

$m^2 + 4m + 3 = 0$

$$m^2+3m+m+3=0$$

$$m(m+3)+1(m+3)=0$$

$$m=-1,-3$$

$$\boxed{C.F = c_1 e^{-x} + c_2 e^{-3x}}$$

P.I $\quad y_p = \dfrac{1}{D^2+4D+3} \cdot e^{2x}$ $\qquad D=2$

$$= \frac{1}{(2)^2+4(2)+3} \cdot e^{2x}$$

$$= \frac{1}{15} e^{2x}$$

$$y = C.F + P.I$$

$$y = c_1 e^{-x} + c_2 e^{-3x} + \frac{1}{15} e^{2x}$$

5. solve the D.E $(D^2+4)y = \sin 3x$.

sol: Given D.E
The operator form is $(D^2+4)y = \sin 3x$

the Auxiliary form $f(m)=0$

$$m^2+4=0 \Rightarrow m=\pm\sqrt{-4}$$

$$m=\pm 2\sqrt{i}$$

$$m=\pm 2i$$

$$m=\pm 2i$$

$$m = 0 \pm 2i$$

$$m = a \pm bi$$

$$C.F = e^{0x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$y_p = \frac{1}{D^2+4} \cdot \sin 3x$$

$$y_p = \frac{1}{-(3)^2+4}\sin 3x \qquad D^2 = (-a)_2 \;-(3)_2$$

$$y_p = \frac{1}{-5}\sin 3x$$

$$y = e^{x}\left[c_1\cos 2x + c_2\sin 2x\right] + \frac{1}{-5}\sin 3x$$

Session 14

3- The motion of a mass spring system without damping is described by the initial value problem $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{2t}$, $x(0) = 0$, $x'(0) = 1$ where x is the distance of the mass from the equilibrium position, downward being taken as positive direction. Calculate the displacement of the motion.

Sol:

$$(D^2+3D+2)x = e^{2t} \qquad D = \frac{d}{dt}$$

$$m^2+3m+2 = 0$$

$$m = -1, -2$$

$$C.F = c_1e^{-t} + c_2e^{-2t}$$

$$x_p = \frac{1}{D^2+3D+2}e^{2t}$$

$$\boxed{D=2}$$

$$= \frac{1}{2^2+3(2)+2}e^{2t}$$

$$x_p = \frac{1}{12}e^{2t}$$

$$x = C.F + P.I$$

$$x = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{12} e^{2t} \longrightarrow ①$$

$$x' = c_1 e^{-t}(-1) + c_2 e^{-2t}(-2) + \frac{1}{12} e^{2t}(2)$$

$$x' = -c_1 e^{-t} - 2c_2 e^{-2t} + \frac{1}{6} e^{2t} \longrightarrow ②$$

$$x(0) = 0 \Rightarrow t = 0 \Rightarrow x = 0 \longrightarrow ①$$

$$\Rightarrow 0 = c_1 + c_2 + \frac{1}{12}$$

$$c_1 + c_2 = -\frac{1}{12} \longrightarrow ③$$

$$x'(0) = 2 \Rightarrow t = 0 \Rightarrow x' = 2$$

$$2 = -c_1 - 2c_2 + \frac{1}{6}$$

$$-c_1 - 2c_2 = 2 - \frac{1}{6}$$

$$-c_1 - 2c_2 = \frac{11}{6} \longrightarrow ④$$

By solving ③ and ④

$$\cancel{c_1} + c_2 = -\frac{1}{12}$$

$$-\cancel{c_1} - 2c_2 = \frac{11}{6}$$

$$-c_2 = \frac{21}{12}$$

$$c_2 = -\frac{\cancel{21}\,7}{\cancel{12}\,4} = -\frac{7}{4}$$

substitue in eqⁿ ①

$$c_1 + (-7/4) = \frac{-1}{12} \qquad \Rightarrow \frac{-1+21}{12} = \frac{\cancel{20}\,5}{\cancel{12}\,3} = \frac{5}{3}.$$

$$c_1 - 7/4 = -\frac{1}{12}$$

$$c_1 = -\frac{1}{12} + \frac{7}{4}$$

$$c_1 = \frac{-(1\times1)+(3\times7)}{12}$$

4. The motion of a mass spring system without damping is described by the initial value problem $\frac{d^2x}{dt^2}+4x = e^t$, $x(0)=2$, $x'(0)=0$, where x is the distance of the mass from the equilibrium position, downward being taken as positive direction. Calculate the displacement of the motion.

Sol: Given $\frac{d^2x}{dt^2} + 4x = e^t$,

$$(D^2+4)x = e^t$$

$$m^2+4=0$$

$$m^2=-4$$

$$m=\pm 2i$$

$$C.F = e^{0.t}\left[c_1\cos 2t + c_2\sin 2t\right]$$

$$C.F = c_1\cos 2t + c_2\sin 2t$$

P.I. $x_p = \frac{1}{D^2+4}e^t$, $\quad \boxed{D=1}$

$$x_p = \frac{1}{5}e^t$$

$$x = C.F + P.I$$

$$x = c_1\cos 2t + c_2\sin 2t + \frac{1}{5}e^t \longrightarrow ①$$

$$x' = c_1(-\sin 2t)(2) + c_2(\cos 2t)(2) + \frac{1}{5}e^t$$

$$x' = -2c_1\sin 2t + 2c_2\cos 2t + \frac{1}{5}e^t \text{ — } ②$$

$x(0) = 2 \Rightarrow t = 0,\ x = 2$

$\Rightarrow 2 = c_1(1) + c_2(0) + \frac{1}{5}e^0$

$c_1 = 2 - \frac{1}{5}$

$c_1 = \frac{9}{5}$

$x'(0) = 0 \Rightarrow t = 0,\ x' = 0$

$(2) \Rightarrow 0 = -2c_1(0) + 2c_2(1) + \frac{1}{5}e^0$

$0 = 2c_2 + \frac{1}{5}$

$c_2 = -\frac{1}{10}$

$x = -\frac{1}{10}\cos 2t + \frac{9}{5}\sin 2t + \frac{1}{5}e^t$

→ The standard form of the 2nd order LCR circuits is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = EMF$

where L is the Inductance

R → Resistance

C → capitor

q → charge.

emf → External voltage (electro motive force)

1. compute the charge on the capacitor in an LRC series circuit at $t = 2$ sec, when inductance 3H, resistance 15Ω and capacitance 1/12 F, $E(t) = 0$ V, $q(0) = 1C$ and $i(0) = 2A$

sol:

$L = 3H$

$R = 15\Omega$ $\quad q(0) = 1C$

$C = \frac{1}{12} F$ $\quad q' = i(0) = 2A.$

$E(t) = 0$

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{c} = EmF$$

$$3 \frac{d^2q}{dt^2} + 15 \frac{dq}{dt} + 12q = 0$$

$$(3D^2 + 15D + 12)q = 0 \qquad D = \frac{d}{dt}$$

$$3m^2 + 15m + 12 = 0.$$

$$m^2 + 5m + 4 = 0$$

$$m^2 + 4m + m + 4 = 0$$

$$m(m+4) 1(m+4)$$

$$(m+1)(m+4)$$

$$m = -1, -4.$$

$$q = c_1 e^{-t} + c_2 e^{-4t} \longrightarrow ①$$

$$q' = -c_1 e^{-t} - 4c_2 e^{-4t} \longrightarrow ②$$

$$q(0) = 1 \Rightarrow t = 0 \quad q = 1$$

$$① \longrightarrow 1 = c_1 + c_2 \longrightarrow ③$$

$$q'(0) = i(0) = 2$$

$$t = 0 \quad q' = 2$$

$$(2) \longrightarrow 2 = -c_1 - 4c_2 \longrightarrow ④$$

$$\cancel{c_1} + c_2 = 1$$

$$-\cancel{c_1} - 4c_2 = 2$$

$$-3c_2 = 3$$

$$c_2 = \frac{\cancel{3}}{-\cancel{3}} = -1 \qquad \boxed{c_2 = -1}$$

$c_1 + (-1) = 1$

$c_1 = 1+1$

$\boxed{c_1 = 2}$

1. Determine the charge on the capacitor in an LC series circuit at t = 2 sec when Indutance 1H, resistance and capacitance 1/16 F, E(t) = sin 3t V, q(0) = 1C and i(0) = 0A.

sol:

$L = 1H$

$c = \frac{1}{16}$

$E(t) = \sin 3t$

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = EMF$$

$$\frac{d^2q}{dt} + 0 + 16q = \sin 3t$$

$$\frac{d^2q}{dt^2} + 16q = \sin 3t$$

$$(D^2+16)q = \sin 3t$$

$$m^2 + 16 = 0$$

$$m^2 = -16$$

$$m = \pm 4i$$

$$C.F = e^{0 \cdot t}\left[c_1 \cos 4t + c_2 \sin 4t\right]$$

$$q_p = \frac{1}{D^2+16} \sin 3t$$

$$D^2 = -(3)^2$$

$$q_p = \frac{1}{(-3)^2+16} \sin 3t$$

$$q_p = \frac{1}{7} \sin 3t$$

$$q = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{7}\sin 3t \longrightarrow ①$$

$$q' = -4 \sin 4t + 4c_2 \cos 4t + \frac{3}{7}\cos 3t \longrightarrow ②$$

$q(0) = 1C$

$q(0) \rightarrow 1 \quad t=0 \Rightarrow q = 1$

$c_1 = 1 \longrightarrow ③$

$i(0) = 0 \quad t=0 \Rightarrow q' = 0.$

$4c_2 + \frac{3}{7} = 0$

$c_2 = -\frac{3}{28}.$