Example: Find the SVD of a matrix $A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$

Step 1:

Find the Eigen values $A^TA$

(1)

Compute $A^TA$ = $A^T = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix}$$

Step 2:

Characteristic Equation.

Eigen values comes from solving

$$\det(A^TA - \lambda I) = 0$$

$$\det \begin{bmatrix} 17-\lambda & 32 \\ 32 & 65-\lambda \end{bmatrix} = 0$$

Step 3: Expand determinant

$$(17-\lambda)(65-\lambda) - (32)(32)$$

$$17(65) - 17\lambda - 65\lambda + \lambda^2 - 1024 = 0$$

$$1105 - 82\lambda + \lambda^2 - 1024 = 0$$

$$\lambda^2 - 82\lambda + 81 = 0$$

Step 4: Solve Quadratic

$$\lambda^2 - 82\lambda + 81 = 0 \quad (ax^2+bx+c=0)$$

$$\lambda = \frac{82 \pm \sqrt{82^2 - 4(81)}}{2} \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{82 \pm \sqrt{6400}}{2} = \frac{82 \pm 80}{2}$$

$$\lambda_1 = \frac{82+80}{2} = \frac{162}{2} = 81 \quad \Big| \quad \lambda_2 = \frac{82-80}{2} = \frac{2}{2} = 1$$

We are solving for eigen vectors of $A^T A$

$$(A^T A - \lambda I)x = 0$$

②

$\lambda = 81$ $\quad A^T A = \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix}$

So,

$$\begin{bmatrix} 17-81 & 32 \\ 32 & 65-81 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -64 & 32 \\ 32 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-64 x x_1 + 32 x_2 = 0 = -64x_1 + 32x_2$

$32x_1 - 16x_2$

$32x_1 = 16x_2$

$2x_1 = x_2$

$32x_2 = 64x_1$

$x_2 = 2x_1$

$2x_1 = x_2$

Express eigen vector

$$v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Normalize (make unit length)

In SVD, we use orthonormal vectors, So we normalize

length of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \sqrt{1^2 + 2^2} = \sqrt{5}$$

So normalized eigen vector $\quad v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So,

Now

$\lambda = 1$ $\quad = \begin{bmatrix} 17-1 & 32 \\ 32 & 65-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\begin{bmatrix} 16 & 32 \\ 32 & 64 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

then

$$16x_1 + 32x_2 = 0$$

$$x_1 = -2x_2$$

So possible eigen vector is $= V = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$V_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

→ Normalize:

length of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$= \sqrt{(-2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}$$

Normalized eigen vector $V_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$

$$V_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Now, we found two normalized eigen vectors

$$V_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad V_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

→ From the eigen vector matrix V

In SVD, the matrix V is formed by Stacking the eigenvectors as columns:

$$V = [V_1 \ V_2] = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

→ take transpose($V^T$)

$$= V^T = \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\right)^T = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\right)$$

Why we need $\sqrt{\lambda}$
in SVD.
$A = U \Sigma V^T$

④

step 3 SVD: finding Singular vectors (the matrix U)

$$u_i = \frac{A v_i}{\sigma_i}$$

$v_i$ = right Singular vectors (eigenvectors of $A^TA$)

$\sigma_i = \sqrt{\lambda_i}$ = Singular values (Square root of eigen values of $A^TA$)

$u_i$ = left singular vectors (eigen vectors of $AA^T$)

So, to Compute $u_i$, we multiply the original matrix A with eigen vector $v_i$ and normalize by diving with $\sigma_i$

step ①

$$u_1 = \frac{A v_1}{\sigma_1}$$

$$A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \quad v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \sigma_1 = 9$$

$\lambda_1 = 81$
$\lambda_2 = 1$
$\sqrt{\lambda_1} = \sqrt{81}$
$\sigma_1 = 9$
$\sqrt{\lambda_2} = \sqrt{1}$
$= 1$
$\sigma_2 = 1$

Now,

$$u_1 = \frac{1}{9}\begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -18 \\ 9 \end{bmatrix}$$

Now divide 9 and $\sqrt{5}$

$$u_1 = \frac{1}{9} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} -18 \\ 9 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Step ②

$$u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \sigma_2 = 1$$

$$u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{1}{1}\begin{bmatrix}-4 & -7\\ 1 & 4\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}-4 & -7\\ 1 & 4\end{bmatrix}\begin{bmatrix}-2\\ 1\end{bmatrix} = \begin{bmatrix}8-7\\ -2+4\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}$$

Now, divide by $\sqrt{5}$

$$u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\ 2\end{bmatrix}$$

Construct $U$

Combine $u_1, u_2$ into matrix $U$:

$$U = [u_1, u_2] = \frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}$$

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}$$

<u>Step 4:</u> Diagonal matrix $S$ with Singular values.

we already found the singular values:

$$\sigma_1 = 9, \ \sigma_2 = 1$$

So diagonal matrix is:

$$S = \begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix} = \begin{bmatrix}9 & 0\\ 0 & 1\end{bmatrix}$$

This is stretching matrix in SVD.

<u>Step 5</u> verify decoposition

$$A = USV^T$$

$$A = \begin{bmatrix}-4 & -7\\ 1 & 4\end{bmatrix} = USV^T$$

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$S = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

① multiply S and $V^T$ $= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 9 & 18 \\ -2 & 1 \end{bmatrix}$$

② Now multiply with U: $\frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 9 & 18 \\ -2 & 1 \end{bmatrix}$

$$= \frac{1}{5} \begin{bmatrix} -20 & -35 \\ 5 & 20 \end{bmatrix}$$

③ Simplify: $\frac{1}{5} \begin{bmatrix} -20 & -35 \\ 5 & 20 \end{bmatrix} = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$