Signals and Communication Systems (24EC2105)

Continuous Time Systems

Continuous Time Systems

1. Classification of Systems
2. Time domain Analysis of LTI systems

    (Convolution Integral)

What is a ‘System’?

A System may be defined as a physical device that operates on a signal.

Examples:

- PA System (Public Address System)
- Digital Computer
- Mobile Phone

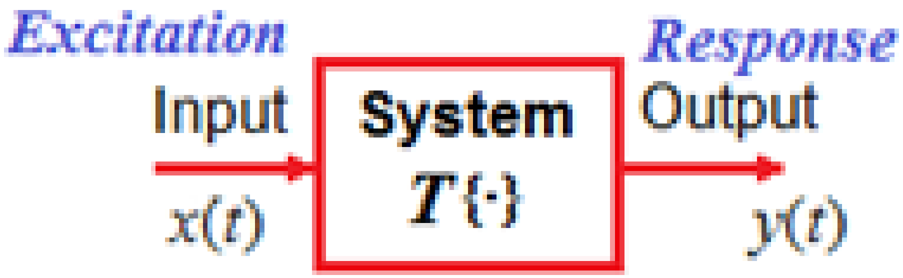
Relation between Signal and System

Signal: Physical quantity

System: Physical device

The input signal is modified according to the characteristics of the system and gives some output.

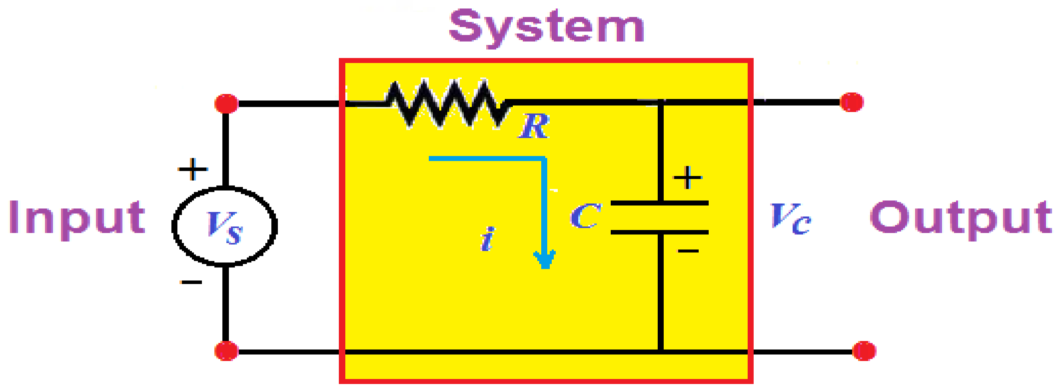
Block diagram Representation of systems



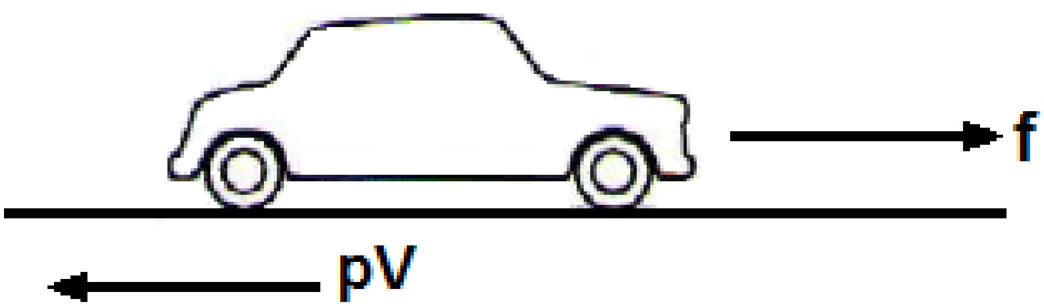
$$y(t) = T\{x(t)\}$$

$$x(t) \rightarrow y(t)$$

Examples



$$\frac{dV_C(t)}{dt} + \frac{1}{RC} V_C(t) = \frac{1}{RC} V_S(t)$$



$$\frac{dV(t)}{dt} + \frac{1}{m} \rho v(t) = \frac{1}{m} f(t)$$

Classification of Systems

Systems with and without Memory (Static or dynamic systems)

A system is said to be *memoryless or static* if its output for at any time depends on the input at the same time but not on the past or future inputs.

$$y(t) = 2x(t) + 3$$

Static or Memoryless

$$y(t) = x(t) - x(t-3)$$

Dynamic or Memory

$$y(t) = e^{x(t)}$$

Static

Systems with and without Memory (Static or dynamic systems)

$y(t) = x(t+3) - x(t-3)$ Dynamic

$y(t) = x(2t)$ Dynamic

$y(t) = x(t)\sin 3t$ Static

$y(t) = x(\sin(t))$ Static

$y(t) = x(-t)$ Dynamic

Causal versus Non-causal systems
(Non-anticipative and anticipative)

A system is said to be *causal*, if the output at any time depends on values of the input at only the present and past times , but does not depend on the future inputs.

$y(t) = 2x(t) + 3$ Causal

$y(t) = x(t) - x(t-3)$ Causal

$y(t) = e^{x(t)}$ Causal

$e^{x(t+2)}$ Non-causal

Causal versus Non-causal systems

$y(t) = x(t+3) - x(t-3)$ Non-causal

$y(t) = x(2t)$ Non-causal

$y(t) = x(t)\sin 3t$ Causal

$y(t) = x(\sin(t))$ Causal

$y(t) = x(-t)$ Non-causal

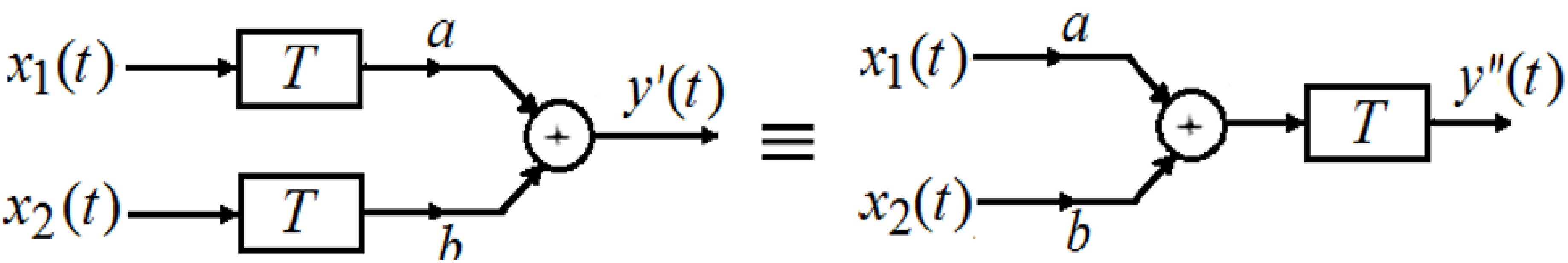
Linear versus nonlinear systems

Definition: A relaxed system T is said to be linear if

$$T\{ax_1(t)+bx_2(t)\} = aT\{x_1(t)\} + bT\{x_2(t)\}$$

for any arbitrary input sequences $x_1(t)$ and $x_2(t)$

and any arbitrary constants a and b.



1. *additivity* property;

   The response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$

2. *scaling* or *homogeneity* property

   The response to $ax_1(t)$ is $ay_1(t)$,

   where a is any complex constant.

Test for linearity $y(t) = 2x(t)$

For input $x_1(t)$, the corresponding output

$$y_1(t) = T\{x_1(t)\} = 2x_1(t)$$

For input $x_2(t)$, the corresponding output

$$y_2(t) = T\{x_2(t)\} = 2x_2(t)$$

Then $y'(t) = y_1(t) + y_2(t) = 2x_1(t) + 2x_2(t)$

For input $x_1(t) + x_2(t)$, the corresponding output

$$y(t)'' = T\{x_1(t) + x_2(t)\} = 2\{x_1(t) + x_2(t)\}$$

Since $y'(t) = y''(t)$ the given system

$y(t) = 2\,x(t)$ is linear system

Linear versus nonlinear systems

$$y(t) = 2x(t) + 3$$

Non-linear

$$y(t) = t\ x(t)$$

linear

$$y(t) = x^2(t)$$

Non-linear

$$y(t) = x(\sin(t))$$

Llinear

$$y(t) = x(t)\sin 3t$$

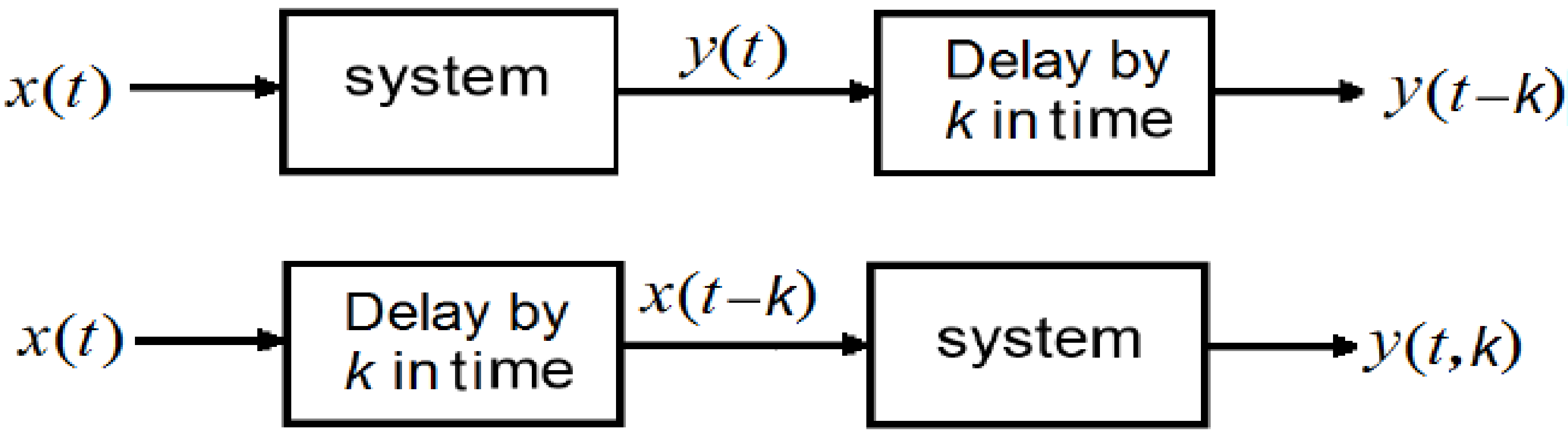
Linear

$$y(t) = x(2t)$$

Linear

Time-invariant versus time-variant systems

- A relaxed system $\boldsymbol{T}$ is said to be *time invariant* or *shift invariant* if and only if



For every input $x(t)$ and for every time shift '$\boldsymbol{k}$'.

The response of the system should not change with time

Test this system for Time Invariance $y(t) = t\ x(t)$

output $y(t) = T\{x(t)\} = t\ x(t)$ is t times the input signal.

Step1: Delay the output by *k* in time.

The resultant signal is $y(t-k) = (t-k)\,x(t-k)$.

Step2: Delay the input by *k* in time

resulting the delayed input signal $x(t-k)$.

Then the output of the system for this delayed input

$x(t-k)$, $y(t,k) = t\ x(t-k)$.

Since $y(t,k) \neq y(t-k)$, the given system is Time Variant.

Time-invariant versus time-variant systems

$$y(t) = x(2t)$$ Time-variant

$$y(t) = x(t-2)$$ Time-invariant

$$y(t) = x(-t)$$ Time-variant

$$y(t) = x(t)\sin 3t$$ Time-variant

Stable versus Unstable systems

A stable system is such that well-behaved outputs are obtained for well-behaved inputs. A system is said to be bounded input bounded output (BIBO) stable, if and only if every bounded input results in a bounded output.

$$|x(t)| \leq B_x < \infty$$

$$|y(t)| \leq B_y < \infty$$

$$y(t) = e^{x(t)}$$

Let $|x(t)| < 1$ or $-B < x(t) < B$ for all t

Then $e^{-B} < |y(t)| < e^{B}$

Thus, the system $y(t) = e^{x(t)}$ is stable.

$$y(t) = t\,x(t)$$

For a constant input $x(t) = 1$,

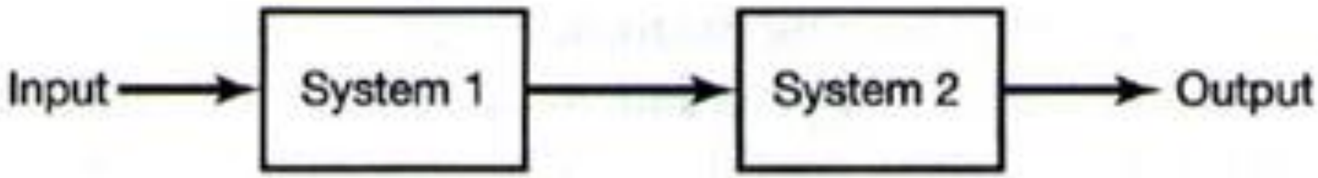
the system $y(t) = t\,x(t)$, produces $y(t) = t$

which is unbounded

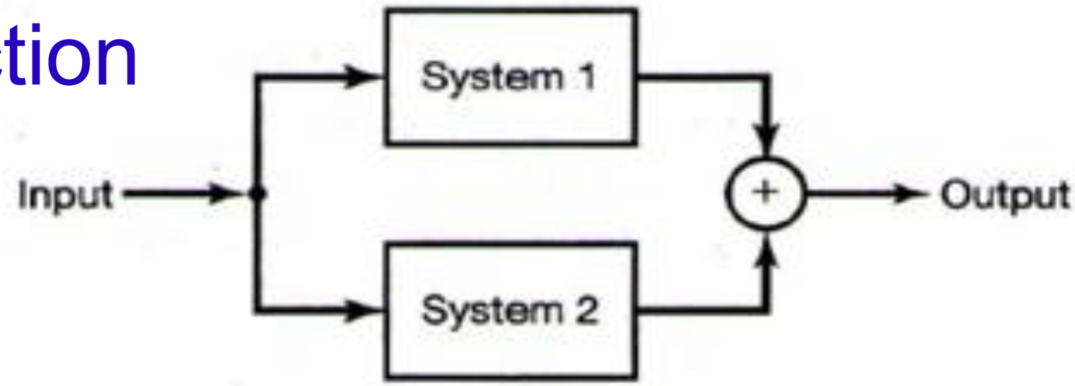
Linear and Time Invariant (LTI) Systems

Interconnections of Systems

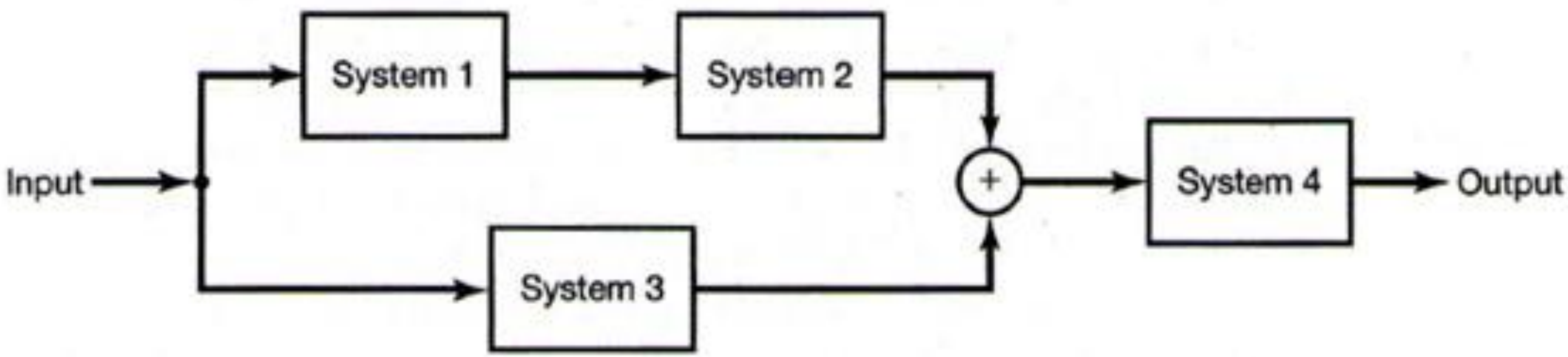
Series (cascade) interconnection



Parallel interconnection

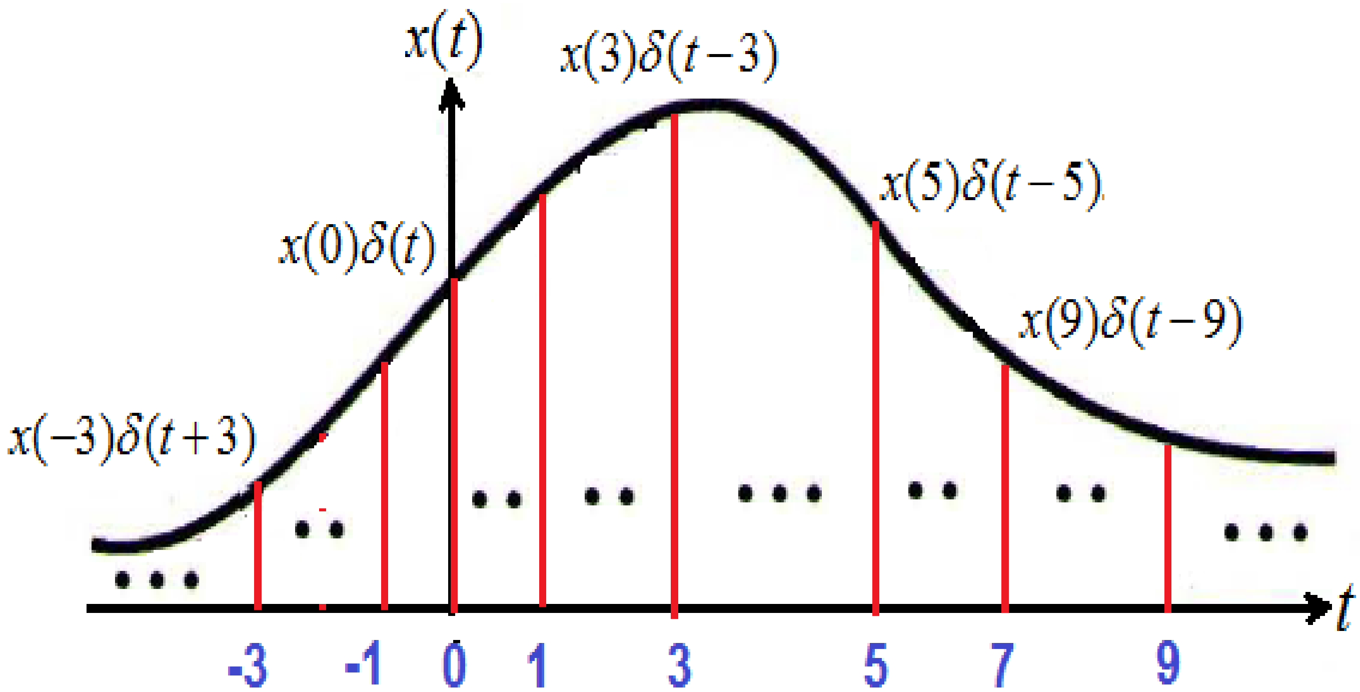


Series-parallel interconnection



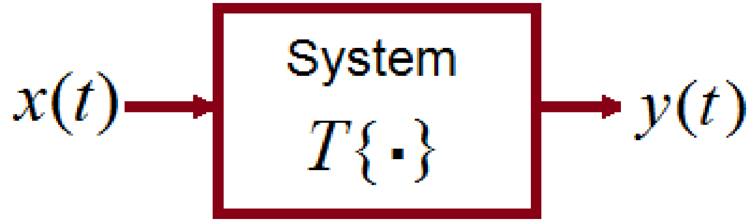
Time domain Analysis of CT systems: **Convolution**

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$



Any signal can be represented by integral sum of weighted shifted impulses

Let $x(t) = \delta(t)$



$$y(t) = T\{x(t)\} = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} x(\tau)\, T\{\delta(t-\tau)\}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

Convolution Algorithm

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

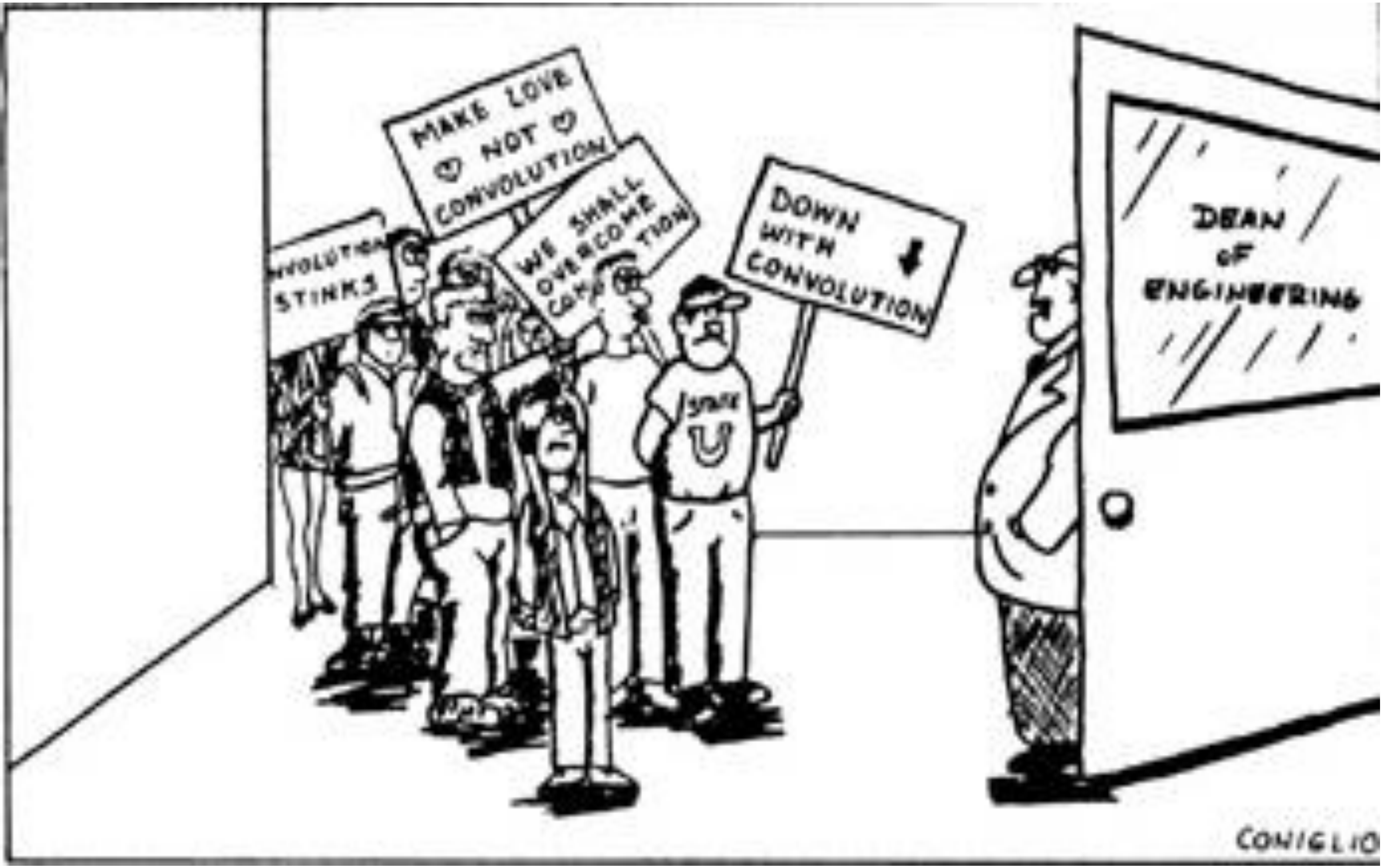
Step1: ***Folding***: Fold $h(\tau)$ about $\tau = 0$ to obtain $h(-\tau)$.

Step2: ***Shifting***: Shift $h(-\tau)$ by t_0 to the right (left) if t_0 is positive (negative), to obtain $h(t_0 - \tau)$

Step3: ***Multiplication***: Multiply $x(\tau)$ **by** $h(t_0 - \tau)$ to obtain the product sequence

$$v_{t_0}(\tau) = x(\tau)h(t_0 - \tau).$$

Step4: ***Integration***: Integrate (sum) all the values of the product sequence $v_{t_0}(\tau)$ to obtain the value of the output at time $t = t_0$.


NVOLUTION STINKS
MAKE LOVE NOT CONVOLUTION
WE SHALL OVERCOME
DOWN WITH CONVOLUTION
STATE U
DEAN OF ENGINEERING
CONIGLIO

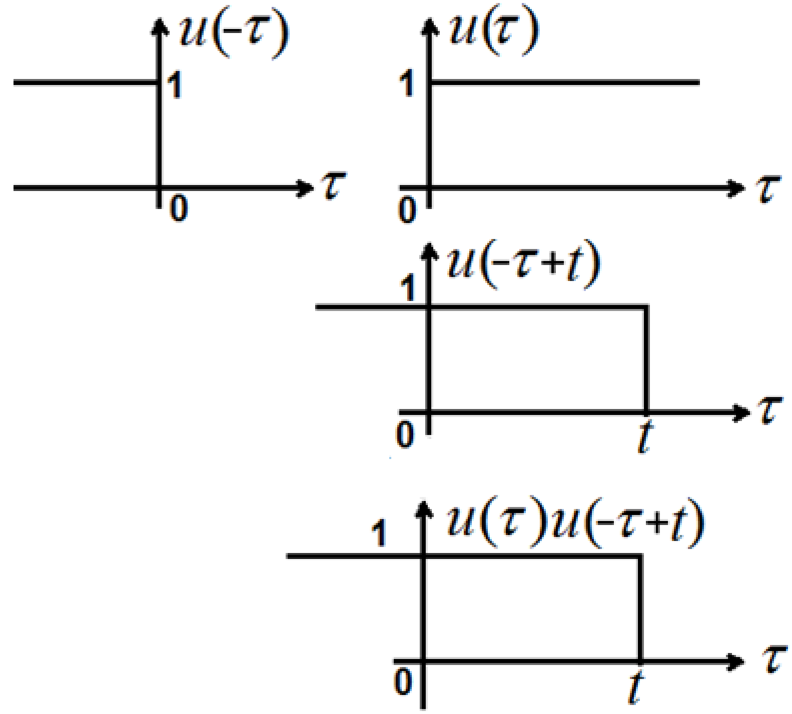
Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where $x(t) = e^{-at}u(t)$ and $h(t) = u(t)$. Find the response of the system.

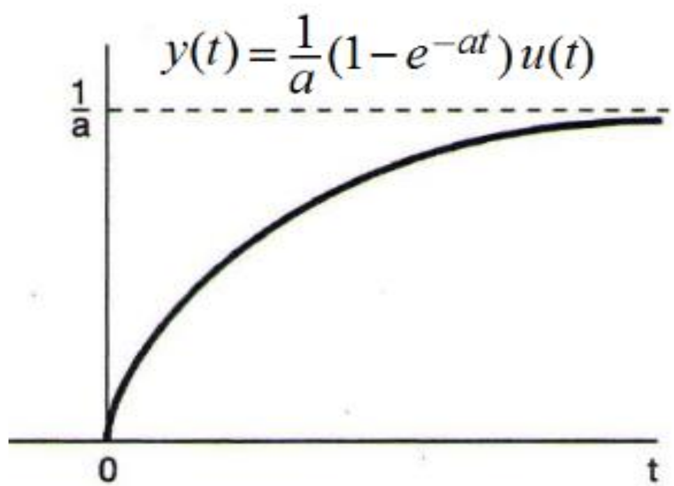
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

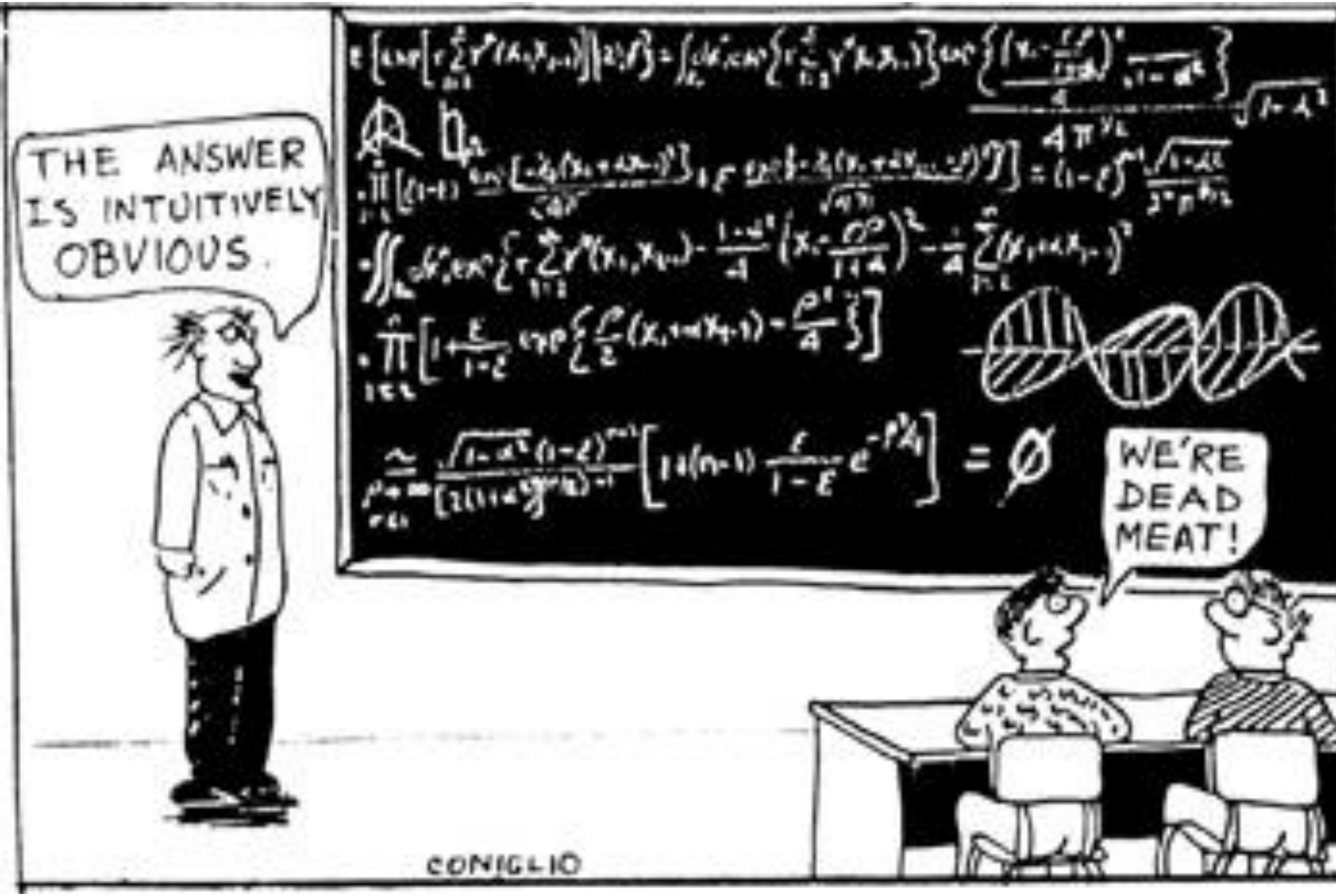
$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-a\tau}\, d\tau = -\frac{1}{a}\left[e^{-a\tau}\right]_0^t$$

$$= \frac{1}{a}(1 - e^{-at})\, u(t)$$






THE ANSWER IS INTUITIVELY OBVIOUS.
WE'RE DEAD MEAT!
CONIGLIO

Find $y(t) = x(t) * h(t)$ by evaluating the convolution integral, where $x(t) = e^{-at}u(t)$ and $h(t) = e^{-bt}u(t)$.

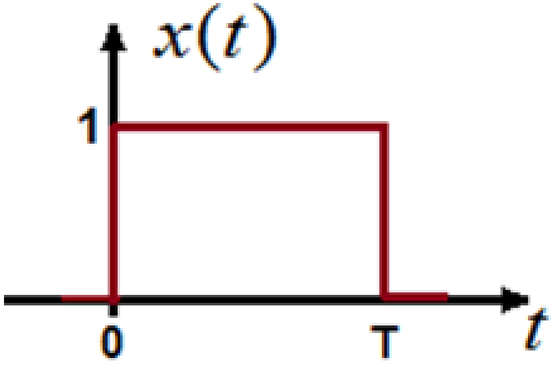
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau$$

$$= \int_0^t e^{-a\tau}\, e^{-b(t-\tau)}\, d\tau = e^{-bt} \int_0^t e^{-(a-b)\tau}\, d\tau$$

$$= e^{-bt} \left[\frac{e^{-(a-b)\tau}}{-(a-b)} \right]_0^t = \frac{\left(e^{-at} - e^{-bt}\right)}{(b-a)} u(t)$$

Consider a rectangular pulse is shown in figure is applied to an LTI system with impulse response $h(t)=e^{-at}u(t)$. Find the response of the system.

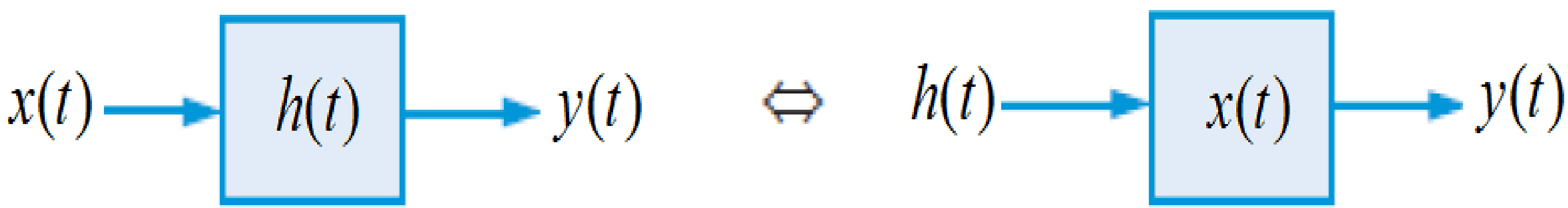


$$y(t)=\frac{1}{a}\left\{1-e^{-at}\right\}u(t)-\frac{1}{a}\left\{1-e^{-a(t-T)}\right\}u(t-T)$$

Properties of LTI systems

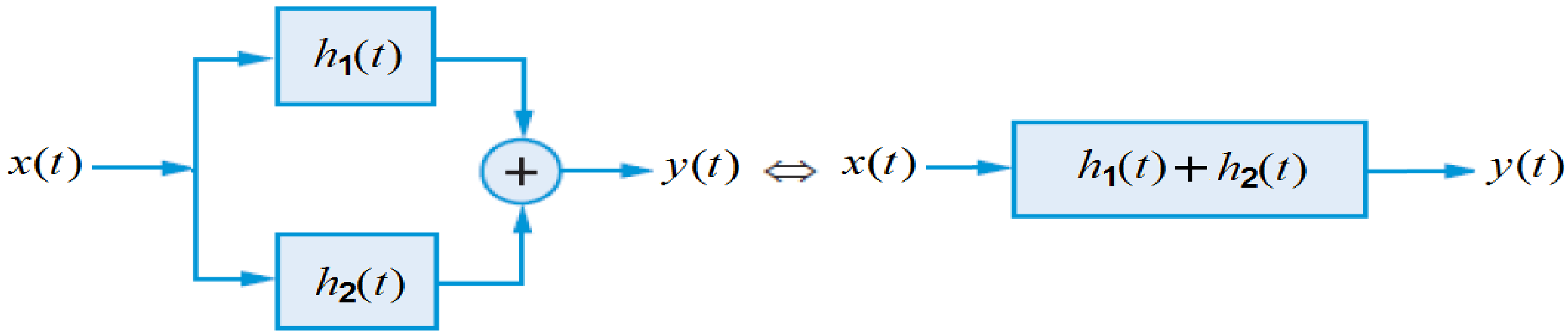
The Commutative Property

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

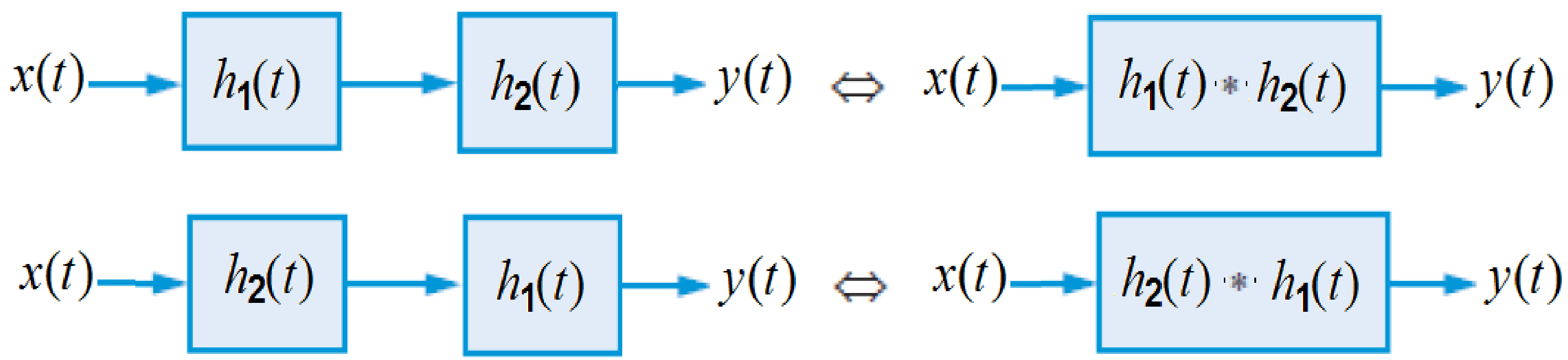


The Distributive Property

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$



The Associative Property



Other Properties

1. $x(t)*k\delta(t)=kx(t)$, where *k* is a constant.

2. $x(t)*\delta(t-t_0)=x(t-t_0)$

3. If $x(t)*h(t)=y(t)$

   then $x(t-a)*h(t-b)=y(t-(a+b))$

$$u(t)*u(t)=tu(t)$$

$$u(t+2)*u(t-1)=(t+1)u(t+1)$$

Causality of LTI system

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{0} h(\tau)\, x(t-\tau)\, d\tau + \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$h(t) = 0$ for $t < 0$.

This is a necessary and sufficient condition for causality.

Stability of LTI system

Let $|x(t)| \le B_x < \infty$ applied to an LTI system

Similarly $y(t)$ is bounded such that $|y(t)| \le B_y < \infty$

$$| y(t) | = | \int_0^\infty h(\tau)\, x(t-\tau)\, d\tau |$$

$$= \int_0^\infty | h(\tau)\, x(t-\tau) |\, d\tau = B_x \int_0^\infty | h(\tau) |\, d\tau$$

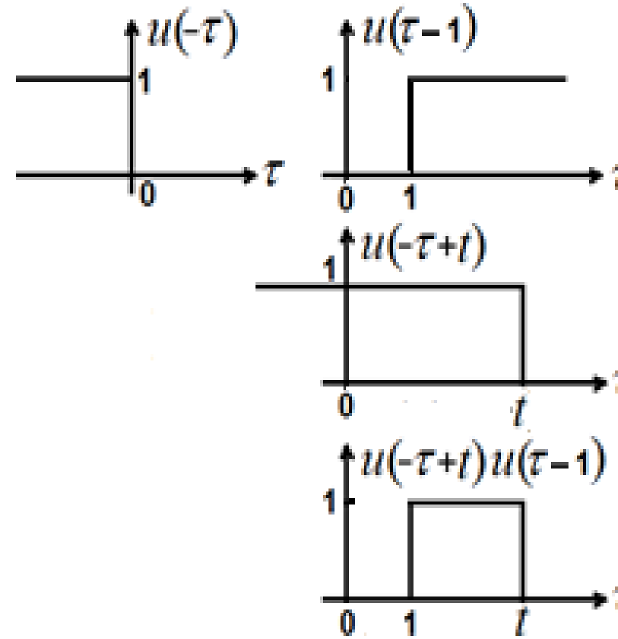
Form this representation the output

sequence $y(t)$ is bounded

If $\int_0^\infty | h(\tau) |\, d\tau < \infty$ or $\int_0^\infty | h(t) |\, dt < \infty$

Examples

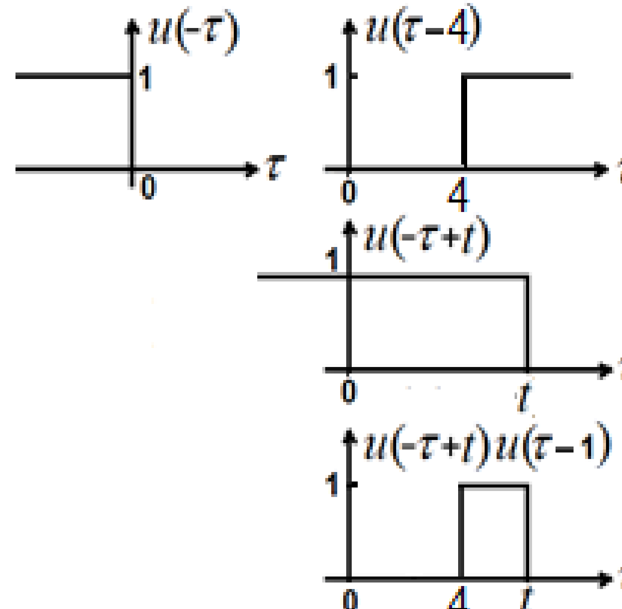
$$x(t) = u(t);\ \ h(t) = 2u(t-1) - 2u(t-4)$$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\ \ d\tau$$

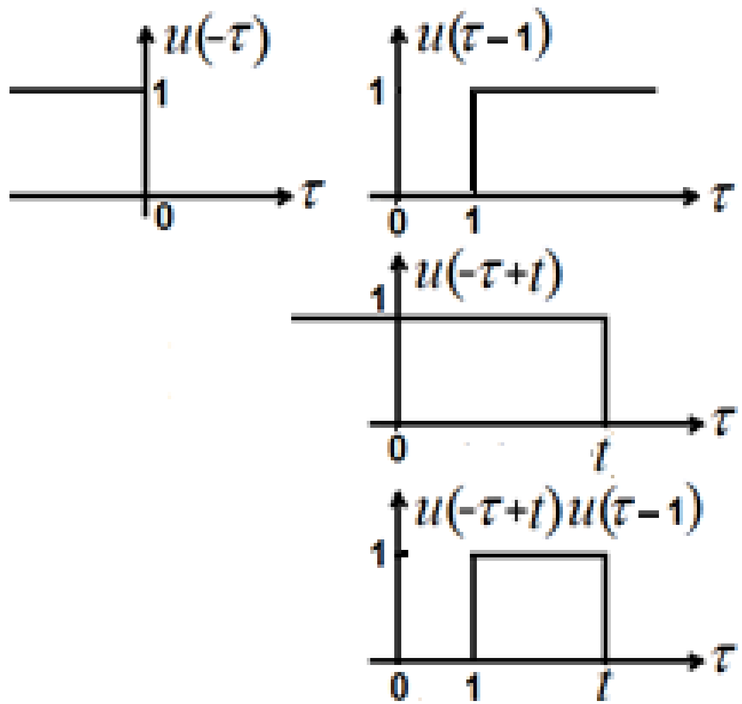
$$= \int_{-\infty}^{\infty} u(t-\tau)\{2u(\tau-1) - 2u(\tau-4)\}\ d\tau$$

$$= \int_{-\infty}^{\infty} u(t-\tau)2u(\tau-1)\ d\tau - \int_{-\infty}^{\infty} u(t-\tau)2u(\tau-4)\ d\tau$$

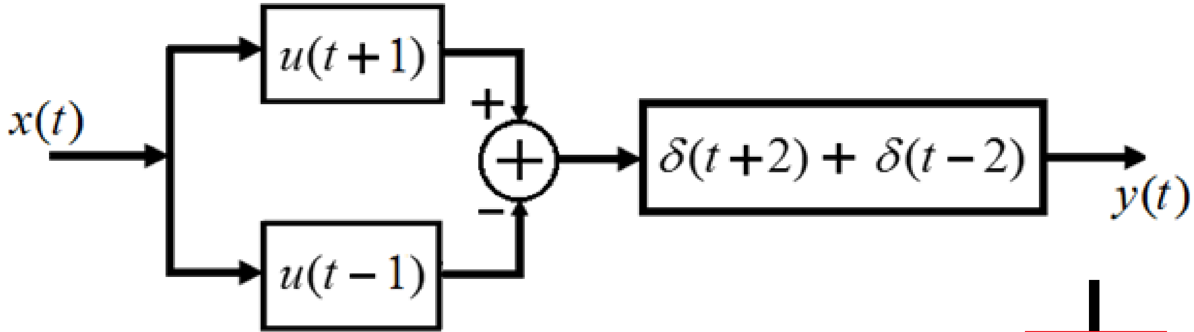


$$= 2\int_1^t 1\, d\tau + 2\int_4^t 1\, d\tau$$

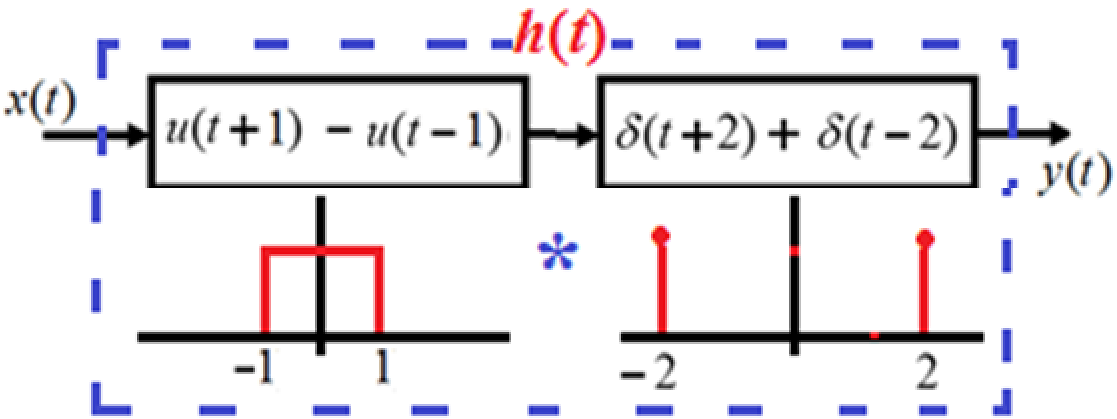
$$= 2(t-1)u(t-1) - 2(t-4)u(t-4)$$

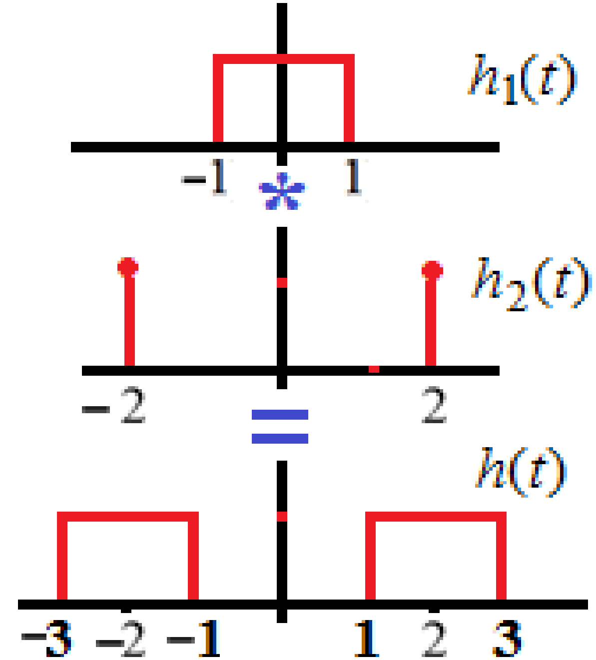

u(-τ)
1
0
τ
u(τ-1)
1
0
1
τ
u(-τ+t)
1
0
t
τ
u(-τ+t)u(τ-1)
1
0
1
t
τ

Three systems are interconnected as shown below



Find and plot output $y(t)$ when $x(t)=\delta(t)$





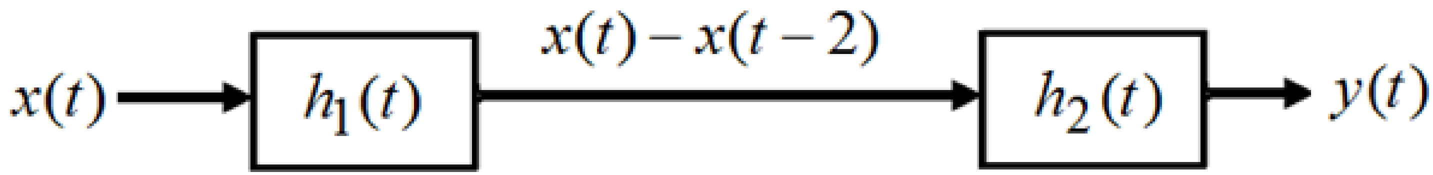
For the given input excitation $x(t)=\delta(t)$,

the response of the system is

$$y(t)=x(t)*h(t)=\delta(t)*h(t)=h(t)$$

$$=\{u(t+3)-u(t+1)\}+\{u(t-1)-u(t-3)\}$$

A cascaded system is shown below



Suppose that $h_1(t)=h_2(t)$, find and plot output $y(t)$ when $x(t)=\delta(t)$

Let $z(t)=x(t)-x(t-2)$

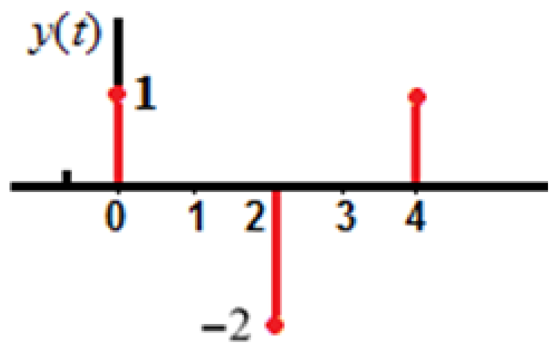
$$y(t)=z(t)-z(t-2)$$

$$=\{x(t)-x(t-2)\}-\{x(t-2)-x(t-4)\}$$

$$=x(t)-2x(t-2)+x(t-4)$$

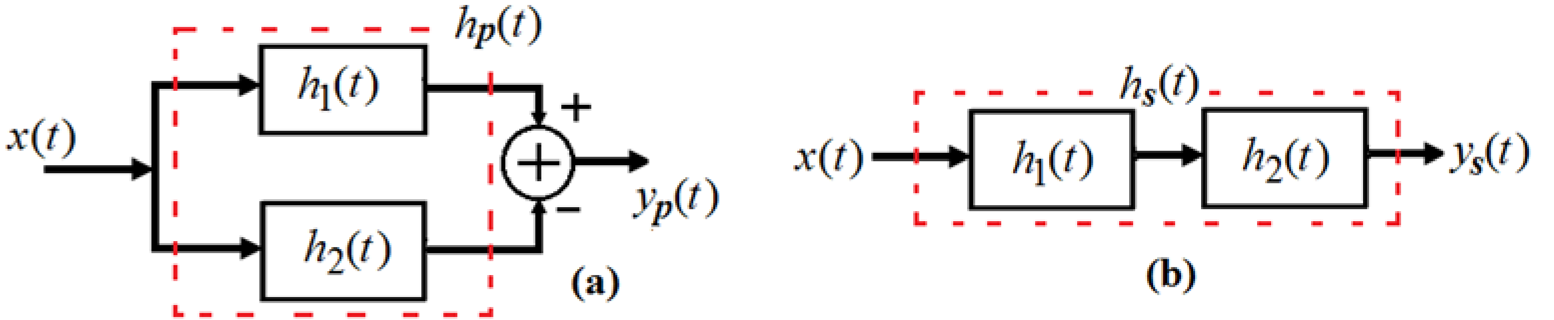
$x(t)=\delta(t)$

$$y(t)=\delta(t)-2\delta(t-2)+\delta(t-4).$$



Two LTI systems have impulse response functions given by

$$h_1(t) = \delta(t-1) \text{ and } h_2(t) = u(t-1)$$



$$h_p(t) = h_1(t) + h_2(t) = \delta(t-1) + u(t-1)$$

$$h_c(t) = h_1(t) * h_2(t) = \delta(t-1) * u(t-1) = u(t-2)$$

Consider a system having impulse response $h(t)=\delta(t+1)+\delta(t-1)$. Determine and sketch the output for the following excitations.

(a) A symmetrical rectangular pulse of unit height and unit width centered at origin

(b) $x(t)=u(t)-u(t-1)$

$$x(t-a)*x(t-b)=y(t-(a+b))$$

$$x(t-a)*x(t-b)=y(t-(a+b))$$

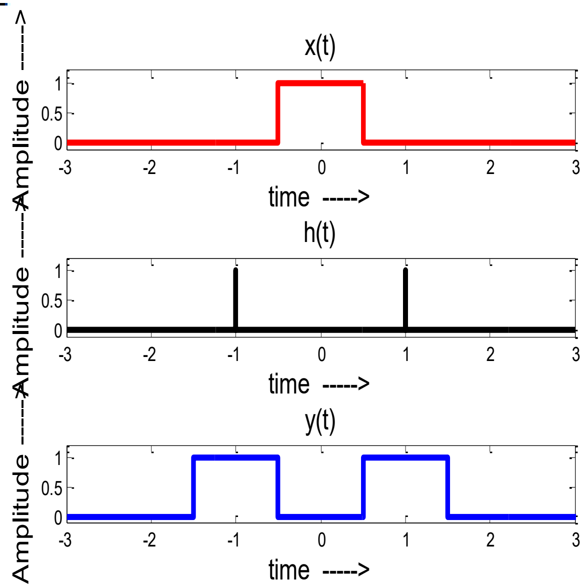
$$y(t)=x(t)*h(t)=\left(u\left(t+0.5\right)-u\left(t-0.5\right)\right)*\left(\delta(t+1)+\delta(t-1)\right)$$

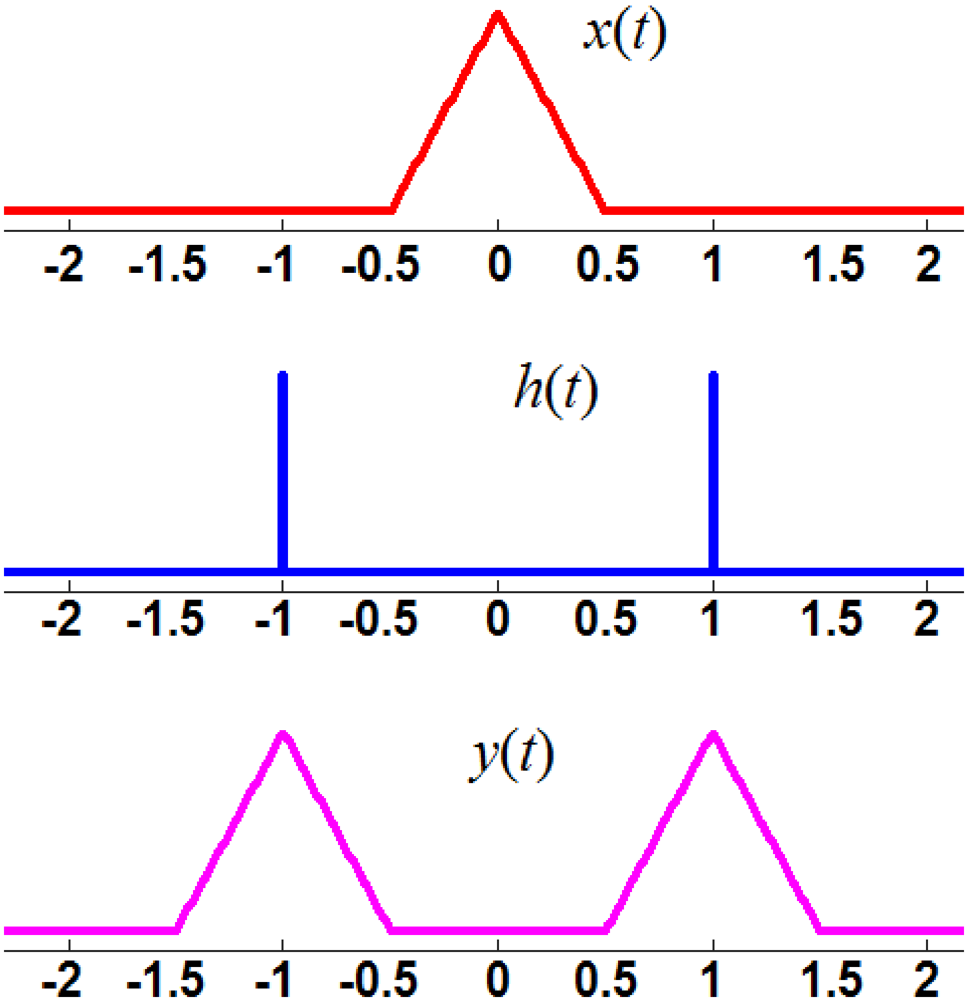
$$=u\left(t+0.5\right)*\delta(t+1)+u\left(t+0.5\right)*\delta(t-1)$$

$$-u\left(t-0.5\right)*\delta(t+1)-u\left(t-0.5\right)*\delta(t-1)$$

$$=u\left(t+1.5\right)+u\left(t-0.5\right)-u\left(t+0.5\right)-u\left(t-1.5\right)$$

$$=\left[u\left(t+1.5\right)-u\left(t+0.5\right)\right]+\left[u\left(t-0.5\right)-u\left(t-1.5\right)\right]$$





x(t)
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
h(t)
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
y(t)
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

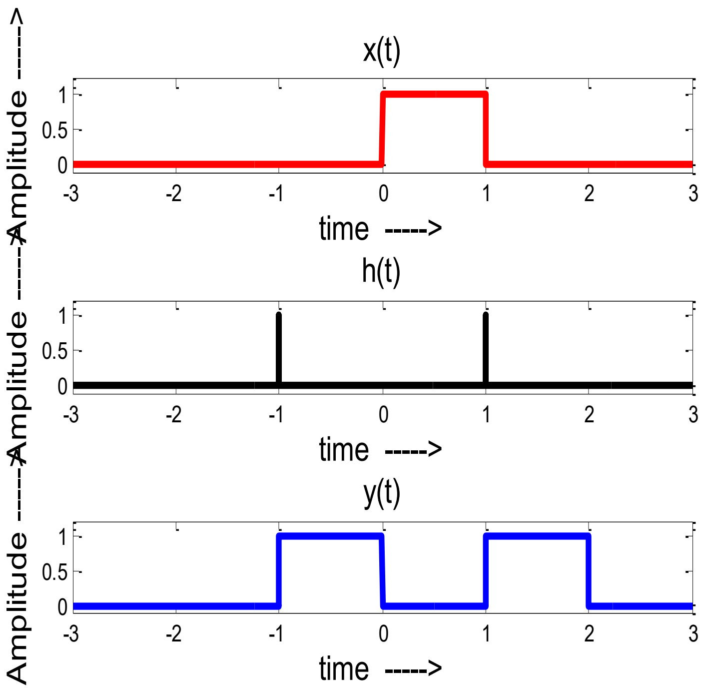
$$x(t) = u(t) - u(t-1) \text{ and } h(t) = [\delta(t+1) + \delta(t-1)]$$

$$y(t) = x(t) * h(t) = [u(t) - u(t-1)] * [\delta(t+1) + \delta(t-1)]$$

$$= u(t) * \delta(t+1) - u(t-1) * \delta(t+1) + u(t) * \delta(t-1) - u(t-1) * \delta(t-1)$$

$$= u(t+1) - u(t) + u(t-1) - u(t-2)$$

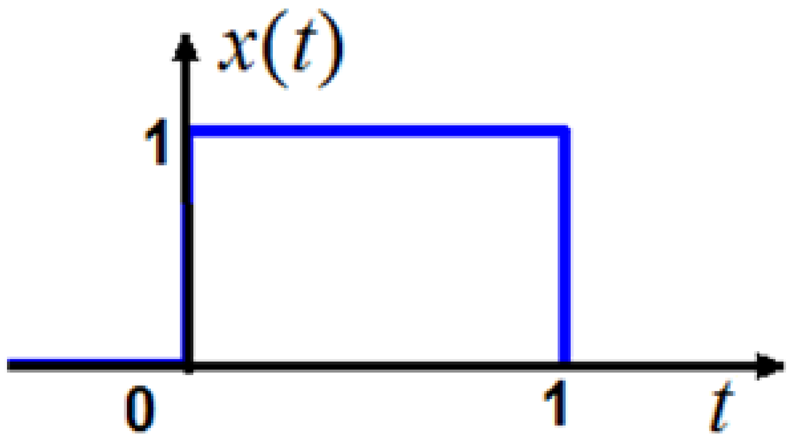
$$= \{u(t+1) - u(t)\} + \{u(t-1) - u(t-2)\}$$

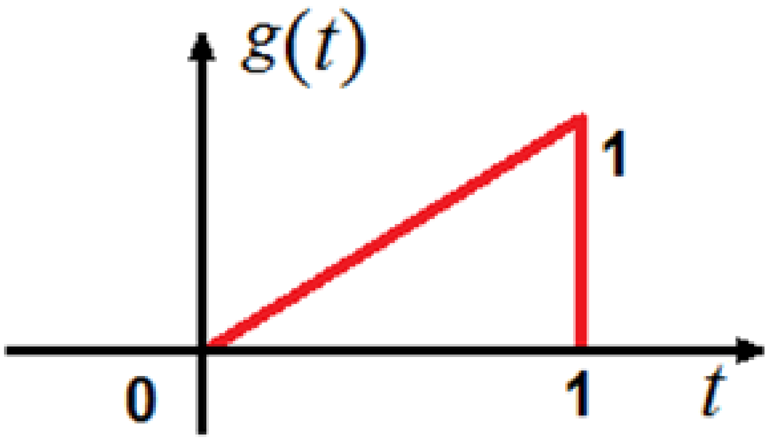


Consider the signals shown below.

Determine analytically the convolution of the following.

(i) $y(t) = x(t) * x(t)$ (ii) $z(t) = g(t) * g(t)$ (iii) $p(t) = x(t) * g(t)$.





End