Signals and Communication Systems

Solution of Linear Constant Coefficient Differential Equations using Laplace Transform

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Unilateral Laplace Transform

The analysis equation of unilateral LT is represented by

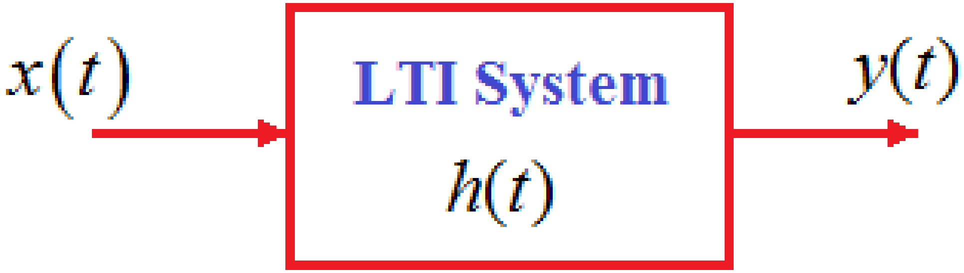
$$X(s) = \int_0^{\infty} x(t) e^{-st} dt .$$

Time differentiation property:

$$\frac{d}{dt} x(t) \xleftrightarrow{\text{LT}} s X(S) - x(0-)$$

where $x(0-)$ is referred to as initial conditions.

Applications of LT: LTI Systems:



$$y(t) = x(t) * h(t)$$

The input output relation of an LTI system is represented by a differential equation:

For example, a second order Linear Constant Coefficient Differential Equation is represented by

$$a_2 \frac{d2}{dt2} y(t) + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_2 \frac{d2}{dt2} x(t) + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

The analysis of Linear Constant Coefficient Differential Equation is much more convenient in LT domain than in time domain.

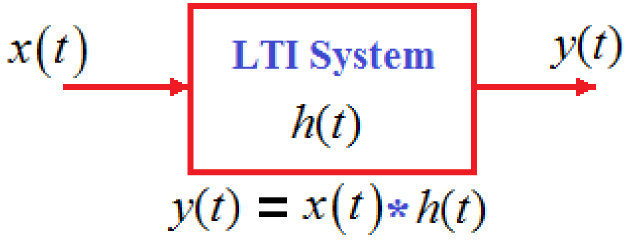
Time Domain: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) * h(t-\tau)\, d\tau$

where $x(t)$ is an input signal or excitation

$h(t)$ is an impulse response of the system.

$y(t)$ Is output or response of the system.

Frequency Domain: $Y(s) = X(s)H(s)$



where $X(s)$ is LT of input signal or excitation

$H(s) = \dfrac{Y(s)}{X(s)}$ is referred to as System Function or Transfer Function

$H(j\Omega)$ Frequency Response of the system.

$Y(s)$ Is LT of output or response of the system.

Remember that $h(t) \xleftrightarrow{\text{L.T}} H(s)$, i.e., impulse response of the system and system function are LT pair.

Causality conditions for LTI System:

1. General Statement: The present output at any time depends on the present and past inputs, but not on the future inputs.
2. In terms of Impulse response: An LTI system is causal if $h(t) = 0$ for $t < 0$.
3. In terms of System Function: The ROC must be the right-side of the right most pole.

Stability condition for LTI System:

1. General Statement: Bounded input Bounded output stable.
2. In terms of Impulse response: The impulse response must be absolutely integrable. $\int_0^{\infty} |h(t)|\, dt < \infty$.
3. In terms of System Function: The ROC must include the $j\Omega$ axis.

Ex1: Consider a differential equation $\frac{d}{dt}y(t)+2y(t)=x(t)$.

(a) Find the System function / transfer function.

(b) Find impulse response of the system.

(c) Check the system for causality and stability using impulse response of the system.

(d) Check the system for causality and stability using system function.

(e) Find step response of the system.

(f) Find the response of the system for an excitation $x(t)=e^{-3t}u(t)$.

Ans: Given that $\frac{d}{dt}y(t)+2y(t)=x(t)$

By taking the LT on both sides of the equation, we get

$$sY(s)+2Y(s)=X(s)$$

$$Y(s)(s+2)=X(s)$$

(a) Then the system function / transfer function

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s+2},\ \ \mathrm{Re}\{s\}>-2$$

(b) The impulse response of the system is computed by taking inverse LT of the system function. Therefore, $h(t)=e^{-2t}u(t)$

(c) Check the system for causality and stability using impulse response of the system.

Causality: Since $h(t) = e^{-2t}u(t)$ is right sided sequence the system $h(t)$ is causal.

Stability: The impulse response must be absolutely integrable,

$$\int_0^\infty |h(t)|\,dt = \int_0^\infty |e^{-2t}u(t)|\,dt = \frac{1}{-2}\left[e^{-2t}\right]_0^\infty .$$

$$= \frac{1}{-2}(0-1) = \frac{1}{2} < \infty$$

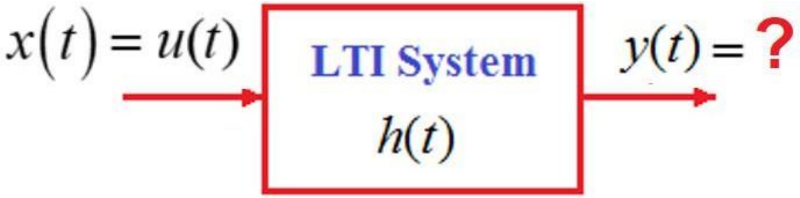
Since absolute integral value is finite i.e., $\frac{1}{2}$, it is a stable system.

(d) Check the system for causality and stability using system function.

Causality: The system function has a pole at s = -2 and ROC $\mathrm{Re}\{s\} > -2$, which is right sided, hence it is causal signal.

Stability: The ROC $\mathrm{Re}\{s\} > -2$ that includes the $j\Omega$ axis, hence it is a stable system.

(e) Find step response of the system.



Ans: Given that $\frac{d}{dt}y(t) + 2y(t) = x(t)$

Step Response: For input $x(t) = u(t)$

$$\frac{d}{dt}y(t) + 2y(t) = x(t) = u(t)$$

By taking LT and simplification, we get

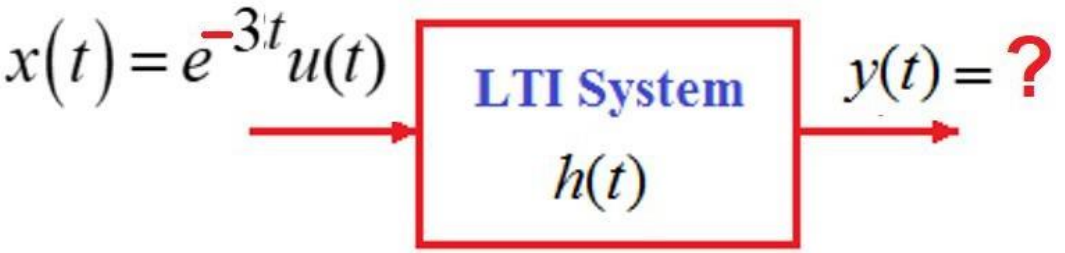
$$\Rightarrow sY(s) + 2Y(s) = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s}\frac{1}{s+2} = \frac{A}{s} + \frac{B}{s+2} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$$

By inverse LT we get the step response

$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t) = \frac{1}{2}(1 - e^{-2t})u(t)$$

(f) Find the response of the system for an excitation $x(t)=e^{-3t}u(t)$.



The response of the system for an excitation $x(t)=e^{-3t}u(t)$

$$\frac{d}{dt}y(t)+2y(t)=x(t)=e^{-3t}u(t)$$

$$\Rightarrow sY(s)+2Y(s)=\frac{1}{s+3}$$

$$\Rightarrow Y(s)=\frac{1}{s+3}\frac{1}{s+2}=\frac{A}{s+2}+\frac{B}{s+3}=\frac{1}{s+2}-\frac{1}{s+3}$$

By inverse LT we get the response of the system

for an excitation $x(t)=e^{-3t}u(t)$ is

$$y(t)=e^{-2t}u(t)-e^{-3t}u(t)$$

Ex2: Consider an LTI system with system function $H(s) = \frac{2s}{s+3}$

a) Describe its differential equation.

b) Find the Impulse response.

c) Test the system for causality and stability.

d) Find the step response of the system.

e) Find the response of the system for an excitation $x(t) = e^{-2t}u(t)$

Ans: Given that system function $H(s) = \frac{2s}{(s+3)}$

(a) The system function can be written as $H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{(s+3)}$

By cross multiplication and simplified, we get

$$Y(s)(s+3) = 2s\ X(s) \Rightarrow sY(s) + 3Y(s) = 2sX(s)$$

By taking inverse LT, we get the differential equation:

$$\frac{d}{dt}y(t) + 3y(t) = 2\frac{d}{dt}x(t)$$

(b) The impulse response can be obtained by taking inverse LT of system function.

Method1: General method: The given system function can be written as

$$H(s) = \frac{2s}{s+3} = 2\frac{s+3-3}{s+3} = 2\left(1 - \frac{3}{s+3}\right)$$

Then by inverse LT, we get $h(t) = 2(\delta(t) - 3e^{-3t}u(t))$

Method2: Using Time differentiation property:

We know that $\dfrac{1}{s+3} \xleftrightarrow{\text{L.T}} e^{-3t}u(t)$

By time differentiation, $2s\dfrac{1}{s+3} \xleftrightarrow{\text{L.T}} 2\dfrac{d}{dt}\left\{e^{-3t}u(t)\right\}$

By formula, $\dfrac{d}{dt}uv = u\dfrac{d}{dt}v + v\dfrac{d}{dt}u$, we get

$$h(t) = 2\left\{e^{-3t}\frac{d}{dt}u(t) + u(t)\frac{d}{dt}e^{-3t}\right\} = 2\left\{e^{-3t}\delta(t) - 3u(t)e^{-3t}\right\}$$

$$= 2\left\{\delta(t) - 3e^{-3t}u(t)\right\}$$

(c) Causality and Stability in terms of Impulse response of the system:

Causality: By observing the impulse response $h(t)=0$ for $t<0$.
Hence it is a causal system.

Stability: $\int_0^{\infty} |h(t)|\, dt = \int_0^{\infty} | 2\left\{\delta(t) - 3u(t)e^{-3t}\right\}|\, dt$

$$= 2\left[\left|1 - \frac{1}{-3}e^{-3t}\right|\right]_0^{\infty} = 2\left[\left|1 + \frac{1}{3}(0-1)\right|\right] = \frac{4}{3} < \infty$$

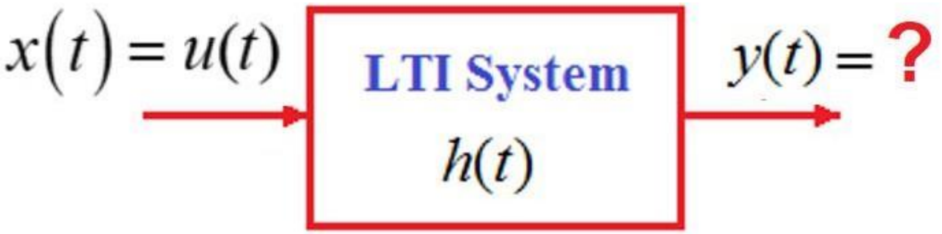
Hence it is a stable system.

(C) Causality and Stability test using system function:

Causality: By observing the system function, a pole at $s=-3$ and ROC is $\mathrm{Re}\{s\} > -3$ which is right-sided. Hence it is a causal system.

Stability: The ROC $\mathrm{Re}\{s\} > -3$ that includes the $j\Omega$ axis, hence it is a stable system.

(d) Step Response: For input $x(t) = u(t) \xleftrightarrow{\text{L.T}} X(s) = \frac{1}{s}$

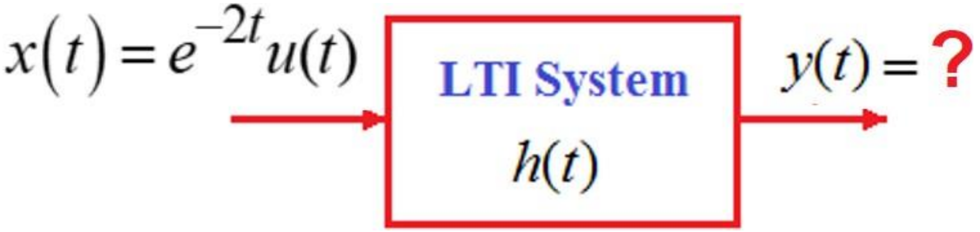


Given that $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{2s}{(s+3)}$

Then $Y(s) = \dfrac{2s}{s+3} X(s) = \dfrac{2s}{s+3}\dfrac{1}{s} = \dfrac{2}{s+3}$

By inverse LT we get the step response, we get $y(t) = 2e^{-3t}u(t)$

(e) The response of the system for an excitation $x(t)=e^{-2t}u(t)$



Given that $H(s)=\dfrac{Y(s)}{X(s)}=\dfrac{2s}{(s+3)}$

For the input excitation $x(t)=e^{-2t}u(t)$, we get

$$Y(s)=\frac{2s}{s+3}X(s)$$

$$=\frac{2s}{s+3}\frac{1}{s+2}=\frac{A}{s+2}+\frac{B}{s+3}$$

$$=6\frac{1}{s+3}-4\frac{1}{s+2}$$

By inverse LT we get the step response, we get

$$y(t)=6e^{-3t}u(t)-4e^{-2t}u(t)$$

Exercise: A LTI system is described by a differential equation $\frac{d}{dt}y(t)+4y(t)=\frac{d}{dt}x(t)$.

Find (a) The transfer function.

(b) Impulse response of the system

(c) Test the system for causality and stability.

(d) Step response of the system

(e) Find the response of the system for an excitation $x(t)=e^{-2t}u(t)$

End