# CO3 and CO4
# Problem

## Example 1

You have 1000 dollars to put in an account with interest rate $R$, compounded annually. That is, if $X_n$ is the value of the account at year $n$, then

$$X_n = 1000(1+R)^n, \quad \text{for } n = 0, 1, 2, \cdots.$$

The value of $R$ is a random variable that is determined when you put the money in the bank, but it does not not change after that. In particular, assume that $R \sim Uniform(0.04, 0.05)$.

a. Find all possible sample functions for the random process $\{X_n, n = 0, 1, 2, \ldots\}$.
b. Find the expected value of your account at year three. That is, find $E[X_3]$.

**Solution**

a. Here, the randomness in $X_n$ comes from the random variable $R$. As soon as you know $R$, you know the entire sequence $X_n$ for $n = 0, 1, 2, \cdots$. In particular, if $R = r$, then

$$X_n = 1000(1 + r)^n, \quad \text{for all } n \in \{0, 1, 2, \cdots\}.$$

Thus, here sample functions are of the form $f(n) = 1000(1 + r)^n$, $n = 0, 1, 2, \cdots$, where $r \in [0.04, 0.05]$. For any $r \in [0.04, 0.05]$, you obtain a sample function for the random process $X_n$.

b. The random variable $X_3$ is given by

$$X_3 = 1000(1+R)^3.$$

If you let $Y = 1 + R$, then $Y \sim Uniform(1.04, 1.05)$, so

$$f_Y(y) = \begin{cases} 100 & 1.04 \leq y \leq 1.05 \\ 0 & \text{otherwise} \end{cases}$$

To obtain $E[X_3]$, we can write

$$\begin{aligned} E[X_3] &= 1000E[Y^3] \\ &= 1000 \int_{1.04}^{1.05} 100y^3 \quad dy \quad \text{(by LOTUS)} \\ &= \frac{10^5}{4} \left[ y^4 \right]_{1.04}^{1.05} \\ &= \frac{10^5}{4} \left[ (1.05)^4 - (1.04)^4 \right] \\ &\approx 1,141.2 \end{aligned}$$

**Definition 5.2 (Law of the unconscious statistician ( LOTUS))** The **"law of the unconscious statistician" (LOTUS)** says that the expected value of a transformed random variable can be found without finding the distribution of the transformed random variable, simply by applying the probability weights of the original random variable to the transformed values.

$$\text{Discrete } X \text{ with pmf } p_X: \qquad \mathrm{E}[g(X)] = \sum_x g(x) p_X(x)$$

$$\text{Continuous } X \text{ with pdf } f_X: \qquad \mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

In probability theory and statistics, a collection of random variables is independent and identically distributed if each random variable has the same probability distribution as the others and all are mutually independent. This property is usually abbreviated as i.i.d., iid, or IID.

**Example 2**

Let $\{X(t), t \in [0, \infty)\}$ be defined as

$$X(t) = A + Bt, \qquad \text{for all } t \in [0, \infty),$$

where $A$ and $B$ are independent normal $N(1,1)$ random variables.

a. Find all possible sample functions for this random process.
b. Define the random variable $Y = X(1)$. Find the PDF of $Y$.
c. Let also $Z = X(2)$. Find $E[YZ]$.

**Solution**

a. Here, we note that the randomness in $X(t)$ comes from the two random variables $A$ and $B$. The random variable $A$ can take any real value $a \in \mathbb{R}$. The random variable $B$ can also take any real value $b \in \mathbb{R}$. As soon as we know the values of $A$ and $B$, the entire process $X(t)$ is known. In particular, if $A = a$ and $B = b$, then

$$X(t) = a + bt, \quad \text{for all } t \in [0, \infty).$$

Thus, here, sample functions are of the form $f(t) = a + bt$, $t \geq 0$, where $a, b \in \mathbb{R}$. For any $a, b \in \mathbb{R}$ you obtain a sample function for the random process $X(t)$.

b. We have

$$Y = X(1) = A + B.$$

Since $A$ and $B$ are independent $N(1,1)$ random variables, $Y = A + B$ is also normal with

$$\begin{aligned} EY &= E[A+B] \\ &= E[A] + E[B] \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(A+B) \\ &= \text{Var}(A) + \text{Var}(B) \quad \text{(since } A \text{ and } B \text{ are independent)} \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Thus, we conclude that $Y \sim N(2,2)$:

$$f_Y(y) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}.$$

c. We have

$$\begin{aligned}
E[YZ] &= E[(A+B)(A+2B)] \\
&= E[A^2 + 3AB + 2B^2] \\
&= E[A^2] + 3E[AB] + 2E[B^2] \\
&= 2 + 3E[A]E[B] + 2 \cdot 2 \quad \text{(since } A \text{ and } B \text{ are independent)} \\
&= 9.
\end{aligned}$$

## Example 3

Consider the random process $\{X_n, n = 0, 1, 2, \cdots\}$, in which $X_i$'s are i.i.d. standard normal random variables.

**Solution**

1. Since $X_n \sim N(0,1)$, we have

$$f_{X_n}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad \text{for all } x \in \mathbb{R}.$$

2. If $m \neq n$, then $X_m$ and $X_n$ are independent (because of the i.i.d. assumption), so

$$\begin{aligned} f_{X_m X_n}(x_1, x_2) &= f_{X_m}(x_1) f_{X_n}(x_2) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}} \\ &= \frac{1}{2\pi} \exp\left\{-\frac{x_1^2 + x_2^2}{2}\right\}, \qquad \text{for all } x_1, x_2 \in \mathbb{R}. \end{aligned}$$

**Example 4**

Find the mean functions for the random processes given in Examples 1 and 2

**Solution**

For $\{X_n, n = 0, 1, 2, \cdots\}$ given in Example 1

$$
\begin{aligned}
\mu_X(n) &= E[X_n] \\
&= 1000E[Y^n] \quad (\text{where } Y = 1 + R \quad \sim \quad Uniform(1.04, 1.05)) \\
&= 1000 \int_{1.04}^{1.05} 100y^n \quad dy \quad (\text{by LOTUS}) \\
&= \frac{10^5}{n+1} \left[ y^{n+1} \right]_{1.04}^{1.05} \\
&= \frac{10^5}{n+1} \left[ (1.05)^{n+1} - (1.04)^{n+1} \right], \quad \text{for all } n \in \{0, 1, 2, \cdots\}.
\end{aligned}
$$

For $\{X(t), t \in [0, \infty)\}$ given in Example 2

$$\begin{aligned}
\mu_X(t) &= E[X(t)] \\
&= E[A + Bt] \\
&= E[A] + E[B]t \\
&= 1 + t, \qquad \text{for all } t \in [0, \infty).
\end{aligned}$$

**Examples 5**

Find the correlation functions and covariance functions for the random processes given in Examples 1 and 2

**Solution**

For $\{X_n, n = 0, 1, 2, \cdots\}$ given in Example 1

$$
\begin{aligned}
R_X(m,n) &= E[X_m X_n] \\
&= 10^6 E[Y^m Y^n] \quad (\text{where } Y = 1 + R \quad \sim \quad Uniform(1.04, 1.05)) \\
&= 10^6 \int_{1.04}^{1.05} 100 y^{(m+n)} \quad dy \quad (\text{by LOTUS}) \\
&= \frac{10^8}{m+n+1} \left[ y^{m+n+1} \right]_{1.04}^{1.05} \\
&= \frac{10^8}{m+n+1} \left[ (1.05)^{m+n+1} - (1.04)^{m+n+1} \right], \quad \text{for all } m, n \in \{0, 1, 2, \cdots\}.
\end{aligned}
$$

To find the covariance function, we write

$$\begin{aligned}
C_X(m,n) &= R_X(m,n) - E[X_m]E[X_n] \\
&= \frac{10^8}{m+n+1}\left[(1.05)^{m+n+1} - (1.04)^{m+n+1}\right] \\
&- \frac{10^{10}}{(m+1)(n+1)}\left[(1.05)^{m+1} - (1.04)^{m+1}\right]\left[(1.05)^{n+1} - (1.04)^{n+1}\right].
\end{aligned}$$

For $\{X(t), t \in [0,\infty)\}$ given in Example 2

$$\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\
&= E[(A+Bt_1)(A+Bt_2)] \\
&= E[A^2] + E[AB](t_1+t_2) + E[B^2]t_1t_2 \\
&= 2 + E[A]E[B](t_1+t_2) + 2t_1t_2 \quad \text{(since } A \text{ and } B \text{ are independent)} \\
&= 2 + t_1 + t_2 + 2t_1t_2, \quad \text{for all } t_1, t_2 \in [0,\infty).
\end{aligned}$$

Finally, to find the covariance function for $X(t)$, we can write

$$\begin{aligned}
C_X(t_1,t_2) &= R_X(t_1,t_2) - E[X(t_1)]E[X(t_2)] \\
&= 2 + t_1 + t_2 + 2t_1t_2 - (1+t_1)(1+t_2) \\
&= 1 + t_1t_2, \quad \text{for all } t_1, t_2 \in [0,\infty).
\end{aligned}$$

## Examples 6

Let $A$, $B$, and $C$ be independent normal $N(1,1)$ random variables. Let $\{X(t), t \in [0,\infty)\}$ be defined as

$$X(t) = A + Bt, \quad \text{for all } t \in [0,\infty).$$

Also, let $\{Y(t), t \in [0,\infty)\}$ be defined as

$$Y(t) = A + Ct, \quad \text{for all } t \in [0,\infty).$$

Find $R_{XY}(t_1,t_2)$ and $C_{XY}(t_1,t_2)$, for $t_1, t_2 \in [0,\infty)$.

## Solution

First, note that

$$\begin{aligned}\mu_X(t) &= E[X(t)] \\ &= EA + EB \cdot t \\ &= 1 + t, \quad \text{for all } t \in [0, \infty).\end{aligned}$$

Similarly,

$$\begin{aligned}\mu_Y(t) &= E[Y(t)] \\ &= EA + EC \cdot t \\ &= 1 + t, \quad \text{for all } t \in [0, \infty).\end{aligned}$$

To find $R_{XY}(t_1,t_2)$ for $t_1,t_2 \in [0,\infty)$, we write

$$\begin{aligned}
R_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] \\
&= E\big[(A+Bt_1)(A+Ct_2)\big] \\
&= E\big[A^2 + ACt_2 + BAt_1 + BCt_1t_2\big] \\
&= E[A^2] + E[AC]t_2 + E[BA]t_1 + E[BC]t_1t_2 \\
&= E[A^2] + E[A]E[C]t_2 + E[B]E[A]t_1 + E[B]E[C]t_1t_2, \quad \text{(by independence)} \\
&= 2 + t_1 + t_2 + t_1t_2.
\end{aligned}$$

To find $C_{XY}(t_1,t_2)$ for $t_1,t_2 \in [0,\infty)$, we write

$$\begin{aligned}
C_{XY}(t_1,t_2) &= R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2) \\
&= \big(2 + t_1 + t_2 + t_1t_2\big) - \big(1+t_1\big)\big(1+t_2\big) \\
&= 1.
\end{aligned}$$

## Example 7

Consider the discrete-time random process $\{X(n), n \in \mathbb{Z} \cdots\}$, in which the $X(n)$'s are i.i.d. with CDF $F_{X(n)}(x) = F(x)$. Show that this is a (strict-sense) stationary process.

## Solution

Intuitively, since $X(n)$'s are i.i.d., we expect that as time evolves the probabilistic behavior of the process does not change. Therefore, this must be a stationary process. To show this rigorously, we can argue as follows. For all real numbers $x_1, x_2, \cdots, x_r$ and all distinct integers $n_1$, $n_2, \cdots$, $n_r$, we have

$$\begin{aligned}
&F_{X(n_1)X(n_2)\cdots X(n_r)}(x_1, x_2, \cdots, x_r) \\
&\quad = F_{X(n_1)}(x_1)F_{X(n_2)}(x_2)\cdots F_{X(n_r)}(x_r) \quad \text{(since the } X(n_i)\text{'s are independent)} \\
&\quad = F(x_1)F(x_2)\cdots F(x_r) \quad \text{(since } F_{X(t_i)}(x) = F(x)\text{)}.
\end{aligned}$$

We also have

$$
\begin{aligned}
&F_{X(n_1+D)X(n_2+D)\cdots X(n_r+D)}(x_1, x_2, \cdots, x_r) \\
&\quad = F_{X(n_1+D)}(x_1)F_{X(n_2+D)}(x_2)\cdots F_{X(n_r+D)}(x_r) \quad \text{(since the } X(n_i+D)\text{'s are independent)} \\
&\quad = F(x_1)F(x_2)\cdots F(x_n) \quad \text{(since } F_{X(n_i+D)}(x) = F(x)\text{)}.
\end{aligned}
$$

**Example 8**

Consider the random process $\{X(t), t \in \mathbb{R}\}$ defined as

$$X(t) = \cos(t + U),$$

where $U \sim Uniform(0, 2\pi)$. Show that $X(t)$ is a WSS process.

**Solution**

We need to check two conditions:

1. $\mu_X(t) = \mu_X$, for all $t \in \mathbb{R}$, and
2. $R_X(t_1, t_2) = R_X(t_1 - t_2)$, for all $t_1, t_2 \in \mathbb{R}$.

We have

$$\begin{aligned}
\mu_X(t) &= E[X(t)] \\
&= E[\cos(t + U)] \\
&= \int_0^{2\pi} \cos(t + u) \frac{1}{2\pi} \, du \\
&= 0, \qquad \text{for all } t \in \mathbb{R}.
\end{aligned}$$

We can also find $R_X(t_1, t_2)$ as follows

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[\cos(t_1 + U)\cos(t_2 + U)] \\
&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U) + \frac{1}{2}\cos(t_1 - t_2)\right] \\
&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U)\right] + E\left[\frac{1}{2}\cos(t_1 - t_2)\right] \\
&= \int_0^{2\pi} \cos(t_1 + t_2 + u)\frac{1}{2\pi}\, du + \frac{1}{2}\cos(t_1 - t_2) \\
&= 0 + \frac{1}{2}\cos(t_1 - t_2) \\
&= \frac{1}{2}\cos(t_1 - t_2), \quad \text{for all } t_1, t_2 \in \mathbb{R}.
\end{aligned}$$

As we see, both conditions are satisfied, thus $X(t)$ is a WSS process.

# Example 9

Let $X(t)$ and $Y(t)$ be two jointly WSS random processes. Consider the random process $Z(t)$ defined as

$$Z(t) = X(t) + Y(t).$$

Show that $Z(t)$ is WSS.

**Solution**

Since $X(t)$ and $Y(t)$ are jointly WSS, we conclude

1. $\mu_X(t) = \mu_X$, $\mu_Y(t) = \mu_Y$,
2. $R_X(t_1, t_2) = R_X(t_1 - t_2)$, $R_Y(t_1, t_2) = R_Y(t_1 - t_2)$,
3. $R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$.

Therefore, we have

$$\begin{aligned} \mu_Z(t) &= E[X(t) + Y(t)] \\ &= E[X(t)] + E[Y(t)] \\ &= \mu_X + \mu_Y. \end{aligned}$$

$$\begin{aligned} R_Z(t_1, t_2) &= E\left[\big(X(t_1) + Y(t_1)\big)\big(X(t_2) + Y(t_2)\big)\right] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] + E[Y(t_1)X(t_2)] + E[Y(t_1)Y(t_2)] \\ &= R_X(t_1 - t_2) + R_{XY}(t_1 - t_2) + R_{YX}(t_1 - t_2) + R_Y(t_1 - t_2). \end{aligned}$$

# Example 10

Consider a random process $X(t)$ and its derivative, $X'(t) = \frac{d}{dt}X(t)$. Assuming that the derivatives are well-defined, show that

$$R_{XX'}(t_1,t_2) = \frac{\partial}{\partial t_2} R_X(t_1,t_2).$$

**Solution**

We have

$$\begin{aligned}
R_{XX'}(t_1,t_2) &= E[X(t_1)X'(t_2)] \\
&= E\left[X(t_1)\frac{d}{dt_2}X(t_2)\right] \\
&= E\left[\frac{\partial}{\partial t_2}\Big(X(t_1)X(t_2)\Big)\right] \\
&= \frac{\partial}{\partial t_2}E[X(t_1)X(t_2)] \\
&= \frac{\partial}{\partial t_2}R_X(t_1,t_2).
\end{aligned}$$

# Mixed Problem

**Problem 1**

Let $Y_1$, $Y_2$, $Y_3$, $\cdots$ be a sequence of i.i.d. random variables with mean $EY_i = 0$ and $\mathrm{Var}(Y_i) = 4$. Define the discrete-time random process $\{X(n), n \in \mathbb{N}\}$ as

$$X(n) = Y_1 + Y_2 + \cdots + Y_n, \quad \text{for all } n \in \mathbb{N}.$$

Find $\mu_X(n)$ and $R_X(m, n)$, for all $n, m \in \mathbb{N}$.

**Solution**

We have

$$\begin{aligned}
\mu_X(n) &= E[X(n)] \\
&= E[Y_1 + Y_2 + \cdots + Y_n] \\
&= E[Y_1] + E[Y_2] + \cdots + E[Y_n] \\
&= 0.
\end{aligned}$$

Let $m \leq n$, then

$$\begin{aligned} R_X(m,n) &= E[X(m)X(n)] \\ &= E\left[X(m)\left(X(m) + Y_{m+1} + Y_{m+2} + \cdots + Y_n\right)\right] \\ &= E[X(m)^2] + E[X(m)]E[Y_{m+1} + Y_{m+2} + \cdots + Y_n] \\ &= E[X(m)^2] + 0 \\ &= \mathrm{Var}(X(m)) \\ &= \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \cdots + \mathrm{Var}(Y_m) \\ &= 4m. \end{aligned}$$

Similarly, for $m \geq n$, we have

$$\begin{aligned} R_X(m,n) &= E[X(m)X(n)] \\ &= 4n. \end{aligned}$$

We conclude

$$R_X(m,n) = 4\min(m,n).$$

**Problem 2**

For any $k \in \mathbb{Z}$, define the function $g_k(t)$ as

$$g_k(t) = \begin{cases} 1 & k < t \leq k+1 \\ 0 & \text{otherwise} \end{cases}$$

Now, consider the continuous-time random process $\{X(t), t \in \mathbb{R}\}$ defined as

$$X(t) = \sum_{k=-\infty}^{+\infty} A_k g_k(t),$$

where $A_1$, $A_2$, $\cdots$ are i.i.d. random variables with $EA_k = 1$ and $\mathrm{Var}(A_k) = 1$. Find $\mu_X(t)$, $R_X(s,t)$, and $C_X(s,t)$ for all $s, t \in \mathbb{R}$.

**Solution**

Note that, for any $k \in \mathbb{Z}$, $g(t) = 0$ outside of the interval $(k, k+1]$. Thus, if $k < t \leq k+1$, we can write

$$X(t) = A_k.$$

Thus,

$$\begin{aligned}\mu_X(t) &= E[X(t)] \\ &= E[A_k] = 1.\end{aligned}$$

So, $\mu_X(t) = 1$ for all $t \in \mathbb{R}$. Now consider two real numbers $s$ and $t$. If for some $k \in \mathbb{Z}$, we have

$$k < s, t \leq k+1,$$

then

$$\begin{aligned}R_X(s,t) &= E[X(s)X(t)] \\ &= E[A_k^2] = 1 + 1 = 2.\end{aligned}$$

On the other hand, if $s$ and $t$ are in two different subintervals of $\mathbb{R}$, that is if

$$k < s \leq k+1, \quad \text{and} \quad l < t \leq l+1,$$

where $k$ and $l$ are two different integers, then

$$\begin{aligned} R_X(s,t) &= E[X(s)X(t)] \\ &= E[A_k A_l] = E[A_k]E[A_l] = 1. \end{aligned}$$

To find $C_X(s,t)$, note that if \

$$k < s,t \leq k+1,$$

then

$$\begin{aligned} C_X(s,t) &= R_X(s,t) - E[X(s)]E[X(t)] \\ &= 2 - 1 \cdot 1 = 1. \end{aligned}$$

On the other hand, if

$$k < s \leq k+1, \quad \text{and} \quad l < t \leq l+1,$$

where $k$ and $l$ are two different integers, then

$$\begin{aligned} C_X(s,t) &= R_X(s,t) - E[X(s)]E[X(t)] \\ &= 1 - 1 \cdot 1 = 0. \end{aligned}$$

## Problem 3

Let $X(t)$ be a continuous-time WSS process with mean $\mu_X = 1$ and

$$R_X(\tau) = \begin{cases} 3 - |\tau| & -2 \le \tau \le 2 \\ 1 & \text{otherwise} \end{cases}$$

a. Find the expected power in $X(t)$.

b. Find $E\left[\left(X(1) + X(2) + X(3)\right)^2\right]$.

## Solution

a. The expected power in $X(t)$ at time $t$ is $E[X(t)^2]$, which is given by

$$R_X(0) = 3.$$

b. We have

$$\begin{aligned}
E\left[\left(X(1)+X(2)+X(3)\right)^2\right] &= E\Big[X(1)^2+X(2)^2+X(3)^2 \\
&\quad + 2X(1)X(2)+2X(1)X(3)+2X(2)X(3)\Big] \\
&= 3R_X(0)+2R_X(-1)+2R_X(-2)+2R_X(-1) \\
&= 3\cdot 3+2\cdot 2+2\cdot 1+2\cdot 2 \\
&= 19.
\end{aligned}$$

**Problem 4**

Let $X(t)$ be a continuous-time WSS process with mean $\mu_X = 0$ and

$$R_X(\tau) = \delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. We define the random process $Y(t)$ as

$$Y(t) = \int_{t-2}^{t} X(u)du.$$

a. Find $\mu_Y(t) = E[Y(t)]$.
b. Find $R_{XY}(t_1, t_2)$.

**Solution**

a. We have

$$\begin{aligned}
\mu_Y(t) &= E\left[\int_{t-2}^{t} X(u)du\right] \\
&= \int_{t-2}^{t} E[X(u)]\,du \\
&= \int_{t-2}^{t} 0\,du \\
&= 0.
\end{aligned}$$

b. We have

$$\begin{aligned}
R_{XY}(t_1,t_2) &= E\left[X(t_1)\int_{t_2-2}^{t_2} X(u)du\right] \\
&= E\left[\int_{t_2-2}^{t_2} X(t_1)X(u)du\right] \\
&= \int_{t_2-2}^{t_2} R_X(t_1-u)\,du \\
&= \int_{t_2-2}^{t_2} \delta(t_1-u)\,du \\
&= \begin{cases} 1 & t_2-2 < t_1 < t_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

## Problem 5

Let $X(t)$ be a Gaussian process with $\mu_X(t) = t$, and $R_X(t_1, t_2) = 1 + 2t_1t_2$, for all $t, t_1, t_2 \in \mathbb{R}$. Find $P\big(2X(1) + X(2) < 3\big)$.

**Solution**

Let $Y = 2X(1) + X(2)$. Then, $Y$ is a normal random variable. We have

$$\begin{aligned} EY &= 2E[X(1)] + E[X(2)] \\ &= 2 \cdot 1 + 2 = 4. \end{aligned}$$

$$\text{Var}(Y) = 4\text{Var}\big(X(1)\big) + \text{Var}\big(X(2)\big) + 4\text{Cov}\big(X(1), X(2)\big).$$

Note that

$$
\begin{aligned}
\mathrm{Var}(X(1)) &= E[X(1)^2] - E[X(1)]^2 \\
&= R_X(1,1) - \mu_X(1)^2 \\
&= 1 + 2 \cdot 1 \cdot 1 - 1 = 2.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X(2)) &= E[X(2)^2] - E[X(2)]^2 \\
&= R_X(2,2) - \mu_X(2)^2 \\
&= 1 + 2 \cdot 2 \cdot 2 - 4 = 5.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(X(1), X(2)) &= E[X(1)X(2)] - E[X(1)]E[X(2)] \\
&= R_X(1,2) - \mu_X(1)\mu_X(2) \\
&= 1 + 2 \cdot 1 \cdot 2 - 1 \cdot 2 = 3.
\end{aligned}
$$

Therefore,

$$
\mathrm{Var}(Y) = 4 \cdot 2 + 5 + 4 \cdot 3 = 25.
$$

We conclude $Y \sim N(4, 25)$. Thus,

$$
\begin{aligned}
P(Y < 3) &= \Phi\left(\frac{3-4}{5}\right) \\
&= \Phi(-0.2) \approx 0.42
\end{aligned}
$$

## Problem 6

Consider a WSS random process $X(t)$ with

$$R_X(\tau) = \begin{cases} 1 - |\tau| & -1 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the PSD of $X(t)$, and $E[X(t)^2]$.

## Solution

First, we have

$$E[X(t)^2] = R_X(0) = 1.$$

We can write triangular function, $R_X(\tau) = \Lambda(\tau)$, as

$$R_X(\tau) = \Pi(\tau) * \Pi(\tau),$$

where

$$\Pi(\tau) = \begin{cases} 1 & -\frac{1}{2} \leq \tau \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \mathcal{F}\{\Pi(\tau) * \Pi(\tau)\} \\ &= \mathcal{F}\{\Pi(\tau)\} \cdot \mathcal{F}\{\Pi(\tau)\} \\ &= [\operatorname{sinc}(f)]^2. \end{aligned}$$

## Problem 7

Let $X(t)$ be a random process with mean function $\mu_X(t)$ and autocorrelation function $R_X(s,t)$
$X(t)$ is not necessarily a WSS process). Let $Y(t)$ be given by

$$Y(t) = h(t) * X(t),$$

where $h(t)$ is the impulse response of the system. Show that

a. $\mu_Y(t) = \mu_X(t) * h(t)$.
b. $R_{XY}(t_1, t_2) = h(t_2) * R_X(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha) R_X(t_1, t_2 - \alpha)\, d\alpha$.

a. We have

$$
\begin{aligned}
\mu_Y(t) &= E[Y(t)] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)\, d\alpha\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)E[X(t-\alpha)]\, d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\mu_X(t-\alpha)\, d\alpha \\
&= \mu_X(t) * h(t).
\end{aligned}
$$

b. We have

$$
\begin{aligned}
R_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] \\
&= E\left[X(t_1)\int_{-\infty}^{\infty} h(\alpha)X(t_2-\alpha)\, d\alpha\right] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)X(t_1)X(t_2-\alpha)\, d\alpha\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)E[X(t_1)X(t_2-\alpha)]\, d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)R_X(t_1,t_2-\alpha)\, d\alpha.
\end{aligned}
$$

**Theorem 10.2**

Let $X(t)$ be a WSS random process and $Y(t)$ be given by

$$Y(t) = h(t) * X(t),$$

where $h(t)$ is the impulse response of the system. Then $X(t)$ and $Y(t)$ are jointly WSS. Moreover,

1. $\mu_Y(t) = \mu_Y = \mu_X \int_{-\infty}^{\infty} h(\alpha)\, d\alpha$;
2. $R_{XY}(\tau) = h(-\tau) * R_X(\tau) = \int_{-\infty}^{\infty} h(-\alpha) R_X(t - \alpha)\, d\alpha$;
3. $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$.

## Problem 8

Prove the third part of Theorem 10.2: Let $X(t)$ be a WSS random process and $Y(t)$ be given by

$$Y(t) = h(t) * X(t),$$

where $h(t)$ is the impulse response of the system. Show that

$$R_Y(s,t) = R_Y(s-t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(s-t-\alpha+\beta)\; d\alpha d\beta.$$

Also, show that we can rewrite the above integral as $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$.

**Solution**

$$
\begin{aligned}
R_Y(s,t) &= E[Y(s)Y(t)] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)X(s-\alpha)\,d\alpha \int_{-\infty}^{\infty} h(\beta)X(t-\beta)\,d\beta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)E[X(s-\alpha)X(t-\beta)]\,d\alpha\,d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(s-t-\alpha+\beta)\,d\alpha\,d\beta.
\end{aligned}
$$

We now compute $h(\tau) * h(-\tau) * R_X(\tau)$. First, let $g(\tau) = h(\tau) * h(-\tau)$. Note that

$$
\begin{aligned}
g(\tau) &= h(\tau) * h(-\tau) \\
&= \int_{-\infty}^{\infty} h(\alpha)h(\alpha-\tau)\,d\alpha.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
g(\tau) * R_X(\tau) &= \int_{-\infty}^{\infty} g(\theta) R_X(\theta - \tau)\, d\theta \\
&= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\alpha) h(\alpha - \theta)\, d\alpha \right] R_X(\theta - \tau)\, d\theta \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\alpha - \theta) R_X(\theta - \tau)\, d\alpha\, d\theta \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R_X(\alpha - \beta - \tau)\, d\alpha\, d\beta \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R_X(\tau - \alpha + \beta)\, d\alpha\, d\beta
\end{aligned}
$$

$$\left(\text{since } R_X(-\tau) = R_X(\tau)\right)$$

# Problem 9

Let $X(t)$ be a WSS random process. Assuming that $S_X(f)$ is continuous at $f_1$, show that $S_X(f_1) \geq 0$.

**Solution**

Let $f_1 \in \mathbb{R}$. Suppose that $X(t)$ goes through an LTI system with the following transfer function

$$H(f) = \begin{cases} 1 & f_1 < |f| < f_1 + \Delta \\ 0 & \text{otherwise} \end{cases}$$

where $\Delta$ is chosen to be very small. The PSD of $Y(t)$ is given by

$$S_Y(f) = S_X(f)|H(f)|^2 = \begin{cases} S_X(f) & f_1 < |f| < f_1 + \Delta \\ \\ 0 & \text{otherwise} \end{cases}$$

Thus, the power in $Y(t)$ is

$$\begin{aligned} E[Y(t)^2] &= \int_{-\infty}^{\infty} S_Y(f)\, df \\ &= 2\int_{f_1}^{f_1+\Delta} S_X(f)\, df \\ &\approx 2\Delta S_X(f_1). \end{aligned}$$

Since $E[Y(t)^2] \geq 0$, we conclude that $S_X(f_1) \geq 0$.

# Problem 10

Let $X(t)$ be a white Gaussian noise with $S_X(f) = \frac{N_0}{2}$. Assume that $X(t)$ is input to an LTI system with

$$h(t) = e^{-t}u(t).$$

Let $Y(t)$ be the output.

a. Find $S_Y(f)$.
b. Find $R_Y(\tau)$.
c. Find $E[Y(t)^2]$.

## Solution

First, note that

$$H(f) = \mathcal{F}\{h(t)\} = \frac{1}{1 + j2\pi f}.$$

a. To find $S_Y(f)$, we can write

$$S_Y(f) = S_X(f)|H(f)|^2 = \frac{N_0/2}{1 + (2\pi f)^2}.$$

b. To find $R_Y(\tau)$, we can write

$$R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(f)\} = \frac{N_0}{4} e^{-|\tau|}.$$

c. We have

$$E[Y(t)^2] = R_Y(0) = \frac{N_0}{4}.$$