Analiză

Riemann integrals + improper integrals

Let $f:[a,b]\to\mathbb{R}$, $a=x_0<x_1<\ldots<x_n=b$

$$P=\{[x_{k-1},x_k]\mid k=\overline{1,n}\}$$

$$\|P\|=\max_{k=\overline{1,n}}\{x_k-x_{k-1}\}$$

Riemann sum

$$\sigma(f,P):=\sum_{k=1}^{n}f(c_k)(x_k-x_{k-1}), \text{ where } c_k\in[x_k,x_{k-1}]$$

Riemann integrability

f is Riemann integrable if $\exists\, I\in\mathbb{R}$ s.t. for any partition P, $\sigma(f,P)$ converges to I as $\|P\|\to0$

$$\lim_{\|P\|\to0}\sigma(f,P)=I=\int_a^b f(x)\,dx$$

! Let f be R. integr. $\Rightarrow A:[a,b]\to\mathbb{R},\ A(x)=\int_a^x f(t)$ is cont.

$$A'(x)=f(x)$$

Trapezium rule

Let f be R. integr. ; $a=x_0<x_1<\ldots<x_n=b$

$\Rightarrow$ area below $f(x)\approx\sum_{k=1}^{n}\frac{f(x_{k-1})+f(x_k)}{2}(x_k-x_{k-1})$

or $\int_a^b f(x)\approx\frac{b-a}{n}\left(\frac{1}{2}f(a)+\sum_{i=1}^{n-1}f(x_k)+\frac{1}{2}f(b)\right)$

Improper integrals

$$\int_a^{\infty} f(x)dx := \lim_{t\to\infty}\int_a^t f(x)dx$$

$$\int_a^{b-0} f(x)\,dx := \lim_{t\nearrow b}\int_a^t f(x)dx$$

$$\int_{a+0}^{b} f(x)dx := \lim_{t\searrow a}\int_t^b f(x)dx$$

! An improper integral is convergent if it is finite.

$$\int_a^{\infty}\frac{1}{x^p}dx \quad \begin{cases} p>1 - \text{converges} \\ p\le 1 \Rightarrow \text{diverges}\end{cases}$$

$$\int_a^{b}\frac{1}{(b-x)^p}dx,\ \int_a^{b}\frac{1}{(x-a)^p}dx \quad \begin{cases} p<1 - \text{converges} \\ p\ge 1 - \text{diverges}\end{cases}$$

Comparison test (?)

Let $a<b\le\infty$; $f,g:[a,b)\to[0,\infty)$. If $\exists\, c\in(a,b)$ s.t. $f(x)\le g(x),\ \forall x\ge c$, then

$$\int_a^b g(x)dx \text{ converges} \Rightarrow \int_a^b f(x)dx \text{ converges}$$

$$\int_a^b f(x)dx \text{ diverges} \Rightarrow \int_a^b g(x)dx \text{ diverges}$$

$$\lim_{x\nearrow b}\frac{f(x)}{g(x)}\in(0,\infty) \Rightarrow \int_a^b f(x)dx \text{ and } \int_a^b g(x) - \text{same nature}$$

Integral test for series

Let $f:[1,\infty)\to[0,\infty)$ be decreasing $\Rightarrow$

$\Rightarrow \int_1^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} f(n)$ have the same nature

Euclidian space

Scalar product

$\langle x,y\rangle = x\cdot y = x_1y_1 + \dots + x_ny_n$

$\langle x,y\rangle = \langle y,x\rangle$

$\langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$

$\langle x,x\rangle > 0, \ \forall x \in \mathbb{R}^n \setminus \{0\}$

$\langle x,y\rangle = x^t M y$

$x\cdot y = x^T y = [x_1 \dots x_n]_{1\times n} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} = x_1y_1 + \dots + x_ny_n$

Norm

$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle x,x\rangle}$ - norm x

$\|x\| = 0$ iff $x = 0$

$\|\alpha x\| = |\alpha|\cdot\|x\|$

$\|x+y\| \le \|x\| + \|y\|$

Cauchy - Schwarz inequality

$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$

Thm For $n \in \{2,3\} \Rightarrow x\cdot y = \|x\|\cdot\|y\|\cdot\cos(\widehat{x,y})$

P-norms

a) $\|x\|_1 := |x_1| + \dots + |x_n|$ (Manhattan norm)

b) $\|x\|_p := (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}, \ p > 1$

c) $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$

Distance

$d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$ = Euclidian distance

$d(x,y) = 0$ iff $x = y$

$d(x,y) = d(y,x), \ \forall x,y \in \mathbb{R}^n$

$d(x,z) \le d(x,y) + d(y,z)$

Neighborhoods. Interior. Closure. Boundary.

A set $A \subseteq \mathbb{R}^n$ is called bounded if $\exists\, r>0$ s.t. $\|x\| \le r, \forall x \in A$

Let $x_0 \in \mathbb{R}^n$ and $r>0$. The open ball of centre x_0 and radius r

$$B(x_0, r) := \{ x \in \mathbb{R}^n \mid \|x - x_0\| < r \},$$

closed ball of centre x_0 and radius r:

$$\bar{B}(x_0, r) := \{ x \in \mathbb{R}^n \mid \|x - x_0\| \le r \}$$

A set $V \subseteq \mathbb{R}^n$ is a neighborhood of $x \in \mathbb{R}^n$ if

$$\exists\, r > 0 \text{ s.t. } B(x, r) \subseteq V$$

Interior: $\mathrm{int}(A) := \{ x \in \mathbb{R}^n \mid \exists\, V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A \}$

Closure: $\mathrm{cl}(A) := \{ x \in \mathbb{R}^n \mid \forall\, V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \}$

Boundary: $\mathrm{bd}(A) := \{ x \in \mathbb{R}^n \mid \forall\, V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \text{ and } V \cap A^c \neq \emptyset \}$

Exp: for $A = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$

$\mathrm{int}(A) = A$, $\mathrm{cl}(A) = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1 \}$

$\mathrm{bd}(A) = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$

! $\mathrm{cl}(A) = A \cup \mathrm{bd}(A)$; $A = \mathrm{int}(A) \Rightarrow$ open ; $\mathrm{int}(A) \subseteq A \subseteq \mathrm{cl}(A)$

$A = \mathrm{cl}(A) \Rightarrow$ closed

! Prove A is open $\Rightarrow A \subseteq \mathrm{int}(A)$

closed $\Rightarrow \mathrm{cl}(A) \subseteq A$

<u>Thm</u> x^k converges to x iff. $\lim_{k \to \infty} x_i^k = x_i, \forall i = \overline{1,n}$

Functions of several variables

Level sets: $L_c := \{x \in A \subseteq \mathbb{R}^n \mid f(x) = c\}$

Limits of functions of several variables

$\lim_{x \to x_0} f(x) = l \in \overline{\mathbb{R}}$ if $\forall V \in \mathcal{V}(l), \exists U \in \mathcal{V}(x_0)$ s.t. $f(x) \in V$, $\forall x \in U \cap (A \setminus \{x_0\})$

sau $\lim_{x \to x_0} f(x) = l \in \mathbb{R}$ if $\forall \varepsilon > 0 \, \exists \delta > 0$ s.t. $|f(x) - l| < \varepsilon$, $\forall x \in A$ with $\|x - x_0\| < \delta$

Thm $\lim_{x \to x_0} f(x) = l \in \overline{\mathbb{R}}$ iff for any sequence (x^k) in $A \setminus \{x_0\}$ with $\lim_{k \to \infty} x^k = x_0$, we have that $\lim_{k \to \infty} f(x^k) = l \in \overline{\mathbb{R}}$

! if $\lim_{(x,y) \to (x_0, y_0)} f(x,y)$ exists $\Rightarrow \lim_{x \to x_0} \lim_{y \to y_0} f(x,y) = \lim_{y \to y_0} \lim_{x \to x_0} f(x,y)$

! $\forall V \in \mathcal{V}(f(x_0)), \exists U \in \mathcal{V}(x_0)$ s.t. $f(x) \in V, \forall x \in U \cap A$ OR
x_0 - accumulation point and $\lim_{x \to x_0} f(x) = f(x_0)$ OR
$x_0 \in A$ - isolated point

$\Downarrow$

f continuous

! Any norm is continuous

Weierstrass

$$\left. \begin{array}{l} A \subseteq \mathbb{R}^n \text{ - closed and bounded} \\ f: A \to \mathbb{R} \text{ - continuous} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} f \text{ is bounded + attains its bounds} \\ \exists \min(f(A)), \max(f(A)) \end{array} \right.$$

Partial derivatives

$\frac{\partial f}{\partial x_i}$ - derivative of f with respect to x_i; all other variables are held fixed

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

$$:= \lim_{h \to 0} \frac{f(x + h e_i) - f(x)}{h}$$

Gradient

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

Exp $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 y + y^2$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right) = (2xy, x^2 + 2y)$$

Differentiability

f is differentiable at $x_0 \in A$ if $\exists$ a vector $\Delta f(x_0) \in \mathbb{R}^n$, called the differential / derivative of f at x_0, s.t.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \Delta f(x_0) \cdot (x - x_0)}{\| x - x_0 \|} = 0$$

$$\iff \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - \Delta f(x_0) \cdot h}{\| h \|} = 0$$

$$\iff f(x) = f(x_0) + \Delta f(x_0) \cdot (x - x_0) + R(x - x_0), \text{ with } \frac{R(x - x_0)}{\| x - x_0 \|} \to 0$$

! Let $A \subseteq \mathbb{R}^n$ be an open set. If $f : A \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$, then f is diff. if $\exists\, Df(x_0) \in \mathbb{R}^{m \times n}$ s.t.

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df(x_0)(x - x_0) \|_m}{\| x - x_0 \|_n} = 0$$

Here $Df(x_0)(x - x_0) \iff [\;]_{m \times n} [\;]_{n \times 1} = [\;]_{m \times 1}$

Scanned with CamScanner

! f constant $\Rightarrow Df(x) = 0$

$f(x) = ax \Rightarrow Df(x) = a$

$f(x) = Ax$ with $A \in \mathbb{R}^{m \times n} \Rightarrow Df(x) = A$

! If $f: A \to \mathbb{R}^m$ is differentiable at $x_0 \Rightarrow f$ cont. at x_0

! If $f: A \to \mathbb{R}$ is differentiable at $x \Rightarrow$

$$Df(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

! If all partial derivatives exist and are continuous at $x \Rightarrow f$ is differentiable at x

__Jacobian matrix__

Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$ be diff. at $x \in A \Rightarrow$

$$Df(x) = J = \begin{bmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_m(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}_{m \times n}$$

! $\nabla(f+g)(x) = \nabla f(x) + \nabla g(x)$

$\nabla(fg)(x) = g(x)\nabla f(x) + f(x)\nabla g(x)$

$\nabla\left(\frac{f}{g}\right)(x) = \frac{g(x)\nabla f(x) - f(x)\nabla g(x)}{g^2(x)}$

$D(f \circ g)(x) = Df(g(x))\, Dg(x)$

Let $g: \mathbb{R} \to \mathbb{R}^n$, $g = (g_1, \dots, g_n)$; $g'(t) = (g_1'(t), \dots, g_n'(t))$

$\Rightarrow (f \circ g)'(t) = \nabla f(g(t)) \cdot g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(g(t)) \cdot g_i'(t)$

__Directional directive__

$$Df_v(x) := \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}, \ h \text{ scalar}$$

If f is diff. at $x \in A \Rightarrow Df_v(x) = \nabla f(x) \cdot v$

Fermat

x - local extremum $\Rightarrow \nabla f(x) = 0$

Direction of steepest ascent/descent

f - diff. with $\nabla f(x) \neq 0 \Rightarrow$

$\nabla f(x)$ = direction of fastest increase (steepest ascent)

$-\nabla f(x) \Rightarrow$ direction of fastest decrease (steepest descent)

Perpendicularity

$\nabla f(x) \perp$ to the level set containing x (v is a tangent vector to the level set) $\Rightarrow D f_v(x) = \nabla f(x) \cdot v = 0$

Tangent line to a level curve

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

Tangent plane to a level surface

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Higher order derivatives. Local extremum conditions

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) =: \frac{\partial^2 f}{\partial x_i^2} =: \partial_i^2 f$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) =: \frac{\partial^2 f}{\partial x_i \partial y_j} = \partial_{i,j}^2 f$$

Schwarz

If $A \subseteq \mathbb{R}^n \to \mathbb{R}$ has continuous second order partial derivatives, then if $i \neq j$

Hessian matrix

$$H(x) = D^2 f(x) = D(\nabla f)(x) = \begin{bmatrix} \nabla\left(\frac{\partial f}{\partial x_1}\right) \\ \nabla\left(\frac{\partial f}{\partial x_2}\right) \\ \vdots \\ \nabla\left(\frac{\partial f}{\partial x_n}\right) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \vdots & & \vdots \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Taylor expansion

Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ be twice differentiable, with continuous second order partial derivatives.

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x-x_0)^T H(x_0)(x-x_0) + R(x-x_0)$$

$$f(x_0+h) = f(x_0) + \underbrace{\nabla f(x_0) \cdot h}_{\text{linear}} + \underbrace{\frac{1}{2} h^T H(x_0) h}_{\text{quadratic}} + R(h)$$

Local extremum conditions

$\nabla f(x_0) = 0$

- $h^T H(x_0) h > 0, \forall h \in \mathbb{R}^n \Rightarrow$ local minimum
- $h^T H(x_0) h < 0, \forall h \in \mathbb{R}^n \Rightarrow$ local maximum

A $n \times n$ matrix A is called:

- positive-definite if $x^T A x > 0, \forall x \in \mathbb{R}^n$
- negative-definite if $x^T A x < 0, \forall x \in \mathbb{R}^n$
- indefinite if $\exists x_1, x_2 \in \mathbb{R}^n$ s.t. $x_1^T A x_1 > 0 > x_2^T A x_2$

! Let A be a symmetric $n \times n$ matrix $\Rightarrow$

- A is positive definite iff all eigenvalues are positive
- A is negative definite iff all eigenvalues are negative
- A is indefinite iff it has both positive and negative eigenvalues

! $H(x)$ - pos. def. $\Rightarrow$ local min.
$H(x)$ - neg. def. $\Rightarrow$ local max.
$H(x)$ - indefinite $\Rightarrow$ saddle point

Double integrals

Rectangular domains

Let $A = [a,b] \times [c,d]$ be a rectangle and consider a partition into smaller rectangles

$$P = \{A_{ij} \mid A_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], i = \overline{1,m}, j = \overline{1,n}\}$$

where $a = x_0 < x_1 < \dots < x_m = b$ and $c = y_0 < y_1 < \dots < y_n = d$.

$\|P\| := \max\left\{ \max_{i=\overline{1,m}} \{x_i - x_{i-1}\}, \max_{j=\overline{1,n}} \{y_j - y_{j-1}\} \right\}$; $\underbrace{(x_{ij}^*, y_{ij}^*) \in A_{ij}}_{\text{attached to } P}$

Riemann sum

$$\sigma(f,P) := \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)(x_i - x_{i-1})(y_j - y_{j-1})$$

f is Riemann integrable if $\lim_{\|P\| \to 0} \sigma(f,P) = \mathcal{I} =: \iint_A f(x,y)\,dx\,dy$

! Let $A_1, A_2 \subset \mathbb{R}^2$ s.t. $A = A_1 \cup A_2$ and $\operatorname{int} A_1 \cap \operatorname{int} A_2 = \emptyset$.

$$\Rightarrow \iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

Fubini

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Exp Let $R = [-1,1] \times [0,1]$ and consider $\iint_R (x^2+y^2)\,dx\,dy$

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{-1}^{1} \int_0^1 (x^2+y^2)\,dy\,dx = \int_{-1}^{1} \left(x^2 + \frac{1}{3}\right) dx = \frac{4}{3}.$$

! Let $f : A = [a,b] \times [c,d] \to \mathbb{R}$ be Riemann integrable.

If $f(x,y) = g(x)h(y) \Rightarrow \iint_A f(x,y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$

Let $D \subset \mathbb{R}^2$ be a bounded set

We say that $f : D \to \mathbb{R}$ is Riemann integrable on D if $\exists$ a rectangle $A \subset \mathbb{R}^2$ s.t. $D \subseteq A$ and the extension function

$\bar{f} : A \to \mathbb{R} = \begin{cases} f(x), & x \in D \\ 0, & x \in A \setminus D \end{cases}$ is Riemann integr. on A. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_A \bar{f}(x,y)\,dx\,dy$$

A set $\Delta \subset \mathbb{R}^2$ is called

- simple with respect to the y-axis if $\exists$ continuous functions $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$:

$$\Delta = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$$

- simple with respect to the x-axis if $\exists$ cont. functions $\tau_1, \tau_2 : [c,d] \to \mathbb{R}$:

$$\Delta = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ \tau_1(y) \le x \le \tau_2(y)\}$$

- simple if it is simple w.r. to both x-axis and y-axis

Let $\Delta \subset \mathbb{R}^2$ be a bounded set and $f : \Delta \to \mathbb{R}$ Riemann integrable on Δ.

- if Δ is y-simple, then

$$\iint_\Delta f(x,y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

- if Δ is x-simple, then

$$\iint_\Delta f(x,y)\,dx\,dy = \int_c^d \int_{\tau_1(y)}^{\tau_2(y)} f(x,y)\,dx\,dy$$

- if Δ is simple, then

$$\iint_\Delta f(x,y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\tau_1(y)}^{\tau_2(y)} f(x,y)\,dx\,dy$$

Change of variables

Let $\Delta, \Delta^* \subseteq \mathbb{R}^2$ and $T : \Delta^* \to \Delta$ bijective and of class C^1 with the Jacobian matrix J. Then for any Riemann integrable $f : \Delta \to \mathbb{R}$, we have:

$$\iint_\Delta f(x,y)\,dx\,dy = \iint_{\Delta^*} f(x(u,v), y(u,v)) \cdot |\det(J)|\,du\,dv$$

Scanned with CamScanner

Polar coordinates

$$\iint_{\Delta} f(x,y)\,dx\,dy = \iint_{\Delta^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

where $J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$, and $|\det J| = r$

Triple integrals

Let $A = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$ and $f: A \to \mathbb{R}$ be Riemann integrable. Then

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz$$

$= \dots$ (all possible orders)

If $D = \{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x),\ \tau_1(x,y) \le z \le \tau_2(x,y)\}$, then

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\tau_1(x,y)}^{\tau_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Change of variables

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{\Delta^*} f(u,v,w)\cdot|\det J|\,du\,dv\,dw,$$

with $J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$

Cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad |\det J| = r$$

Spherical coordinates

$x = (r\sin\varphi)\cos\theta$, $y = (r\sin\varphi)\sin\theta$, $z = r\cos\varphi$

$$J = \begin{bmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{bmatrix}$$

$|\det J| = r^2\sin\varphi$

! The ball $B(0,R)$ is given by $r \in [0,R]$, $\theta \in [0,2\pi]$, $\varphi \in [0,\pi]$ in spherical coords.