Seminar 6

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Planes and lines in 3D

π plane, $A_0(x_0, y_0, z_0) \in \pi$

$\vec{v}(x_v, y_v, z_v), \vec{w}(x_w, y_w, z_w) \parallel \pi$

$\vec{\pi} = \{\vec{v} \in V^3 \mid \vec{v} \parallel \pi\} = \langle(\vec{v}, \vec{w})\rangle$

Parametric equations : $\lambda, \mu \in \mathbb{R}$

$$\begin{cases} x = x_0 + \lambda x_v + \mu x_w \\ y = y_0 + \lambda y_v + \mu y_w \\ z = z_0 + \lambda z_v + \mu z_w \end{cases}$$

Cartesian equation (determinant point)

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = 0$$

Cartesian equation (implicit form)

$Ax + By + Cz + D = 0$

$\vec{\pi}^{\perp} = \langle \vec{v} \times \vec{w} \rangle = \langle (A, B, C) \rangle$

ℓ line, $A_0(x_0, y_0, z_0) \in \ell$; $\vec{v}(x_v, y_v, z_v) \parallel \ell$

Parametric equation

$$\begin{cases} x = x_0 + \lambda x_v \\ y = y_0 + \lambda y_v \\ z = z_0 + \lambda z_v \end{cases}$$

Cartesian equation (symmetric form)

① $x_v, y_v, z_v \neq 0$

$$\frac{x - x_0}{x_v} = \frac{y - y_0}{y_v} = \frac{z - z_0}{z_v}$$

② $x_v = 0$, $y_v, z_v \neq 0$

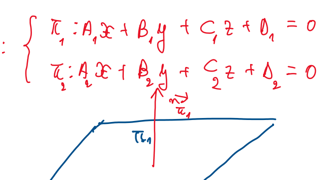
$$\begin{cases} x = x_0 \\ \frac{y - y_0}{y_v} = \frac{z - z_0}{z_v} \end{cases}$$

③ $x_v = y_v = 0$, $z_v \neq 0$

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases}$$

Cartesian equation (implicit form)

$$\ell : \begin{cases} \pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0 \\ \pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$



$\vec{\ell} = \langle \vec{v}_{\pi_1} \times \vec{v}_{\pi_2} \rangle = \langle (A_1, B_1, C_1) \times (A_2, B_2, C_2) \rangle$

1. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$, → scot 2 vectori

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$, → $a_1(1,0,0)$ $a_2(0,1,0)$

d) π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz, $a_1(0,1,0)$, $a_2(0,0,1)$

e) π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$, → scot al doilea vector $\vec{M_1 M_2}$

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox, $O(0,0,0)$, $P(1,0,0)$

$\vec{OA}(1,5,7)$

$\vec{PA}(0,5,7)$

① a) Cartesian equation

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_{a_1} & y_{a_1} & z_{a_1} \\ x_{a_2} & y_{a_2} & z_{a_2} \end{vmatrix} = \begin{vmatrix} x - 1 & y & z - 2 \\ 3 & -1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 1 - x + 9z - 18 - 3x + 3 - 3y = 0$$

$-4x - 3y + 9z - 14 = 0$

$$\begin{cases} x = 1 + 3\lambda \\ y = 0 - \lambda + 3\mu \\ z = 2 + \lambda + \mu \end{cases} \Leftrightarrow \begin{cases} x = 1 + 3\lambda \\ y = -\lambda + 3\mu \\ z = 2 + \lambda + \mu \end{cases}$$

2. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2 + 3u - 4v$, $y = 4 - v$, $z = 2 + 3u$;

b) $\pi : x = u + v$, $y = u - v$, $z = 5 + 6u - 4v$. ⇒ $v_1(1,1,6)$, $v_2(1,-1,-4)$, $A(0,0,5)$

$$\begin{vmatrix} x & y & z - 5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0$$

π: $x = 2 + \mu \cdot 3 + v(-4)$

$y = 4 + \mu \cdot 0 + v \cdot (-1)$

$z = 2 + \mu \cdot 3 + v \cdot 0$

$v_1(3,0,3)$

$v_2(-4,-1,0)$

$A(2,4,2)$

$$\Rightarrow \begin{vmatrix} x - 2 & y - 4 & z - 2 \\ -4 & -1 & 0 \\ 3 & 0 & 3 \end{vmatrix} = 0$$

3. Determine parametric equations for the plane π in the following cases:

a) $3x - 6y + z = 0$; (Dai valori ca țăranul la x, y sau z)

b) $2x - y - z - 3 = 0$;

a) $3x - 6y + z = 0$

$$\begin{cases} x = \lambda \\ y = \mu \\ z = 6\mu - 3\lambda \end{cases}$$

$\vec{\pi} \langle (1, 0, -3), (0, 1, 6) \rangle$

b) $2x - y - z - 3 = 0$

$$\begin{cases} x = \lambda \\ y = \mu \\ z = 2\lambda - \mu - 3 \end{cases}$$

$\vec{\pi} \langle (1, 0, 2), (0, 1, -1) \rangle$

16. Consider the lines $\ell_1 : x = 1 + t, y = 1 + 2t, z = 3 + t, t \in \mathbb{R}$ and $\ell_2 : x = 3 + s, y = 2s, z = -2 + s, s \in \mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

$$\ell_1 : \begin{cases} x = 1 + 1 \cdot t \\ y = 1 + 2 \cdot t \\ z = 3 + 1 \cdot t \end{cases} \Rightarrow \vec{t}(1,2,1), \; A(1,1,3)$$

$$\ell_2 : \begin{cases} x = 3 + s \\ y = 2s \\ z = -2 + s \end{cases} \Rightarrow \vec{s}(1,2,1), \; B(3,0,-2)$$

$\vec{AB}(2,-1,-5)$

$\vec{s}(1,2,1)$

$A(1,1,3)$

⇒ Cartesian eq (Det form)

$$\begin{vmatrix} x - 1 & y - 1 & z - 3 \\ 2 & -1 & -5 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

5. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.

$M\left(\frac{x_C + x_D}{2}, \frac{y_C + y_D}{2}, \frac{z_C + z_D}{2}\right) = (3, -2, 6)$

$\vec{AB}(-1, 2, 5)$

$\vec{AM}(1, -1, 6)$

6. Show that a parallelepiped with faces in the planes $2x + y - 2z + 6 = 0$, $2x - 2y + z - 8 = 0$ and $x + 2y + 2z + 1 = 0$ is rectangular.

$\pi_1 : 2x + y - 2z + 6 = 0$

$\pi_2 : 2x - 2y + z - 8 = 0$

$\pi_3 : x + 2y + 2z - 1 = 0$

$$\vec{\pi_1} \times \vec{\pi_2} = (2, 1, -2) \times (2, -2, 1) = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} \vec{k} =$$

$= -3\vec{i} - 6\vec{j} - 6\vec{k} = \langle (1, 2, 2) \rangle$

$\vec{\pi_2} \times \vec{\pi_3} = \langle (-2, 1, 2) \rangle$

$\vec{\pi_1} \times \vec{\pi_3} = \langle (2, -2, 1) \rangle$

Calculăm soma ⇒ $\|\vec{\pi_2} \times \vec{\pi_3}\| = \|\vec{\pi_1} \times \vec{\pi_3}\| = \|\vec{\pi_1} \times \vec{\pi_2}\|$ ⇒ cub

7. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

$$\begin{vmatrix} x_D - x_A & y_D - y_A & z_D - z_A \\ x_D - x_B & y_D - y_B & z_D - z_B \\ x_D - x_C & y_D - y_C & z_D - z_C \end{vmatrix} = 0 \Leftrightarrow A, B, C, D \text{ coplanar}$$

8. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

$OA = \sqrt{11}$

$d(O, \pi) = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

$\Rightarrow \frac{|d|}{\sqrt{a^2 + b^2 + c^2}} = \sqrt{11} \Rightarrow |d| = 11$

$\Rightarrow d = \pm 11$

$$\vec{OA}(1, -1, 3) \Rightarrow \begin{cases} x = \lambda \\ y = -\lambda \\ z = 3\lambda \end{cases}$$

$\ell \langle (1, -1, 3) \rangle$

$\pi_1 : x - y + 3z + 11 = 0$

9. Determine the distance between the planes $x - 2y - 2z + 7 = 0$ and $2x - 4y - 4z + 17 = 0$.

$A(-7, 0, 0) \in \pi_1 \Rightarrow d(\pi_1, \pi_2) = d(A, \pi_2)$

$d(A, \pi_2) = \frac{|2 \cdot (-7) + 17|}{\sqrt{2^2 + 4^2 + 4^2}} = \frac{3}{6} = \frac{1}{2}$

10. Determine the relative positions of the planes in the following cases

a) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : x + 2y - 3z - 1 = 0$.

b) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : 2x + y + 3z - 2 = 0$, $\pi_3 : x + 2y + 3z + 2 = 0$.

c) $\begin{cases} x + 2y + 3z - 1 = 0 \\ x + 2y - 3z - 1 = 0 \end{cases}$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & -3 & 1 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 - L_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 6 & 0 \end{array}\right) \Rightarrow \text{rank } \bar{A} = 2 \Rightarrow \text{the planes intersect}$$

$$\begin{cases} x + 2y + 3z - 1 = 0 \\ x + 2y - 3z - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + 3z - 1 = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y = 1 \\ z = 0 \end{cases}$$

Eq of the line: $\begin{cases} x + 2y = 0 \\ z = 0 \end{cases}$

d) π_1 and π_3 are parallel

$$\begin{cases} x + 2y + 3z - 1 = 0 \\ 2x + y + 3z - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} -x + y = 1 \\ z = y - \frac{1}{3} \end{cases} \quad ??$$

11. Show that the planes

$$\pi_1 : 3x + y + z - 1 = 0, \quad \pi_2 : 2x + y + 3z + 2 = 0, \quad \pi_3 : -x + 2y + z + 4 = 0$$

have a point in common.

$\exists A(x, y, z) \Leftrightarrow \det \neq 0$

$$\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = -15 \neq 0$$

12. Show that the pairwise intersection of the planes

$$\pi_1 : 3x + y + z - 5 = 0, \quad \pi_2 : 2x + y + 3z + 2 = 0, \quad \pi_3 : 5x + 2y + 4z + 1 = 0$$

are parallel lines.

$\pi_1 : 3x + y + z - 5 = 0 \Rightarrow \ell_1^{\perp} \langle (3, 1, 1) \rangle$

$\pi_2 : 2x + y + 3z + 2 = 0 \Rightarrow \ell_2^{\perp} \langle (2, 1, 3) \rangle$

$$(3, 1, 1) \times (2, 1, 3) = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} \vec{i} + \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} \vec{k} =$$

$= \vec{i} - 7\vec{j} + \vec{k} \Rightarrow \langle (1, -7, 1) \rangle$

Caută punct comun

$x = 0$

$$\begin{cases} y + z = 5 \\ y + 3z = -2 \end{cases} \Leftrightarrow \begin{cases} z = \frac{-7}{2} \\ y = \frac{17}{2} \end{cases}$$

$$\begin{cases} x = t \\ y = -\frac{7}{2} - 7t \\ z = \frac{17}{2} + t \end{cases}$$

13. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

a)

$$\frac{x - 2}{3} = \frac{y}{-2} = \frac{z - 3}{-2} = \lambda \Leftrightarrow \begin{cases} x = 2 + 3\lambda \\ y = -2\lambda \\ z = 3 - 2\lambda \end{cases}$$

$$\begin{cases} \frac{x - 2}{3} = \frac{y}{-2} \\ \frac{y}{-2} = \frac{z - 3}{-2} \end{cases} \Leftrightarrow \begin{cases} 2x + 3y - 4 = 0 \\ y - z + 3 = 0 \end{cases}$$

b) $\ell \parallel Oz \Rightarrow B(0, 0, z_v)$

$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$