Seminar 9

1. Study the limits of the following functions when $(x, y) \to (0, 0)$:

   (a) $\dfrac{x^2 - y^2}{x^2 + y^2}$. (b) $\dfrac{x + y}{x^2 + y^2}$ (c) $\dfrac{x^3 + y^3}{x^2 + y^2}$. (d) $\dfrac{\sin x - \sin y}{x - y}$.

2. Compute the partial derivatives (and specify where they exist) for the following functions:

   (a) $f(x, y) = e^{-(x^2+y^2)}$.
   (b) $f(x, y) = \cos x \cos y - \sin x \sin y$.
   (c) $f(x, y) = \|(x, y)\| = \sqrt{x^2 + y^2}$.
   (d) $f(x, y, z) = x^2yz + ye^z$.

3. Let $f : \mathbb{R}^2 \to R, f(x, y) = xy$. Using the definition, prove that $Df(x_0, y_0) = (y_0, x_0)$.

4. Prove that

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

   is continuous and has partial derivatives, but it is not differentiable in the origin.

5. Find the gradient of the function f at the point a for the following:

   (a) $f(x, y) = e^{-x} \sin(x + 2y)$, $a = (0, \frac{\pi}{4})$.
   (b) $f(x, y) = \arctan(\frac{y}{x})$, $a = (1, 1)$.
   (c) $f(x, y, z) = e^{xyz}$, $a = (0, 0, 0)$.
   (d) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $a = (1, 1, 1)$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x, y) = f(x^2 + y^2),\ \forall\, (x, y) \in \mathbb{R}^2.$$

   Prove that

$$y\frac{\partial g}{\partial x}(x, y) = x\frac{\partial g}{\partial y}(x, y).$$

1. a) $\frac{x^2-y^2}{x^2+y^2}$, let $y = mx$

$$= \frac{x^2 - m^2x^2}{x^2+m^2x^2} = \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)} = \frac{1-m^2}{1+m^2}$$ depends on m

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Or. take • $y=0 \Rightarrow \lim_{x\to 0} \frac{x^2}{x^2} = 1$ (left/right)

• $x = 0 \Rightarrow \lim_{y\to 0} \frac{-y^2}{y^2} = -1$ (up/down)

$\Bigg\} \Rightarrow \nexists \lim$

b) $\frac{x+y}{x^2+y^2}$

• $y=0$, $\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ doesn't exist

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$$

c) $\frac{x^3+y^3}{x^2+y^2}$

• $y = 0$, $\lim_{x\to 0} \frac{x^3}{x^2} = 0$, The same for $x=0$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \le \underbrace{|x|+|y|}_{\downarrow 0}$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2}$$

$$\le \underbrace{\frac{|x|\cdot x^2}{x^2+y^2}}_{\le 1} + \underbrace{\frac{|y|\cdot y^2}{x^2+y^2}}_{\le 1}$$

$$\frac{x^2}{x^2+y^2} \le 1$$

$$\le \boxed{|x|+|y|}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

d) $$\frac{\sin x - \sin y}{x-y} = \frac{2\cos\frac{x+y}{2}\sin\frac{x-y}{2}}{x-y} = \cos\frac{x+y}{2} \longrightarrow 1$$

$\to 1$

2. a) $f(x,y) = e^{-(x^2+y^2)}$

$$\frac{df}{dx} f(x,y) = e^{-(x^2+y^2)} \cdot (-2x) \qquad (y \text{ is const})$$

$$\frac{df}{dy} f(x,y) = e^{-(x^2+y^2)} \cdot (-2y) \qquad \text{Defined } \forall (x,y) \in \mathbb{R}^2$$

b) $f(x,y) = \cos x \cos y - \sin x \sin y$
$= \cos(x+y)$

$$\frac{df}{dx} f(x,y) = -\sin(x+y)$$

Defined $\forall (x,y) \in \mathbb{R}^2$

$$\frac{df}{dy} f(x,y) = -\sin(x+y)$$

c) $f(x,y) = \|(x,y)\| = \sqrt{x^2+y^2}$

$$\frac{df}{dx} \sqrt{x^2+y^2} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{df}{dy} \sqrt{x^2+y^2} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y = \frac{y}{\sqrt{x^2+y^2}}$$

Defined $\forall (x,y) \neq (0,0)$

d) $f(x,y,z) = x^2yz + ye^z$

$$\frac{df}{dx} (x^2yz + ye^z) = 2xyz$$

$$\frac{df}{dy}(x^2yz + ye^z) = x^2z + e^z$$

$$\frac{df}{dz}(x^2yz + ye^z) = x^2y + ye^z$$

Defined $\forall\ (x,y,z) \in \mathbb{R}^3$

3. $f(x,y) = xy$ Prove that $Df(x_0, y_0) = (y_0, x_0)$

! $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0) - Df(x_0)\cdot(x - x_0)}{\|x - x_0\|} = 0$

$x \in \mathbb{R}^n$

$x_0 \in \mathbb{R}^n$

$$L = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \overbrace{Df(x_0,y_0)}^{(y_0,x_0)}(x - x_0, y - y_0)}{\|(x - x_0, y - y_0)\|} = 0$$

$$f(x,y) - f(x_0,y_0) - (y_0,x_0)\cdot(x - x_0, y - y_0) = xy - x_0y_0 - y_0(x - x_0) - x_0(y - y_0)$$

$$= xy - \cancel{x_0y_0} - y_0x + \cancel{x_0y_0} - x_0y + x_0y_0$$

$$= xy - x\cdot y_0 - x_0y + x_0y_0$$

$$= (x - x_0)\cdot(y - y_0)$$

$$\|(x - x_0, y - y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$L = \lim_{(x,y)\to(x_0,y_0)} \frac{(x - x_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \lim_{(u,v)\to(0,0)} \frac{u\cdot v}{\sqrt{u^2 + v^2}}$$

! $u^2 + v^2 \geq 2|u\cdot v|$, $\dfrac{1}{\sqrt{u^2+v^2}} \leq \dfrac{1}{\sqrt{2}\sqrt{|uv|}}$

$$\Rightarrow 0 \le \frac{|uv|}{\sqrt{u^2+v^2}} \le \frac{\sqrt{|uv|}}{\sqrt{2}} \to 0$$

$$\Rightarrow L = 0$$

$$Df(x_0, y_0) = \nabla f(x_0, y_0) = \left(\frac{df}{dx}(x_0,y_0), \frac{df}{dy}(x_0,y_0) \right)$$

$$= (y_0, x_0)$$

4. $$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

- Continuity : $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$$

$\Rightarrow$ f cont. at $(0,0)$

$$0 \le \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{|xy|}}{\sqrt{2}}$$

- Partial deriv. at $(0,0)$

$$\frac{df}{dx}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{df}{dy}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$$

- Assume that f is diff at $(0,0)$, then $Df(0,0) = \nabla f(0,0)$

$$= \left(\frac{df}{dx}(0,0), \frac{df}{dy}(0,0) \right) = (0,0)$$

But $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - \overset{0}{\widetilde{f(0,0)}} - \overset{0}{\widetilde{(0,0)\cdot(x,y)}}}{\|(x,y)\|}$

$$= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Take $y = mx$, $\lim\limits_{x\to 0} \dfrac{m\cancel{x^2}}{(1+m^2)\cancel{x^2}} = \dfrac{m}{1+m^2}$ depends on m

$\Rightarrow \nexists \lim$

$\Rightarrow f$ is not diff. at $(0,0)$

5. a) $f(x,y) = e^{-x}\sin(x+2y)$ $\qquad a = (0, \frac{\pi}{4})$

$$\frac{df}{dx}\, e^{-x}\sin(x+2y) = -e^{-x}\cdot\sin(x+2y) + e^{-x}\cos(x+2y)$$
$$= e^{-x}(\cos(x+2y) - \sin(x+2y))$$

$$\frac{df}{dy}\, e^{-x}\sin(x+2y) = 2e^{-x}\cos(x+2y)$$

$$\nabla f(x,y) = (e^{-x}\cos(x+2y) - e^{-x}\sin(x+2y),\ 2e^{-x}\cos(x+2y))$$

$$\Rightarrow f(0, \tfrac{\pi}{4}) = (\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2},\ 2\cos\tfrac{\pi}{2})$$
$$= (-1, 0)$$

b) $f(x,y) = \arctan\left(\frac{y}{x}\right)$, $\quad a = (1,1)$

$$\frac{df}{dx}\arctan\left(\frac{y}{x}\right) = \frac{1}{\left(\frac{y}{x}\right)^2+1} \cdot \left(\frac{y}{x}\right)'_x$$
$$= \frac{\cancel{x^2}}{y^2+x^2} \cdot \left(-\frac{y}{\cancel{x^2}}\right) = -\frac{y}{y^2+x^2}$$

$$\frac{df}{dy} \arctan\left(\frac{y}{x}\right) = \frac{x^2}{y^2+x^2} \cdot \frac{1}{x} = \frac{x}{y^2+x^2}$$

$$\nabla f(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

c) $f(x,y,z) = e^{xyz}$ $\qquad a = (0,0,0)$

$$\frac{df}{dx} e^{xyz} = e^{xyz} \cdot yz$$

$$\frac{df}{dy} e^{xyz} = e^{xyz} \cdot xz \qquad \Rightarrow \nabla f(0,0,0) = (0,0,0)$$

$$\frac{df}{dz} e^{xyz} = e^{xyz} \cdot xy$$

d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $\qquad a = (1,1,1)$

$$\frac{df}{dx} = \frac{x}{\sqrt{}} \qquad \frac{df}{dy} = \frac{y}{\sqrt{}} \qquad \frac{df}{dz} = \frac{z}{\sqrt{}}$$

$$\nabla f(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$