Homework

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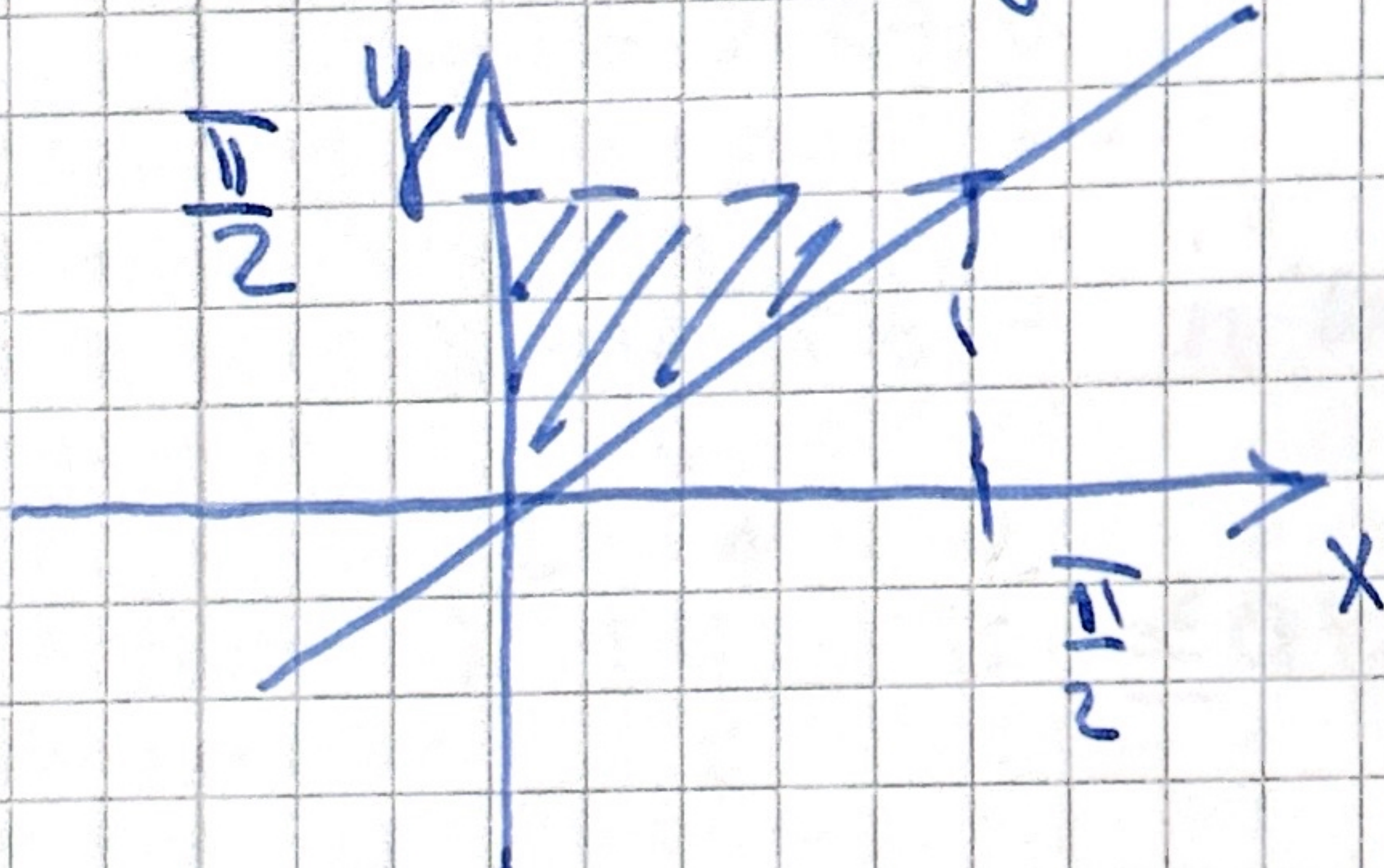
913

1.

c) $\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\, dy\, dx$



$$\begin{cases} x \le y \le \frac{\pi}{2} \\ 0 \le x \le \frac{\pi}{2} \end{cases} \Rightarrow \begin{cases} 0 \le y \le \frac{\pi}{2} \\ 0 \le x \le y \end{cases}$$

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\, dy\, dx = \int_0^{\frac{\pi}{2}}\int_0^{y} \frac{\sin y}{y}\, dx\, dy =$$

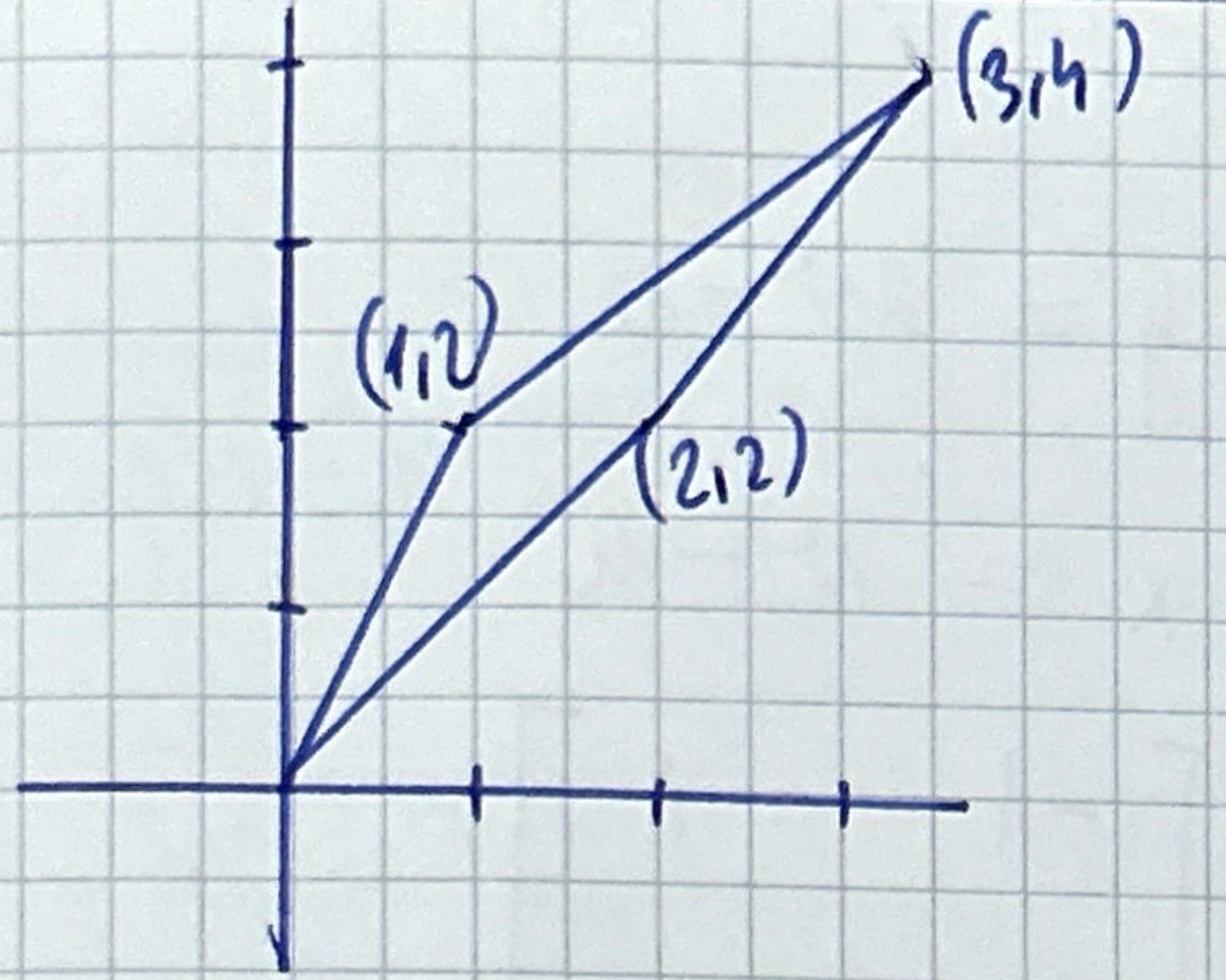
$$= \int_0^{\frac{\pi}{2}} \frac{\sin y}{y}\cdot x \Big|_0^{y} dy = \int_0^{\frac{\pi}{2}} \left(\frac{\sin y}{\cancel{y}}\cdot \cancel{y} - \frac{\sin y}{y}\cdot 0\right) dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y\, dy = -\cos y \Big|_0^{\frac{\pi}{2}} = -\left(\cos\frac{\pi}{2} - \cos 0^\circ\right) =$$

$$= -0 + 1 = 1.$$

2.

d)



we compute the equation of the sides of the parallelogram

I for $(0,0), (1,2)$

$$y = 2x$$

II for $(0,0), (2,2)$

$$y = x$$

III for $(2,2), (3,4)$

$$m = \frac{4-2}{3-2} = 2$$

$$y - 2 = m(x-2)$$
$$y = 2x - 4 + 2$$
$$y = 2x - 2$$

IV for $(1,2), (3,4)$

$$m = \frac{4-2}{3-1} = 1$$

$$y - 2 = m(x-1)$$
$$y - 2 = x - 1$$
$$y = x + 1$$

taking I and III, $\begin{cases} y - 2x = 0 \\ y - 2x = -2 \end{cases}$

taking II and IV, $\begin{cases} y - x = 0 \\ y - x = 1 \end{cases}$

let $u = y - 2x$, $u \in [-2, 0]$

$v = y - x$, $v \in [0, 1]$

$$\rightarrow \begin{cases} u - 2v = -y \Rightarrow y = 2v - u \\ u - v = -x \Rightarrow x = v - u \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\det J = -2 + 1 = -1 \quad ; \quad D^* = \{ (x,y) \in \mathbb{R}^2 \mid (x,y) \in [-2,0] \times [0,1] \}$$

$$\iint_D xy \, dx \, dy = \iint_{D^*} (v-u)(2v-u) \cdot |\det J| \, du \, dv =$$

$$= \int_0^1 \int_{-2}^0 (2v^2 - vu - 2uv + u^2) \, du \, dv =$$

$$= \int_0^1 \int_{-2}^0 (2v^2 - 3uv + u^2) \, du \, dv =$$

$$= \int_0^1 \left(2v^2 u - \frac{3}{2} u^2 v + \frac{u^3}{3} \right) \Big|_{-2}^0 dv =$$

$$= \int_0^1 \left[\left(2v^2 \cdot 0 - \frac{3}{2} 0 \cdot v + \frac{0}{3} \right) - \left(2 \cdot v^2 \cdot (-2) - \frac{3}{2} \cdot 4 \cdot v + \frac{-8}{3} \right) \right] dv$$

$$= \int_0^1 \left(4v^2 + 6v + \frac{8}{3} \right) dv =$$

$$= 4 \cdot \frac{v^3}{3} + \frac{6}{2} v^2 + \frac{8}{3} v \Big|_0^1 =$$

$$= \frac{4}{3} \cdot 1 + \frac{6}{2} + \frac{8}{3} = \frac{4 + 9 + 8}{3} = \frac{21}{3} = 7$$

3.

d) $\iint_D \ln(x^2+y^2)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 \mid a^2 < x^2+y^2 < b^2,$
$0 < a < b,\ x, y > 0\}$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ \theta \in [0, \frac{\pi}{2}] \end{array}\right| \Rightarrow \left.\begin{array}{l} x^2+y^2 = r^2 \\ a^2 < x^2+y^2 < b^2 \end{array}\right| \Rightarrow \left.\begin{array}{l} a^2 < r^2 < b^2 \\ r > 0 \end{array}\right| \Rightarrow$$

$$\Rightarrow a < r < b$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \Rightarrow$$

$$\Rightarrow |\det J| = |r\cos^2\theta + r\cdot\sin^2\theta| = r$$

$$D^* = \{(r,\theta) \in \mathbb{R}^2 \mid (r,\theta) \in [a,b] \times [0, \tfrac{\pi}{2}]\}$$

$$\iint_D \ln(x^2+y^2)\,dx\,dy = \iint_{D^*} \ln r^2 \cdot r\,dr\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_a^b 2\ln r \cdot r\,dr \right) d\theta =$$

$$= \frac{\pi}{\cancel{2}} \cdot \cancel{2} \cdot \int_a^b \ln r \cdot r\,dr = \pi\left(\ln r \cdot \frac{r^2}{2}\Big|_a^b - \int_a^b \frac{1}{\cancel{r}} \cdot \frac{r^{\cancel{2}}}{2}\,dr \right) =$$

$$f(x) = \ln r \Rightarrow f'(x) = \frac{1}{r}$$

$$g'(x) = r \Rightarrow g(x) = \frac{r^2}{2}$$

$$= \pi\left(\ln b \cdot \frac{b^2}{2} - \ln a \cdot \frac{a^2}{2} - \frac{r^2}{4}\Big|_a^b \right) = \pi\left(\frac{b^2}{2}\ln b - \frac{a^2}{2}\ln a - \frac{a^2}{4} + \frac{b^2}{4} \right)$$