Seminar 11

1. In the real vector space $\mathbb{R}^3$ consider the bases $B = (v_1, v_2, v_3) = ((1,0,1),(0,1,1),(1,1,1))$ and $B' = (v_1', v_2', v_3') = ((1,1,0),(-1,0,0),(0,0,1))$. Determine the matrices of change of basis $T_{BB'}$ and $T_{B'B}$, and compute the coordinates of the vector $u = (2,0,-1)$ in both bases.

exam **2.** In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x,y) = (3x+3y, 2x+4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

$$\textbf{5.}\ \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}. \quad \textbf{6.}\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$\textbf{7.}\ \begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix} (x, y \in \mathbb{R}^*). \quad \textbf{8.}\ \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} (x \in \mathbb{R}).$$

9. Let $A \in M_2(\mathbb{R})$ and let λ_1, λ_2 be the eigenvalues of A in $\mathbb{C}$. Prove that:
(i) $\lambda_1 + \lambda_2 = Tr\,(A)$ and $\lambda_1 \cdot \lambda_2 = det\,(A)$, where $Tr\,(A)$ denotes the trace of A, that is, the sum of the elements of the principal diagonal. Generalization.
(ii) A has all the eigenvalues in $\mathbb{R} \iff (Tr\,(A))^2 - 4 \cdot det\,(A) \geq 0$.
(iii) Show that A is a root of its characteristic polynomial.

10. Let $A \in M_2(\mathbb{R})$ be such that $det(A + iI_2) = 0$. Show that $det(A + 2I_2) = 5$.

V, V' K vect. spaces

$\mathcal{B}, \mathcal{B}'$ bases of V, V'

$\mathcal{B} = (v_1, \dots, v_n)$

$f \in \mathrm{Hom}_K(V, V')$

$$[f]_{\mathcal{B},\mathcal{B}'} = \left([f(v_1)]_{\mathcal{B}'} \quad \dots \quad [f(v_n)]_{\mathcal{B}'} \right)$$

$f_1, f_2 \in \mathrm{Hom}_K(V, V')$ $\mathcal{B}, \mathcal{B}'$ bases

$$[f_1 + f_2]_{\mathcal{B},\mathcal{B}'} = [f_1]_{\mathcal{B},\mathcal{B}'} + [f_2]_{\mathcal{B},\mathcal{B}'}$$

$\forall \alpha \in K$

$$[\alpha f]_{\mathcal{B},\mathcal{B}'} = \alpha \cdot [f]_{\mathcal{B},\mathcal{B}'}$$

$f \in \mathrm{Hom}_K(V, V')$

$g \in \mathrm{Hom}_K(V', V'')$

$\mathcal{B}, \mathcal{B}', \mathcal{B}''$ bases of V, V', V''

$g \circ f \in \mathrm{Hom}_K(V, V'')$

$$[g \circ f]_{\mathcal{B},\mathcal{B}''} = [g]_{\mathcal{B}',\mathcal{B}''} \cdot [f]_{\mathcal{B},\mathcal{B}'}$$

Changing bases for a lin. map:

V, V' K vector space

$f \in \mathrm{Hom}_K(V, V')$

B_1, B_2 bases of V $\qquad$ B_1', B_2' bases of V'

$$[f]_{B_2,B_2'} = [id]_{B_1',B_2'}\,[f]_{B_1,B_1'}\,[id]_{B_2,B_1}$$

identity function

$$[id]_{B_2,B_1} = T_{B_1,B_2}$$ = base exchange matrix from B_1 to B_2

$$[id]_{B_1,B_2} = [id]^{-1}_{B_2,B_1}$$ THIS IS NOT TRUE IN GENERAL!

Convert vectors from a basis to another:

$\forall\, v \in V$:

$$[v]_{B'} = [id]_{B,B'} \cdot [v]_B$$

$u = (2, 0, -1)$ in both bases.

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

$$[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B$$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$[id]_{B,B'}$

$(1,2) = a\cdot(1,0) + b(2,1)$

$\Rightarrow \begin{cases} a+2b=1 \\ b=2 \end{cases} \Rightarrow a=-3 \Rightarrow [id((1,2))]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$(1,3) = a(1,0) + b(2,1)$

$\Rightarrow \begin{cases} a+2b=1 \\ b=3 \end{cases} \Rightarrow a=-5 \Rightarrow [id((1,3))]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

$$[id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$[id]_{B',B}$

$(1,0) = a\cdot(1,2) + b(1,3)$

$\Rightarrow \begin{cases} a+b=1 & | \cdot(-3) \\ 2a+3b=0 \end{cases} \Rightarrow \begin{cases} -3a-3b=-3 \\ 2a+3b=0 \end{cases} +$

$-a=-3$

$\Rightarrow a=3$

$\Rightarrow b=-2 \quad [id((1,0))]_{B'} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$(2,1) = a(1,2) + b(1,3)$

$\Rightarrow \begin{cases} a+b=2 & | \cdot(-3) \\ 2a+3b=1 \end{cases} \Rightarrow \begin{cases} -3a-3b=-6 \\ 2a+3b=1 \end{cases} +$

$-a=-5$

$\Rightarrow a=5$

$\Rightarrow b=-3 \Rightarrow [id((2,1))]_{B'} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

$$\Rightarrow [id]_{B',B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f+g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} = \begin{pmatrix} -19 & -30 \\ 12 & 19 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{E\,B'} \cdot [g]_{B',E} \qquad E = ((1,0),(0,1))$$

can use this but it's complicated

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B}$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

matrices of change of basis.)

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ and $B' = (1, X - b, (X - b)^2)$ $(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

$$[T]_{EB}$$

$$1 = t_1 \cdot 1 + t_2(x - a) + t_3(x^2 - 2xa + a^2)$$

$$= t_1 - at_2 + a^2 t_3 + t_2 \cdot x - 2xat_3 + t_3 x^2$$

$$\Rightarrow \begin{cases} t_3 = 0 \\ t_2 - 2at_3 = 0 \\ t_1 - at_2 + a^2 t_3 = 1 \end{cases} \Rightarrow \begin{cases} t_3 = 0 \\ t_2 = 0 \\ t_1 = 1 \end{cases}$$

$$[T(1)]_{EB} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x = t_1 - at_2 + a^2 t_3 + t_2 \cdot x - 2xat_3 + t_3 x^2$$

$$\Rightarrow \begin{cases} t_3 = 0 \\ t_2 - 2at_3 = 1 \\ t_1 - at_2 + a^2 t_3 = 0 \end{cases} \Rightarrow \begin{cases} t_3 = 0 \\ t_2 = 1 \\ t_1 = a \end{cases}$$

$$\Rightarrow [T(x)]_{EB} = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$$

$$x^2 = t_1 - at_2 + a^2t_3 + x(t_2 - 2at_3) + x^2 \cdot t_3$$

$$\Rightarrow \begin{cases} t_3 = 1 \\ t_2 - 2at_3 = 0 \\ t_1 - at_2 + a^2 t_3 = 0 \end{cases} \Rightarrow \begin{cases} t_3 = 1 \\ t_2 = 2 \\ t_1 = a \end{cases}$$

$$\Rightarrow [T(x^2)]_{EB} = \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}$$

$$T_{EB} = \begin{pmatrix} 1 & a & a \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x, y) = (3x + 3y, 2x + 4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Step 1: Write f in a convenient basis (usually E)

$$[f]_E = ([f(e_1)]_E \ [f(e_2)]_E)$$

$$f(e_1) = f(1,0) = (3,2)$$
$$f(e_2) = f(0,1) = (3,4)$$
$$\Rightarrow [f]_E = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$$

Step 2: Find the characteristic polynomial of $A = [f]_E$

$$P_A(x) = \det(A - xI_n)$$

$$= \begin{vmatrix} 3-x & 3 \\ 2 & 4-x \end{vmatrix} = x^2 - 7x + 6$$

Step 3: The eigen values of A are the distinct roots of the poly.

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49-24}}{2} = \frac{7 \pm 5}{2} \Rightarrow \begin{matrix} \lambda_1 = 6 \\ \lambda_2 = 1 \end{matrix}$$

Step 4: For every eigenvalue λ, we have the eigenspace

$$S(\lambda) = \{ v \in V \mid f(v) = \lambda v \}$$

the set of eigenvectors corresponding to λ and also 0

$$S(\lambda) = \{ v \in \mathbb{R}^2 \mid [f]_E \cdot [v]_E = \lambda \cdot [v]_E \}$$

If $\lambda = \lambda_1 = 6 \Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{cases} 3x + 3y = 6x \\ 2x + 4y = 6y \end{cases} \Rightarrow y = x$

$\Rightarrow S(\lambda_1) = \{ (x, x) \mid x \in \mathbb{R} \} = \langle (1,1) \rangle$

If $\lambda = \lambda_2 = 1 \Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{cases} 3x + 3y = x \\ 2x + 4y = y \end{cases} \Rightarrow \begin{cases} 2x + 3y = 0 \\ 2x + 3y = 0 \end{cases} \Rightarrow y = \frac{-2x}{3}$

$S(\lambda_2) = \{ (x, -\frac{2x}{3}) \mid x \in \mathbb{R} \} = \langle (1, -\frac{2}{3}) \rangle =$

$\langle (3, -2) \rangle$

$\Rightarrow$ we have a basis of eigenvectors for $\mathbb{R}^2$

$$B = ((1,1), (3,-2))$$

If f lin. map and B is a basis of eigenvalues

$$[f]_B = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$