Seminar 10

1. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for $\mathbb{R}^3$.

exam **2.** Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

3. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ be defined by

$$f(e_1) = (1, 2, 3, 4), f(e_2) = (4, 3, 2, 1), f(e_3) = (-2, 1, 4, 1)$$

on the elements of the canonical basis of $\mathbb{R}^3$. Determine:
(i) $f(v)$ for every $v \in \mathbb{R}^3$.
(ii) the matrix of f in the canonical bases.
(iii) a basis and the diemnsion of $Ker\, f$ and $Im\, f$.

exam **4.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

5. Consider the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ and its bases $E = (1, X, X^2)$ and $B = (1, X - 1, X^2 + 1)$. Consider $\varphi \in End_{\mathbb{R}}(\mathbb{R}_2[X])$ defined by

$$\varphi(a_0 + a_1X + a_2X^2) = (a_0 + a_1) + (a_1 + a_2)X + (a_0 + a_2)X^2.$$

Determine the matrices $[\varphi]_E$ and $[\varphi]_B$.

6. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1, 2), (1, 3))$ and $B' = (v'_1, v'_2) = ((1, 0), (2, 1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f + g]_B$ and $[f \circ g]_{B'}$.

7. Consider the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of $\mathbb{R}^2$ and show that f is an automorphism.

8. Let V be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine $|V|$, $|End_K(V)|$ and $|Aut_K(V)|$.
[*Hint: use the isomorphism between $End_K(V)$ and $M_n(K)$, where $dim_K(V) = n$.*]

Main idea of linear algebra

V, V' K- vector space

$B = (v_1, \dots, v_n)$ basis of V

$B' = (v'_1, \dots, v'_m)$ basis of V'

$f: V \to V'$

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \ \cdots \ [f(v_n)]_{B'} \right) \longrightarrow \text{if } w = a_1 v_1 + a_2 v_2 + \dots + a_m v_m \quad [w]_{B'} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

* applying a linear map is just multiplying a vector with a matrix

$\forall\ v \in V$

$$\boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by linear map

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$= ((1,0),(0,1))$

apply f to B and write as B' or E'

$$v_1 = (1,1,0) \qquad f(v_1) = f(1,1,0) = (1,-1)$$
$$v_2 = (0,1,1) \quad \Rightarrow \quad f(v_2) = f(0,1,1) = (1,0)$$
$$v_3 = (1,0,1) \qquad f(v_3) = f(1,0,1) = (0,-1)$$

$$(1,-1) = a\cdot(1,0) + b\cdot(0,1) = (a,b) \Rightarrow \begin{cases} a = 1 \\ b = -1 \end{cases} \Rightarrow [f(v_1)]_{E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1)$$

$$(1,0) = a(1,0) + b\cdot(0,1) \Rightarrow \begin{cases} a = 1 \\ b = 0 \end{cases} \Rightarrow [f(v_2)]_{E'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2)$$

$$(0,-1) = a\cdot(1,0) + b(0,1) \Rightarrow \begin{cases} a = 0 \\ b = -1 \end{cases} \Rightarrow [f(v_3)]_{E'} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3)$$

(1)(2)(3) $\Rightarrow [f]_{B,e'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

* advantage of using the canonical basis is that you can "just take the vector and write it as the column

$$(1,1) = a\cdot(1,1) + b\cdot(1,-2) = (a+b, a-2b) \Rightarrow \begin{cases} a+b=1 \\ a-2b=-1 \end{cases} (-)$$

$$3b = 2 \Rightarrow b = \frac{2}{3}, \quad a = \frac{1}{3} \Rightarrow [f(v_1)]_{B'} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$(1,0) = a\cdot(1,1) + b\cdot(1,-2) \Rightarrow \begin{cases} a+b=1 \\ a-2b=0 \end{cases}$$

$$3b = 1 \Rightarrow b = \frac{1}{3}, \quad a = \frac{2}{3} \Rightarrow [f(v_2)]_{B'} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$(0,-1) = a(1,1) + b(1,-2) \Rightarrow \begin{cases} a+b=0 = a=-b \\ a-2b=-1 \Rightarrow a=-1+2b \end{cases} (-)$$

$$3b = 1 \Rightarrow b = \frac{1}{3}, \quad a = \frac{-1}{3} \Rightarrow [f(v_3)]_{B'} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow [f(v)]_{B,B'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

linear map $f: \mathbb{R}^4 \to \mathbb{R}^4$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}. \quad [f(v)]_E = [f]_E \cdot [v]_E$$

means source and target basis is the same E

(*i*) Show that $v = (1,4,1,-1) \in Ker\, f$ and $v' = (2,-2,4,2) \in Im\, f$.
(*ii*) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(*iii*) Define f. ($f(x,y,z,t)$)

(i) $v \in Ker\, f \Leftrightarrow \boxed{f(v) = 0 \Leftrightarrow [f(v)]_E = 0} \Leftrightarrow [f]_E [v]_E = 0$

$$\Rightarrow [f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+4-3-2 \\ -1+4+1-4 \\ 2+4-5-1 \\ 1+8-4-5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$v' \in \operatorname{Im} f \Leftrightarrow \exists\, u \in \mathbb{R}^4$ s.t $f(u) = v' \Leftrightarrow \exists\, u = (x,y,z,t) \in \mathbb{R}^4$ s.t $[f(u)]_E = [v']_E$

$\Leftrightarrow [f]_E \cdot [u]_E = [v']_E \Leftrightarrow$ the system given by

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$ is comp.

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

row echelon form $\Rightarrow$ system is comp $\Rightarrow \exists\, u$ s.t. $f(u) = v'$

$v' \in \operatorname{Im} f$

(ii) $\operatorname{Ker} f = \{ v = (x,y,z,t) \mid f(v) = 0 \} = \{ v = (x,y,z,t) \mid [f(v)]_E = 0 \} =$

$$= \left\{ [f(v)]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x+y-3z+2t=0 \\ y-z+3t=0 \end{cases}$$

$$\begin{matrix} z=\alpha \\ t=\beta \end{matrix} \Rightarrow \begin{cases} x+y=3\alpha-2\beta \\ y=\alpha-3\beta \end{cases} \Rightarrow x=3\alpha-2\beta-(\alpha-3\beta)=2\alpha+\beta$$

$$\Rightarrow \operatorname{Ker} f=\{(2\alpha+\beta,\alpha-3\beta,\alpha,\beta) \mid \alpha,\beta\in\mathbb{R}\}$$

$$(2\alpha+\beta,\alpha-3\beta,\alpha,\beta)=(2\alpha,\alpha,\alpha,0)+(\beta,-3\beta,0,\beta)=$$

$$=\alpha(2,1,1,0)+\beta(1,-3,0,1)=\langle(2,1,1,0),(1,-3,0,1)\rangle$$

$\Rightarrow \operatorname{Ker} f=\langle(2,1,1,0),(1,-3,0,1)\rangle$

not proportional $\Rightarrow$ lin. independent $\Big\} \Rightarrow ((2,1,1,0),(1,-3,0,1))$ - basis for $\operatorname{Ker} f$

$\dim \operatorname{Ker} f=2 \Rightarrow \operatorname{null} f=2$

$$\operatorname{im} f=\{w=(a,b,c,d) \mid \exists\, u=(x,y,z,t) \text{ s.t. } f(u)=w\}=$$

$$=\{w=(a,b,c,d) \mid \exists\, u=(x,y,z,t) \text{ s.t. } [f]_e\cdot[u]_e=[w]_e\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \xrightarrow[\substack{L_3\leftarrow L_3-2L_1 \\ L_4\leftarrow L_4-L_1}]{L_2\leftarrow L_2+L_1} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & 3 & c-2a \\ 0 & 1 & -1 & -3 & d-a \end{array}\right) \xrightarrow{L_4\leftrightarrow L_2}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & -3 & d-a \\ 0 & -1 & 1 & 3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \xrightarrow[L_4\leftarrow L_4-2L_2]{L_3\leftarrow L_3+L_2} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & -3 & d-a \\ 0 & 0 & 0 & 0 & c-2a+d-a \\ 0 & 0 & 0 & 0 & a+b-2(d-a) \end{array}\right)$$

compatible iff $\begin{cases} -3a+c+d=0 \\ -a+b-2d=0 \end{cases} \Rightarrow \begin{cases} -3a=-\alpha-\beta \Rightarrow a=\dfrac{\alpha+\beta}{3} \\ -a+b=-2\beta \Rightarrow b=-2\beta+\dfrac{\alpha+\beta}{3} \end{cases}$

$c=\alpha$

$d=\beta$

$$b=\frac{\alpha-5\beta}{3}$$

$$\Rightarrow \quad w = \left(\frac{\alpha+\beta}{3}, \frac{\alpha-5\beta}{3}, \alpha, \beta\right)$$

$\beta-\alpha$?

$$\Rightarrow \quad \operatorname{Im} f = \left\langle \left(\frac{1}{3}, -1, 1, 0\right), \left(\frac{1}{3}, 1, 0, 1\right) \right\rangle \text{ basis } \Rightarrow \dim \operatorname{Im} f = 2$$

(iii) $$[f(x,y,z,t)]_{\mathcal{E}} = [f]_{\mathcal{E}} \cdot [(x,y,z,t)]_{\mathcal{E}} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$$

$$= \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-5z+t \\ x+2y-4z+5t \end{pmatrix}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+2t)\cdot e_1 + (-x+y+z+4t)\cdot e_2 + (2x+y-5z+t)\cdot e_3 + (x+2y-4z+5t)\cdot e_4 =$$

$$= (x+y-3z+2t,\ -x+y+z+4t,\ 2x+y-5z+t,\ x+2y-4z+5t)$$

ex: $f(1,2) = (1,0)$ $\qquad f(2,3) = (0,1)$

$$B = ((1,2),(2,3)) \Rightarrow [f]_{B,\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$f(7,8) = ?$

$$[f(7,8)]_{\mathcal{E}} = [f]_{\mathcal{E}} \cdot \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$[f]_{\mathcal{E}} = [f]_{B,\mathcal{E}} \cdot [id]_{\mathcal{E},B}$$

$$[id]_{\mathcal{E},B} = [id]_{B,\mathcal{E}}^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1}$$

$$[f]_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\Rightarrow [f(7,8)]_{\mathcal{E}} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$$