Seminar 9

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

6. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

Write the matrix of the list X in the basis B, determine an echelon form for it and deduce that X is linearly dependent.

For the following exercises, for a list X of vectors in a canonical vector space $\mathbb{R}^n$, use that dim $< X >$ is equal to the rank of an echelon form C of the matrix consisting of the components of the vectors of X, and a basis of $< X >$ is given by the non-zero rows of C.

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine *dim* $< X >$ and a basis of $< X >$.

8. In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine *dim* $< X >$ and a basis of $< X >$.

9. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S =< (1, 0, 4), (2, 1, 0), (1, 1, -4) >\,,$$

$$T =< (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > \;.$$

10. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S =< (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) >\,,$$

$$T =< (2, 5, -6, -5), (-1, 2, -7, -3) > \;.$$

$A, B \in \mathcal{M}_{m,n}(K)$

If $A \sim B$, then $\operatorname{rank} A = \operatorname{rank} B$

So if E is the echelon form of A, then: $\operatorname{rank} A = \operatorname{rank} E$

$=$ nr of non-zero rows in E

9.1. $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ echelon form

$\Rightarrow \operatorname{rank} A = 3$

3. $A = \begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

$1 - \alpha\beta + x\alpha = 0$

$\alpha(x - \beta) = -1$

$x - \beta = -\frac{1}{\alpha}$

$x = \beta - \frac{1}{\alpha}$

$\overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - \beta L_1}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix}$

$\overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$

- if $\alpha = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$ echalon form

$\Rightarrow \text{rank } A = 3$

- if $\alpha \neq 0 \Rightarrow \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$

$L_3 \leftarrow L_3 + (\beta - \frac{1}{\alpha}) L_2$

$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3+\frac{2}{\alpha}-5\beta & 4-\frac{1}{\alpha}-2\beta \end{pmatrix}$

- $3 - 3\beta + \left(\beta - \frac{1}{\alpha}\right)(-2)$

$= 3 - 3\beta - 2\beta + \frac{2}{\alpha} = 3 + \frac{2}{\alpha} - 5\beta$

- $4 - 3\beta + \left(\beta - \frac{1}{\alpha}\right) = 4 - \frac{1}{\alpha} - 2\beta$

$\begin{cases} 3 + \frac{2}{\alpha} - 5\beta = 0 \\ 4 - \frac{1}{\alpha} - 2\beta = 0 \quad | \cdot 2 \end{cases} \Leftrightarrow \begin{cases} 3 + \frac{2}{\alpha} - 5\beta = 0 \\ 8 - \frac{2}{\alpha} - 4\beta = 0 \end{cases} +$

$11 - 9\beta = 0$

$\beta = \frac{11}{9}$

$\Rightarrow 3 + \frac{2}{\alpha} - \frac{55}{9} = 0$

$\frac{2}{\alpha} = \frac{55}{9} - 3 \Rightarrow \frac{2}{\alpha} = \frac{55-27}{9}$

$\frac{2}{\alpha} = \frac{28}{9} \Rightarrow \alpha = \frac{2 \cdot 9}{28} = \frac{9}{14}$

$$\Rightarrow \operatorname{rank} A = \begin{cases} 2, \text{ if } \alpha = \frac{5}{14} \quad \beta = \frac{11}{5} \\ 3, \text{ otherwise} \end{cases}$$

OR

• $\alpha \neq 0 \longrightarrow \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$

$\underset{\sim}{L_2 \leftarrow \frac{1}{\alpha} L_2}$ $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & \frac{-2}{\alpha} & \frac{1}{\alpha} \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$

$\underset{\sim}{L_3 \leftarrow L_3 - (1-\alpha\beta)L_2}$ $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & -\frac{2}{\alpha} & \frac{1}{\alpha} \\ 0 & 0 & 3+\frac{2}{\alpha}-5\beta & 4-\frac{1}{\alpha}-2\beta \end{pmatrix}$

• $3-3\beta+\frac{2}{\alpha}-2\beta = 3+\frac{2}{\alpha}-5\beta$

• $4-3\beta-\frac{1}{\alpha}(1-\alpha\beta) = 4-3\beta-\frac{1}{\alpha}+\beta = 4-\frac{1}{\alpha}-2\beta$

$$\boxed{\begin{gathered} A \in M_n(K) \\ (A \mid I_n) \underset{\text{Gauss-Jordan}}{\sim \cdots \sim} (I_n \mid A^{-1}) \end{gathered}}$$

5. find A^{-1}

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$$

$$(A \mid I_3) \sim \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -2 & 1 & 0 \\ 0 & -8 & -7 & -3 & 0 & 1 \end{array}\right) \xrightarrow{L_2 \leftarrow -\frac{1}{5} L_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -8 & -7 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \leftarrow L_3 + 8L_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & 1 \end{array}\right) \xrightarrow[L_1 \leftarrow L_1 - 3L_3]{L_2 \leftarrow L_2 - L_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{2}{5} & \frac{24}{5} & -3 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{7}{5} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & 1 \end{array}\right)$$

$$\xrightarrow{L_1 \leftarrow L_1 - 4L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & -\frac{4}{5} & 1 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{7}{5} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{2}{5} & -\frac{4}{5} & 1 \\ \frac{1}{5} & \frac{7}{5} & -1 \\ \frac{1}{5} & -\frac{8}{5} & 1 \end{pmatrix}$$

We can use Gaussian elimination to extract a basis out of a list of generators (if we place the generators as rows to a matrix, bring it to the echelon form, the rows will form a basis)

9.10. Find a basis for S, T, $S+T$ and the dim for S, T, $S+T$, $S \cap T$

$$S = \langle (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) \rangle$$

$$T = \langle (2, 5, -6, -5), (-1, 2, -7, -3) \rangle$$

$$M_1 = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[L_3 \leftarrow L_3 + L_1]{L_2 \leftarrow L_2 - 3L_1} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 2 & 0 & -3 \end{pmatrix}$$

$L_3 \leftarrow L_3 + \frac{2}{5} L_2$
$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \end{pmatrix}$$
echalon form $\Rightarrow$ rank $M_S = 3$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 4 & -\frac{1}{2} \end{pmatrix} \Rightarrow \dim S = 3$$

$((1,2,-1,-2), (0,1,0,-\frac{3}{2}), (0,0,4,-\frac{1}{2}))$ basis for S

$$M_t = \begin{pmatrix} 2 & 5 & -6 & -5 \\ -1 & 2 & -7 & -3 \end{pmatrix} \overset{L_2 \leftarrow L_2 + \frac{1}{2}L_1}{\sim} \begin{pmatrix} 2 & 5 & -6 & -5 \\ 0 & \frac{9}{2} & -10 & -\frac{11}{2} \end{pmatrix}$$

$\Rightarrow$ echalon form $\Rightarrow$ rank $M_t = 2$ $\Rightarrow \dim T = 2$

$((2,5,-6,-5), (0, \frac{9}{2}, -10, \frac{11}{2}))$ basis for T

$$M_{St} = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 4 & -\frac{1}{2} \\ 2 & 5 & -6 & -5 \\ 0 & \frac{9}{2} & -10 & \frac{11}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 2 & 5 & -6 & -5 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & \frac{9}{2} & -10 & \frac{11}{2} \\ 0 & 0 & 4 & -\frac{1}{2} \end{pmatrix}$$

$$\overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -4 & -1 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & \frac{9}{2} & -10 & \frac{11}{2} \\ 0 & 0 & 4 & -\frac{1}{2} \end{pmatrix} \underset{L_4 \leftarrow L_4 - \frac{9}{2}L_2}{\overset{L_3 \leftarrow L_3 - L_2}{\sim}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 4 & -\frac{1}{2} \\ 0 & 0 & 8 & 10 \\ 0 & 0 & 4 & -\frac{1}{2} \end{pmatrix}$$

$$\underset{L_5 \leftarrow L_5 - L_3}{\overset{L_4 \leftarrow L_4 - 2L_3}{\sim}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 4 & -\frac{1}{2} \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Calcule greșite

$\ldots \quad \dim(S+T) = 3$

$$\Rightarrow \quad \dim(S\cap T) = \dim S + \dim T - \dim(S+T)$$
$$= 3 + 2 - 3 = 2$$