Seminar 10

08.05.2023 10:10

Eigenvalues. Eigenvectors.

1. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

$$A = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = -1$$

$$\lambda = -1: \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow V(\lambda_1) = \{(x,y)\in\mathbb{R}^2 \mid x = 0\} = \{(0,y) \mid y\in\mathbb{R}\} = \langle(0,1)\rangle \Rightarrow m_g(\lambda_1) = 1$$

$$\lambda = 1: \begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow V(\lambda_2) = \{(x,y)\in\mathbb{R}^2 \mid y = 0\} = \{(x,0) \mid x\in\mathbb{R}\} = \langle(1,0)\rangle \Rightarrow m_g(\lambda_2) = 1$$

M is diagonalizable if $\exists\, S \in GL_n(\mathbb{R})$:

$M = S^{-1}\Lambda S$, when $\Lambda = \begin{pmatrix}\lambda_1 & 0 & & 0\\ 0 & \lambda_2 & & 0\\ & & \ddots & \\ 0 & 0 & & \lambda_n\end{pmatrix} \Leftrightarrow \exists$ a basis of eigenvectors of M for $\mathbb{R}^n \Leftrightarrow$

$\Leftrightarrow \forall\, \lambda$ eigenvalue of M

$m_a(\lambda) = m_g(\lambda)$

algebraic multiplicity — geometric multiplicity

b) $B = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1 \Rightarrow m_a(\lambda) = 2$

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow V(\lambda) = \{(x,y) \mid y = 0\} = \langle(1,0)\rangle \Rightarrow m_g(\lambda) = 1 \Rightarrow$$

$\Rightarrow B$ is not diagonalizable

c) $C = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \Rightarrow \lambda = 1$ with multiplicity 2

Same as above

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix}-1 & 2 & -2\\ -2 & -2 & -1\\ -2 & 1 & 2\end{bmatrix} \quad\text{and}\quad B = \frac{1}{11}\begin{bmatrix}-9 & -2 & 6\\ 6 & -6 & 7\\ 2 & 9 & 6\end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

$A \in M_{n\times n}(\mathbb{R})$ orthogonal if $A^t A = I_n$

$O(n) \to$ the set of orthogonal matrices

$SO(n) \to$ the set of special orthogonal matrices

$A \in SO(n) \Leftrightarrow \begin{cases} A \in O(n) \Leftrightarrow A^t\cdot A = I_n\\ \det A = 1\end{cases}$

$A \in SO(2) \Leftrightarrow A$ has the form $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \theta\in\mathbb{R}$

① $\cos\theta = \dfrac{\mathrm{tr}(\mathrm{lin}\,\phi)}{2}, \phi \in E^2$

② $\cos\theta = \dfrac{\mathrm{tr}(\mathrm{lin}\,\phi) - 1}{2}, \phi \in E^3$

$$A = \frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & -2 & -1\\ -2 & 1 & 2\end{pmatrix} \Rightarrow \det A = \frac{1}{27}(4+4+4+8+8-1) = 1 \Rightarrow A \in SO(3)$$

If f is a rotation, then its axis given by

$\mathrm{Fix}(f) = \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z) = (x,y,z)\} = \{(x,y,z)\in\mathbb{R}^3 \mid A\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}\}$

$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$

$$\mathrm{tr}(A) = \frac{1}{3}(-1-2+2) = -\frac{1}{3} \Rightarrow \cos\theta = \frac{-\frac{1}{3} - 1}{2} = -\frac{2}{3}$$

$$\theta = \pi - \arccos\frac{2}{3}$$

$$A\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix} \Leftrightarrow A\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} - \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$A\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & -2 & 1\\ -2 & 1 & 2\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-x+2y-2z\\ -2x-2y+z\\ -2x+y+2z\end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix}-x+2y-2z\\ -2x-2y+z\\ -2x+y+2z\end{pmatrix} - \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \Big| \cdot 3$$

$$\begin{pmatrix}-4x+2y-2z\\ -2x-5y+z\\ -2x+y-z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}-2 & 1 & -1 & 0\\ -2 & -5 & 1 & 0\\ -2 & 1 & -1 & 0\end{array}\right) \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \left(\begin{array}{ccc|c}-2 & 1 & -1 & 0\\ 0 & -6 & 2 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \Leftrightarrow \begin{cases}-2x+y-z = 0\\ -6y+2z = 0\end{cases} \Leftrightarrow \begin{cases}x = -\alpha\\ y = \alpha\\ z = 3\alpha\end{cases} \Rightarrow \mathrm{Fix}(A) = \langle(-1,1,3)\rangle$$

10. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}\mathbf{x} + \begin{bmatrix}1\\ -2\end{bmatrix}.$$

$$f(x) = \frac{1}{5}\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix} v + \begin{pmatrix}1\\ -2\end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}1\\ -2\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} \Big| \cdot 5 \Rightarrow \text{Find } \mathrm{Fix}(f(x))$$

$$\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}5\\ -10\end{pmatrix} = 5\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} - \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-5\\ 10\end{pmatrix}$$

$$\begin{pmatrix}-2 & -4\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-5\\ 10\end{pmatrix} \Leftrightarrow \begin{cases}-2x-4y = -5\\ 4x-2y = 10\end{cases} \Leftrightarrow \begin{cases}x = \frac{5}{2}\\ y = 0\end{cases}$$

$$R_\theta(P) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}[P]$$

$$R_\theta(P) = \begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}\cdot\begin{pmatrix}\frac{5}{2}\\ 0\end{pmatrix}$$

Rotation with θ of a point P around $A(x_0, y_0)$

$R = (O, i, j)$

$R' = (A, i, j)$

$$[Q]_{R'} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot[P]_R$$

$$[Q]_R = \underbrace{M_{R,R'}}_{I_2}\cdot[Q]_{R'} + [A]_R$$

$$[Q]_R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot[P]_{R'} + [A]_R$$

$$[P]_{R'} = M_{R'R}[P]_R - [O]_{R'}$$

$$[Q]_R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot([P]_R - [O]_{R'}) + [A]_R$$

$$[Q]_{R'} = [\overrightarrow{AO}]_{R'} = -\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

$$[A]_R = [\overrightarrow{OA}]_R = \begin{pmatrix}x_0\\ y_0\end{pmatrix} \Rightarrow$$

$$\Rightarrow [R_{A,\theta}(P)] = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x + x_0\\ y + y_0\end{pmatrix} + \begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

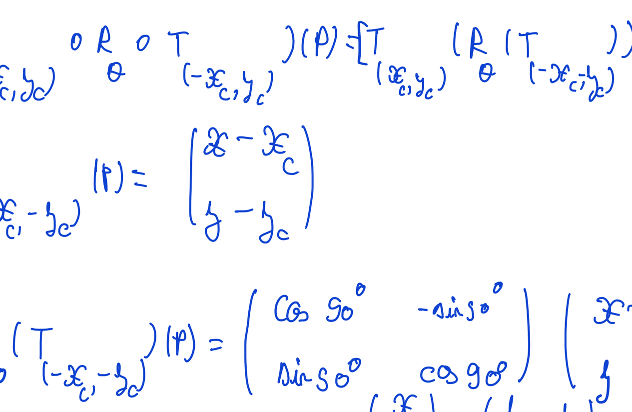
$$R_{A,\theta} = T_{(x_0, y_0)} \circ R_\theta \circ T_{(-x_0, y_0)}$$

Rotation Matrix: $[R_\theta(P)] = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot[P]$

Translation Matrix: $[T_v(P)] = [P] + [v]$

$$[(T_v \circ R_\theta)(P)] = [T_v(R_\theta(P))] = [R_\theta(P)] + [v] = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot[P] + [v]$$

7. The vertices of a triangle are $A(1,1)$, $B(4,1)$ and $C(2,3)$. Determine the image of the triangle ABC under a rotation by 90° around C followed by an orthogonal reflection relative to the line AB.



$$R_{90°,C}(P) = (T_{(x_c, y_c)} \circ R_\theta \circ T_{(-x_c, -y_c)})(P)$$

$$(T_{(x_c, y_c)} \circ R_\theta \circ T_{(-x_c, y_c)})(P) = [T_{(x_c, y_c)}(R_\theta(T_{(-x_c, -y_c)}))](P) =$$

$$T_{(-x_c, -y_c)}(P) = \begin{pmatrix}x - x_c\\ y - y_c\end{pmatrix}$$

$$R_{90}(T_{(-x_c, -y_c)})(P) = \begin{pmatrix}\cos 90° & -\sin 90°\\ \sin 90° & \cos 90°\end{pmatrix}\begin{pmatrix}x - x_c\\ y - y_c\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x - x_c\\ y - y_c\end{pmatrix} = \begin{pmatrix}y_c - y\\ x - x_c\end{pmatrix}$$

$$T_{(x_c, y_c)}(R_{90°}(T_{(-x_c, -y_c)}))(P) = \begin{pmatrix}x_c\\ y_c\end{pmatrix} + \begin{pmatrix}y_c - y\\ x - x_c\end{pmatrix} = \begin{pmatrix}x_c + y_c - y\\ -x_c + y_c + x\end{pmatrix}$$

Let $M = T_{(x_c, y_c)}(R_{90°}(T_{(-x_c, -y_c)}))$

$$M(A) = \begin{pmatrix}2+3-1\\ -2+3+1\end{pmatrix} = \begin{pmatrix}4\\ 2\end{pmatrix} \Rightarrow A'(4,2)$$

$$M(B) = \begin{pmatrix}2+3-1\\ -2+3+4\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix} \Rightarrow B'(4,5)$$

For reflection: $AB \parallel O_x \Rightarrow A''(4,0)$

$B''(4,-3)$

$C''(2,-1)$

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$$\left.\begin{aligned}(R_\theta \circ T_v)^{-1} &= T_v^{-1} \circ R_\theta^{-1}\\ T_v^{-1} &= T_{-v}\\ R_\theta^{-1} &= R_{-\theta}\end{aligned}\right\} \Rightarrow (R_\theta \circ T_v)^{-1} = T_{-v}(R_{-\theta})$$

$$(R_{-\frac{\pi}{3}} \circ T_v)^{-1} = \begin{pmatrix}-2\\ 5\end{pmatrix} + \begin{pmatrix}\cos\frac{\pi}{3}x - \sin\frac{\pi}{3}y\\ \sin\frac{\pi}{3}x + \cos\frac{\pi}{3}y\end{pmatrix}$$

11. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

$$f(x) = \frac{1}{\sqrt{13}}\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix}x + \begin{pmatrix}1\\ -2\end{pmatrix} \Rightarrow v = \begin{pmatrix}1\\ -2\end{pmatrix}$$

$$\mathrm{lin}\, f(x) = \frac{1}{\sqrt{13}}\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix}x$$

$$A = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}}\\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}}\end{pmatrix} \Rightarrow \mathrm{tr}\, A = 2\cdot\frac{2}{\sqrt{13}}$$

$$\cos\theta = \frac{\mathrm{tr}\, A}{2} \Rightarrow \cos\theta = \frac{2}{\sqrt{13}} \Rightarrow \sin\theta = \frac{3}{\sqrt{13}} \quad (\sqrt{1-\cos^2\theta})$$

$$f^{-1} = (T_v \circ R_\theta)^{-1} = R_{-\theta} \circ T_{-v} = \begin{pmatrix}\cos -\theta & -\sin -\theta\\ \sin -\theta & \cos -\theta\end{pmatrix}\begin{pmatrix}x-1\\ y+2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x-1\\ y+2\end{pmatrix} = \frac{1}{\sqrt{13}}\begin{pmatrix}2 & 3\\ -3 & 2\end{pmatrix}\begin{pmatrix}x-1\\ y+2\end{pmatrix}$$