Seminar 4

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Cross (vector) product. Box (mixt) product

$\vec{v}, \vec{w} \in V^3$

If $\vec{v} \parallel \vec{w}$, $\vec{v} \times \vec{w} = \vec{0}$

$\vec{v} \nparallel \vec{w}$, then

① $\vec{v} \times \vec{w} = <\vec{v}, \vec{w}>^{\perp}$

$(\vec{v} \times \vec{w} \perp \vec{v}, \vec{v} \times \vec{w} \perp \vec{w})$

② $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\vec{v}, \vec{w})$

③ orientation is given by right hand rule

$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ orthonormal basis

$\hookrightarrow$ right oriented if $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$

left oriented if $\vec{v}_3 = -\vec{v}_1 \times \vec{v}_2$

If $(\vec{i}, \vec{j}, \vec{k})$ right oriented orthonormal basis

and $\vec{v}(x_v, y_v, z_v)$, $\vec{w}(x_w, y_w, z_w)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = \begin{vmatrix} y_v & z_v \\ y_w & z_w \end{vmatrix} \cdot \vec{i} + \begin{vmatrix} z_v & x_v \\ z_w & x_w \end{vmatrix} \vec{j} + \begin{vmatrix} x_v & z_v \\ x_w & z_w \end{vmatrix} \vec{k}$$

Properties

① $(\alpha \vec{v}_1 + \beta \vec{v}_2) \times \vec{w} = \alpha(\vec{v}_1 \times \vec{w}) + \beta(\vec{v}_2 \times \vec{w})$

② $\vec{v} \times (\alpha \vec{w}_1 + \beta \vec{w}_2) = \alpha(\vec{v} \times \vec{w}_1) + \beta(\vec{v} \times \vec{w}_2)$

③ $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

2. With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a}+\mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a}+\mathbf{b}) \times (2\mathbf{a}-\mathbf{b}).$$

a) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} + 6\vec{k} - 2\vec{j} + \vec{k} + 4\vec{i} + 3\vec{j} = 5\vec{i} + \vec{j} + 7\vec{k} = (5, 1, 7)$

b) $2a + b = (6, -2, -4) + (1, 2, -1) = (7, 0, -5)$

$(2a+b) \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & -5 \\ 1 & 2 & -1 \end{vmatrix} = 14\vec{k} - 5\vec{j} + 10\vec{i} + 7\vec{j} = 10\vec{i} + 2\vec{j} + 14\vec{k}$

$(2a+b) \times b = (10, 2, 14)$

c) $2a - b = (6, -2, -4) - (1, 2, -1) = (5, -4, -3)$

$(2a+b) \times (2a-b) = 4(a \times a) + 2(b \times a) - 2(a \times b) - b \times b = -4(a \times b) = -(20, 4, 28)$

5. Consider the points $A(1,2,0)$, $B(3,0,-3)$ and $C(5,2,6)$ with respect to an orthonormal coordinate system.

a) Determine the area of the triangle ABC.

b) Determine the distance from C to AB.

a) $\vec{AB} = (2, -2, -3)$

$\vec{AC} = (4, 0, 6)$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} = -12\vec{i} - 12\vec{j} + 8\vec{k} - 12\vec{j} = -12\vec{i} - 24\vec{j} + 8\vec{k}$

$\|\vec{AB} \times \vec{AC}\| = \sqrt{12^2 + 24^2 + 8^2} = 4\sqrt{9 + 36 + 4} = 28 \Rightarrow S_{\Delta ABC} = 14$

b) $dist(C, AB) = \frac{2 \cdot A_{\Delta ABC}}{\|\vec{AB}\|} = \frac{28}{\sqrt{17}} = \frac{28\sqrt{17}}{17}$

8. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Double cross product

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$

$\vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3) = \begin{vmatrix} \vec{v}_2 & \vec{v}_3 \\ \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_1 \cdot \vec{v}_3 \end{vmatrix} = (\vec{v}_1 \cdot \vec{v}_3) \cdot \vec{v}_2 - (\vec{v}_1 \cdot \vec{v}_2) \cdot \vec{v}_3$

$(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_3 = \begin{vmatrix} \vec{v}_2 & \vec{v}_1 \\ \vec{v}_2 \cdot \vec{v}_3 & \vec{v}_1 \cdot \vec{v}_3 \end{vmatrix} = (\vec{v}_1 \cdot \vec{v}_3) \vec{v}_2 - (\vec{v}_2 \cdot \vec{v}_3) \cdot \vec{v}_1$

$(a \times b) \times c = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\vec{b} & \vec{c}\vec{a} \end{vmatrix} = -\vec{b} - 5\vec{a} =$

$\vec{c} \cdot \vec{b} = -3 + 2 + 6 = 5$

$\vec{c} \cdot \vec{a} = -1$

13. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

Box Product

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V^3$

$[\vec{v}_1, \vec{v}_2, \vec{v}_3] = (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$

$\vec{v}_1(x_1, y_1, z_1)$

$\vec{v}_2(x_2, y_2, z_2) \Rightarrow [\vec{v}_1, \vec{v}_2, \vec{v}_3] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$\vec{v}_3(x_3, y_3, z_3)$

A, B, C, D coplanar $\Leftrightarrow \begin{vmatrix} x_D - x_A & y_D - y_A & z_D - z_A \\ x_D - x_B & y_D - y_B & z_D - z_B \\ x_D - x_C & y_D - y_C & z_D - z_C \end{vmatrix} = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 0 & -2 \\ 3 & -1 & 2 \end{vmatrix} = -8 + 6 - 2 + 4 = 0 \Rightarrow A, B, C, D$ coplanar

1. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

$a(1, 2, -2) \Rightarrow a_1 = 1, a_2 = 2, a_3 = -2$

$b(7, 4, 6) \Rightarrow b_1 = 7, b_2 = 4, b_3 = 6$

$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$

$a \times b = \begin{vmatrix} 2 & -2 \\ 4 & 6 \end{vmatrix} i - \begin{vmatrix} 1 & -2 \\ 7 & 6 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 7 & 4 \end{vmatrix} \vec{k}$

$a \times b = 20\vec{i} - 20\vec{j} - 10\vec{k}$

3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC}(1.5,2,1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

$Area_{Parall}(\vec{AB}, \vec{AC}) = \|\vec{AB} \times \vec{AC}\| = \sqrt{\begin{vmatrix} 6 & 0 \\ 1,5 & 2 \end{vmatrix}^2 + \begin{vmatrix} 6 & 1 \\ 1,5 & 1 \end{vmatrix}^2 + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}^2} =$

$= \sqrt{144 + 4,5^2 + 2^2}$

$= \sqrt{144 + 20,25 + 4} =$

$= \sqrt{168,25}$

$AB = \sqrt{1,5^2 + 2^2 + 1^2} = \sqrt{2,25 + 4 + 1} = \sqrt{7,25}$

$d = \sqrt{\frac{168,25}{7,25}} = \sqrt{\frac{673}{29}}$

4. Consider the vectors $\mathbf{a}(2,3,-1)$ and $\mathbf{b}(1,-1,3)$ with respect to an orthonormal basis.

a) Determine the vector subspace $\langle \mathbf{a}, \mathbf{b} \rangle^{\perp}$.

b) Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51$.

$<a, b>^{\perp} = \vec{a} \times \vec{b}$

$\vec{a} \times \vec{b} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \vec{k} = 8\vec{i} - 7\vec{j} - 5\vec{k} \Rightarrow \vec{a} \times \vec{b} = <(8, -7, -5)>$

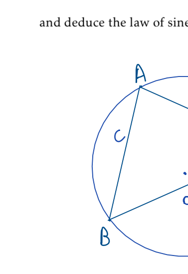
b) $p \perp a \Rightarrow a \cdot p = 0 \Leftrightarrow \begin{cases} 2p_1 + 3p_2 - p_3 = 0 \\ p_1 - p_2 + 3p_3 = 0 \\ 2p_1 - 3p_2 + 4p_3 = 51 \end{cases}$

$p \perp b \Rightarrow b \cdot p = 0 \Leftrightarrow$

$p \cdot (2\vec{i} - 3\vec{j} + 4\vec{k}) = 51 \Leftrightarrow$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & -3 & 4 & 51 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - 2L_1]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 5 & -7 & 0 \\ 0 & -1 & -2 & 51 \end{array}\right) \xrightarrow{L_3 \leftarrow 5L_3 + L_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & -17 & 255 \end{array}\right) \Leftrightarrow \begin{cases} p_1 - p_2 + 3p_3 = 0 \\ 5p_2 - 7p_3 = 0 \\ -17p_3 = 255 \end{cases} \Leftrightarrow \begin{cases} p_1 = 24 \\ p_2 = 21 \\ p_3 = -15 \end{cases} \Rightarrow \vec{p}(24, 21, -15)$$

6. Let $ABCD$ be a quadrilateral in $\mathbb{E}^3$ and let E, F be the midpoints of $[AB]$ and $[CD]$ respectively. Denote by K, L, M and N the midpoints of the segments $[AF]$, $[CE]$, $[BF]$ and $[DE]$ respectively. Prove that $KLMN$ is a parallelogram.



$(x_K, y_K, z_K) = \frac{1}{2}(x_A + x_F, y_A + y_F, z_A + z_F) = \frac{1}{4}(2x_A + x_D + x_C, 2y_A + y_D + y_C)$

$(x_L, y_L, z_L) = \frac{1}{4}(2x_C + x_A + x_B, 2y_C + y_B + y_A, 2z_C + z_B + z_A)$

$(x_M, y_M, z_M) = \frac{1}{4}(2x_B + x_D + x_C, 2y_B + y_D + y_C, 2z_B + z_D + z_C)$

$(x_N, y_N, z_N) = \frac{1}{4}(2x_D + x_A + x_B, 2y_D + y_A + y_B, 2z_D + z_A + z_B)$

$\vec{KL} = \frac{1}{4}(x_C + x_B - x_D - x_A, y_C + y_B - y_D - y_A, z_C + z_B - z_D - z_A)$

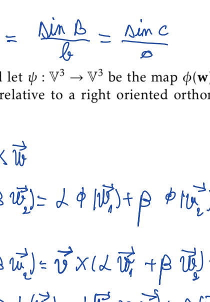
$\vec{NM} = \frac{1}{4}(x_C + x_B - x_D - x_A, y_C + y_B - y_D - y_A, z_C + z_B - z_D - z_A)$

$\Rightarrow \vec{KL} = \vec{NM}$

7. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

and deduce the law of sines in a triangle.



$\vec{AB} \times \vec{BC} = (\vec{AO} + \vec{OB}) \times (\vec{BO} + \vec{OC}) =$

$= \vec{AO} \times \vec{BO} + \vec{OB} \times \vec{BO} + \vec{AO} \times \vec{OC} + \vec{OB} \times \vec{OC} =$

$= \vec{AO} \times \vec{BO} + \vec{AO} \times \vec{OC} + \vec{OB} \times \vec{OC}$

$\vec{BC} \times \vec{CA} = (\vec{BO} + \vec{OC}) \times (\vec{CO} + \vec{OA}) =$

$= \vec{BO} \times \vec{CO} + \underbrace{\vec{OC} \times \vec{CO}}_{0} + \vec{BO} \times \vec{OA} + \vec{OC} \times \vec{OA} =$

$= \vec{OB} \times \vec{OC} + \vec{AO} \times \vec{BO} + \vec{AO} \times \vec{OC}$

La fel și $\vec{CA} \times \vec{AB}$

$bc \sin A = ac \sin B = ab \sin C \,\Big|\, \cdot \frac{1}{abc}$

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

9. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

$\phi(\vec{w}) = \vec{v} \times \vec{w}$

$\phi(\alpha \vec{w}_1 + \beta \vec{w}_2) = \alpha \phi(\vec{w}_1) + \beta \phi(\vec{w}_2) \Leftrightarrow \phi$ linear map

$\phi(\alpha \vec{w}_1 + \beta \vec{w}_2) = \vec{v} \times (\alpha \vec{w}_1 + \beta \vec{w}_2) = \alpha(\vec{v} \times \vec{w}_1) + \beta(\vec{v} \times \vec{w}_2)$

$\alpha \phi(\vec{w}_1) + \beta \phi(\vec{w}_2) = \alpha(\vec{v} \times \vec{w}_1) + \beta(\vec{v} \times \vec{w}_2)$

11. Prove the following identities:

a) the Jacobi identity,

b) the Lagrange identity,

c) the formula for the cross product of two cross products.

a) The Jacobi identity: $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$

$(a \times b) \times c = (a \cdot c) b - (b \cdot c) a \rightarrow$ Grassman's vector product formula

$(b \times c) \times a = (b \cdot a) c - (c \cdot a) b = (a \cdot b) c - (a \cdot c) b$

$(c \times a) \times b = (b \cdot c) a - (a \cdot b) c$

(+)

$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = \cancel{(a \cdot c) b} - \cancel{(b \cdot c) a} + \cancel{(a \cdot b) c} - \cancel{(a \cdot c) b} + \cancel{(b \cdot c) a} - \cancel{(a \cdot b) c}$

b) The Lagrange Identity: $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d)$

$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} =$

$= (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}$

$c \times d = \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} \vec{i} - \begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix} \vec{j} + \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \vec{k}$

$c \times d = (c_2 d_3 - c_3 d_2) \vec{i} + (c_3 d_1 - c_1 d_3) \vec{j} + (c_1 d_2 - c_2 d_1) \vec{k}$

$(a \times b) \cdot (c \times d) = (a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2) + (a_3 b_1 - a_1 b_3)(c_3 d_1 - c_1 d_3) + (a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)$

$(a \cdot c)(b \cdot d) = (a_1 c_1 + a_2 c_2 + a_3 c_3)(b_1 d_1 + b_2 d_2 + b_3 d_3)$

$(b \cdot c)(a \cdot d) = (b_1 c_1 + b_2 c_2 + b_3 c_3)(a_1 d_1 + a_2 d_2 + a_3 d_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i d_i b_j c_j$

12. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{k}$ and $\mathbf{c} = \mathbf{k}$. Determine if

a) $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a basis of $\mathbb{V}^3$,

b) if it is a basis, decide if it is left or right oriented.

a) (a, b, c) basis of $V^3 \Leftrightarrow a, b, c$ lin indep

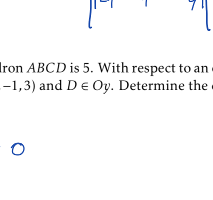
$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = -1 \Rightarrow (a, b, c)$ is left oriented

14. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v}, \mathbf{u}, \mathbf{w}]$.

$\phi(v) = \begin{vmatrix} v_1 & v_2 & v_3 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 3v_1 - v_2$

$M(\phi) = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

15. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.



$\vec{DA} = (-2, -2, -2) = -2 \cdot (1, 1, 1)$

$\vec{DB} = (1, 4, 1)$

$\vec{DC} = (-1, 1, -4)$

$V_{DABC} = \left| \frac{1}{6} [\vec{DA}, \vec{DB}, \vec{DC}] \right| = \frac{1}{3} \left\| \begin{matrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ -1 & 1 & -4 \end{matrix} \right\| = \frac{1}{3} |-16 + 1 - 1 + 4 - 1 + 4| = 3$

16. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D \in Oy$. Determine the coordinates of D.

$D \in O_y \Rightarrow x_D = 0, z_D = 0$

$\vec{AD} = (-2, y_D - 1, 1)$

$\vec{AB} = (1, 1, 2)$

$\vec{AC} = (0, -2, 4)$

$V_{ABCD} = \frac{1}{6} \left\| \begin{matrix} -2 & y_D - 1 & 1 \\ 1 & 1 & 2 \\ 0 & -2 & 4 \end{matrix} \right\| = \frac{1}{6} |-8 - 2 - 8 - 4(y_D - 1)| = \frac{1}{6} \underbrace{18 + 4(y_D - 1)} = 3 + \frac{2}{3}(y_D - 1) = 5 \Leftrightarrow \frac{2}{3}(y_D - 1) = 2 \Leftrightarrow y_D = 4$