Seminar 8

1. Prove that for any $x, y \in \mathbb{R}^n$ the following identities hold:
   (a) $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$ (the parallelogram identity).
   (b) $\langle x, y\rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$.

2. ★ For $x, y \in \mathbb{R}^n$ prove that the following statements are equivalent:
   (a) $\langle x, y\rangle = 0$.
   (b) $\|x+y\| = \|x-y\|$.
   (c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

3. Find the orthogonal projection of a vector $v \in \mathbb{R}^2$ onto a vector $a \in \mathbb{R}^2$.

4. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix with angle θ in $\mathbb{R}^2$.

5. Consider the p-norm $\|x\|_p := \big(|x_1|^p + \ldots + |x_n|^p\big)^{\frac{1}{p}}$, $p \geq 1$ and $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$. Draw the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1, 2, \infty\}$.

6. ★ [Python] Represent the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1.25, 1.5, 3, 8\}$ by random sampling and plotting the points that are inside the unit ball.

7. Find the interior, the closure and the boundary for each of the following sets:
   (a) $[0,1) \times (1,2]$.
   (b) $\{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\}$.
   (c) $\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$.
   (d) $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1,\ x \leq 1\}$.

8. Draw the level sets $L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$ and sketch the graph for the following.
   (a) $f(x,y) = x + y$ and $c \in \{0, \pm 1\}$.
   (b) $f(x,y) = 2x^2 + y^2$ and $c \in \{0, 1, 4\}$.
   (c) $f(x,y) = \sqrt{x^2 + y^2}$ and $c \in \{0, 1, 2\}$.
   (d) $f(x,y) = x^2 - y^2$ and $c \in \{0, \pm 1\}$.

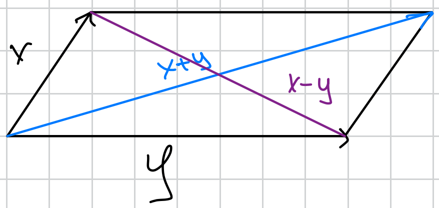
Homework questions are marked with ★.

1. Prove that for any $x, y \in \mathbb{R}^n$ the following identities hold:

(a) $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$ (the parallelogram identity).

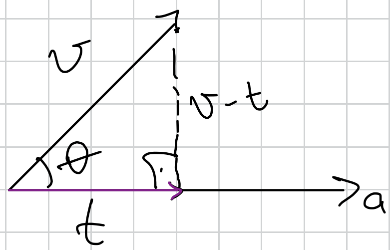
(b) $\langle x, y\rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$.

a) $$\|x+y\|^2 + \|x-y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y)$$
$$= \|x\|^2 + \|y\|^2 + \cancel{2\cdot x\cdot y} + \|x\|^2 + \|y\|^2 - \cancel{2\, x\cdot y}$$
$$= 2\left(\|x\|^2 + \|y\|^2\right)$$



b) $$\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\right)$$
$$= \frac{1}{4}\left(\cancel{\|x\|^2} + \cancel{\|y\|^2} + 2\langle x,y\rangle \cancel{- \|x\|^2} \cancel{- \|y\|^2} + 2\langle x,y\rangle\right)$$
$$= \langle x, y\rangle$$

3. $v \in \mathbb{R}^2$ $\qquad a \in \mathbb{R}^2$



Let $t = pr_a v = \alpha \cdot \vec{a}$

$$\cos\theta = \frac{\|\alpha \cdot a\|}{\|v\|} = \frac{\alpha\|a\|}{\|v\|}$$

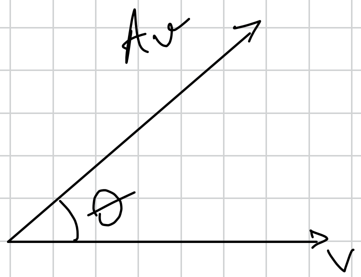
$$v\cdot a = \|v\|\cdot\|a\|\cdot\cos\theta$$
$$\Rightarrow \cos\theta = \frac{v\cdot a}{\|v\|\cdot\|a\|}$$

$$\Rightarrow \frac{\alpha\cdot\|a\|}{\cancel{\|v\|}} = \frac{v\cdot a}{\cancel{\|v\|}\cdot\|a\|} \Rightarrow \alpha = \frac{v\cdot a}{\|a\|^2}$$

$$t = \alpha\cdot a = \frac{v\cdot a}{\|a\|}\cdot\underbrace{\frac{1}{\|a\|}\cdot a}_{\text{unit vector}}$$

4. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotation matrix ?

$a \cdot b = a_1 b_1 + a_2 b_2$



$\widehat{v, Av} \stackrel{?}{=} \theta$

$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}$

$v \in \mathbb{R}^2$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$Av = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \cdot v_1 - \sin\theta \, v_2 \\ \sin\theta \, v_1 + \cos\theta \, v_2 \end{bmatrix}$$

$$v \circ Av = v_1 \cdot (\cos\theta \, v_1 - \sin\theta \, v_2) + v_2 (\sin\theta \, v_1 + \cos\theta \, v_2)$$

$$= \cos\theta \, v_1^2 - \sin\theta \, \cancel{v_1 v_2} + \sin\theta \, \cancel{v_1 v_2} + \cos\theta \, v_2^2$$

$$= \cos\theta \, (v_1^2 + v_2^2) = \cos\theta \cdot \|v\| \cdot \|Av\|$$

$$\Rightarrow \theta = \widehat{v, Av}$$

$\|v\| = \sqrt{v_1^2 + v_2^2}$

$$\|Av\|^2 = (\cos\theta \, v_1 - \sin\theta \, v_2)^2 + (\sin\theta \, v_1 + \cos\theta \, v_2)^2$$

$$= \cos^2\theta \, v_1^2 + \sin^2\theta \, v_2^2 + \cos^2\theta \, v_2^2 + \sin^2\theta \, v_1^2$$

$$= v_1^2 + v_2^2 = \|v\|^2$$

$$\Rightarrow \|Av\| = \|v\|$$

5. $\|x\|_p := (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$, $p \geq 1$ $\quad p \in \{1, 2, \infty\}$

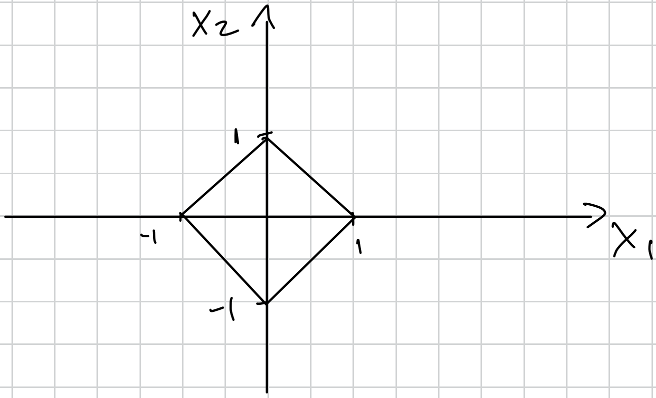
$\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$

Unit ball: $\|x\| = 1$

$\underline{p=1}$: $\|x\|_1 = |x_1| + |x_2|$

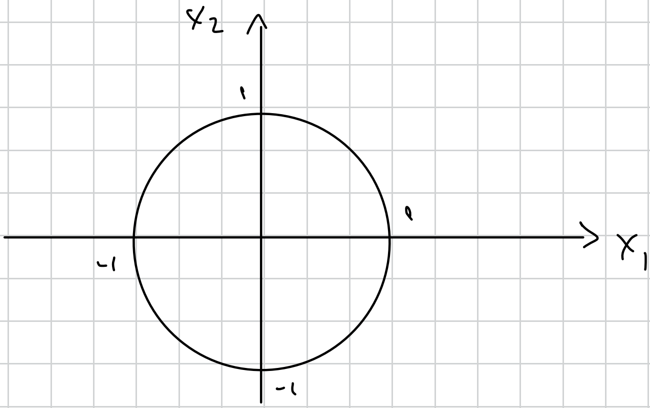
$\|x\|_1 = 1 = |x_1| + |x_2|$

- If $x_1, x_2 > 0$, $x_1 + x_2 = 1 \Rightarrow x_2 = 1 - x_1$
- If $x_1 > 0, x_2 < 0$, $x_1 - x_2 = 1$, $x_2 = x_1 - 1$
- If $x_1 < 0, x_2 > 0$, $-x_1 + x_2 = 1$, $x_2 = 1 + x_1$
- If $x_1, x_2 < 0$, $-x_1 - x_2 = 1$, $x_2 = -1 - x_1$



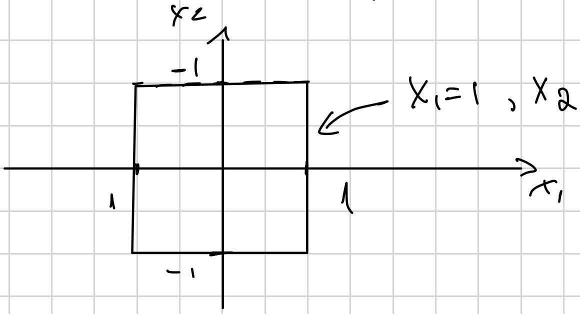
$\underline{p=2}$: $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

$\|x\|_2 = 1 \Leftrightarrow x_1^2 + x_2^2 = 1$



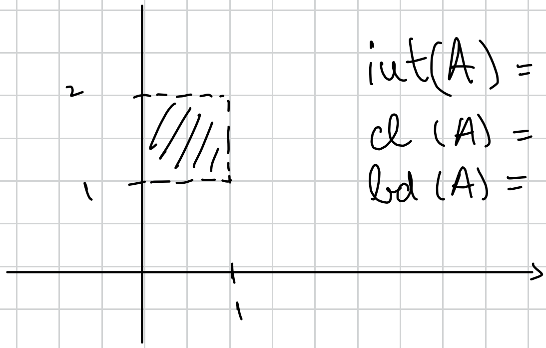
$\underline{p=\infty}$: $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

$\|x\|_\infty = 1 \Leftrightarrow \max\{|x_1|, |x_2|\} = 1 \Rightarrow x_1, x_2 \in [-1, 1]$



$x_1 = 1$, $x_2 \in [-1, 1]$

7. a) $A = [0,1) \times (1,2]$

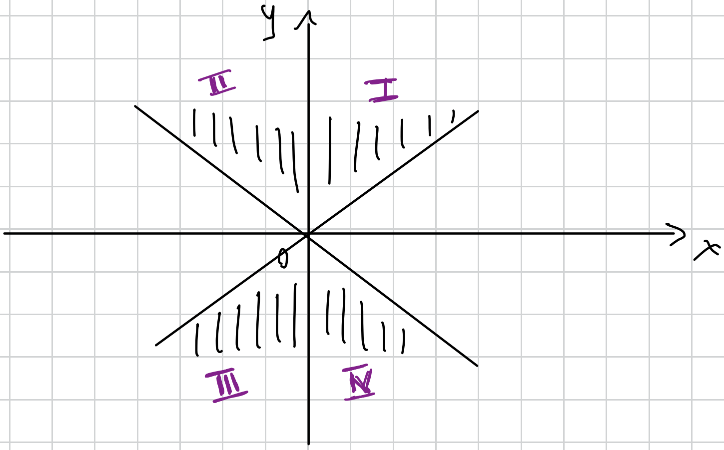


$\text{int}(A) = (0,1) \times (1,2)$

$\text{cl}(A) = [0,1] \times [1,2]$

$\text{bd}(A) = [0,1] \times \{1\} \cup [0,1] \times \{2\} \cup \{0\} \times [1,2] \cup \{1\} \times [1,2]$

b) $A = \{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\}$



$x, y < 0 \Rightarrow |x| < |y| \Leftrightarrow x > y$ III

$x > 0, y < 0 \Rightarrow x < -y \Leftrightarrow y < -x$ IV

$x > 0, y > 0 \Rightarrow x < y$ I

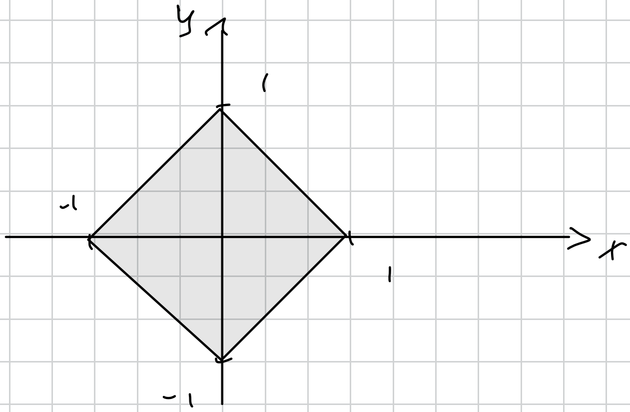
$x < 0, y > 0 \Rightarrow -x < y \to y > -x$ II

$\text{int}(A) = A$

$\text{cl}(A) = A \cup \{(x,x) \mid x \in \mathbb{R}\} \cup \{(x,-x) \mid x \in \mathbb{R}\} = \{(x,y) \in \mathbb{R}^2, |x| \leq |y|\}$

$\text{bd}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| = |y|\}$

c) $A = \{(x,y) \mid |x| + |y| < 1\}$



$\text{int}(A) = A$

$\text{cl}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$

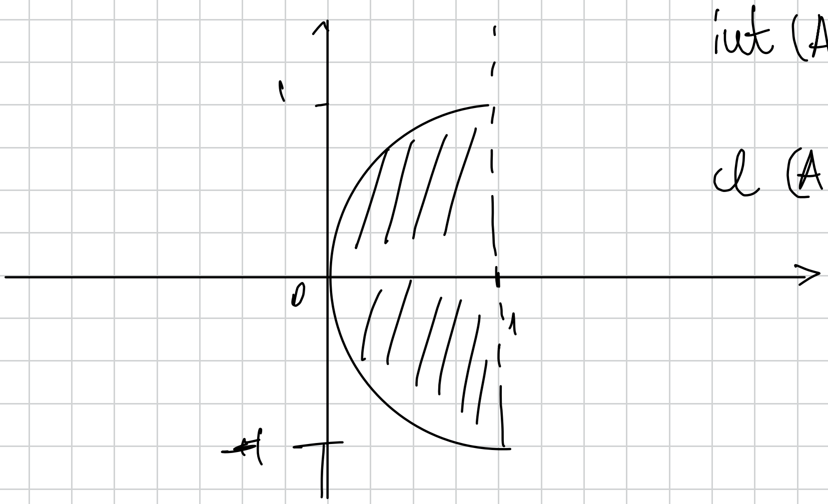
$\text{bd}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$

d) $A = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1, x \leq 1\}$

$x^2+y^2=R^2 \to$ the origin is $O(0,0)$

$(x-x_0)^2+(y-y_0)^2=R^2 \to$ the centre is $A(x_0,y_0)$

$\Rightarrow x_0=1 \quad y_0=0 \quad R=1$



$int(A)=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2<1, x<1\}$

$cl(A)=A$

$bd(A)=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2=1, x=1\}$

8. $L_c=\{(x,y)\in\mathbb{R}^2 \mid f(x,y)=c\}$

a) $f(x,y)=x+y \quad , \quad c\in\{0,\pm1\}$