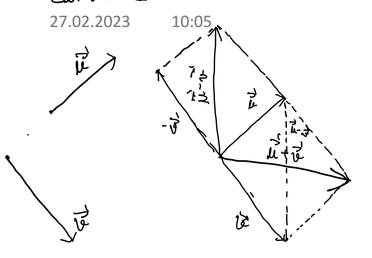
Seminar 1

27.02.2023 10.03



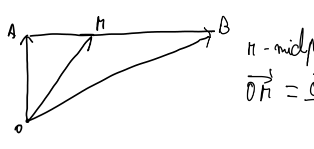
**1.** Let $A_1,\dots,A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\dots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_1}$.

$\vec{A_1A_2}+\vec{A_2A_3}=\vec{A_1A_3}$

$\vec{A_1A_3}+\vec{A_3A_4}=\vec{A_1A_4}$

$\vdots$

$\vec{A_1A_n}+\vec{A_nA_1}=\vec{0}$



n - midpoint

$\vec{On}=\frac{\vec{OA}+\vec{OB}}{2}$

**2.** In each of the following cases, decide if the indicated vectors **u**, **v**, **w** can be represented with the vertices of a triangle:

a) $\mathbf{u}(7,3)$, $\mathbf{v}(-2,-8)$, $\mathbf{w}(-5,5)$,

b) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$,

c) $\|\mathbf{u}\|=7$, $\|\mathbf{v}\|=3$, $\|\mathbf{w}\|=11$.

d) $\mathbf{u}(1,0,1)$, $\mathbf{w}(0,1,0)$, $\mathbf{w}(2,2,2)$.

a) $\vec{u}(7,3)$, $\vec{v}(-2,-8)$, $\vec{w}(-5,5)$

$\vec{u}+\vec{v}+\vec{w}=(7,3)+(-2,-8)+(-5,5)=(0,0)$ ⇒ The triangle exists

b) $\vec{u}(7,3)$, $\vec{v}(2,8)$, $\vec{w}(-5,5)$

$\vec{u}-\vec{v}+\vec{w}=(7,3)-(2,8)+(-5,5)=(0,0)$ ⇒ The triangle exists

c) $\|u\|=7$, $\|v\|=3$, $\|w\|=11$

$\|u\|+\|v\|=10$, $\|w\|=11$ ⇒ $\|u\|+\|v\|<\|w\|$ ⇒ The triangle does not exist

d) $\vec{u}(1,0,1)$, $\vec{v}(0,1,0)$, $\vec{w}(2,2,2)$

It can not be represented. No matter the sign of $\vec{u}$, $\vec{v}$ and $\vec{w}$

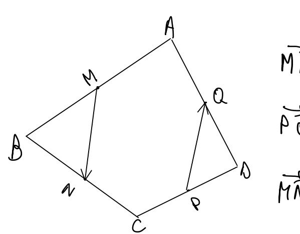
$\begin{cases} 0+1\neq 2\\ 0-1\neq 2\\ 0+1\neq -2\\ 0-1\neq -2\end{cases}$

The triangle does not exist

**4.** Let $ABCD$ be a quadrilateral. Let $M$, $N$, $P$, $Q$ be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN}+\overrightarrow{PQ}=0.$$

Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.



$\vec{MN}=\vec{MB}+\vec{BN}=\frac{1}{2}(\vec{AB}+\vec{BC})=\frac{1}{2}\vec{AC}$

$\vec{PQ}=\vec{PD}+\vec{DQ}=\frac{1}{2}(\vec{CD}+\vec{DA})=\frac{1}{2}\vec{CA}$

$\vec{MN}+\vec{PQ}=\frac{1}{2}(\vec{AC}+\vec{CA})=\vec{0}$

$\left.\begin{array}{l}\vec{MN}+\vec{PQ}=\vec{0}\\ \vec{MN}=\vec{QP}\\ M\neq Q\\ N\neq P\end{array}\right\}$ ⇒ MNPQ parallelogram

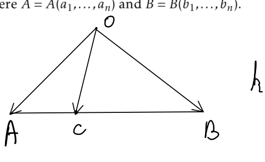
**7.** Let $k=\frac{|CA|}{|CB|}$ be the ratio in which the point $C\in[AB]$ divides the segment $[AB]$. Show that for any point $O$ we have

$$\overrightarrow{OC}=\frac{1}{1+k}(\overrightarrow{OA}+k\overrightarrow{OB}).$$

Deduce that $C$ has coordinates

$$\left(\frac{a_1+kb_1}{1+k},\frac{a_2+kb_2}{1+k},\dots,\frac{a_n+kb_n}{1+k}\right)$$

where $A=A(a_1,\dots,a_n)$ and $B=B(b_1,\dots,b_n)$.



$k=\frac{CA}{CB}$ ⇒ $\frac{|CB|}{|CA|}=\frac{1}{k}$

$\frac{|CB|}{|CA|+|CB|}=\frac{1}{k+1}$

$\frac{|CB|}{|AB|}=\frac{1}{k+1}$

$\vec{CB}=\frac{1}{k+1}\vec{AB}$

$\vec{BC}=\frac{1}{k+1}\vec{BA}$

$\vec{OC}=\vec{OB}+\vec{BC}$

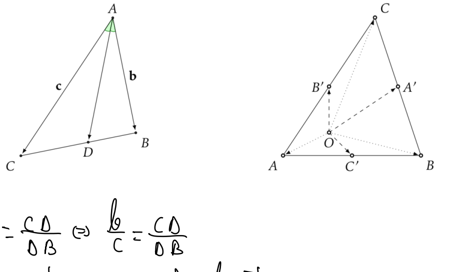
$\vec{OC}=\vec{OB}+\frac{1}{k+1}(\vec{BO}+\vec{OA})$

$\vec{OC}=\frac{1}{k+1}(\vec{OA}+k\vec{OB})$

$C=\frac{1}{k+1}(A+kB)$

$C=\frac{A+kB}{k+1}$ ⇒ $\begin{cases} C_1=\frac{a_1+kb_1}{k+1}\\ C_2=\frac{a_2+kb_2}{k+1}\\ \vdots\\ C_n=\frac{a_n+kb_n}{k+1}\end{cases}$

**8.** Let $ABC$ be a triangle and let $D\in[BC]$ be such that $AD$ is an angle bisector. Express $\overrightarrow{AD}$ in terms of $\mathbf{b}=\overrightarrow{AB}$ und $\mathbf{c}=\overrightarrow{AC}$.



$\frac{AC}{AB}=\frac{CD}{DB}$ ⇔ $\frac{b}{c}=\frac{CD}{DB}$

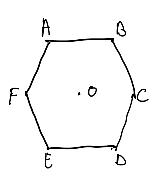
$\frac{CD}{DB}=k$ ⇒ $\vec{AD}=\frac{1}{k+1}(\vec{AC}+k\vec{AB})$

$\vec{AD}=\frac{1}{\frac{b}{c}+1}\left(\vec{AC}+\frac{b}{c}\vec{AB}\right)=\frac{c}{b+c}\left(\vec{AC}+\frac{b}{c}\vec{AB}\right)$

**3.** Let $ABCDEF$ be a regular hexagon centered at $O$.

a) Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of $\overrightarrow{OE}$ and $\overrightarrow{OF}$.

b) Show that $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=3\overrightarrow{AD}$.



$\vec{OA}=\vec{EF}=-\vec{OE}+\vec{OF}$

$\vec{OB}=\vec{EO}$

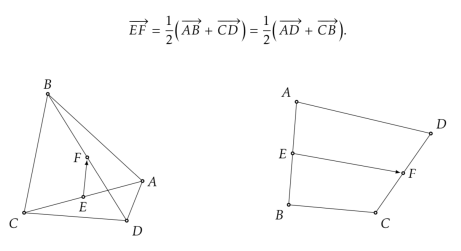
$\vec{OC}=\vec{FO}$

$\vec{OD}=\vec{OE}-\vec{OF}$

b) $\left.\begin{array}{l}\vec{AB}+\vec{AE}=\vec{AD}\\ \vec{AC}+\vec{AF}=\vec{AD}\end{array}\right\}$ ⇒ $\vec{AB}+\vec{AC}+\vec{AD}+\vec{AE}+\vec{AF}=3\vec{AD}$

**5.** Let $ABCD$ be a quadrilateral. Let $E$ be the midpoint of $[AC]$ and let $F$ be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{CD})=\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{CB}).$$



$\vec{EF}=\vec{EC}+\vec{CB}+\vec{BF}=$

$=\frac{1}{2}\vec{AC}+\vec{CB}+\frac{1}{2}\vec{BD}$

$=\frac{1}{2}(\vec{AC}+\vec{CB})+\frac{1}{2}(\vec{CB}+\vec{BD})=$

$=\frac{1}{2}(\vec{AB}+\vec{CD})=$

$=\frac{1}{2}(\vec{AD}+\cancel{\vec{DB}}+\vec{CB}+\cancel{\vec{BD}})=$

$=\frac{1}{2}(\vec{AD}+\vec{CB})$

**6.** Let $ABCD$ be a quadrilateral. Let $E$ be the midpoint of $[AB]$ and let $F$ be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF}=\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{BC}).$$

Deduce that the length of the midsegment in a trapezoid is the arithmetic mean of the lengths of the bases.

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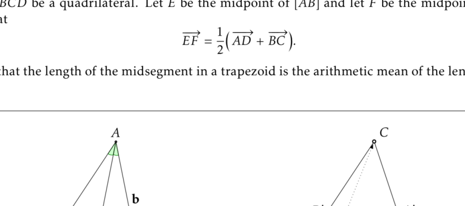
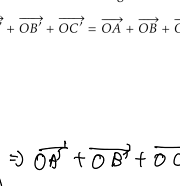


**9.** Let $A'$, $B'$ and $C'$ be midpoints of the sides of a triangle $ABC$. Show that for any point $O$ we have

$$\overrightarrow{OA'}+\overrightarrow{OB'}+\overrightarrow{OC'}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}.$$

$\left.\begin{array}{l}\vec{OA'}=\frac{1}{2}(\vec{OC}+\vec{OB})\\ \vec{OB'}=\frac{1}{2}(\vec{OA}+\vec{OC})\\ \vec{OC'}=\frac{1}{2}(\vec{OA}+\vec{OB})\end{array}\right\}$ ⇒ $\vec{OA'}+\vec{OB'}+\vec{OC'}=\vec{OA}+\vec{OB}+\vec{OC}$

**10.** Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.



$\left.\begin{array}{l}\frac{CN}{NA}=1\\ \frac{AP}{PB}=1\end{array}\right\}$ $\xRightarrow{\text{Ceva's converse}}$ $CP\cap AM\cap BN=\{G\}$

**11.** In each of the following cases, decide if the given points are collinear:

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$.

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$.

c) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$.

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$.

a) $\left.\begin{array}{l}m_{PQ}=\frac{2+5}{-1-3}=-\frac{7}{4}\\ m_{QR}=\frac{5-2}{-5+1}=-\frac{7}{4}\end{array}\right\}$ ⇒ P, Q, R collinear

b) $\left.\begin{array}{l}m_{AB}=\frac{-5}{-10}=\frac{1}{2}\\ m_{BC}=\frac{16}{30}=\frac{8}{15}\end{array}\right\}$ ⇒ A, B, C not collinear

c) $d_{PQ}+d_{QR}=d_{PR}$

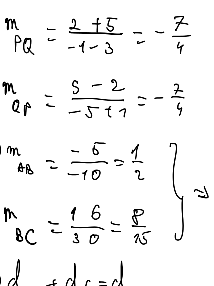
d) La fel

**12.** Let $ABCD$ be a tetrahedron. Determine the sums

a) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$,

b) $\overrightarrow{AD}+\overrightarrow{BC}+\overrightarrow{DB}$,

c) $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{BC}+\overrightarrow{DA}$.



a) $\vec{AB}+\vec{BC}+\vec{CD}=\vec{AC}+\vec{CD}=\vec{AD}$

b) $\vec{AD}+\vec{BC}+\vec{DB}=\vec{AB}+\vec{BC}=\vec{AC}$

c) $\vec{AB}+\vec{CD}+\vec{BC}+\vec{DA}=\vec{AC}+\vec{CA}=\vec{0}$

**13.** Let $ABCD$ be a tetrahedron. Show that

$$\overrightarrow{AD}+\overrightarrow{BC}=\overrightarrow{BD}+\overrightarrow{AC}.$$

$\vec{AD}+\vec{BC}=\vec{BD}+\vec{AC}$ ⇔

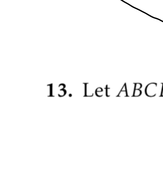
⇔ $\vec{AD}+\vec{DB}+\vec{BC}+\vec{CA}=\vec{0}$ =

⇔ $\vec{0}=\vec{0}$

**14.** Let $SABCD$ be a pyramid with apex $S$ and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC}+\overrightarrow{SD}=4\overrightarrow{SO}$$

where $O$ is the center of the parallelogram.



$\vec{SO}=\vec{SA}+\vec{AO}$

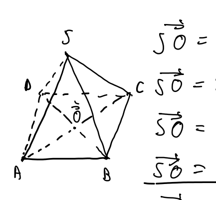
$\vec{SO}=\vec{SB}+\vec{BO}$

$\vec{SO}=\vec{SC}+\vec{CO}$

$\vec{SO}=\vec{SD}+\vec{DO}$

$4\vec{SO}=\vec{SA}+\vec{SB}+\vec{SC}+\vec{SD}+\underbrace{\vec{AO}+\vec{CO}}_{\vec{0}}+\underbrace{\vec{BO}+\vec{DO}}_{\vec{0}}$

**15.** Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x=1$, $y=3$ and $z=-2$.



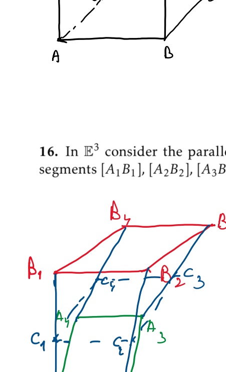
$D(0,0,0)$ $H(0,0,-2)$

$A(1,0,0)$ $E(1,0,-2)$

$B(1,3,0)$ $F(1,3,-2)$

$C(0,3,0)$ $G(0,3,-2)$

**16.** In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.



$\vec{C_1C_2}=\vec{C_1B_1}+\vec{B_1B_2}+\vec{B_2C_2}=$

$=\frac{1}{2}(\vec{A_1B_1}+\vec{B_2A_2})+\vec{B_1B_2}$

$\vec{C_4C_3}=\frac{1}{2}(\vec{A_4B_4}+\vec{B_3A_3})+\vec{B_4B_3}$

$\vec{C_1C_2}=\vec{C_4C_3}$ ⇔

⇔ $\vec{A_1B_1}+\vec{B_2A_2}=\vec{A_4B_4}+\vec{B_3A_3}$ ⇔

⇔ $\vec{A_1B_4}+\cancel{\vec{B_4B_1}}+\cancel{\vec{B_2B_3}}+\vec{B_3A_2}=\cancel{\vec{A_4A_1}}+\vec{A_1B_4}+\vec{B_3A_2}+\cancel{\vec{A_2A_3}}$ ⇔

⇔ $\vec{A_1B_4}+\vec{B_3A_2}=\vec{A_1B_4}+\vec{B_3A_2}$ (A)