Seminar 7

1. Compute the following limits using Riemann integrals:

   (a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$.

   (b) ★ $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\cdots+n\sqrt[n]{e^n}}{n^2}$.

   (c) $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$.

   (d) ★ $\displaystyle\lim_{n\to\infty}\sqrt[n]{\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}}$.

2. Study the Riemann integrability of the function $f:[0,1]\to\mathbb{R}$,

$$f(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

3. Compute the following improper integrals:

   (a) $\displaystyle\int_1^2\frac{1}{x(x-2)}\,\mathrm{d}x$.

   (b) $\displaystyle\int_0^\infty xe^{-x^2}\,\mathrm{d}x$.

   (c) $\displaystyle\int_0^1\frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$.

   (d) ★ $\displaystyle\int_0^\infty e^{-x}\sin x\,\mathrm{d}x$.

4. Study the convergence of the following improper integrals:

   (a) $\displaystyle\int_1^\infty\frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x$.

   (b) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{\cos x}\,\mathrm{d}x$.

   (c) $\displaystyle\int_1^\infty\frac{\ln x}{x\sqrt{x^2-1}}\,\mathrm{d}x$.

5. Using the integral test, study the convergence of the following series:

   (a) $\displaystyle\sum_{n\geq 1}\frac{1}{n^p}$, $p>0$.

   (b) $\displaystyle\sum_{n\geq 2}\frac{1}{n(\ln n)^2}$.

   (c) $\displaystyle\sum_{n\geq 2}\frac{\ln n}{n^2}$.

6. ★ [Python] The integral $\displaystyle\int_{-\infty}^{\infty}e^{-x^2}\,\mathrm{d}x$ represents the area under the bell curve $y=e^{-x^2}$ and it is related to the normal (Gaussian) probability distribution. It is essential in probability theory and has a wide range of applications. Considering intervals of the form $[-a,a]$, for increasing $a>0$, show numerically (e.g. trapezium rule) that $\displaystyle\int_{-\infty}^{\infty}e^{-x^2}\,\mathrm{d}x=\sqrt{\pi}$.

Homework questions are marked with ★.

Riemann Integral / Sum

ex: if we have $[0,1]$

$$0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < \frac{n}{n} = 1$$

$$\boxed{\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f(x)\,dx}$$

1. a) $\lim\limits_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right)$

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \to \int_0^1 f(x)\,dx$$

$$f\left(\frac{k}{n}\right) = \frac{1}{1+\frac{k}{n}}, \quad f(x) = \frac{1}{1+x}$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1 = \ln 2$$

c) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln\frac{n!}{n^n}} \to e^{-1} = \frac{1}{e}$

$$\frac{1}{n} \ln \frac{n!}{n^n} = \frac{1}{n} \ln \frac{1\cdot 2\cdot \dots \cdot n}{n\cdot n\cdot \dots \cdot n} = \frac{1}{n} \sum_{k=1}^{n} \underbrace{\ln \frac{k}{n}}_{f\left(\frac{k}{n}\right)} \to \int_0^1 \ln x\,dx$$

$$\Rightarrow f(x) = \ln x$$

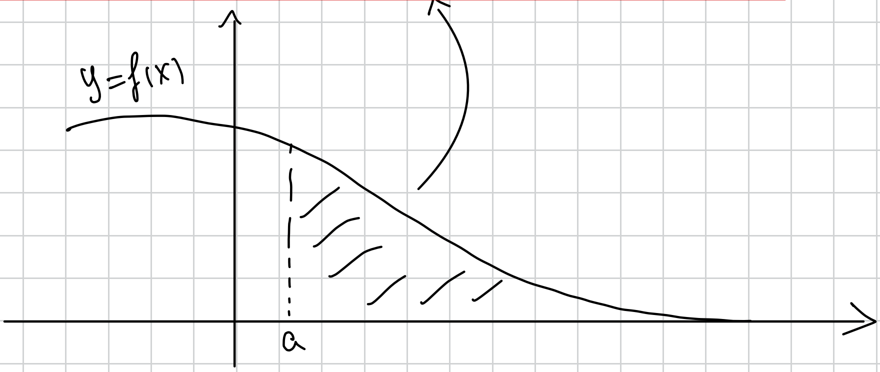
$$\int_0^1 \ln x\,dx = \underbrace{x\ln x\Big|_0^1}_{0} - 1 = -1$$

2. Let $a = x_0 < x_1 < \dots < x_n = b$ a partition on $[0,1]$

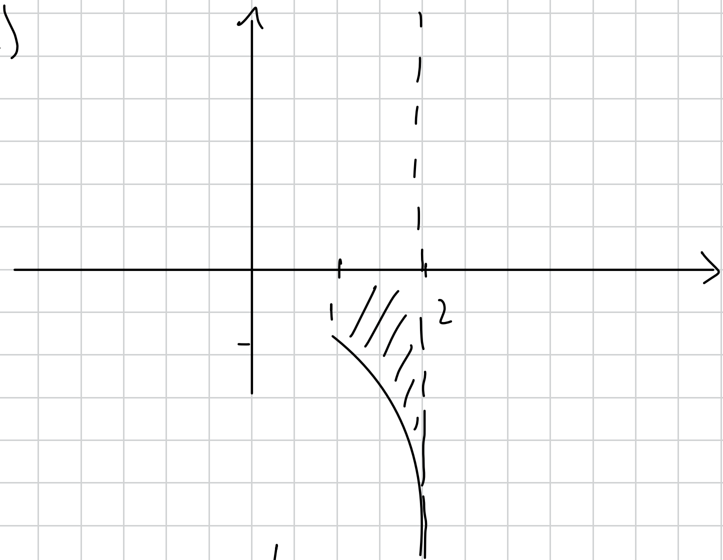
Let $\overline{c_k} \in \mathbb{Q}$, $\underline{c_k} \in \mathbb{R}\setminus\mathbb{Q}$, $\overline{c_k}, \underline{c_k} \in [x_{k-1}, x_k]$

$$\left.\begin{aligned} \overline{\sigma}(f, P, \overline{c_k}) &= \sum_{k=1}^{n} 1\cdot(x_k - x_{k-1}) = 1 \\ \underline{\sigma}(f, P, \underline{c_k}) &= \sum_{k=1}^{n} 0\cdot(x_k - x_{k-1}) = 0 \end{aligned}\right] \neq \Rightarrow \nexists \int_0^1 f(x)\,dx$$

3. $$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\uparrow\infty} \int_a^t f(x)\,dx}$$



a)



$$\int_1^2 \frac{1}{x(x-2)} = \lim_{t\uparrow 2} \int_1^t \frac{1}{x(x-2)}\,dx$$

$$\int_1^t \frac{1}{x(x-2)}\,dx = \frac{1}{2}\int_1^t \left(\frac{1}{x} - \frac{1}{x-2}\right)dx$$

$$= -\frac{1}{2}\ln x \Big|_1^t + \frac{1}{2}\ln|x-2|\Big|_1^t$$

$$= -\frac{1}{2}\ln t + \frac{1}{2}\ln(2-t)$$

$$\Rightarrow \lim_{t\uparrow 2}\int_1^t \frac{1}{x(x-2)}\,dx = \lim_{x\uparrow 2}\left(-\frac{1}{2}\ln t + \frac{1}{2}\ln(2-t)\right)$$

b) $$\int_0^\infty x e^{-x^2}\,dx \qquad t = x^2 \qquad \frac{dt}{dx} = 2x \quad , dt = 2x\,dx$$

$$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\cdot 2x\,dx = \frac{1}{2}\int_0^\infty e^{-t}\,dt$$

$$\frac{1}{2}\lim_{u\to\infty}\int_0^u e^{-t}\,dt = \frac{1}{2}(-e^{-t})\Big|_0^u = \lim_{u\to\infty}\Big(\underbrace{-\frac{1}{2}e^{-u}}_{0} + \frac{1}{2}\Big) = \frac{1}{2}$$

We can also just write : $\frac{1}{2}\int_0^\infty e^{-t}\,dt = -\frac{1}{2}e^{-t}\Big|_0^\infty$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

c) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ $\qquad \lim_{x\searrow 0} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0_+} = -\infty$

$$\lim_{t\searrow 0} \int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\searrow 0}\left(\underbrace{-2\ln t\cdot\sqrt{t}}_{0} - 4 + \underbrace{4t}_{0}\right) = -4$$

$$\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = 2\int_t^1 \ln x\cdot(\sqrt{x})'\,dx = 2\ln x\cdot\sqrt{x}\Big|_t^1 - 2\int_t^1 \frac{1}{x}\cdot\sqrt{x}\,dx$$

$$= -2\ln t\cdot\sqrt{t} - 4\sqrt{x}\Big|_t^1$$

$$= -2\ln t\cdot\sqrt{t} - 4 + 4t$$

$$\lim_{t\searrow 0}\sqrt{x}\ln t = \lim_{t\searrow 0}\frac{\ln t}{\frac{1}{\sqrt{t}}} = \lim_{t\searrow 0}\frac{\frac{1}{\cancel{t}}}{\frac{-\frac{1}{2\sqrt{t}}}{\cancel{t}}} = \lim_{x\searrow 0} -2\sqrt{t} = 0$$

4. a) $\int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,dx$

$$\sqrt{1+x^2} \approx x \quad , \quad \frac{1}{x\sqrt{1+x^2}} \approx \frac{1}{x^2} \quad , \quad \int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty}+1 = 1$$

$$\Downarrow$$

$\int_1^\infty \frac{1}{x\sqrt{1+x^2}}$ "like" $\int_1^\infty \frac{1}{x^2}\,dx < \infty \Rightarrow$ convergence

c) $\int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}}\,dx$

$$\frac{\ln x}{x\sqrt{x^2-1}} \approx \frac{\ln x}{x^2} < \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}} \qquad \alpha\in(0,1)$$

$\Rightarrow 2-\alpha > 1 \Rightarrow \int_1^\infty \frac{1}{x^{2-\alpha}}\,dx$ converges

$\Rightarrow \int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}} < \int_1^\infty \frac{1}{x^{2-\alpha}} < \infty \Rightarrow$ conv

b) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx$

$$\frac{1}{\cos x} \approx \frac{1}{\left(\frac{\pi}{2}-x\right)^p} \quad \text{for some } \boxed{p=1}$$

$$\frac{1}{\cos x} = \frac{1}{\sin\left(\frac{\pi}{2}-x\right)} \approx \frac{1}{\frac{\pi}{2}-x} \quad \text{as } x \nearrow \frac{\pi}{2}$$

$$\lim_{x \nearrow \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x} = 1$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx \text{ "like" } \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\,dx = +\infty \Rightarrow \text{div.}$$

Integral Test: $f : [a,\infty) \to [0,\infty)$ decreasing

$\int_a^\infty f(x)\,dx$ has the same nature as $\sum_{n=[a]}^\infty f(n)$

for $a=1 \Rightarrow \int_1^\infty f(x)\,dx$ "like" $\sum_{n=1}^\infty f(n)$

5. a) $\sum_{n\geq 1} \frac{1}{n^p}$, $p>0$

$$\sum_{n\geq 1} \frac{1}{n^p} \text{ "like" } \int_1^\infty \frac{1}{x^p}\,dx$$

$$\begin{cases} -\text{ conv. for } p>1 \\ -\text{ div for } p\leq 1 \end{cases}$$

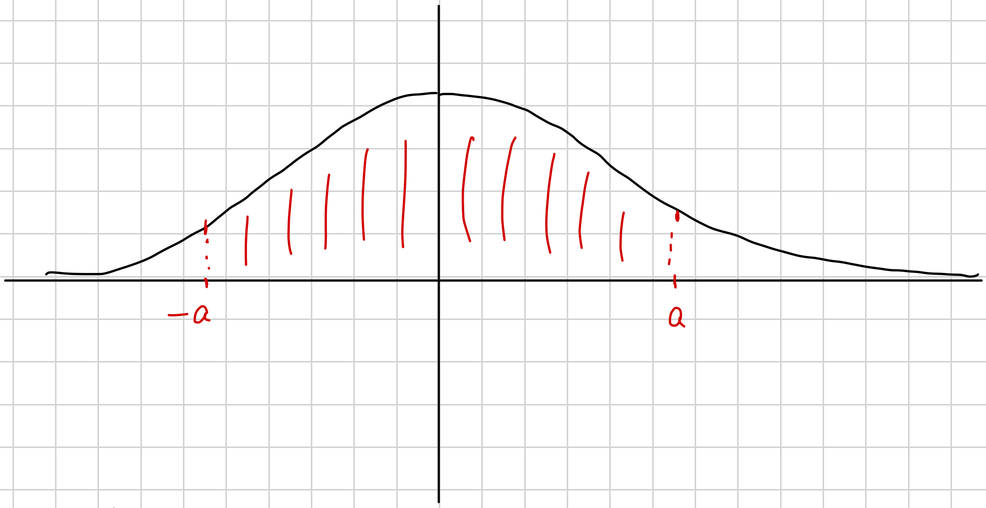
b) $\sum_{n\geq 2} \frac{1}{n\ln^2 n}$ "like" $\int_2^\infty \frac{1}{x\ln^2 x}\,dx \overset{t=\ln x}{=\!=} \int_{\ln 2}^\infty \frac{1}{t^2}\,dt$

$$= -\frac{1}{t}\Big|_{\ln 2}^\infty = \frac{1}{\ln 2} < \infty \Rightarrow \text{conv.}$$

c) $$\int_2^\infty \frac{\ln x}{x^2}\,dx = \int_2^\infty \ln x\left(-\frac{1}{x}\right)'dx = \underbrace{\ln x\left(-\frac{1}{x}\right)\Big|_2^\infty}_{=\frac{\ln 2}{2}} + \underbrace{\int_2^\infty \frac{1}{x^2}\,dx}_{<\infty}$$

$< \infty \Rightarrow$ conv.

6. $\int_{-\infty}^{\infty} e^{-x^2}\,dx$



$\int e^{-x^2}\,dx$ is impossible to compute with elementary functions

$\rightarrow$ approximate with the trapezium rule

take a lot of points a