Seminar 12

1. (*i*) Which of the following received words contain detectable errors when using the (3,2)-parity check code: 110, 010, 001, 111, 101, 000?

(*ii*) Decode the following words using the (3,1)-repeating code to correct errors: 111, 011, 101, 010, 000, 001. Which of them contain detectable errors?

2. Are $1 + X^3 + X^4 + X^6 + X^7$ and $X + X^2 + X^3 + X^6$ code words in the polynomial (8,4)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$?

3. Write down all the words in the (6,3)-code generated by $p = 1 + X^2 + X^3 \in \mathbb{Z}_2[X]$.

4. A code is defined by the generator matrix $G = \left(\frac{P}{I_3}\right) \in M_{5,3}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Write down the parity check matrix and all the code words.

5. Determine the minimum Hamming distance between the code words of the code with generator matrix $G = \left(\frac{P}{I_4}\right) \in M_{9,4}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Discuss the error-detecting and error-correcting capabilities of this code, and write down the parity check matrix.

6. Encode the following messages using the generator matrix of the (9,4)-code of Exercise **5.**: 1101, 0111, 0000, 1000.

Determine the generator matrix and the parity check matrix for:

7. The (4,1)-code generated by $p = 1 + X + X^2 + X^3 \in \mathbb{Z}_2[X]$.

8. The (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.

n-k bits	k bits
check digits	message

$\gamma : \mathbb{Z}_2^k \longrightarrow \mathbb{Z}_2^n$ encoder

$\mathbb{Z}_2 = \{0,1\}$

Linear code:

$\gamma : \mathbb{Z}_2^k \longrightarrow \mathbb{Z}_2^n$
linear

$$[\gamma]_{E,E'} = \left([\gamma(e_1)]_{E'} \cdots [\gamma(e_k)]_{E'} \right)$$

$$=: G \text{ generator matrix}$$

↳ use it to encode

$$[\gamma(m)]_{E'} = G \cdot [m]_E$$

$$G = \begin{pmatrix} P \\ \hline I_k \end{pmatrix} \in M_{n,k}(\mathbb{Z}_2)$$

$$H = (I_{n-k} \mid P)$$

↳ parity check matrix

$$v \in \mathcal{C} \iff H \cdot [v]_E = 0$$

$$\mathcal{C} = \operatorname{Im} \gamma \leq \mathbb{Z}_2^n$$

$d_H(v, v')$ = nr of positi
that v and v'
disagree on

↓

Humming distance

$= w(v - v')$ = nr of
1's in $v - v'$

ex: $d_H(\underline{1}10\underline{1}\underline{0}, \underline{0}10\underline{0}\underline{1}) = 3$

$= w(10011) = 3$

$$d(\mathcal{C}) = \min_{\in \mathcal{C}} d_H(v, v')$$

We can detect at most $d(\mathcal{C})-1$ errors and we can correct at most $\left\lfloor \frac{d(\mathcal{C})-1}{2} \right\rfloor$ errors

$d(G) =$ min nr of columns in H that add up to 0

Write down the parity check matrix and all the code words.

5. Determine the minimum Hamming distance between the code words of the code with generator matrix $G = \left(\frac{P}{I_4}\right) \in M_{9,4}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Discuss the error-detecting and error-correcting capabilities of this code, and write down the parity check matrix.

6. Encode the following messages using the generator matrix of the (9,4)-code of Exercise **5.**: 1101, 0111, 0000, 1000.

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H = \left(\begin{array}{ccccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$\nexists$ a 0 column $\Rightarrow d(G) > 1$

$\nexists$ identical column $\Rightarrow d(G) > 2$

$$C_1 + C_3 + C_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow d(G) = 3$$

We can detect at most 2 errors

We can correct at most 1 error

encoding:

$$[\gamma(m)]_{E'} = G \cdot [m]_E$$

$$G\,[(1101)]_E = G \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$G\left[(0000)\right]_E = G\cdot\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$G\left[(0111)\right]_E = G\cdot\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\1\\1\\0\\1\\1\\1\end{pmatrix}$$

$$G\left[(1000)\right]_E = G\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\\1\\1\\0\\0\\0\end{pmatrix}$$

(n,k) polynomial code generated by $p \in \mathbb{Z}_2[x]$

$m = \overline{a_0 a_1 \ldots a_k ..}$

message

ex: $n=5$ $k=3$ $p = x^2+1$

$m = 101$

If $\deg p = n-k$, then the code is <u>linear</u>

Step 1: Encode m as a polynomial

$$\boxed{m = \overline{a_0 \ldots a_k} \rightsquigarrow f_m = a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}}$$

$m = 101 \rightsquigarrow f_m = 1+x^2$

Step 2: Multiply f_m by X^{n-k}

$$\boxed{F_m = f_m \cdot X^{n-k}}$$

$$F_m = (1+X^2)\,X^2 = X^4 + X^2$$

Step 3: Divide F_m by P (Euclidian division)

$$\begin{array}{r|l} X^4 + X^2 & X^2+1 \\ \cline{2-2} -X^4 - X^2 & X^2 \\ \cline{1-1} 0 & \end{array}$$

$$R_m = 0$$

Step 4: Compute $g_m = F_m + R_m$

$$g_m = X^4 + X^2 + 0$$
$$= X^4 + X^2$$

Step 5: Convert the poly to a vector

$$g_m = X^2 + X^4 \rightsquigarrow v = 00\underbrace{101}_{m}$$

8. The (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.

Find G and H, $d(\mathcal{C})$ and discuss the capability

$e_1 = 100$
$e_2 = 010$
$e_3 = 001$

- $m = 100$

$$f_m = 1 \qquad F_m = X^4$$

$$\begin{array}{r|l} X^4 & X^4+X^3+X^2+1 \\ \cline{2-2} -X^4-X^3-X^2-1 & 1 \\ \cline{1-1} X^3+X^2+1 & \end{array}$$

$$R_m = X^3 + X^2 + 1$$

$$g_u = x^4 + x^3 + x^2 + 1 \rightsquigarrow v_1 = 1011100$$

- $u = 010 \rightsquigarrow f_u = x \qquad F_u = x^5$

$$\begin{array}{r|l} x^5 & x^4+x^3+x^2+1 \\ \hline -x^5-x^4-x^3 \quad -x & x+1 \\ \hline x^4+x^3 \quad +x & \\ -x^4-x^3-x^2-1 & \\ \hline x^2+x+1 & \end{array}$$

$$\Rightarrow R_u = x^2+x+1$$

$$\Rightarrow g_u = x^5 + x^2 + x + 1 \rightsquigarrow v_2 = 1110010$$

- $u = 001 \rightsquigarrow f_u = x^2 \qquad F_u = x^6$

$$\begin{array}{r|l} x^6 & x^4+x^3+x^2+1 \\ \hline -x^6+x^5+\cancel{x}^4+x^2 & x^2+x \\ \hline x^5+x^4+x^2 & \\ -x^5-x^4-x^3-x & \\ \hline x^3+x^2+x & \end{array}$$

$$R_u = x^3+x^2$$

$$\Rightarrow g_u = x^6 + x^3 + x^2 + x \rightsquigarrow v_3 = 0111001$$

$$\Rightarrow G = (v_1 \mid v_2 \mid v_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} P \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} I_3 \end{matrix}$$

$$\Rightarrow P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I_4} & \begin{matrix} \vdots \\ \vdots \end{matrix} & \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{matrix} \end{pmatrix}$$

$\nexists$ a zero column $\Rightarrow d(G) > 1$

there are no id. columns $\Rightarrow d(G) > 2$

there are no 2 columns that add up to a third $\Rightarrow d(G) > 3$

$C_1 + C_3 + C_4 + C_5 = 0 \Rightarrow d(\mathcal{C}) \leq 4$

$\Rightarrow d(\mathcal{C}) = 4$

$\Rightarrow$ we can detect 3 error
we can correct 1 errors