Seminar 11

1. Find the second-order Taylor polynomial for the following functions at the given points:

   (a) $f(x, y) = \sin(x + 2y)$ at $(0, 0)$.

   (b) $f(x, y) = e^{x+y}$ at $(0, 0)$ and $(1, -1)$.

   (c) $f(x, y) = \sin(x)\sin(y)$ at $(\pi/2, \pi/2)$.

   (d) $f(x, y) = e^{-(x^2+y^2)}$ at $(0, 0)$.

2. Compute the Hessian matrix and its eigenvalues for the following:

   (a) $f(x, y) = (y - 1)e^x + (x - 1)e^y$ at $(0, 0)$.

   (b) $f(x, y) = \sin(x)\cos(y)$ at $(\pi/2, 0)$.

3. Find and classify the critical points for each of the following functions:

   (a) $f(x, y) = x^3 - 3x + y^2$.

   (b) $f(x, y) = x^3 + y^3 - 6xy$.

   (c) $f(x, y) = x^4 + y^4 - 4(x - y)^2$.

   (d) $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

4. Let A be a symmetric $n \times n$ matrix and the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Ax$. Prove that $\nabla f(x) = Ax$ and $H(x) = A$. *Hint: use the Taylor expansion.*

5. Let A be an $m \times n$ matrix, b a vector in R^m and the least squares minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2.$$

   Prove that the solution x^* of this problem satisfies (the so-called normal equations)

$$A^T Ax^* = A^T b.$$

6. ★[Python] Let A be a 2×2 matrix and let the quadratic function $f : \mathbb{R}^2 \to R$, $f(x) = \frac{1}{2}x^T Ax$.

   (a) Give a matrix A such that f has a unique minimum.

   (b) Give a matrix A such that f has a unique maximum.

   (c) Give a matrix A such that f has a unique saddle point.

   In each case plot the 3d surface, three contour lines and the gradient at three different points.

Homework questions are marked with ★.

1. a) $f(x,y) = \sin(x+2y)$ at $(0,0)$

$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0)$$
$$+ \frac{1}{2}(x-x_0, y-y_0)\, H(x_0,y_0) \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$f(0,0) = 0$

$\nabla f(x,y) = (1,2)$

$$\frac{\partial f}{\partial x} = \cos(x+2y) \qquad \frac{\partial f}{\partial y} = 2\cos(x+2y)$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+2y) \qquad \frac{\partial^2 f}{\partial y^2} = -4\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2\sin(x+2y)$$

$$H(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T_2(0,0) = 0 + (1,2)(x,y) = x+2y$$

$t = x+2y \qquad \sin t = t - \frac{t^3}{3!} + \dots, \quad T_2(t) = t$

b) $f(x,y) = e^{x+y}$ at $(0,0)$, $(1,-1)$

$$t = x+y \qquad e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

$$T_2(t) = 1 + t + \frac{t^2}{2}$$

$$T_2(x,y) = 1 + x + y + \frac{(x+y)^2}{2}$$

$f(0,0) = e^0 = 1$

$f(1,-1) = e^0 = 1$

$\nabla f(0,0) = (1,1)$ $\qquad$ $\nabla f(1,-1) = (1,1)$

$\frac{\partial f}{\partial x} = e^{x+y}$ $\qquad$ $\frac{\partial f}{\partial y} = e^{x+y}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = e^{x+y}$$

$H(0,0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$H(1,-1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$T_2(0,0) = 1 + (1,1)(x,y) + \frac{1}{2}(x,y)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + x + y + \frac{1}{2}(x,y) \cdot \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

$$= 1 + x + y + \frac{1}{2}(x,y) \cdot (x+y,\ x+y)$$

$$= 1 + x + y + \frac{1}{2}(x(x+y) + y(x+y))$$

$$= 1 + x + y + \frac{(x+y)^2}{2}$$

$$T_2(1,-1) = 1 + (1,1)(x-1,\ y+1) + \frac{1}{2}(x-1, y+1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y+1 \end{pmatrix}$$

$$= 1 + x - 1 + y + 1 + \frac{1}{2}(x-1, y+1)\begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

$$= 1 + x + y + \frac{(x+y)^2}{2}$$

c) $f(x,y) = \sin x \sin y$ at $\left(+\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$= \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\cos t = \boxed{1 - \frac{t^2}{2!}} + \frac{t^4}{4!} - \dots \quad , \forall t \in \mathbb{R}$$

$t = x-y \Rightarrow \cos(x-y) = 1 - \frac{(x-y)^2}{2} + \dots$

$t = x+y \Rightarrow \cos(x+y) = 1 - \frac{(x+y)^2}{2} + \dots$

$$T_2(x,y) = \frac{1}{2}\left[1 - \frac{(x-y)^2}{2} - 1 + \frac{(x+y)^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{(\cancel{x}+y-\cancel{x}+y)(x+\cancel{y}+x-\cancel{y})}{2}\right]$$

$$= x \cdot y$$

$$f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \sin\left(+\frac{\pi}{2}\right)\sin\frac{\pi}{2}$$

$$= +1$$

$$\nabla f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (0,0)$$

$\frac{\partial f}{\partial x} = \cos x \sin y$ $\qquad$ $\frac{\partial f}{\partial y} = \sin x \cos y$

d) $f(x,y) = e^{-(x^2+y^2)}$ at $(0,0)$

$f(0,0) = e^0 = 1$

$\nabla f(0,0) = (0,0)$

$$\frac{\delta f}{dx} = e^{-(x^2+y^2)} \cdot (-2x) \qquad \frac{\delta f}{dy} = e^{-(x^2+y^2)} (-2y)$$

$$\frac{\delta^2 f}{dx^2} = (-2) \cdot e^{-(x^2+y^2)} + (-2x) \cdot e^{-(x^2+y^2)} \cdot (-2x)$$

$$= -2e^{-(x^2+y^2)} + 4x^2 \cdot e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)} (4x^2-2)$$

$$\frac{\delta^2 f}{dy^2} = e^{-(x^2+y^2)} (4y^2-2)$$

$$\frac{\delta^2 f}{dx\,dy} = \frac{\delta^2 f}{dy\,dx} = e^{-(x^2+y^2)} (-2x)(-2y)$$

$$= 4xy \cdot e^{-(x^2+y^2)}$$

$$H(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$T_2(x,y) = 1 + \frac{1}{2}(x,y) \cdot \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + \frac{1}{2}(x,y) \cdot \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

$$= 1 + \frac{1}{2}(-2x^2 - 2y^2)$$

$$= 1 - x^2 - y^2$$

(a) $f(x, y) = (y-1)e^x + (x-1)e^y$ at $(0, 0)$.

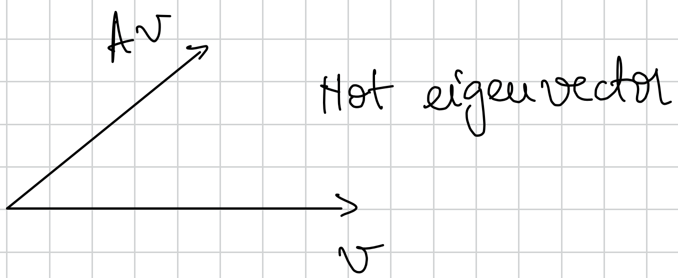
$$\frac{\partial f}{\partial x} = (y-1)e^x + e^y \qquad \frac{\partial^2 f}{\partial x^2} = (y-1)e^x$$

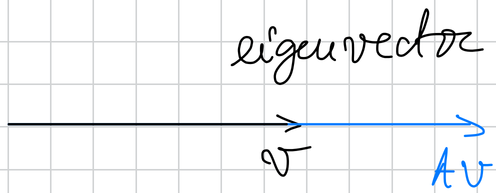
$$\frac{\partial f}{\partial y} = e^x + (x-1)e^y \qquad \frac{\partial^2 f}{\partial y^2} = (x-1)e^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = e^x + e^y$$

$$H(0,0) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

eigenvalues $\qquad A \cdot v = \lambda v$





$$(A - \lambda I)v = 0 \quad , v \neq 0$$

$$\det(A - \lambda I) = 0$$

$$H - \lambda I = \begin{pmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} \Rightarrow \det = (-1-\lambda)^2 - 4$$
$$= \lambda^2 + 2\lambda - 3$$

$$\det(H - \lambda I) = 0$$

$$\lambda^2 + 2\lambda - 3 = 0$$

$$\Delta = 4 + 12 = 16 \qquad \lambda_{1,2} = \frac{-2 \pm 4}{2} \begin{cases} \lambda_1 = -3 \\ \lambda_2 = 1 \end{cases}$$ eigenvalues

3. a) $f(x,y) = x^3 - 3x + y^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 3 \qquad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

$$\nabla f(x,y) = (3x^2 - 3, 2y)$$

$$\nabla f(x,y) = (0,0) \iff \begin{cases} x = \pm 1 \\ y = 0 \end{cases}$$

$$H(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \qquad H(-1,0) = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda_1 = 6 \quad \lambda_2 = 2 > 0$ $\qquad$ $\lambda_1 = -6 \quad \lambda_2 = 2$

$\Rightarrow (1,0)$ local min. $\qquad$ $\Rightarrow (-1,0)$ saddle point

d) $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$

$$\frac{\partial f}{\partial x} = 2x - y + 1 \qquad \frac{\partial f}{\partial y} = 2y - x \qquad \frac{\partial f}{\partial z} = 2z - 2$$

$$\nabla f(x,y,z) = (0,0,0) \iff \begin{cases} 2x - y + 1 = 0 \\ -x + 2y = 0 \\ z = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x - y = -1 \\ -2x + 4y = 0 \\ z = 1 \end{cases} \quad)(+) \quad \Leftrightarrow \begin{cases} 2x - y = -1 \\ 3y = -1 \\ z = 1 \end{cases} \quad \Leftrightarrow \begin{cases} 2x + \frac{1}{3} = -1 \\ y = -\frac{1}{3} \\ z = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -\frac{2}{3} \\ y = -\frac{1}{3} \\ z = 1 \end{cases}$$

$$\frac{\partial^2 f}{dx^2} = 2 \qquad \frac{\partial^2 f}{dy^2} = 2 \qquad \frac{\partial^2 f}{dz^2} = 2$$

$$\frac{\partial^2 f}{dx dy} = \frac{\partial^2 f}{dy dx} = -1 \qquad \frac{\partial^2 f}{dx dz} = \frac{\partial^2 f}{dz dx} = 0$$

$$\frac{\partial^2 f}{dy dz} = \frac{\partial^2 f}{dz dy} = 0$$

$$H\left(-\frac{2}{3}, -\frac{1}{3}, 1\right) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(H - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)^3 - (2-\lambda) = 0$$

$$(2-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$$\lambda_1 = 2 > 0 \qquad \lambda^2 - 4\lambda + 3 = 0$$

$$\delta = 16 - 12 = 4$$

$$\lambda_{2,3} = \frac{4 \pm 2}{2} \quad \begin{cases} \lambda_2 = 3 > 0 \\ \lambda_3 = 1 > 0 \end{cases}$$

$\Rightarrow H(x,y,z)$ is positive def. $\Rightarrow \left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ local minimum

4. $$\boxed{f(x+h) = f(x) + \nabla f(x) \cdot h + \frac{1}{2} h^T H(x) \circ h + \ldots}$$

Taylor expansion

$$f(x) = \frac{1}{2} x^T \cdot Ax$$

<u>Method 1</u>: $\frac{\delta f}{dx_1} = ?$

<u>Method 2</u>: $f(x+h) = \frac{1}{2} (x+h)^T \cdot A(x+h)$

$$= \frac{1}{2} x^T \cdot A(x) + \frac{1}{2} x^T \cdot A(h) + \frac{1}{2} h^T A(x) + \frac{1}{2} h^T A(h)$$

$$x^T \cdot Ah = \langle x, Ah \rangle = x \circ (Ah)$$

$$= \langle Ah, x \rangle \qquad \langle a, b \rangle = a^T b$$

$$= Ah^T \cdot x \qquad (A \cdot B)^T = A^T \cdot B^T$$

$$= h^T \cdot A^T x$$

$$= h^T \cdot Ax$$

$$= \langle h, Ax \rangle$$

$$f(x+h) = f(x) + \underbrace{Ax}_{\nabla f(x)} \cdot h + \frac{1}{2} h^T \underbrace{A}_{H(x)} h$$

5.

Let A be an $m \times n$ matrix, b a vector in R^m and the least squares minimization problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|^2.$$

Prove that the solution x^* of this problem satisfies (the so-called normal equations)

$$A^T A x^* = A^T b.$$

$$f(x) = \| Ax - b \|^2 = \langle Ax-b, Ax-b \rangle$$

$$f(x) \longrightarrow \min, \quad \nabla f(x) = 0$$

$$f(x) = \langle Ax, Ax \rangle - 2 \langle Ax, b \rangle + \langle b, b \rangle$$

$$\langle Ax, Ax \rangle = (Ax)^T \cdot Ax = A^T \cdot x^T \cdot A \cdot x$$

$$= x^T (A^T \cdot A) \cdot x$$

$$f(x) = x^T (A^T \cdot A) x - 2 \underbrace{b^T A x}_{\langle x, A^T b \rangle = x \cdot (A^T b)} + \|b\|^2$$

$$\nabla f(x) = 2 A^T A x - 2 A^T b = 0$$

$$\Rightarrow \quad A^T A x - A^T b = 0$$

$$A^T A x = A^T b$$