Name
Date
Course

Solving Polynomial Equations When the Degree Is Greater than Two

Polynomial Equation: An equation that sets a polynomial equal to zero.

Examples: $x^3+4x^2-7x+8=0$, $0=-9x^2+x-2$

Three Names for the Solutions of a Polynomial Equation

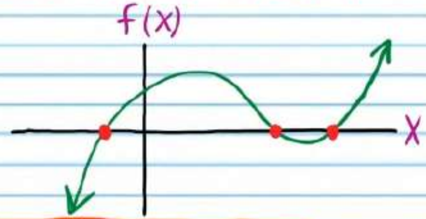
I) Zeros

II) Roots

III) X-Intercepts (unless the solution is complex)

Example:

$f(x)=x^3-8x^2+11x+20$



The Rational Zeros Theorem

If a polynomial equation has rational solutions, then the solutions can be found by dividing the factors of the last term by the factors of the coefficient of the first term. Positive and negative numbers must be considered. This theorem applies only when the coefficients of the polynomial are integers (i.e. whole numbers).

Example: $8x^4-3x^3+4x^2-10x+21=0$

Use the Rational Zeros Theorem to find all of the potential rational solutions of each polynomial equation.

① $0=4x^3-x^2+4x-10$

$p: 10 \Longrightarrow 1, 2, 5, 10$

$q: 4 \Longrightarrow 1, 2, 4$

$\frac{p}{q}: \boxed{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{4}, \pm 10}$

② $5x^4+7x^3-9x^2+16x+17=0$

$p: 17 \Longrightarrow 1, 17$

$q: 5 \Longrightarrow 1, 5$

$\frac{p}{q}: \boxed{\pm 1, \pm\frac{1}{5}, \pm 17, \pm\frac{17}{5}}$

③ $0=9x^5+2x^4-x-48$

$p: 48 \Longrightarrow 1, 2, 3, 4, 6, 8, 12, 16, 24, 48$

$q: 9 \Longrightarrow 1, 3, 9$

$\frac{p}{q}: \boxed{\pm 1, \pm\frac{1}{3}, \pm\frac{1}{9}, \pm 2, \pm\frac{2}{3}, \pm\frac{2}{9}, \pm 3, \pm 4, \pm\frac{4}{3}, \pm\frac{4}{9}, \pm 6, \pm 8, \pm\frac{8}{3}, \pm\frac{8}{9}, \pm 12, \pm 16, \pm\frac{16}{3}, \pm\frac{16}{9}, \pm 24, \pm 48}$

④ $2x^6+19x^5+2x^4+15x^3-8x^2+4x-12=0$

$p: 12 \Longrightarrow 1, 2, 3, 4, 6, 12$

$q: 2 \Longrightarrow 1, 2$

$\frac{p}{q}$: $\boxed{\pm 1, \pm \frac{1}{2}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm 4, \pm 6, \pm 12}$

⑤ $0=8x^3+7x^2+x+7$

$p: 7 \Longrightarrow 1, 7$

$q: 8 \Longrightarrow 1, 2, 4, 8$

$\frac{p}{q}$: $\boxed{\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \pm 7, \pm \frac{7}{2}, \pm \frac{7}{4}, \pm \frac{7}{8}}$

⑥ $x^4-x^3-x^2-x+50=0$

$p: 50 \Longrightarrow 1, 2, 5, 10, 25, 50$

$q: 1 \Longrightarrow 1$

$\frac{p}{q}$: $\boxed{\pm 1, \pm 2, \pm 5, \pm 10, \pm 25, \pm 50}$

⑦ Use the Rational Zeros Theorem to Solve the equation below.

$3x^4-16x^3+11x^2+22x-8=0$

$p: 8 \Longrightarrow 1, 2, 4, 8$

$q: 3 \Longrightarrow 1, 3$

$\frac{p}{q}$: $\pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 4, \pm \frac{4}{3}, \pm 8, \pm \frac{8}{3}$

$x=1 \Longrightarrow 3(1)^4-16(1)^3+11(1)^2+22(1)-8=0$

$3(1)-16(1)+11(1)+22-8=0$

$3-16+11+22-8=0$

$12=0$

X

$x=-1 \Longrightarrow 3(-1)^4-16(-1)^3+11(-1)^2+22(-1)-8=0$

$3(1)-16(-1)+11(1)-22-8=0$

$3+16+11-22-8=0$

$0=0$

✓

The Polynomial Degree Zeros Theorem: Every polynomial with degree n has exactly n zeros.
* The zeros can be real, complex, distinct, or repeated.

The Polynomial Degree Factor Theorem: Every polynomial with degree n can be factored into n factors of the form $x - c_i$, where c_i can be real, complex, distinct, or repeated.

Example: $5x^{3} - 36x^2 + 28x + 48 = 5\overset{1}{(x+\tfrac{4}{5})}\overset{2}{(x-2)}\overset{3}{(x-6)}$

The Factor Theorem: Let f be a polynomial function. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

$$\frac{3x^4 - 16x^3 + 11x^2 + 22x - 8}{x+1}$$

$$x+1 \enclose{longdiv}{3x^4 - 16x^3 + 11x^2 + 22x - 8}$$

$3x^4 - 16x^3 + 11x^2 + 22x - 8$

$$\begin{array}{r|rrrrr} -1 & 3 & -16 & 11 & 22 & -8 \\ & & -3 & 19 & -30 & 8 \\ \hline & 3 & -19 & 30 & -8 & 0 \end{array}$$

$3x^3 - 19x^2 + 30x - 8$

Therefore,

$$3x^4 - 16x^3 + 11x^2 + 22x - 8 = (x+1)(3x^3 - 19x^2 + 30x - 8)$$

The Remainder Theorem: Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Repeated zero(s)?

$x = -1 \implies 3(-1)^3 - 19(-1)^2 + 30(-1) - 8 = 0$

$3(-1) - 19(1) - 30 - 8 = 0$

$-3 - 19 - 30 - 8 = 0$

X

$x = 2 \implies 3(2)^3 - 19(2)^2 + 30(2) - 8 = 0$

$3(8) - 19(4) + 60 - 8 = 0$

$0 = 0$ ✓

$$\begin{array}{r|rrrr} 2 & 3 & -19 & 30 & -8 \\ & & 6 & -26 & 8 \\ \hline & 3 & -13 & 4 & 0 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$3x^2-13x+4$$

Factor Theorem $\Rightarrow$ If $f(2)=0$, then $x-2$ is a factor of $f(x)$.

Therefore,

$$3x^4-16x^3+11x^2+22x-8 = (x+1)(x-2)(3x^2-13x+4)$$
$$= (x+1)(x-2)(3x-1)(x-4)$$

Solutions of the Original Equation:

$$3x^4-16x^3+11x^2+22x-8=0$$
$$(x+1)(x-2)(3x-1)(x-4)=0$$
$$\boxed{x=-1,2,\tfrac{1}{3},4}$$

⑧ Use the Rational Zeros Theorem to find the roots of the polynomial function below.

$$f(x)=2x^3-11x^2+12x+9$$

$p: 9 \Rightarrow 1,3,9$

$q: 2 \Rightarrow 1,2$

$\frac{p}{q}: \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}, \pm 9, \pm \frac{9}{2}$

$x=1 \Rightarrow 2(1)^3-11(1)^2+12(1)+9=0$
$2(1)-11(1)+12+9=0$
$2-11+12+9=0$
$12=0$
X

$x=-1 \Rightarrow 2(-1)^3-11(-1)^2+12(-1)+9=0$
$2(-1)-11(1)-12+9=0$
$-2+11-12+9=0$
$6=0$
X

$x=3 \Rightarrow 2(3)^3-11(3)^2+12(3)+9=0$

$-8)$

$2(27)-11(9)+36+9=0$

$54-99+36+9=0$

$0=0$ ✓

Factor Theorem $\Rightarrow$ If $f(3)=0$, then $x-3$ is a factor of $f(x)$.

$\dfrac{2x^3-11x^2+12x+9}{x-3}$

$x-3\,\overline{)\,2x^3-11x^2+12x+9}$

$$\begin{array}{r|rrrr} 3 & 2 & -11 & 12 & 9 \\ & & 6 & -15 & -9 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

$\downarrow \quad \downarrow \quad \downarrow$

$2x^2-5x-3$

Therefore

$2x^3-11x^2+12x+9 = (x-3)(2x^2-5x-3)$

$= (x-3)(2x+1)(x-3)$

Roots: $\boxed{x=3, -\frac{1}{2}, 3}$

The Multiplicity of a Zero: If a zero is repeated m times, then the multiplicity of the zero is defined to be m.

Example: $f(x)=(x-7)^2(x+2)^3(x-4)$

7 is a zero of multiplicity 2.

-2 is a zero of multiplicity 3.

4 is a zero of multiplicity 1.

⑨ Use the Rational Zeros Theorem to find the zeros of the polynomial function below and write the multiplicity of each zero.

$f(x)=x^5-4x^4-12x^3-2x^2+11x+6$

$p: 6 \Rightarrow 1, 2, 3, 6$

$q: 1 \Rightarrow 1$

$\frac{p}{q}: \pm1, \pm2, \pm3, \pm6$

$x=1 \Rightarrow (1)^5-4(1)^4-12(1)^3-2(1)^2+11(1)+6=0$

$1-4-12-2+11+6=0$

$0=0$ ✓

$$\begin{array}{r|rrrrrr} 1 & 1 & -4 & -12 & -2 & 11 & 6 \\ & & 1 & -3 & -15 & -17 & -6 \\ \hline & 1 & -3 & -15 & -17 & -6 & 0 \end{array}$$

$x^4-3x^3-15x^2-17x-6$

Repeated zero(s)? $x=1 \Rightarrow$ ✗

the

$x = -1 \Rightarrow$ ✓

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & -15 & -17 & -6 \\ & & -1 & 4 & 11 & 6 \\ \hline & 1 & -4 & -11 & -6 & 0 \end{array}$$

$x^3 - 4x^2 - 11x - 6$

Repeated zero(s)? $x = -1 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} -1 & 1 & -4 & -11 & -6 \\ & & -1 & 5 & 6 \\ \hline & 1 & -5 & -6 & 0 \end{array}$$

$x^2 - 5x - 6$

Therefore,

$$x^5 - 4x^4 - 12x^3 - 2x^2 + 11x + 6 = (x-1)(x+1)(x+1)(x^2-5x-6)$$
$$= (x-1)(x+1)(x+1)(x-6)(x+1)$$
$$= (x-1)(x+1)^3(x-6)$$

Zeros: $\boxed{1, -1, -1, -1, 6}$

1 is a zero of multiplicity 1.
-1 is a zero of multiplicity 3.
6 is a zero of multiplicity 1.

⑩ Use the Rational Zeros Theorem to find the x-intercepts of the function below, and write the multiplicity of each zero.

$f(x) = 5x^3 + 2x^2 - 63x + 36$

$p: 36 \Rightarrow 1, 2, 3, 4, 6, 9, 12, 18, 36$

$q: 5 \Rightarrow 1, 5$

$\frac{p}{q}: \pm 1, \pm \frac{1}{5}, \pm 2, \pm \frac{2}{5}, \pm 3, \pm \frac{3}{5} \rightsquigarrow$

$x = 1 \Rightarrow$ ✗

$x = -1 \Rightarrow$ ✗

$x = 2 \Rightarrow$ ✗

$x = -2 \Rightarrow$ ✗

$x = 3 =$ ✓

$$\begin{array}{r|rrrr} 3 & 5 & 2 & -63 & 36 \\ & & 15 & 51 & -36 \\ \hline & 5 & 17 & -12 & 0 \end{array}$$

$5x^2 + 17x - 12$

Therefore,

$$5x^3 + 2x^2 - 63x + 36 = (x-3)(5x^2+17x-12)$$
$$= (x-3)(5x-3)(x+4)$$

x-Intercepts: $\boxed{3, \frac{3}{5}, -4}$

3 is a zero of multiplicity 1.
$\frac{3}{5}$ is a zero of multiplicity 1.
-4 is a zero of multiplicity 1.

the zeros
the

⑪ Use the Rational Zeros Theorem to Solve the equation below.

- $4x^3-31x^2+61x-30=0$

$p: 30 \Rightarrow 1, 2, 3, 5, 6, 10, 15, 30$

$q: 4 \Rightarrow 1, 2, 4$

$\frac{p}{q}: \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}$

$x=1 \Rightarrow$ X $\quad x=-1 \Rightarrow$ X $\quad x=2 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} 2 & 4 & -31 & 61 & -30 \\ & & 8 & -46 & 30 \\ \hline & 4 & -23 & 15 & 0 \end{array}$$

$4x^2-23x+15$

Therefore,

$4x^3-31x^2+61x-30 = (x-2)(4x^2-23x+15)$

$= (x-2)(4x-3)(x-5)$

Solutions: $\boxed{x=2, \frac{3}{4}, 5}$

⑫ Use the Rational Zeros Theorem to find the roots of the polynomial function below.

$f(x)=3x^4+5x^3-25x^2-45x-18$

$p: 18 \Rightarrow 1, 2, 3, 6, 9, 18$

$q: 3 \Rightarrow 1, 3$

$\frac{p}{q}: \pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 3$

$x=1 \Rightarrow$ X

$x=-1 \Rightarrow$ ✓

$$\begin{array}{r|rrrrr} -1 & 3 & 5 & -25 & -45 & -18 \\ & & -3 & -2 & 27 & 18 \\ \hline & 3 & 2 & -27 & -18 & 0 \end{array}$$

$3x^3+2x^2-27x-18$

Repeated zero(s):

$x=-1 \Rightarrow$ X $\quad x=2 \Rightarrow$ X $\quad x=-2 \Rightarrow$ X

$x=3 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} 3 & 3 & 2 & -27 & -18 \\ & & 9 & 33 & 18 \\ \hline & 3 & 11 & 6 & 0 \end{array}$$

$3x^2+11x+6$

nd the
write the

$$\begin{array}{cc} 63 & 36 \\ 51 & -36 \\ \hline -12 & 0 \end{array}$$

-12

2)

4)

Therefore,

$$3x^4+5x^3-25x^2-45x-18 = (x+1)(x-3)(3x^2+11x+6)$$
$$= (x+1)(x-3)(3x+2)(x+3)$$

Roots: $\boxed{-1,\ 3,\ -\frac{2}{3},\ -3}$

(13) Use the Rational Zeros Theorem to find the zeros of the polynomial function below and write the multiplicity of each zero.

$f(x) = 2x^5 - x^4 - 7x^3 - x^2 + 5x + 2$

$p: 2 \Rightarrow 1, 2$

$q: 2 \Rightarrow 1, 2$

$\frac{p}{q}: \pm 1, \pm \frac{1}{2}, \pm 2$

$x = 1 \Rightarrow$ ✓

$$\begin{array}{r|rrrrrr} 1 & 2 & -1 & -7 & -1 & 5 & 2 \\ & & 2 & 1 & -6 & -7 & -2 \\ \hline & 2 & 1 & -6 & -7 & -2 & 0 \end{array}$$

$2x^4 + x^3 - 6x^2 - 7x - 2$

Repeated zero(s)?: $x = 1 \Rightarrow$ ✗

$x = -1 \Rightarrow$ ✓

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -6 & -7 & -2 \\ & & -2 & 1 & 5 & 2 \\ \hline & 2 & -1 & -5 & -2 & 0 \end{array}$$

$2x^3 - x^2 - 5x - 2$

$2x^3-x^2-5x-2$

Repeated zero(s)?:

$x=-1 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} -1 & 2 & -1 & -5 & -2 \\ & & -2 & 3 & 2 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

$2x^2-3x-2$

Therefore,

$$2x^5-x^4-7x^3-x^2+5x+2 = (x-1)(x+1)(x+1)(2x^2-3x-2)$$
$$= (x-1)(x+1)(x+1)(2x+1)(x-2)$$
$$= (x-1)(x+1)^2(2x+1)(x-2)$$

zeros: $\boxed{1, -1, -1, -\frac{1}{2}, 2}$

1 is a zero of multiplicity 1.
-1 is a zero of multiplicity 2.
$-\frac{1}{2}$ is a zero of multiplicity 1.
2 is a zero of multiplicity 1.

(14) Use the Rational Zeros Theorem to find the x-intercepts of the function below, and write the multiplicity of each zero.

$f(x) = 6x^6-19x^5+17x^4+2x^3-8x^2+x+1$

$p: 1 \Rightarrow 1$

$q: 6 \Rightarrow 1, 2, 3, 6$

$\frac{p}{q}: \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$

$x=1 \Rightarrow$ ✓

$$\begin{array}{r|rrrrrrr} 1 & 6 & -19 & 17 & 2 & -8 & 1 & 1 \\ & & 6 & -13 & 4 & 6 & -2 & -1 \\ \hline & 6 & -13 & 4 & 6 & -2 & -1 & 0 \end{array}$$

$6x^5-13x^4+4x^3+6x^2-2x-1$

$x=1 \Rightarrow$ ✓

$$\begin{array}{r|rrrrrr} 1 & 6 & -13 & 4 & 6 & -2 & -1 \\ & & 6 & -7 & -3 & 3 & 1 \\ \hline & 6 & -7 & -3 & 3 & 1 & 0 \end{array}$$

$6x^4-7x^3-3x^2+3x+1$

$x=1 \Rightarrow ✓$

$$\begin{array}{r|rrrrr} 1 & 6 & -7 & -3 & 3 & 1 \\ & & 6 & -1 & -4 & -1 \\ \hline & 6 & -1 & -4 & -1 & 0 \end{array}$$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

$6x^3-x^2-4x-1$

$x=1 \Rightarrow ✓$

$$\begin{array}{r|rrrr} 1 & 6 & -1 & -4 & -1 \\ & & 6 & 5 & 1 \\ \hline & 6 & 5 & 1 & 0 \end{array}$$

$6x^2+5x+1$

Therefore

$$\begin{aligned} 6x^6-19x^5+17x^4+2x^3-8x^2+x+1 &= (x-1)(x-1)(x-1)(x-1)(6x^2+5x+1) \\ &= (x-1)(x-1)(x-1)(x-1)(2x+1)(3x+1) \\ &= (x-1)^4(2x+1)^1(3x+1)^1 \end{aligned}$$

X-Intercepts: $\boxed{1, -\frac{1}{2}, -\frac{1}{3}}$

1 is a zero of multiplicity 4.

$-\frac{1}{2}$ is a zero of multiplicity 1.

$-\frac{1}{3}$ is a zero of multiplicity 1.