Name
Date
Course

Polynomials (Part II): Irrational Zeros, Complex Zeros, & Alternative Methods for Solving Polynomial Equations

Review on Radical Expressions

Factor out the perfect squares in the expressions below. $\boxed{\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}}$

① $\sqrt{12}$
$\sqrt{4\cdot 3}$
$\sqrt{4}\cdot\sqrt{3}$
$2\cdot\sqrt{3}$
$\boxed{2\sqrt{3}}$

② $\sqrt{18}$
$\sqrt{9\cdot 2}$
$\sqrt{9}\cdot\sqrt{2}$
$3\cdot\sqrt{2}$
$\boxed{3\sqrt{2}}$

③ $\sqrt{27}$
$\sqrt{9\cdot 3}$
$\sqrt{9}\cdot 3$
$3\cdot\sqrt{3}$
$\boxed{3\sqrt{3}}$

④ $\sqrt{8}$
$\boxed{2\sqrt{2}}$

⑤ $\sqrt{90}$
$\sqrt{9\cdot 10}$
$\sqrt{9}\cdot\sqrt{10}$
$\boxed{3\sqrt{10}}$

⑥ $\sqrt{28}$
$\sqrt{4\cdot 7}$
$\boxed{2\sqrt{7}}$

⑦ $\sqrt{44}$
$\sqrt{4\cdot 11}$
$\boxed{2\sqrt{11}}$

⑧ $\sqrt{54}$
$\sqrt{9\cdot 6}$
$\boxed{3\sqrt{6}}$

Review on Imaginary Numbers and Complex Numbers

Imaginary Numbers

$i = \sqrt{-1}$

$i^2 = -1$

Examples

$i, -5i, 7i, i\sqrt{3}, \pi i, \frac{1}{2}i$

Complex Numbers

Real Imaginary

$a + bi$

Examples

$5+3i, -4+i, \sqrt{7}-2i, -9-i\sqrt{8}, \frac{3}{4}+\frac{5}{7}i, \pi+2i, 10, 6i, i$

Write each expression in terms of i.

⑨ $\sqrt{-3} = \sqrt{-1\cdot 3} = \sqrt{-1}\cdot\sqrt{3} = i\cdot\sqrt{3} = \boxed{i\sqrt{3}}$

⑩ $\sqrt{-81} = \sqrt{-1\cdot 81} = \sqrt{-1}\cdot\sqrt{81} = i\cdot 9 = \boxed{9i}$

⑪ $\sqrt{-2} = \boxed{i\sqrt{2}}$

⑫ $\sqrt{-4} = i\cdot 2 = \boxed{2i}$

⑬ $\sqrt{-5} = \boxed{i\sqrt{5}}$

⑭ $\sqrt{-49} = i\cdot 7 = \boxed{7i}$

⑮ $\sqrt{-7} = \boxed{i\sqrt{7}}$

⑯ $\sqrt{-100} = i\cdot 10 = \boxed{10i}$

Review on the Difference between Two Squares

$(x-k)(x+k) = x^2+kx-kx-k^2 = x^2-k^2$

Factor the following expressions.

(17) $x^2-25 = \boxed{(x+5)(x-5)}$

(18) $4x^2-9 = \boxed{(2x+3)(2x-3)}$

(19) $x^2-49 = \boxed{(x+7)(x-7)}$

(20) $36x^2-1 = \boxed{(6x+1)(6x-1)}$

Polynomial Equation: An equation that sets a polynomial equal to zero.

Examples: $x^3+4x^2-7x+8=0,\quad 0=-9x^2+x-2$

Three Names for the Solutions of a Polynomial Equation

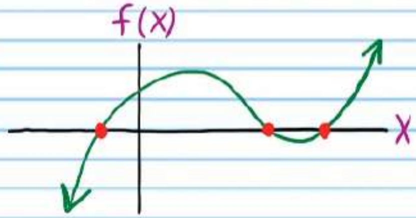
I) Zeros

II) Roots

III) X-Intercepts (unless the solution is complex)

Example:

$f(x) = x^3-8x^2+11x+20$



The Rational Zeros Theorem

If a polynomial equation has rational solutions, then the solutions can be found by dividing the factors of the last term by the factors of the coefficient of the first term. Positive and negative numbers must be considered. ☆ This theorem applies only when the coefficients of the polynomial are integers (i.e. whole numbers).

Example: $8x^4-3x^3+4x^2-10x+21=0$

The Polynomial Degree Zeros Theorem: Every polynomial with degree n has exactly n zeros. ☆ The zeros can be real, complex, distinct, or repeated.

The Polynomial Degree Factor Theorem: Every polynomial with degree n can be factored into n factors of the form $x-c_i$, where c_i can be real, complex, distinct, or repeated.

Example: $5x^{3}-36x^2+28x+48 = 5\overset{1}{(x+\frac{1}{5})}\overset{2}{(x-2)}\overset{3}{(x-6)}$

The Factor Theorem: Let f be a polynomial function. If $f(c)=0$, then $x-c$ is a factor of $f(x)$.

The Remainder Theorem: Let f be a polynomial function. If $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

Finding Irrational Zeros and Imaginary Zeros

Find the zeros of the polynomial functions below and write each function in factored form.

(21) x^3-4x-3

$p: 3 \Rightarrow 1, 3$

$q: 1 \Rightarrow 1$

$\frac{p}{q}: \pm 1, \pm 3$

$x=1 \Rightarrow$ ✗, $x=-1 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -4 & -3 \\ & & -1 & 1 & 3 \\ \hline & 1 & -1 & -3 & 0 \end{array}$$

$x^3-4x-3 = (x+1)(x^2-x-3)$

$= \boxed{(x+1)\left(x-\frac{1+\sqrt{13}}{2}\right)\left(x-\frac{1-\sqrt{13}}{2}\right)}$

Zero(s): $\boxed{-1, \frac{1+\sqrt{13}}{2}, \frac{1-\sqrt{13}}{2}}$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(-3)}}{2(1)}$

$x = \frac{1 \pm \sqrt{1+12}}{2}$

$x = \frac{1 \pm \sqrt{13}}{2}$

(22) $f(x) = -5x^3 - 7x^2 - 12$

$p: 12 \Rightarrow 1, 2, 3, 4, 6, 12$

$q: 5 \Rightarrow 1, 5$

$\frac{p}{q}: \pm 1, \pm \frac{1}{5}, \pm 2, \pm \frac{2}{5}$

$x = 1 \Rightarrow$ X, $x = -1 \Rightarrow$ X, $x = 2 \Rightarrow$ X, $x = -2 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} -2 & -5 & -7 & 0 & -12 \\ & & 10 & -6 & 12 \\ \hline & -5 & 3 & -6 & 0 \end{array}$$

$-5x^2 + 3x - 6$

$-5x^3 - 7x^2 - 12 = (x+2)(-5x^2 + 3x - 6)$

$= \boxed{(x+2)\left(x - \frac{3+i\sqrt{111}}{10}\right)\left(x - \frac{3-i\sqrt{111}}{10}\right)}$

Zero(s): $\boxed{-2, \frac{3+i\sqrt{111}}{10}, \frac{3-i\sqrt{111}}{10}}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-3 \pm \sqrt{3^2 - 4(-5)(-6)}}{2(-5)}$

$x = \frac{-3 \pm \sqrt{9 - 120}}{-10}$

$x = \frac{-3 \pm \sqrt{-111}}{-10}$

$x = \frac{-3 \pm i\sqrt{111}}{-10}$

$x = \frac{3 \pm i\sqrt{111}}{10}$

(23) $f(x) = -x^3 + 7x^2 - 20$

$p: 20 \Rightarrow 1, 2, 4, 5, 10, 20$

$q: 1 \Rightarrow 1$

$\frac{p}{q}: \pm 1, \pm 2, \pm 4 \rightsquigarrow$

$x = 1 \Rightarrow$ ✗, $x = -1 \Rightarrow$ ✗, $x = 2 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} 2 & -1 & 7 & 0 & -20 \\ & & -2 & 10 & 20 \\ \hline & -1 & 5 & 10 & 0 \end{array}$$

$-x^2 + 5x + 10$

$-x^3 + 7x^2 - 20 = (x-2)(-x^2+5x+10)$

$= \boxed{(x-2)\left(x - \frac{5+\sqrt{65}}{2}\right)\left(x - \frac{5-\sqrt{65}}{2}\right)}$

Zero(s): $\boxed{2, \frac{5+\sqrt{65}}{2}, \frac{5-\sqrt{65}}{2}}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-5 \pm \sqrt{5^2 - 4(-1)(10)}}{2(-1)}$

$x = \frac{-5 \pm \sqrt{25+40}}{-2}$

$x = \frac{-5 \pm \sqrt{65}}{-2}$

$x = \frac{5 \pm \sqrt{65}}{2}$

(24) $f(x) = 3x^3 - 19x + 24$

$p: 24 \Rightarrow 1, 2, 3, 4, 6, 8, 12, 24$

$q: 3 \Rightarrow 1, 3$

$\frac{p}{q}: \pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 3 \longrightarrow$

$x = 1 \Rightarrow$ ✗, $x = -1 \Rightarrow$ ✗, $x = 2 \Rightarrow$ ✗

$x = -2 \Rightarrow$ ✗, $x = 3 \Rightarrow$ ✗, $x = -3 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} -3 & 3 & 0 & -19 & 24 \\ & & -9 & 27 & -24 \\ \hline & 3 & -9 & 8 & 0 \end{array}$$

$3x^2 - 9x + 8$

$3x^3 - 19x + 24 = (x+3)(3x^2 - 9x + 8)$

$= \boxed{(x+3)\left(x - \frac{9+i\sqrt{15}}{6}\right)\left(x - \frac{9-i\sqrt{15}}{6}\right)}$

Zero(s): $\boxed{-3, \frac{9+i\sqrt{15}}{6}, \frac{9-i\sqrt{15}}{6}}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(3)(8)}}{2(3)}$

$x = \frac{9 \pm \sqrt{81 - 96}}{6}$

$x = \frac{9 \pm \sqrt{-15}}{6}$

$x = \frac{9 \pm i\sqrt{15}}{6}$

Using Alternative Methods to Solve Polynomial Equations

(25) $f(x) = 9x^4 + 4x^2 - 28$

$= (9x^2 - 14)(x^2 + 2)$

$9x^2 - 14 = 0$ or $x^2 + 2 = 0$

$9x^2 = 14$ $\qquad$ $x^2 = -2$

$x^2 = \frac{14}{9}$ $\qquad$ $x = \pm\sqrt{-2}$

$x = \pm \frac{\sqrt{14}}{3}$ $\qquad$ $x = \pm i\sqrt{2}$

$9x^4 + 2x^2 - 32 = \boxed{\left(x - \frac{\sqrt{14}}{3}\right)\left(x + \frac{\sqrt{14}}{3}\right)(x - i\sqrt{2})(x + i\sqrt{2})}$

Zero(s): $\boxed{\frac{\sqrt{14}}{3}, -\frac{\sqrt{14}}{3}, i\sqrt{2}, -i\sqrt{2}}$

(26) $f(x) = 2x^4 + 3x^2 - 20$

$= (2x^2 - 5)(x^2 + 4)$

$2x^2 - 5 = 0$ or $x^2 + 4$

$x = \pm\sqrt{\frac{5}{2}}$ $\qquad$ $x^2 = -4$

$x = \pm \frac{\sqrt{10}}{2}$ $\qquad$ $x = \pm\sqrt{-4}$

$x = \pm 2i$

$2x^4 + 3x^2 - 20 = \boxed{\left(x - \frac{\sqrt{10}}{2}\right)\left(x + \frac{\sqrt{10}}{2}\right)(x - 2i)(x + 2i)}$

Zero(s): $\boxed{\frac{\sqrt{10}}{2}, -\frac{\sqrt{10}}{2}, 2i, -2i}$

Find the zeros of the following polynomial functions using the grouping method.

27) $f(x) = 5x^3 - 3x^2 + 30x - 18$

$= x^2(5x-3) + 6(5x-3)$

$= (5x-3)(x^2+6)$

$5x-3=0$ or $x^2+6=0$

$x = \frac{3}{5}$ $\quad x^2 = -6$

$x = \pm\sqrt{-6}$

$x = \pm i\sqrt{6}$

$5x^3-3x^2+30x-18 = \boxed{(x-\frac{3}{5})(x-i\sqrt{6})(x+i\sqrt{6})}$

Zero(s): $\boxed{\frac{3}{5}, i\sqrt{6}, -i\sqrt{6}}$

28) $f(x) = 6x^3+x^2-48x-8$

$= x^2(6x+1) - 8(6x+1)$

$= (6x+1)(x^2-8)$

$6x+1=0$ or $x^2-8=0$

$x = -\frac{1}{6}$ $\quad x^2 = 8$

$x = \pm 2\sqrt{2}$

$6x^3+x^2-48x-8 = \boxed{(x+\frac{1}{6})(x-2\sqrt{2})(x+2\sqrt{2})}$

Zero(s): $\boxed{-\frac{1}{6}, 2\sqrt{2}, -2\sqrt{2}}$

(29) $f(x)=4x^5+12x^4-x-3$

$$=4x^4(x+3)-1(x+3)$$

$$=(x+3)(4x^4-1)$$

$$=(x+3)(2x^2-1)(2x^2+1)$$

$x+3=0$ or $2x^2-1=0$ or $2x^2+1=0$

$x=-3$ $\quad$ $x=\pm\sqrt{\frac{1}{2}}$ $\quad$ $2x^2=-1$

$x=\pm\frac{1}{\sqrt{2}}$ $\quad$ $x^2=-\frac{1}{2}$

$x=\pm\frac{\sqrt{2}}{2}$ $\quad$ $x=\pm\sqrt{-\frac{1}{2}}$

$$x=\pm i\frac{\sqrt{1}}{\sqrt{2}}$$

$$x=\pm i\frac{1}{\sqrt{2}}$$

$$x=\pm i\frac{\sqrt{2}}{2}$$

$4x^5+12x^4-x-3=\boxed{(x+3)\left(x-\frac{\sqrt{2}}{2}\right)\left(x+\frac{\sqrt{2}}{2}\right)\left(x-i\frac{\sqrt{2}}{2}\right)\left(x+\frac{i\sqrt{2}}{2}\right)}$

Zero(s): $\boxed{-3,\ \frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ \frac{i\sqrt{2}}{2},\ -\frac{i\sqrt{2}}{2}}$

(30) $f(x) = 4x^5 - 8x^4 - x + 2$

$= 4x^4(x-2) - 1(x-2)$

$= (x-2)(4x^4-1)$

$= (x-2)(2x^2-1)(2x^2+1)$

$x-2=0$ or $2x^2-1=0$ or $2x^2+1=0$

$x=2$ $\quad x^2 = \frac{1}{2}$ $\quad x = \pm\frac{i\sqrt{2}}{2}$

$x = \pm\frac{\sqrt{2}}{2}$

$4x^5 - 8x^4 - x + 2 = \boxed{(x-2)\left(x+\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)\left(x-\frac{i\sqrt{2}}{2}\right)\left(x+\frac{i\sqrt{2}}{2}\right)}$

Zero(s): $\boxed{2, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, -\frac{i\sqrt{2}}{2}}$

Factoring the Sum or Difference of Two Cubes

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

(31) $f(x)=x^4-x^3-27x+27$

$= x^3(x-1)-27(x-1)$

$= (x-1)(x^3-27)$

$p: 27 \Rightarrow 1,3,9,27$

$q: 1 \Rightarrow 1$

$\frac{p}{q}: \pm1, \pm3, \pm9, \pm27$

$x=1 \Rightarrow$ ✗

$x=-1 \Rightarrow$ ✗

$x=3 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} 3 & 1 & 0 & 0 & -27 \\ & & 3 & 9 & 27 \\ \hline & 1 & 3 & 9 & 0 \end{array}$$

x^2+3x+9

$x^4-x^3-27x+27=(x-1)(x-3)(x^2+3x+9)$

$=\boxed{(x-1)(x-3)\left(x-\frac{-3+3i\sqrt{3}}{2}\right)\left(x-\frac{-3-3i\sqrt{3}}{2}\right)}$

Zero(s): $\boxed{1, 3, \frac{-3+3i\sqrt{3}}{2}, \frac{-3-3i\sqrt{3}}{2}}$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-3\pm\sqrt{3^2-4(1)(9)}}{2(1)}$$

$$x=\frac{-3\pm\sqrt{-27}}{2}$$

$\sqrt{-27}=\sqrt{-1\cdot27}$

$=\sqrt{-1}\cdot\sqrt{27}$

$=i\cdot3\sqrt{3}$

$=3i\sqrt{3}$

$$x=\frac{-3\pm3i\sqrt{3}}{2}$$

- (32) $f(x) = x^4 + 2x^3 - 8x - 16$

$$= x^3(x+2) - 8(x+2)$$

$$= (x+2)(x^3 - 8)$$

$p: 8 \Rightarrow 1, 2, 4, 8$

$q: 1 \Rightarrow 1$

$\frac{p}{q}: \pm 1, \pm 2, \pm 4, \pm 8$

$x = 1 \Rightarrow$ ✗

$x = -1 \Rightarrow$ ✗

$x = 2 \Rightarrow$ ✓

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -8 \\ & & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$

$x^2 + 2x + 4$

$$x^4 + 2x^3 - 8x - 16 = (x+2)(x-2)(x^2+2x+4)$$

$$= \boxed{(x+2)(x-2)(x-(-1+i\sqrt{3}))(x-(-1-i\sqrt{3}))}$$

Zeros (s): $\boxed{-2, 2, -1+i\sqrt{3}, -1-i\sqrt{3}}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$\sqrt{-12} = \sqrt{-1 \cdot 12} = \sqrt{-1} \cdot \sqrt{12}$$
$$= i \cdot \sqrt{12}$$
$$= i \cdot 2 \cdot \sqrt{3}$$
$$= 2i\sqrt{3}$$

$$x = \frac{-2 \pm 2i\sqrt{3}}{2}$$

$$x = -1 \pm i\sqrt{3}$$