Visual effects of special relativity

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Outline

Stating the problem

What do we want to achieve?

Create a program which can simulate kinematics of special relativity and the associated visual effects. It has to be fast - generate $\sim$60 frames per second, so that smooth motion can be shown.

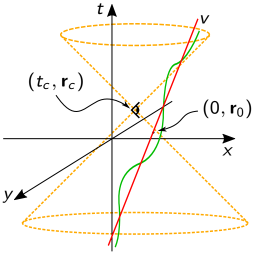
The program will serve as a tool to test our understanding of special relativity.

Theory

Assumptions

- Observer (camera) is in an inertial frame.
- Every other considered frame is inertial.
- Objects in the other frames can perform any transformation: they can move, rotate, change shape with time, etc.
- Neglect relativistic Doppler effect.
- Approximate the objects by sampling points on their surfaces - to speed up calculations.

Initial information



Space-time diagram (S)

If we call the frame of reference of the camera S and frame of reference in which we construct the object S', the known information is:

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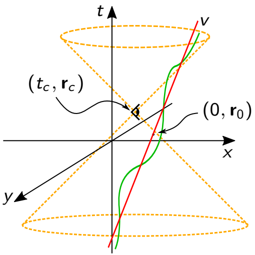
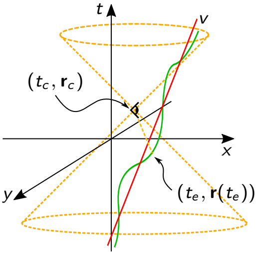


Space-time diagram (S)

If we call the frame of reference of the camera S and frame of reference in which we construct the object S', the known information is:

- Position of the camera $\mathbf{r}_c$ at the time t_c in the S frame.
- Position of every element of the object $\mathbf{r}'(t')$ at any given time t' in the frame S'.
- Velocity $\mathbf{v}$ at which the frame S' moves with respect to S
- Position of origin of the S' frame $\mathbf{r}_0$ in the S frame at the time $t = 0,\ t' = 0$.

Space-time interval



Space-time diagram (S)

Because of the finite speed of light, given a point with a trajectory $\mathbf{r}(t)$, the location at which the camera sees it at the time t_c is not $\mathbf{r}(t_c)$ but $\mathbf{r}(t_e)$, where t_e is a time at which the light ray was emitted from this point.

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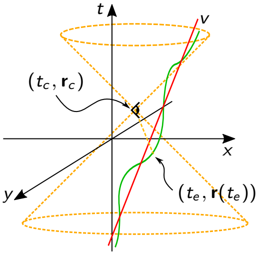

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$$(t_c - t_e)^2 - (\mathbf{r}_c - \mathbf{r}(t_e))^2 = 0$$

3D Lorentz transform

The Lorentz transformation in 3D is:

$$
\begin{aligned}
t' &= \gamma\left(t - r_{\|} v\right) \\
\mathbf{r}'_{\|} &= \gamma\left(\mathbf{r}_{\|} - \mathbf{v}t\right) \\
\mathbf{r}'_{\perp} &= \mathbf{r}_{\perp}
\end{aligned}
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Where $\mathbf{r}_{\|}$ is the component of $\mathbf{r}$ along $\mathbf{v}$ and $\mathbf{r}_{\perp}$ is a component perpendicular to it.

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Where $\mathbf{r}_{\|}$ is the component of $\mathbf{r}$ along $\mathbf{v}$ and $\mathbf{r}_{\perp}$ is a component perpendicular to it. If we define $\mathbf{n} = \frac{\mathbf{v}}{|\mathbf{v}|}$, these can be easily related to $\mathbf{r}$ and $\mathbf{v}$, by finding a projection: $\mathbf{r}_{\|} = (\mathbf{r} \cdot \mathbf{n})\, \mathbf{n} = \frac{(\mathbf{r} \cdot \mathbf{v})}{\mathbf{v}^2} \mathbf{v}$ and $\mathbf{r}_{\perp} = \mathbf{r} - \mathbf{r}_{\|}$.

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$$t' = \gamma\left(t - \mathbf{r}\cdot\mathbf{v}\right)$$
$$\mathbf{r}' = \mathbf{r} + \mathbf{v}\left((\gamma - 1)\frac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}^2} - \gamma t\right)$$

Derivation of the equation

Taking the inverse Lorentz transform, the problem reduces to solving a set of the three following equations:

$$
\begin{aligned}
&(t_c - t_e)^2 = (\mathbf{r}_c - \mathbf{r}(t_e))^2 \\
&t_e = \gamma\left(t'_e + \mathbf{r}'(t'_e) \cdot \mathbf{v}\right) \\
&\mathbf{r}(t_e) = \mathbf{r}_0 + \mathbf{r}'(t'_e) + \mathbf{v}\left((\gamma - 1)\frac{\mathbf{r}'(t'_e) \cdot \mathbf{v}}{\mathbf{v}^2} + \gamma t'_e\right)
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In the last equation I have added $\mathbf{r}_0$ - position of the origin of the S' frame at $t = 0$, $t' = 0$ in S frame.

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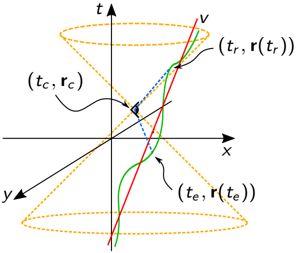
In the last equation I have added $\mathbf{r}_0$ - position of the origin of the S' frame at $t = 0$, $t' = 0$ in S frame. Eliminating t_e and $\mathbf{r}(t_e)$:

$$
t_c - \gamma\left(t'_e + \mathbf{r}'(t'_e)\cdot\mathbf{v}\right) = \left|\mathbf{r}'(t'_e) + \mathbf{v}\left((\gamma - 1)\tfrac{\mathbf{r}'(t'_e)\cdot\mathbf{v}}{\mathbf{v}^2} + \gamma t'_e\right) + \mathbf{r}_0 - \mathbf{r}_c\right|
$$

Solving the equation



Space-time diagram (S)

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We have to solve the equation for t'_e remembering that there are two solutions - t'_r in the future and t'_e in the past.

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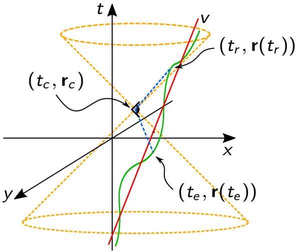

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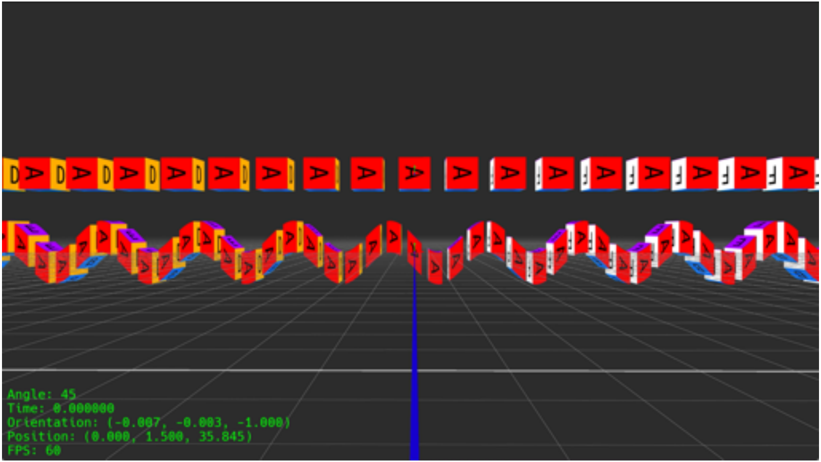
We have to solve the equation for t'_e remembering that there are two solutions - t'_r in the future and t'_e in the past. We solve the equation numerically to find t'_e and then calculate $\mathbf{r}(t_e)$ from the Lorentz transform. Once we have $\mathbf{r}(t_e)$, we transform it into 2D image coordinates and render the image showing each of the points of an object.

Results

Lorentz contraction and loss of simultaneity

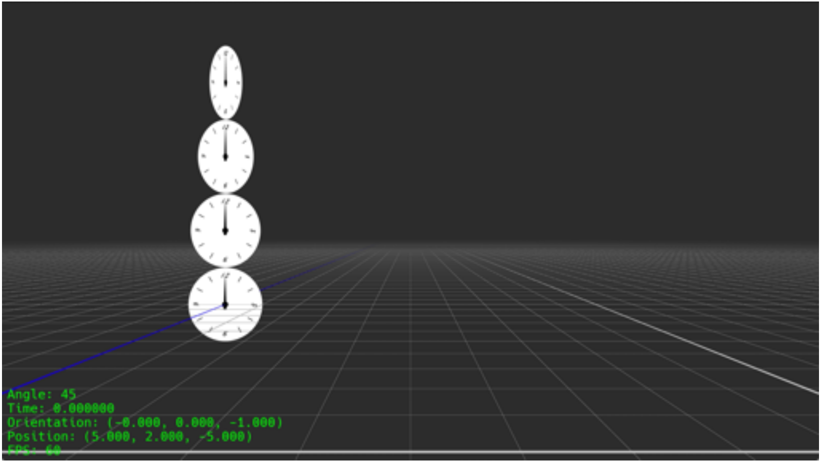
To obtain the effects known from special relativity, the **actual positions** of the objects in the frame S will be shown. **The grid spacing is one light-second**.



A sequence of boxes oscillating in y-axis. The boxes at the bottom are in a frame moving along x-axis with $v = 0.9$. The boxes at the top are constructed in the S frame for reference.

Time dilation

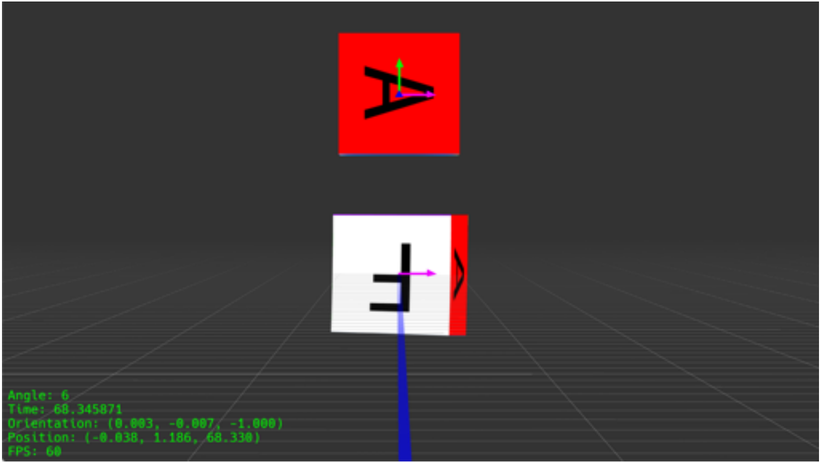
Clocks moving along x-axis with speeds $v = 0.33$, $v = 0.66$, $v = 0.90$.



Terrell rotation

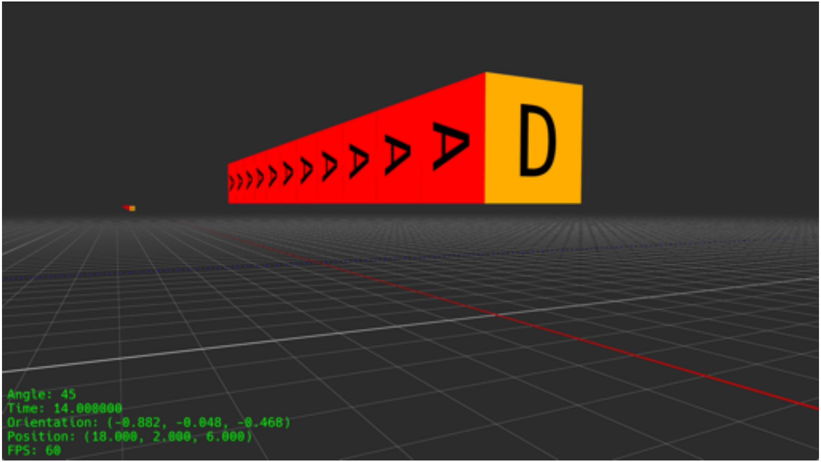
From now on we **include the effects of finite speed of light propagation**.

A box moving at $v = 0.99$ along x-axis, seen from a long distance. The box appears to be rotated - the phenomena called Terrell rotation.



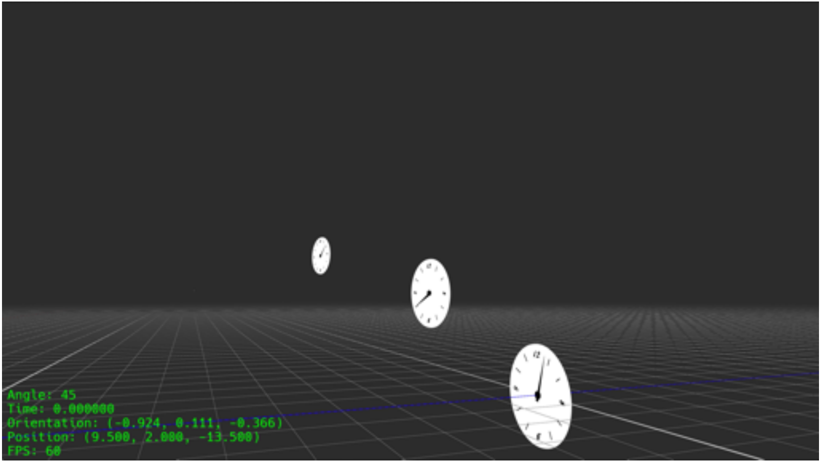
Apparent Lorentz contraction

A sequence of boxes moving at $v = 0.99$ along x-axis, seen from a short distance:



Apparent time dilation

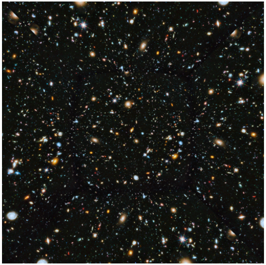
Clocks moving along x-axis with speeds $v = 0.33$, $v = 0.66$, $v = 0.99$, with finite light propagation taken into account.



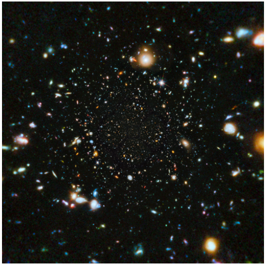
Relativistic aberration

Camera moving at different velocities, looking in the direction of motion. The camera at $v = 0$ relative to the stars observes them to be spread uniformly. The stars start to gather in front of the camera as we increase the velocity - the relativistic aberration can be observed.

Stationary camera



$v = 0.9$

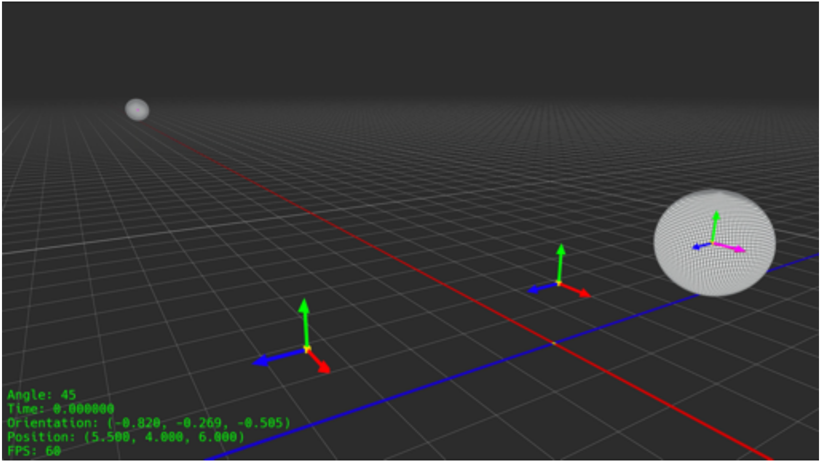


$v = 0.999$

Texture credit: NASA, June 03, 2014 2:15PM (EDT) Release ID: 2014-27

Relativistic spheres

From left to right: spheres moving at $v = 0.99$, $v = 0.9$ and a stationary sphere. Note that the spheres always retain their shape.



Outcomes

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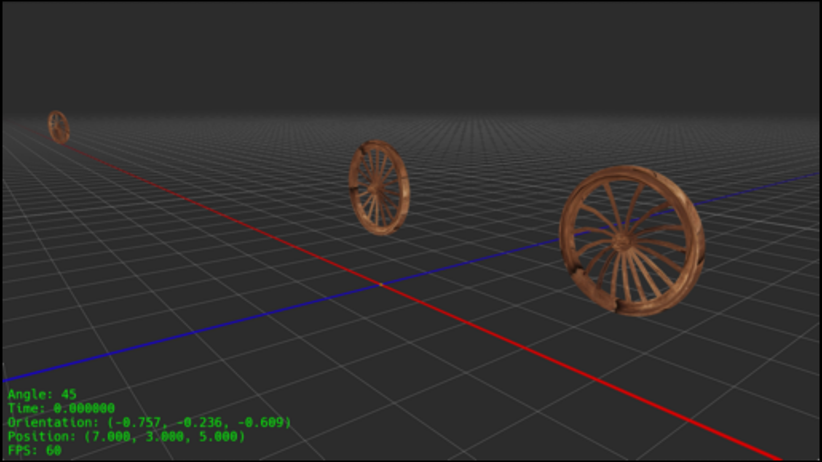
- Derived equation for visible position of the object at any given time.
- Created the program to simulate visual effects of special relativity in real-time.
- Successfully simulated kinematics of objects moving in inertial frames at relativistic velocities. Tested Lorentz contraction, time dilation, loss of simultaneity and effects coming from finite speed of propagation of light such as Terrell rotation and relativistic aberration.



Angle: 45
Time: 0.000000
Orientation: (-0.757, -0.236, -0.609)
Position: (7.000, 3.000, 5.000)
FPS: 60