I

MATHEMATICAL MORPHOLOGY

Point to be covered

- **Basic morphological concepts**
- **Dilation**
- **Erosion**
- **Opening and closing**
- **Hit or miss transformation**
- **Boundary extraction**
- **Thinning and skeleton**

Mathematical Morphology

- It is a tool for extracting image components, that are useful in the representation and description of region shape
- Usually applied to binary images
- Based on set theory

Basic morphological concepts

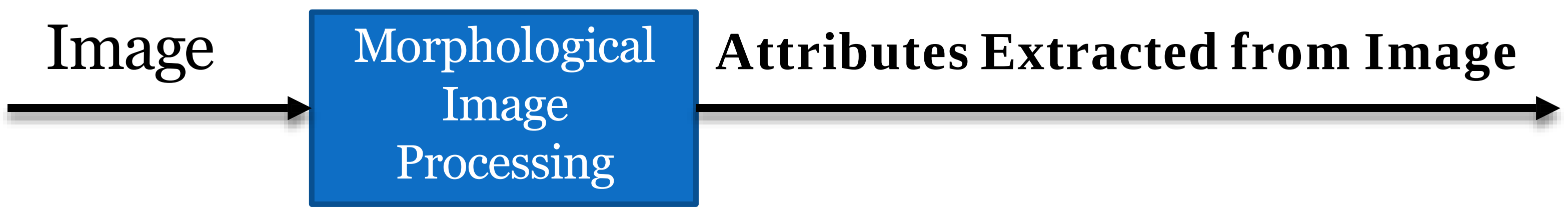
- Sets in mathematical morphology represent objects in an image:

❑ **binary image** (0 =black, 1=white) :

The element of the set is the coordinates (x,y) of pixel belong to the object Z^2

❑ **gray-scaled image** :

The element of the set is the coordinates (x,y) of pixel belong to the object and the gray levels Z^3



Applications of Morphology

- Boundaries extraction
- Skeletons
- Convex hull
- Morphological filtering
- Thinning

Basic Concepts in Set Theory

- Subset

  $A \subseteq B$
- Union

  $A \cup B$
- Intersection

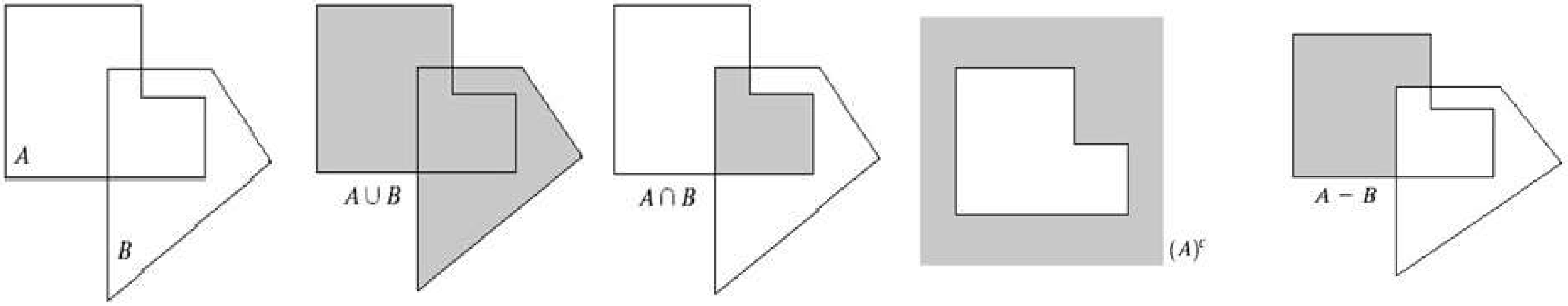
  $A \cap B$

  disjoint / mutually exclusive $A \cap B = \emptyset$
- Complement $A^c \equiv \{ w \mid w \notin A \}$
- Difference $A - B \equiv \{ w \mid w \in A, w \notin B \} = A \cap B^c$
- Reflection $\hat{B} \equiv \{ w \mid w = -b, \quad \forall b \in B \}$
- Translation $(A)z \equiv \{ c \mid c = a + z, \quad \forall a \in A \}$

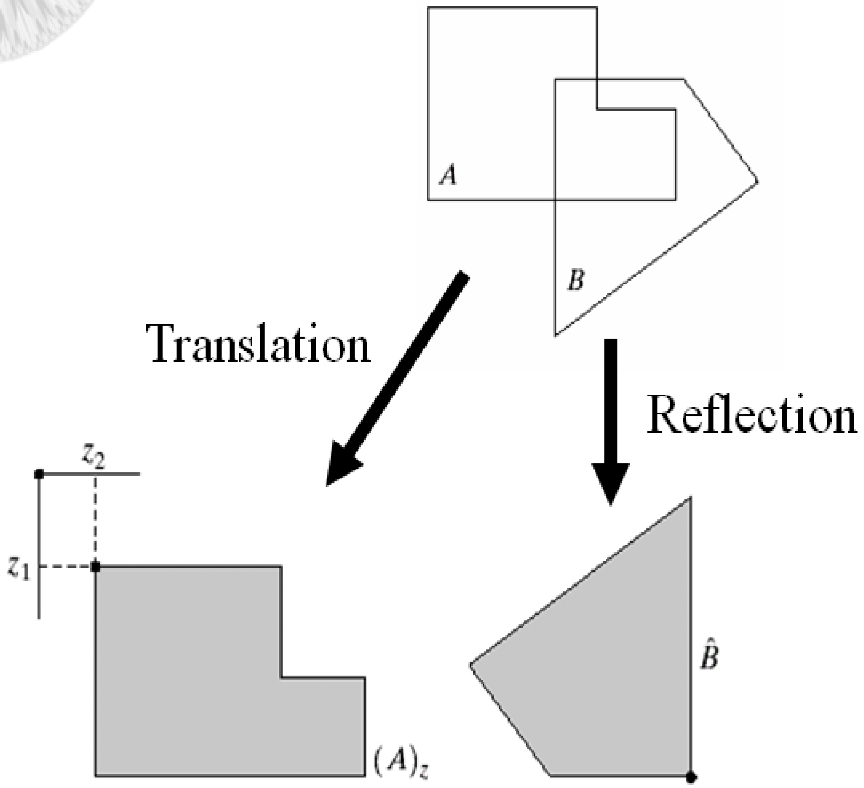
Set Theory

BASICS:

- UNION = AUB
- INTERSECTION = A∩B
- COMPLIMENT = $(A)^c$
- DIFFERENCE = A-B



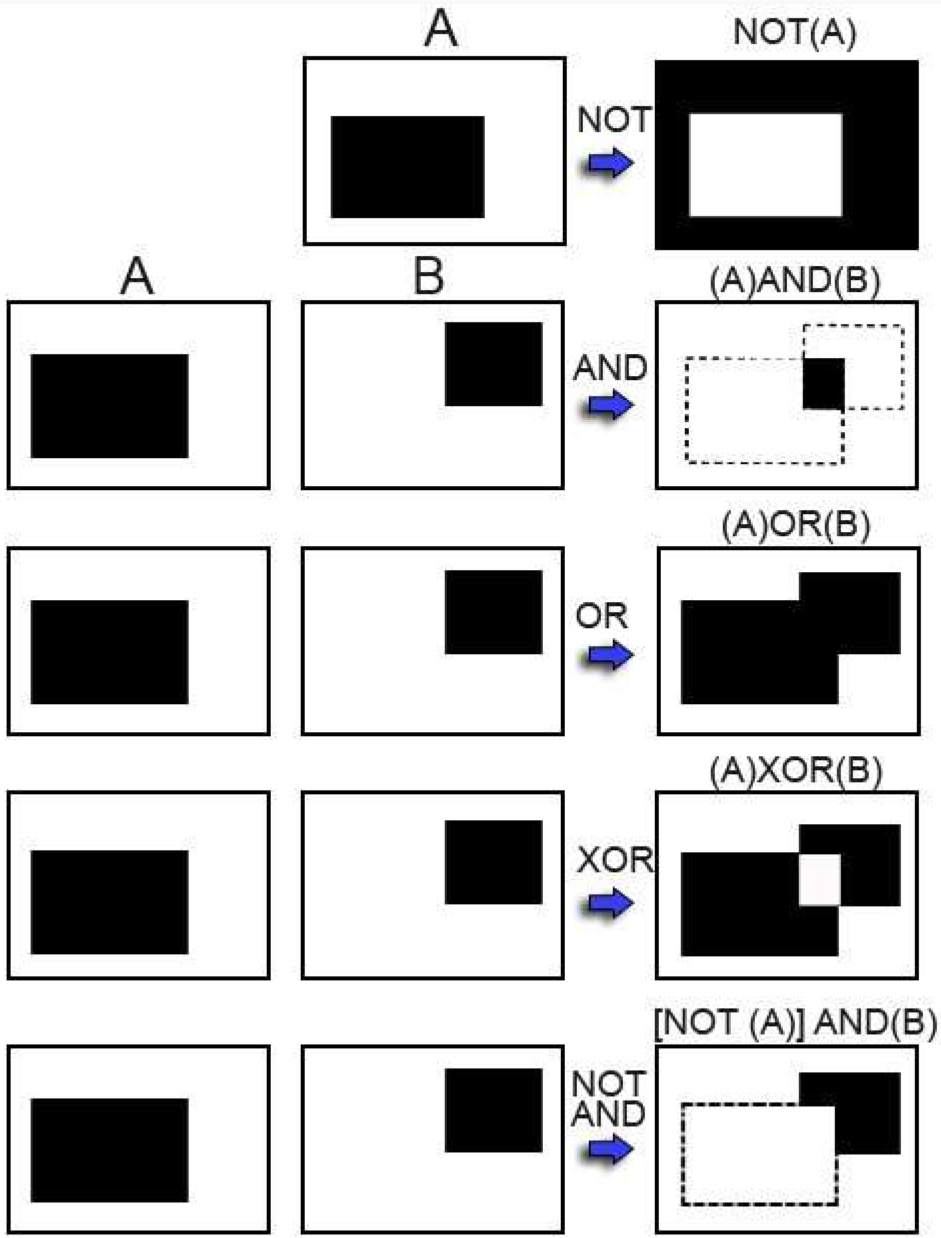
Some Basic Concepts from Set Theory



a b

FIGURE 9.2
(a) Translation of A by z.
(b) Reflection of B. The sets A and B are from Fig. 9.1.

LOGIC OPERATIONS REPRESENTATION



Structuring Elements (SE)

- Structuring Element (SE):
- Small sets or subimages used to probe an image under study for properties of interest.
- The origin of the SE is indicated by a **black dot.**
- When SE is symmetric and no dot is shown the origin is at the center of symmetry.
- When SE is symmetric B = (reflection of B)
- When working with images
  - SE must be in a Rectangular Arrays (padding with the smallest possible number of background elements)
  - Images must be in a Rectangular Arrays (padding with the smallest possible number of background elements)
  - For Images a background border is provided to accommodate the entire SE when its origin is on the border

Dilation

- Dilation is used for expanding an element A by using structuring element B

- With A and B as sets in Z^2, the dilation of A by B is defined as

$$A \oplus B = \left\{ z \mid (\hat{B})_z \cap A \neq \phi \right\}$$

where $\hat{B}$: the reflection of B about its origin and shifting this reflection by z

- The dilation of A by B is the set of all displacements, z, such that $\hat{B}$ and A overlap by at least one element. Thus,

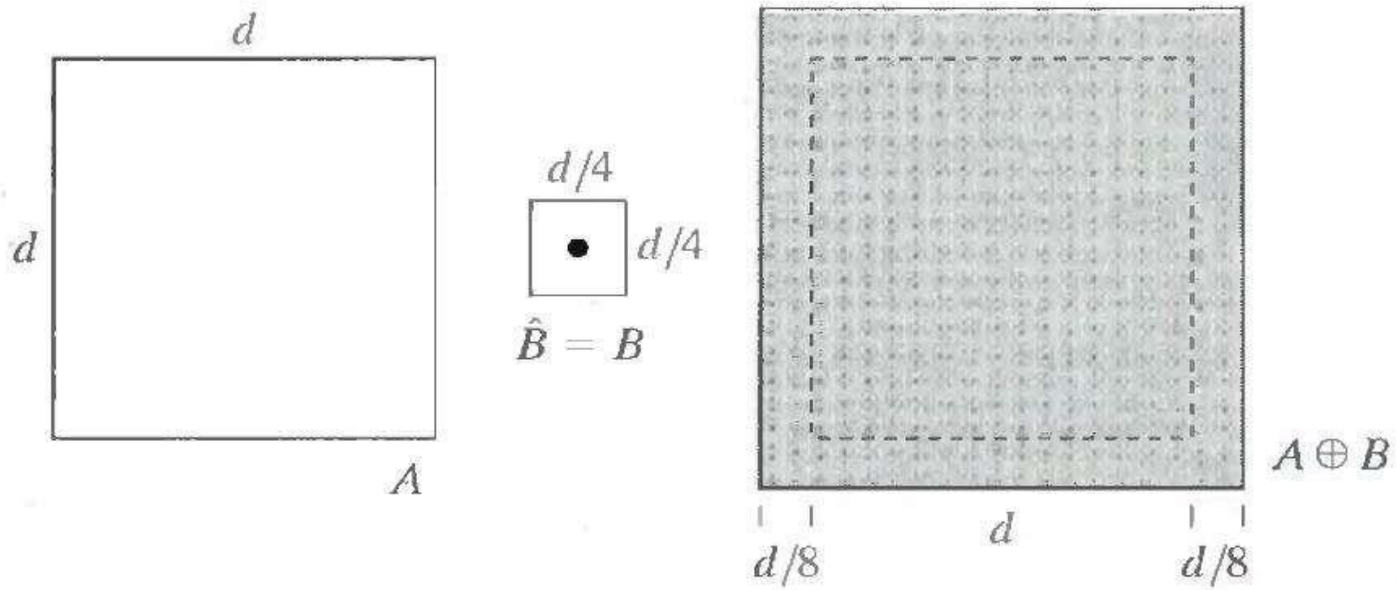
$$A \oplus B = \left\{ z \mid [(\hat{B})_z \cap A] \subseteq A \right\}$$

- Set B is referred to as the structuring element in dilation.

Dilation – Example 1

a b c

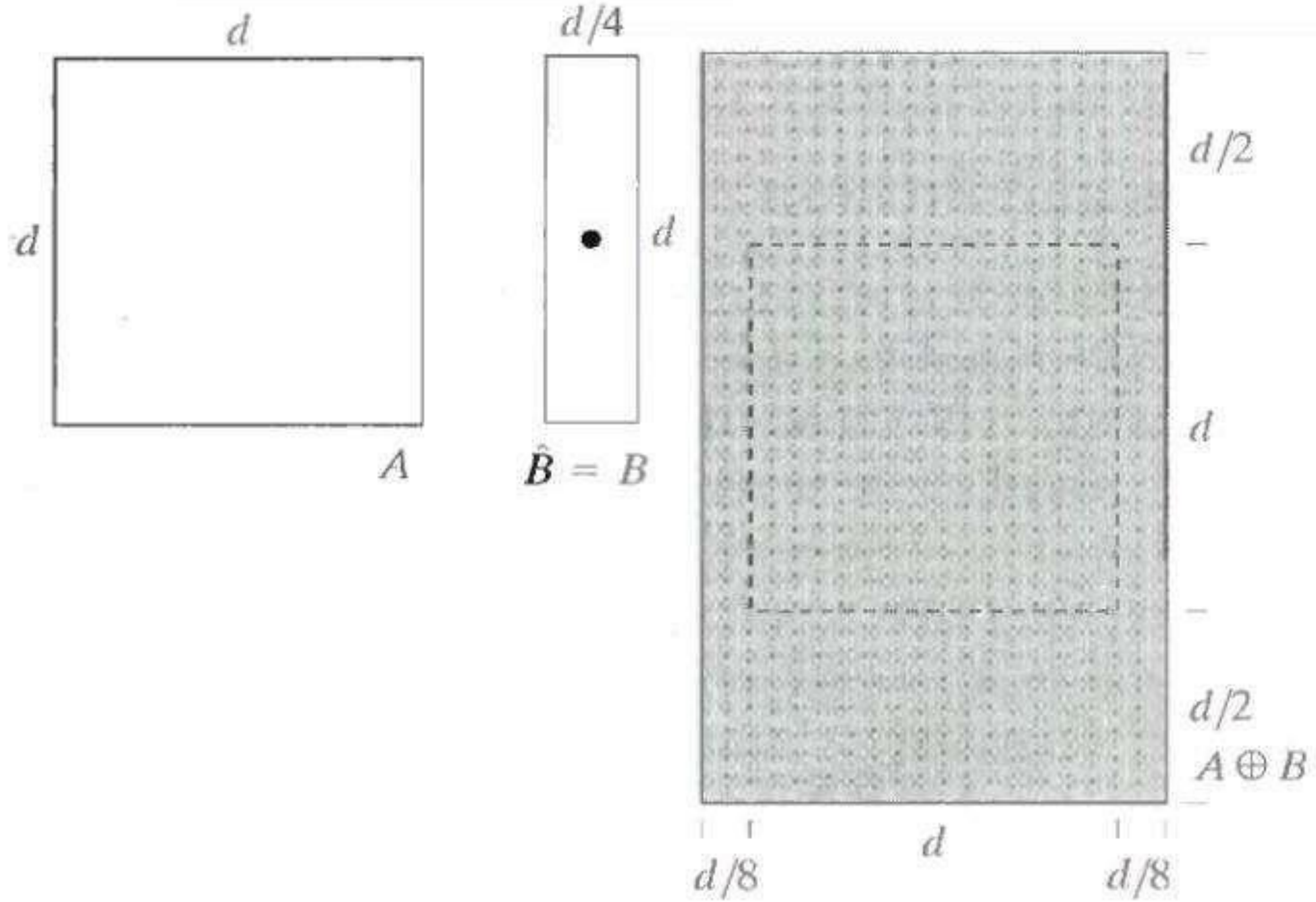
FIGURE 9.4
(a) Set A.
(b) Square structuring element (dot is the center).
(c) Dilation of A by B, shown shaded.



Dilation – Example 2

a d e

(d) Elongated structuring element.
(e) Dilation of A using this element.



Suppose that the structuring element is a 3×3 square.

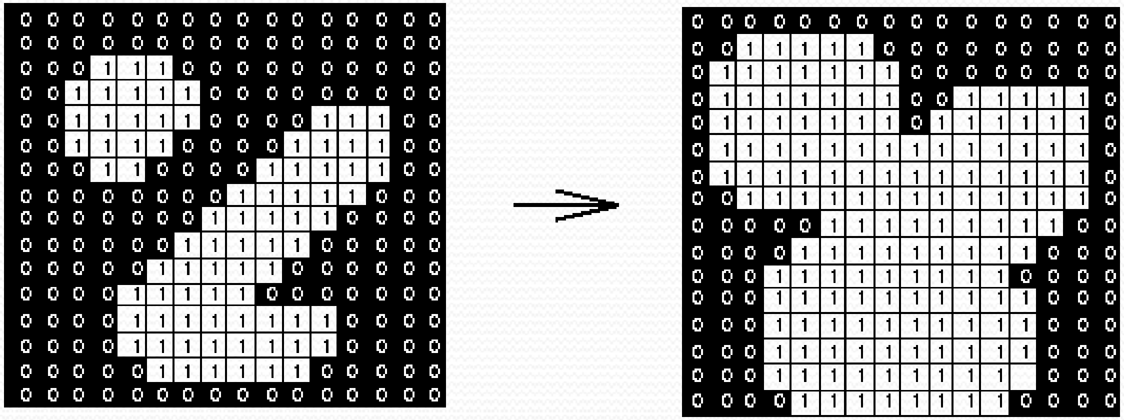
Note that in subsequent diagrams, foreground pixels are represented by 1's and background pixels by 0's.

To compute the dilation of a binary input image by this structuring element, we superimpose the structuring element on top of the input image so that the origin of the structuring element coincides with the input pixel position.

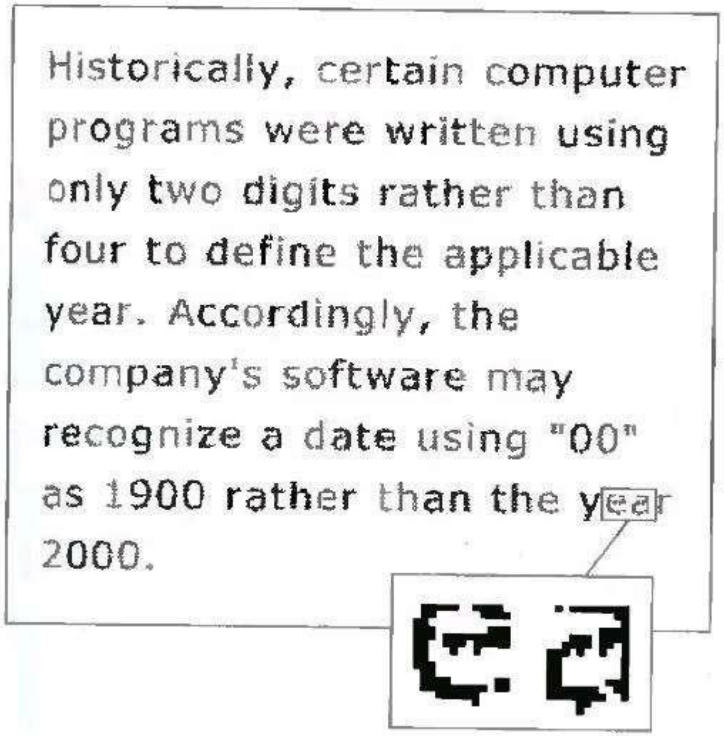
If the center pixel in the structuring element coincides with a foreground pixel in the image underneath, then the input pixel is set to the foreground value.

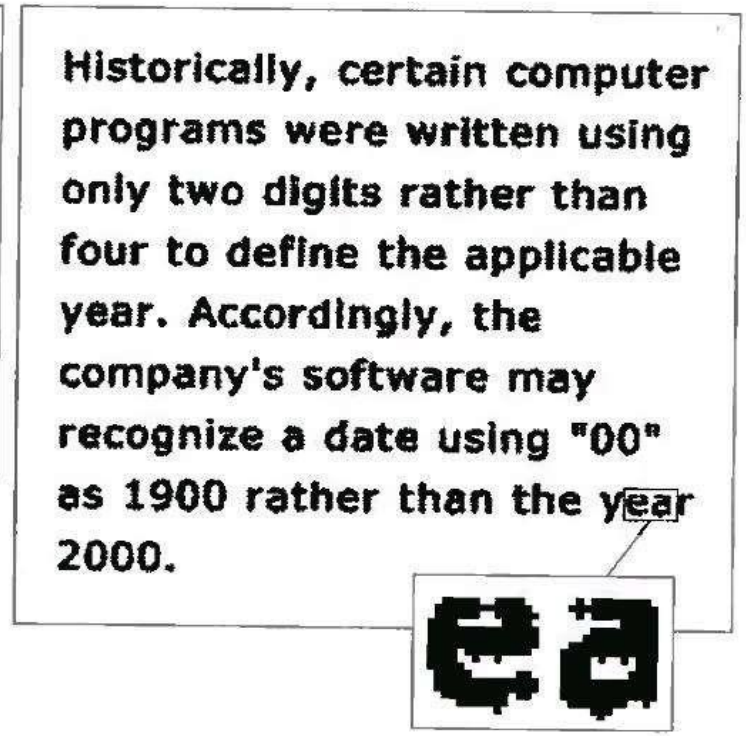
1	1	1
1	1	1
1	1	1

Structuring element



Dilation – A More interesting Example (bridging gaps)





a c
b

0	1	0
1	1	1
0	1	0

FIGURE 9.5 (a) Sample text of poor resolution with broken characters (magnified view). (b) Structuring element. (c) Dilation of (a) by (b). Broken segments were joined.

Erosion

- Erosion is used for shrinking of element A by using element B

- For sets A and B in Z^2, the erosion of A by B is defined as

$$A \ominus B = \left\{ z \mid (\hat{B})_z \subseteq A \right\}$$

where $\hat{B}$: the reflection of B about its origin and shifting this reflection by z

- The erosion of A by B is the set of all points z, such that B, translated by z is contained in A.

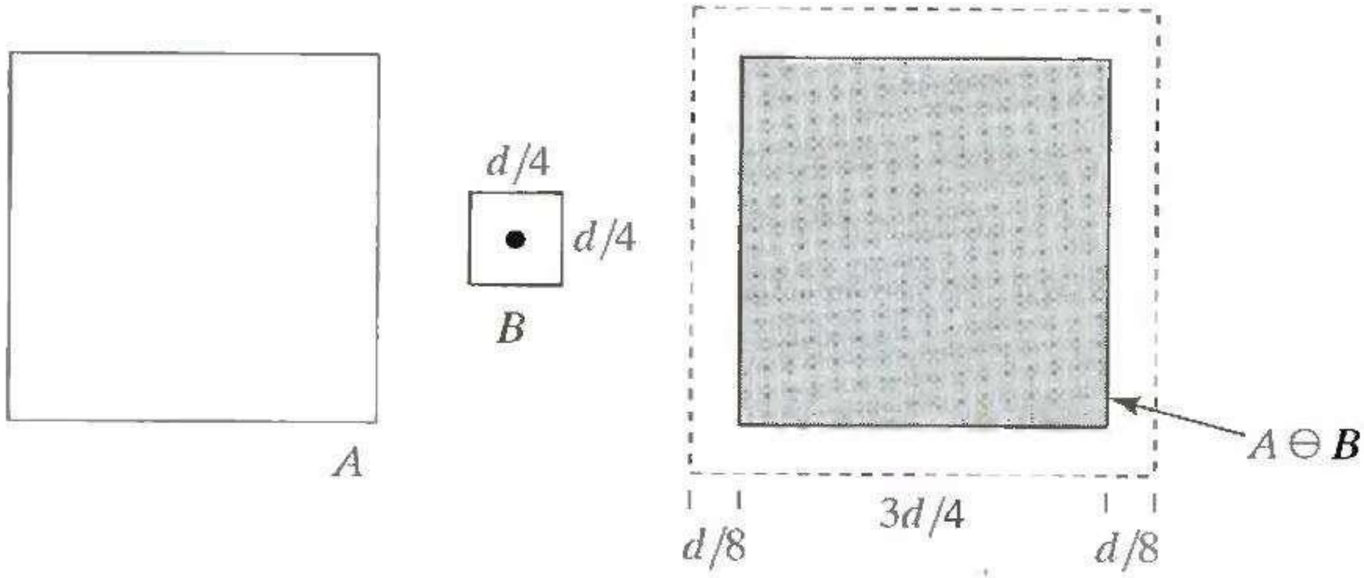
$$(A \ominus B)^c = A^c \oplus \hat{B}$$

$$\begin{aligned}(A \ominus B)^c &= \left\{ z \mid (B)_z \subseteq A \right\}^c \\ &= \left\{ z \mid (B)_z \subseteq A^c = \phi \right\}^c \\ &= \left\{ z \mid (B)_z \subseteq A^c \neq \phi \right\} \quad = A^c \oplus \hat{B}\end{aligned}$$

Erosion

- Erosion can be used to
  - **Shrink** or **thin** the objects in binary images
  - **Remove image components(how?)**
    - Erosion is a morphological filtering operation in which image details smaller than the structuring elements are filtered(removed)

Erosion – Example 1



a b c

FIGURE 9.6 (a) Set A. (b) Square structuring element. (c) Erosion of A by B, shown shaded

Erosion – Example 2

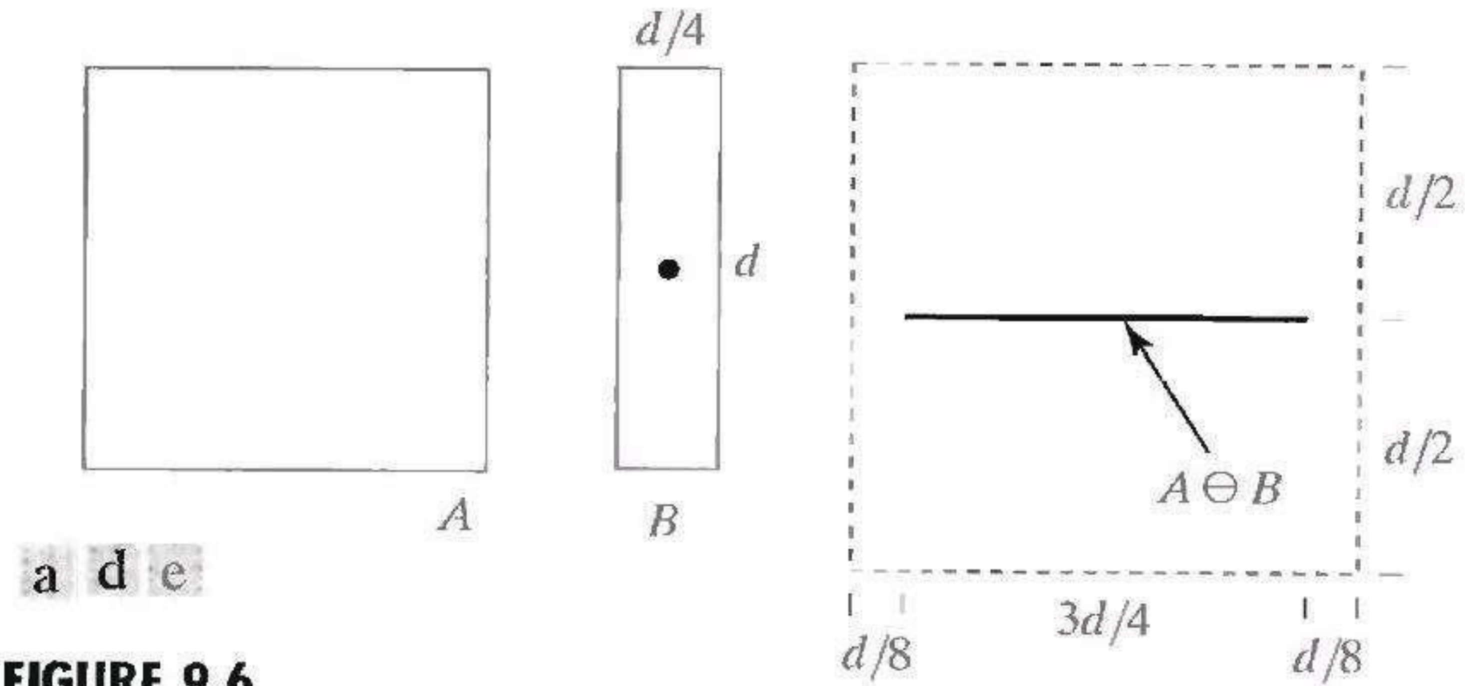


FIGURE 9.6

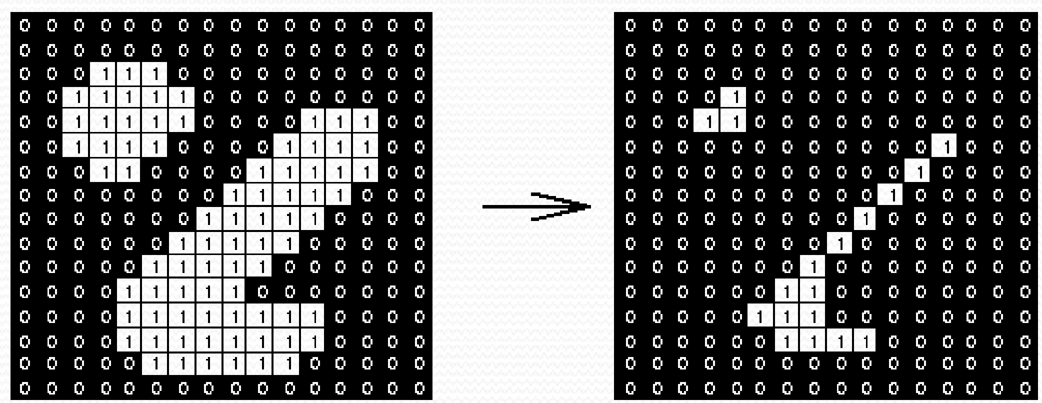
(a) Set A. (d) Elongated structuring element. (e) Erosion of A using this element.

Suppose that the structuring element is a 3×3 square; foreground pixels are represented by 1's and background pixels by 0's.

The structuring element is now superimposed over each foreground pixel (input pixel) in the image. If all the pixels below the structuring element are foreground pixels then the input pixel retains it's value. But if any of the pixels is a background pixel then the input pixel gets the background pixel value.

1	1	1
1	1	1
1	1	1

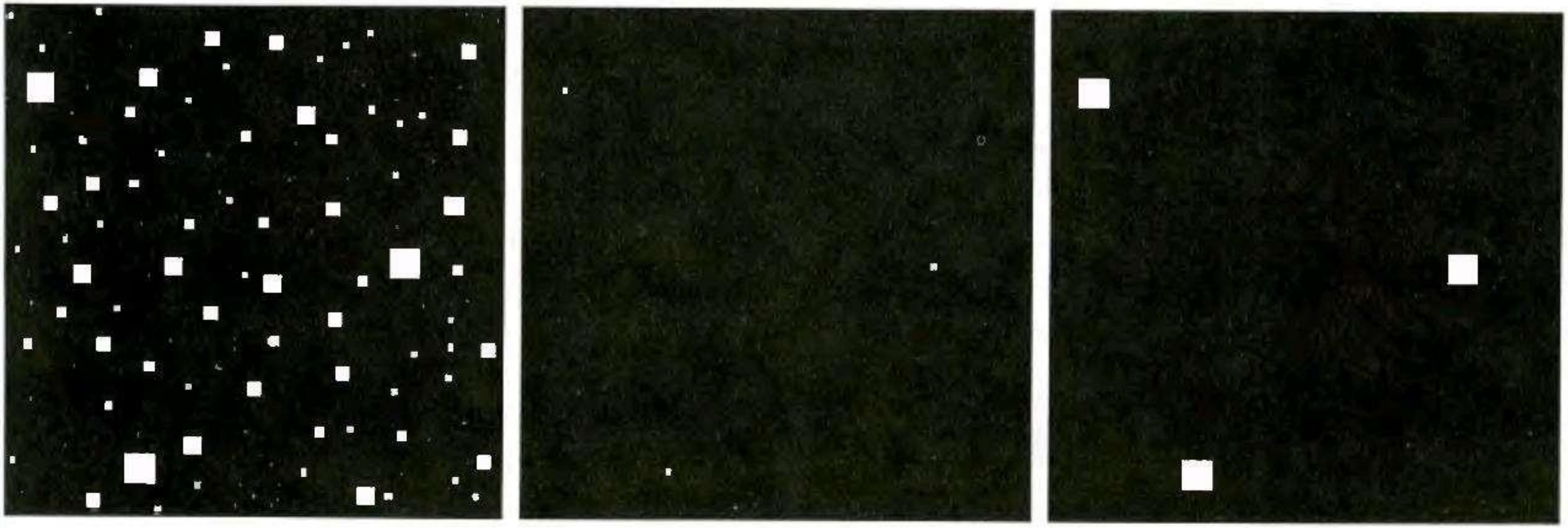
Structuring element



Dilation vs. Erosion

- **Dilation**:
  - **Grows** or **thickens** object in a binary image
  - Bridges gaps
  - Fills small holes of sufficiently small size
- **Erosion**:
  - **Shrinks** or **thins** objects in binary images
  - **Removes image components**
  - Erode away the boundaries of regions of foreground pixels
  - Areas of foreground pixels shrink in size, and holes within those areas become larger

Dilation and Erosion summary



a b c

FIGURE 9.7 (a) Image of squares of size 1, 3, 5, 7, 9, and 15 pixels on the side. (b) Erosion of (a) with a square structuring element of 1's, 13 pixels on the side. (c) Dilation of (b) with the same structuring element.

Dilation	Erosion
It increases the size of the objects.	It decreases the size of the objects.
It fills the holes and broken areas.	It removes the small anomalies.
It connects the areas that are separated by space smaller than structuring element.	It removes the objects smaller than the structuring element.
It increases the brightness of the objects.	It reduces the brightness of the bright objects.
Distributive, duality, translation and decomposition properties are followed.	It also follows the different properties like duality etc.
It is XOR of A and B.	It is dual of dilation.
It is used prior in Closing operation.	It is used later in Closing operation.
It is used later in Opening operation.	It is used prior in Opening operation.

Opening and Closing

- **Opening**
  - smoothes the contour of an object, breaks narrow strips, and eliminates thin projections.
  - The opening *A* by *B* is the erosion of *A* by *B*, followed by a dilation of the result by *B*

$$A \circ B = (A \ominus B) \oplus B$$

$$A \circ B = \cup \{(B)_z \mid (B)_z \subseteq A\}$$

- **Closing**
  - smoothes sections of contours, fuses narrow breaks and long thin gulfs, eliminates small holes and fills gaps in contours

$$A \bullet B = (A \oplus B) \ominus B$$

- These operations are dual to each other
- These operations are can be applied few times, but has <u>effect only once</u>

Opening

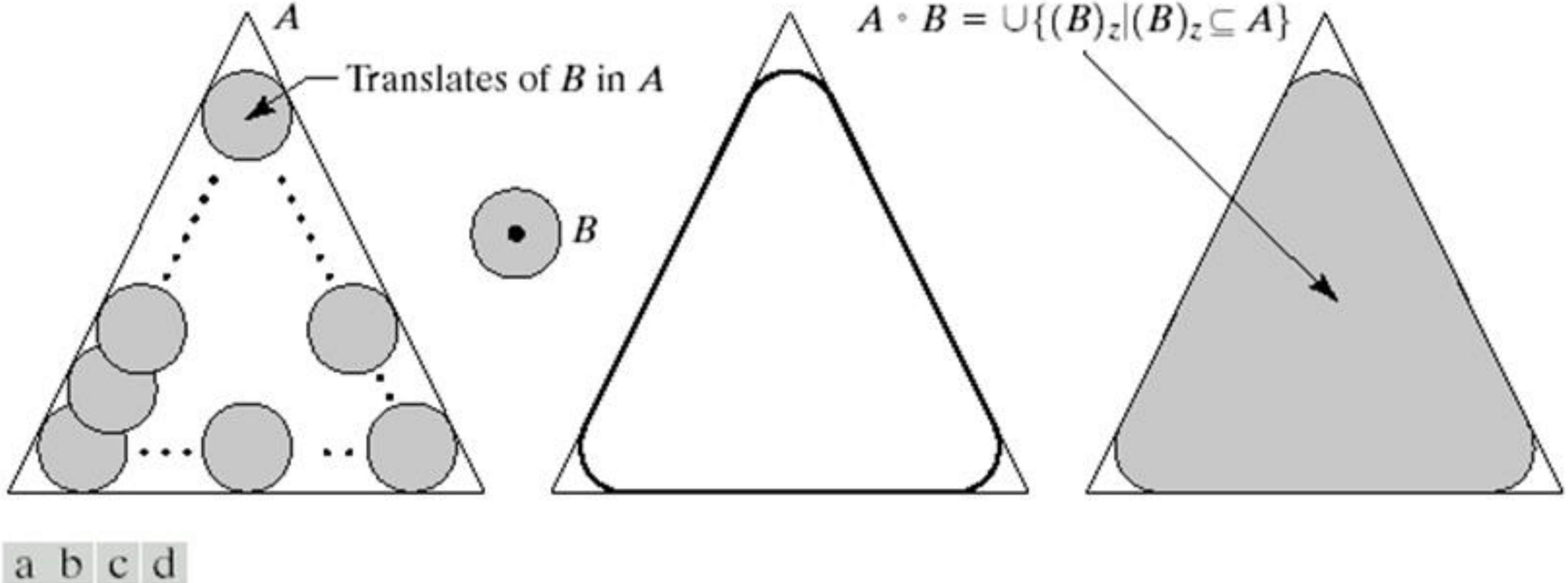
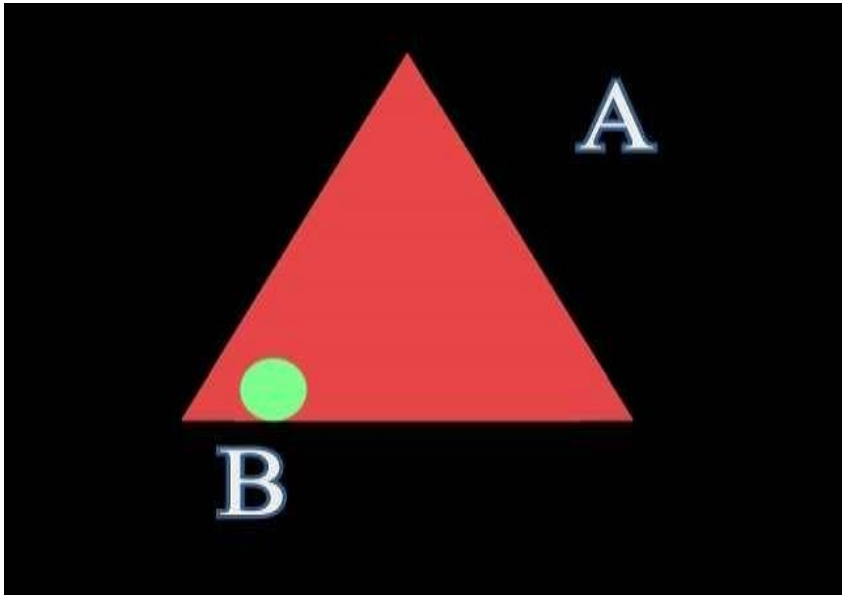


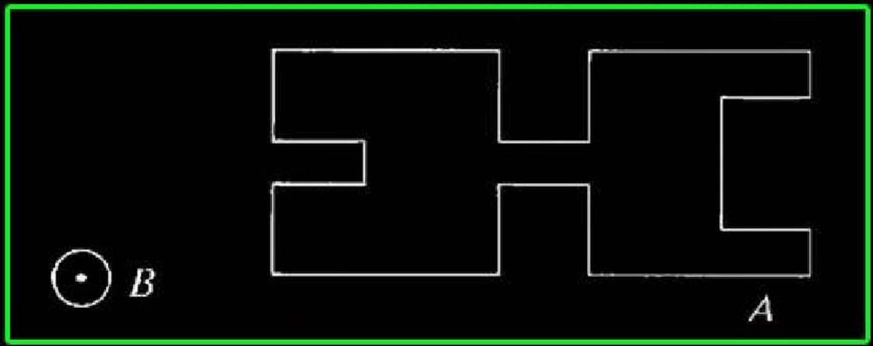
FIGURE 9.8 (a) Structuring element B "rolling" along the inner boundary of A (the dot indicates the origin of B). (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded).

Opening

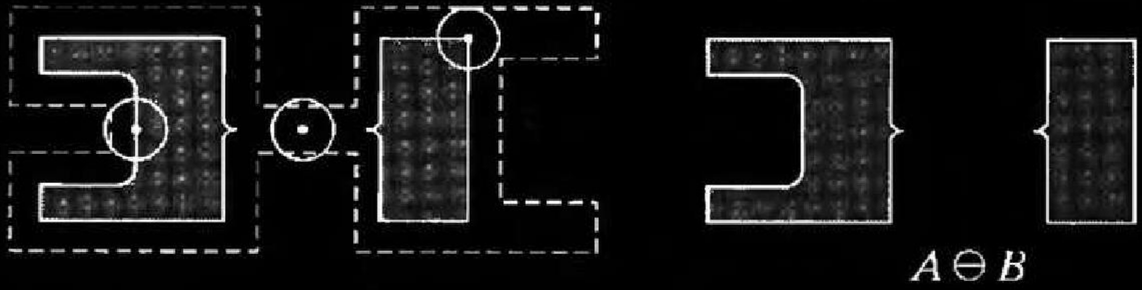
- First – erode A by B, and then dilate the result by B
- In other words, opening is **the unification of all B objects Entirely Contained in A**



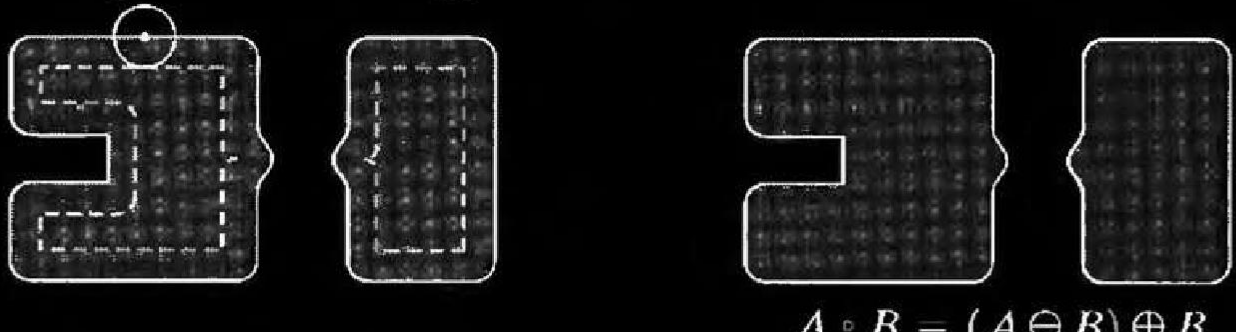
$$A \circ B = (A \ominus B) \oplus B$$



Erosion



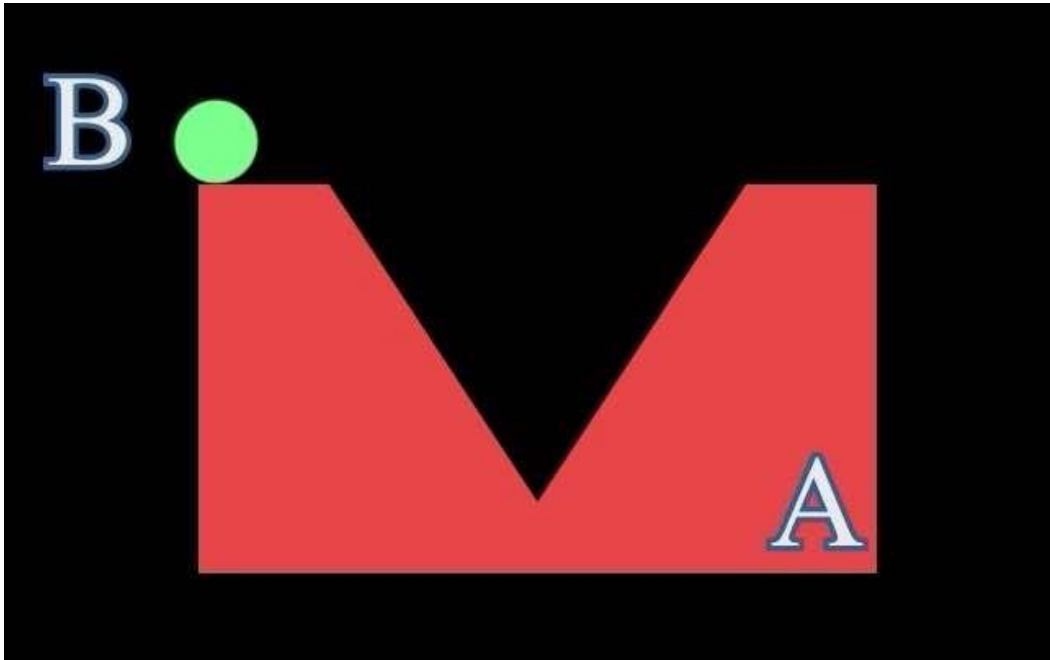
Opening

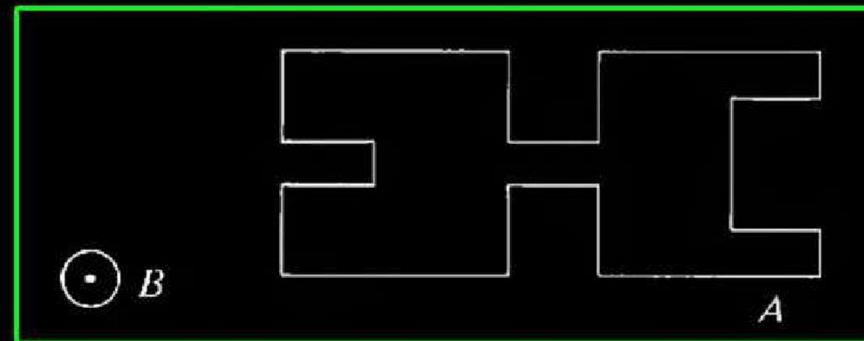


Closing

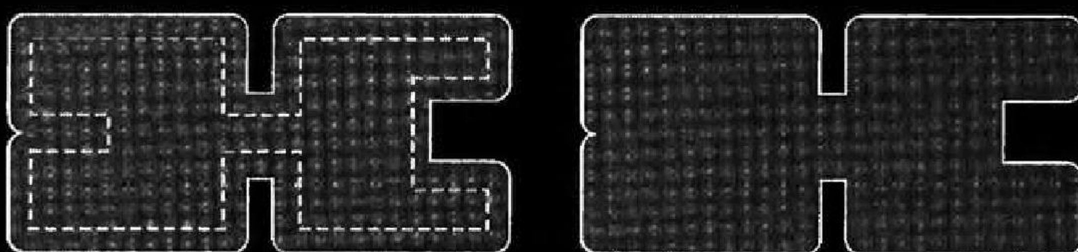
- First – dilate A by B, and then erode the result by B
- In other words, closing is the group of points, which the intersection of object B around them with object A – is not empty

$$A \cdot B = (A \oplus B) \ominus B$$



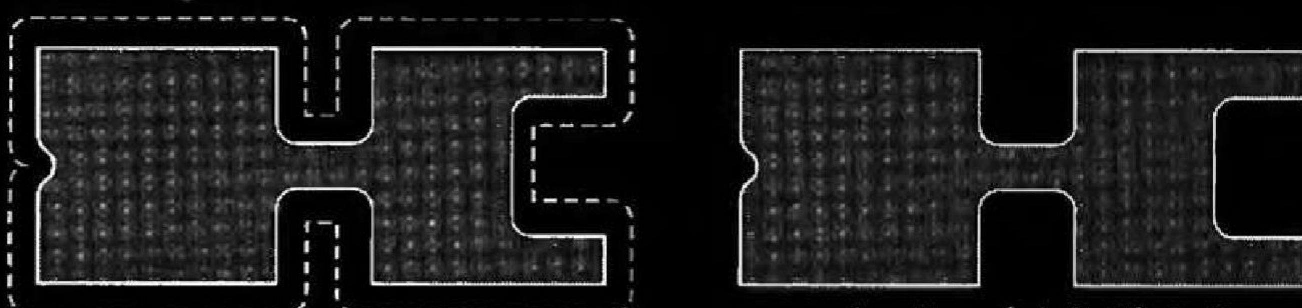


Dilation



$A \oplus B$

Closing



$A \bullet B = (A \oplus B) \ominus B$

Opening And Closing

The opening operation satisfies the following properties:

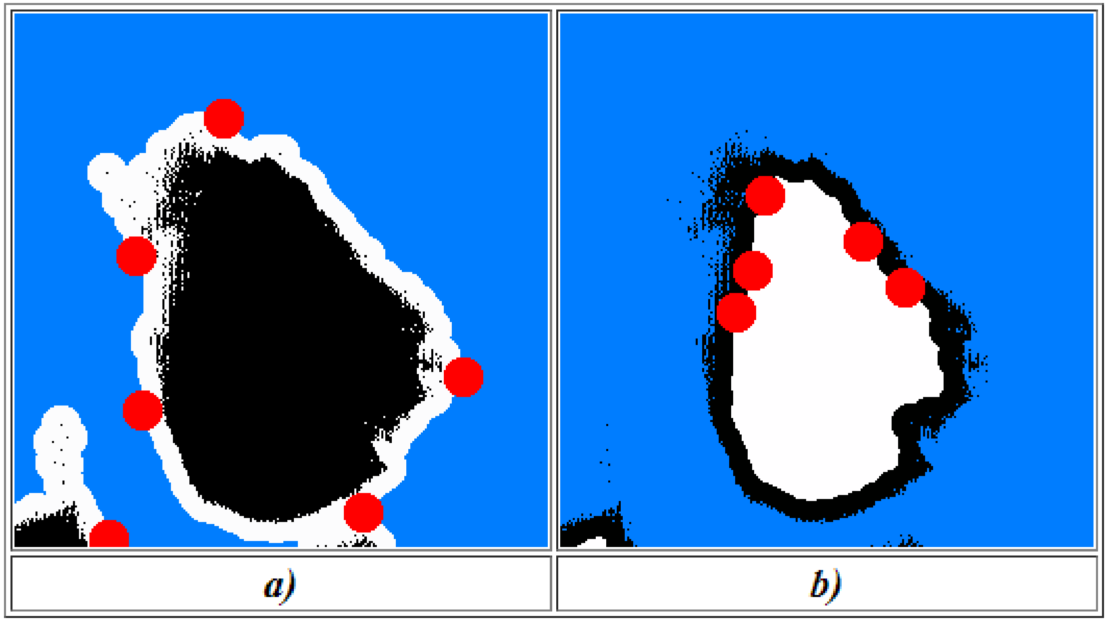
- $A \circ B$ is a subset (subimage) of A.
- If C is a subset of D, then $C \circ B$ is a subset of $D \circ B$

$$(A \circ B) \circ B = A \circ B$$

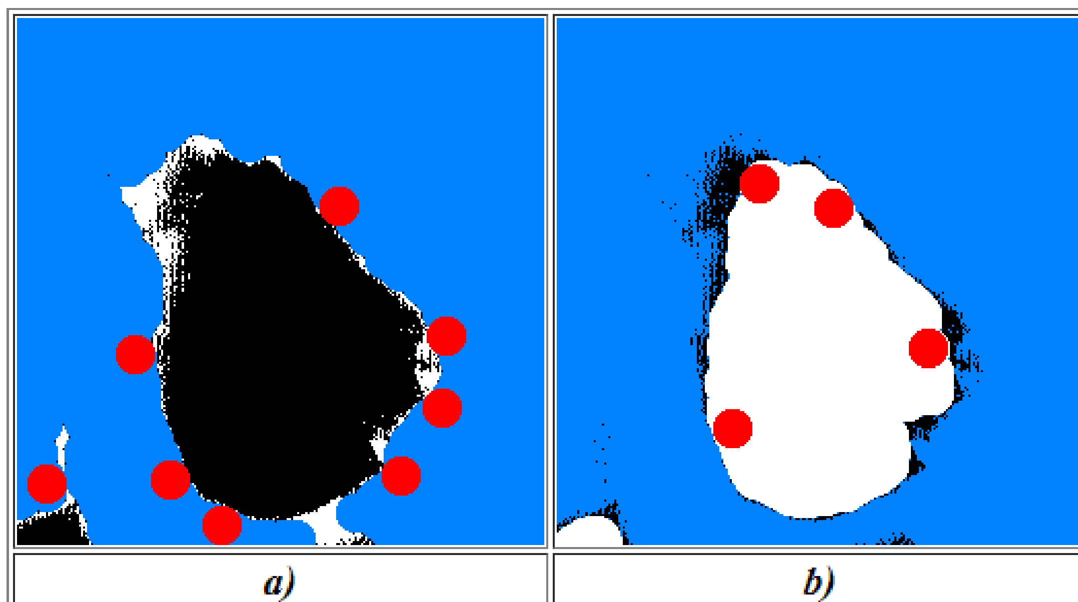
The closing operation satisfies the following properties:

- A is a subset (subimage) of $A \bullet B$
- If C is a subset of D, then $C \bullet B$ is a subset of $D \bullet B$

$$(A \bullet B) \bullet B = A \bullet B$$

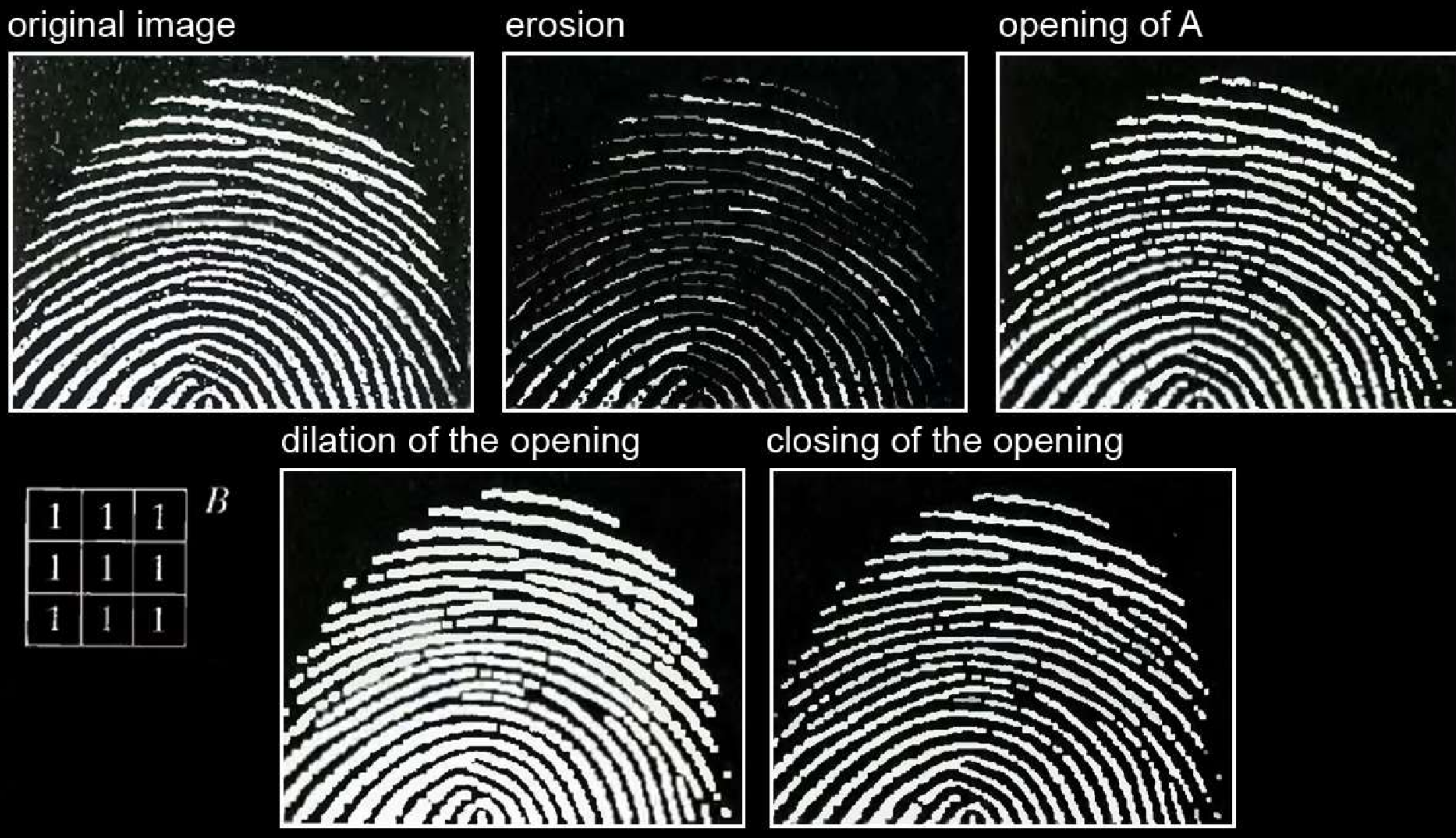


a)Dilation; b)Erosion



a)Closing; b)Opening

Use of opening and closing for morphological filtering



The Hit-or-Miss Transformation

- A basic morphological tool for **shape detection**.
- Let the origin of each shape be located at its center of gravity.
- If we want to find the location of a shape , say – **X** , at (larger) image, say – **A** :
  - Let X be enclosed by a small window, say – **W**.
  - The ***local background*** of X with respect to W is defined as the *set difference* (**W - X**).
  - Apply *erosion* operator of A by X, will get us the set of locations of the origin of X, such that X is completely contained in A.
  - It may be also viewed geometrically as the set of all locations of the origin of X at which X found a match (**hit**) in A.

The Hit-or-Miss Transformation

- Apply *erosion* operator on the *complement of A* by the *local background* set (W – X).
- Notice, that the set of locations for which X **exactly** fits inside A is the ***intersection*** of these two last operators above.

  This intersection is precisely the location sought.

Formally:

If B denotes the set composed of X and it's background –

B = (B1,B2) ; B1 = X , B2 = (W-X).

The match (or set of matches) of B in A, denoted $A \circledast B$ is:

$$A \circledast B = (A \ominus X) \cap \left[A^c \ominus (W - X)\right] \qquad B = (X, W - X)$$

The Hit-or-Miss Transformation

- Useful to identify specified configuration of pixels, such as, isolated foreground pixels or pixels at end of lines (end points)

The Hit-or-Miss Transformation

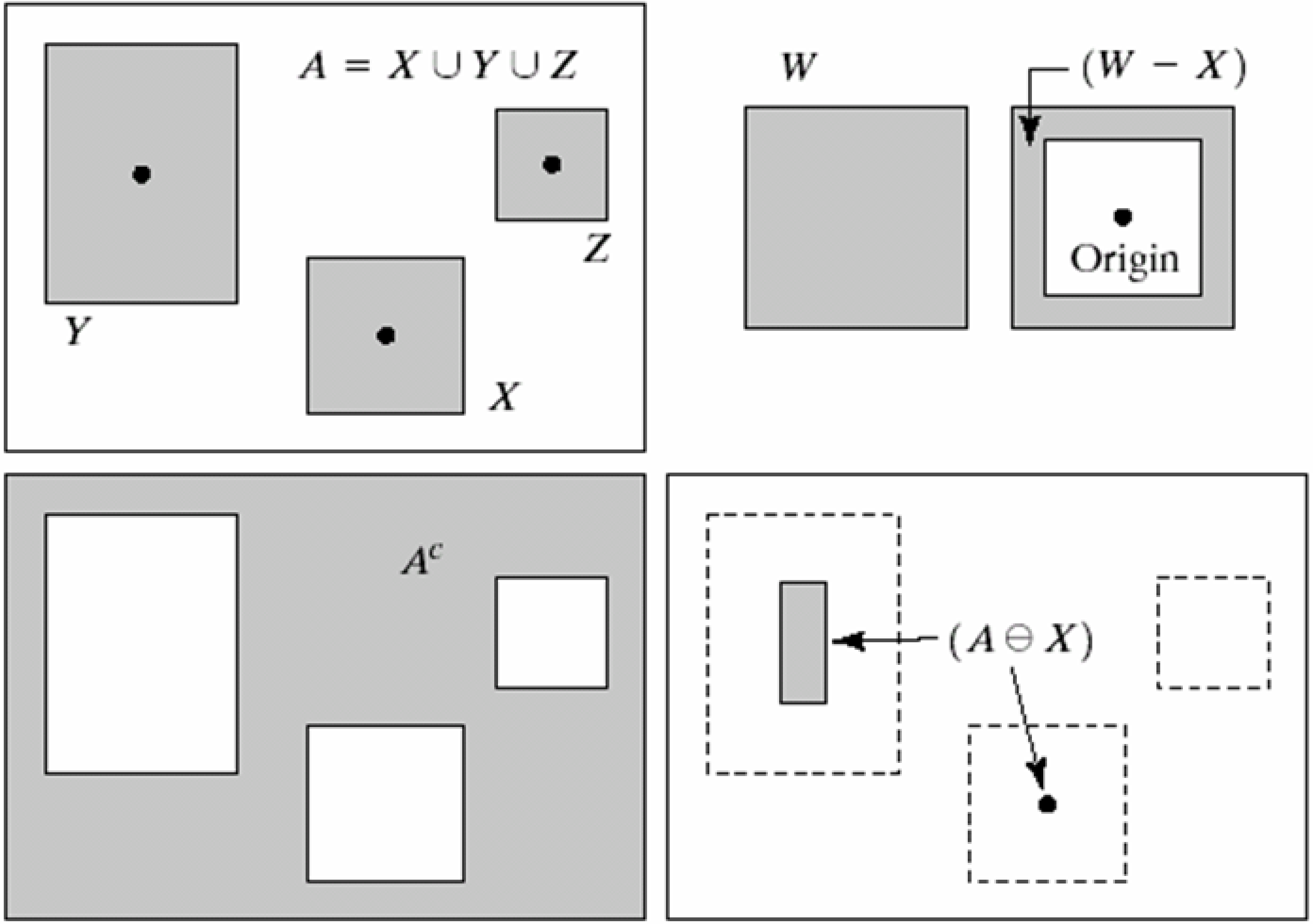


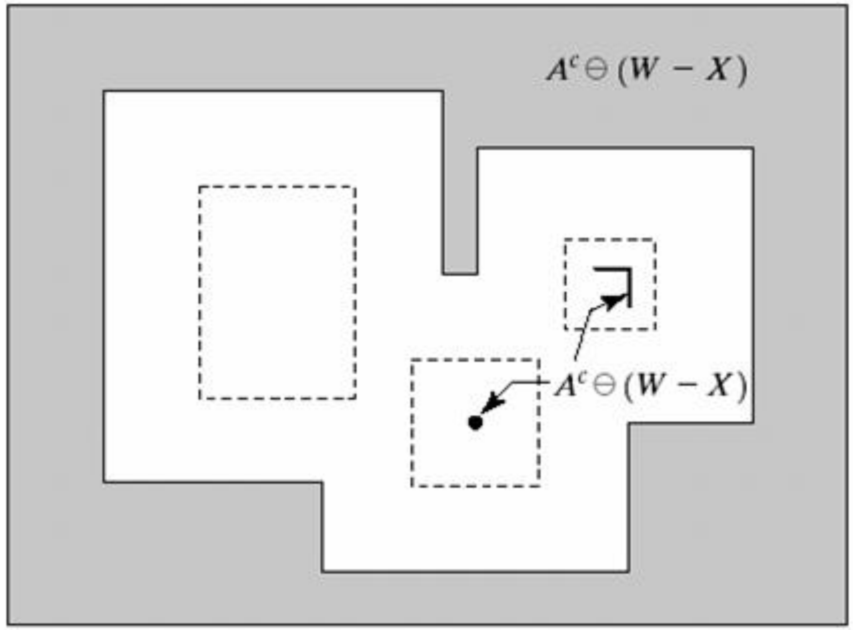
FIGURE 9.12 (a) Set A. (b) A window, W, and the local background of X with respect to W, $(W - X)$. (c) Complement of A. (d) Erosion of A by X. (e) Erosion of A^c by $(W - X)$. (f) Intersection of (d) and (e), showing the location of the origin of X, as desired.

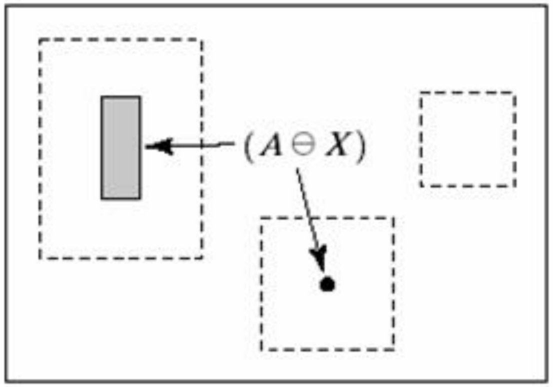
a, b

c, d

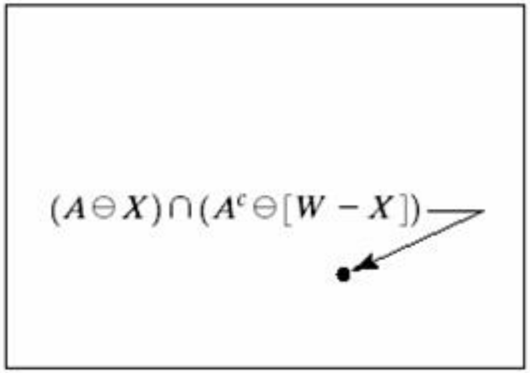
The Hit-or-Miss Transformation

e





d



f

a b
c d
e
f

FIGURE 9.12 (a) Set A. (b) A window, W, and the local background of X with respect to W, $(W - X)$. (c) Complement of A. (d) Erosion of A by X. (e) Erosion of A^c by $(W - X)$. (f) Intersection of (d) and (e), showing the location of the origin of X, as desired.

Illustration

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	1	1	1	0	0	0	0	0	0
0	1	1	1	0	0	0	0	0	0	0	0	1	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

B_1

$$\begin{array}{ccc} & 1 & \\ 1 & \boxed{1} & 1 \\ & 1 & \end{array}$$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	0	0	0	1	1	1	1	1	1
1	0	0	0	1	1	1	1	1	1	1	1	0	0	1	1
1	1	0	1	1	1	1	1	1	1	1	1	0	0	0	1
1	1	1	1	1	0	1	1	1	1	1	1	1	0	1	1
1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

B_2

$$\begin{array}{ccc} 1 & & 1 \\ & \boxed{} & \\ 1 & & 1 \end{array}$$

Complement of Original Image and B2

1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	0	1	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	1
1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	1	1	1	0	0	0	0	1
1	0	1	0	0	0	0	0	1	1	1	0	0	0	0	0
1	1	1	1	0	1	0	1	1	1	1	1	0	1	0	1
1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1
1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1

Erosion of A complement
And B2

Intersection of eroded images

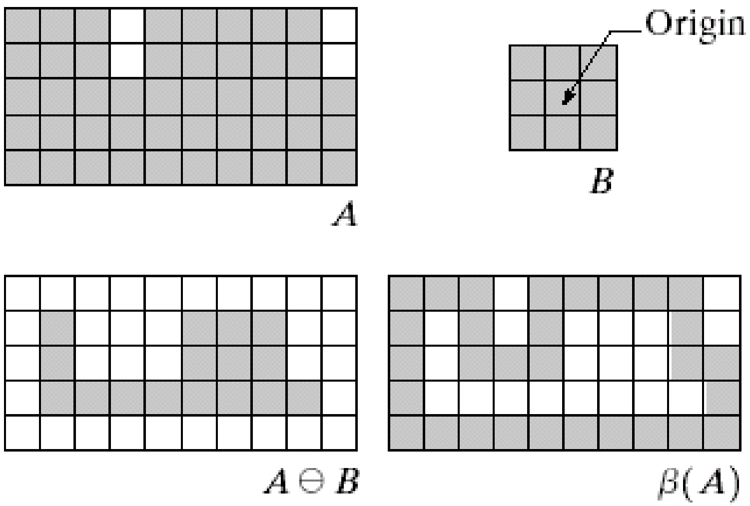
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Boundary Extraction

- Extracting the boundary (or outline) of an object is often extremely useful
- The boundary of a set *A*,

$$\beta(A) = A - (A \ominus B)$$

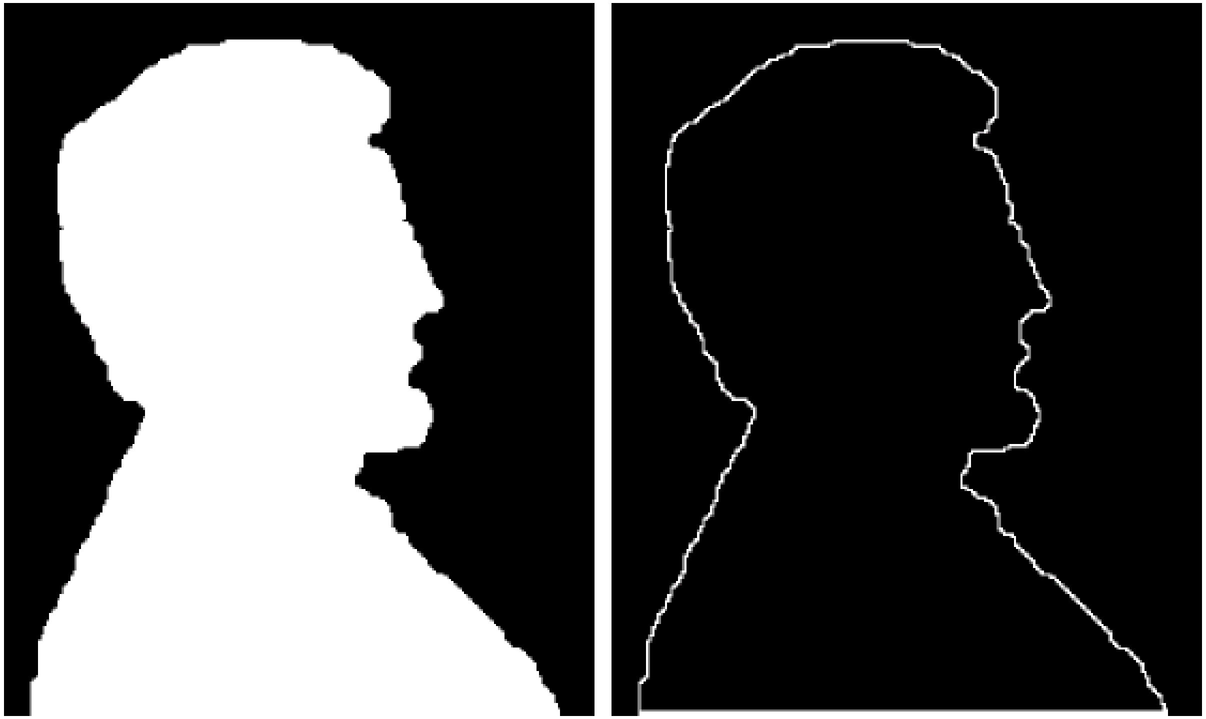
where *B* is a suitable structuring element.



a b
c d

FIGURE 9.13 (a) Set A. (b) Structuring element B. (c) A eroded by B. (d) Boundary, given by the set difference between A and its erosion.

Boundary Extraction



a b

FIGURE 9.14 (a) A simple binary image, with 1's represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

Thinning

- The thinning of a set *A* by a structuring element *B* can be defined in terms of the hit-or-miss transform:

$$A \otimes B = A - (A \circledast B)$$

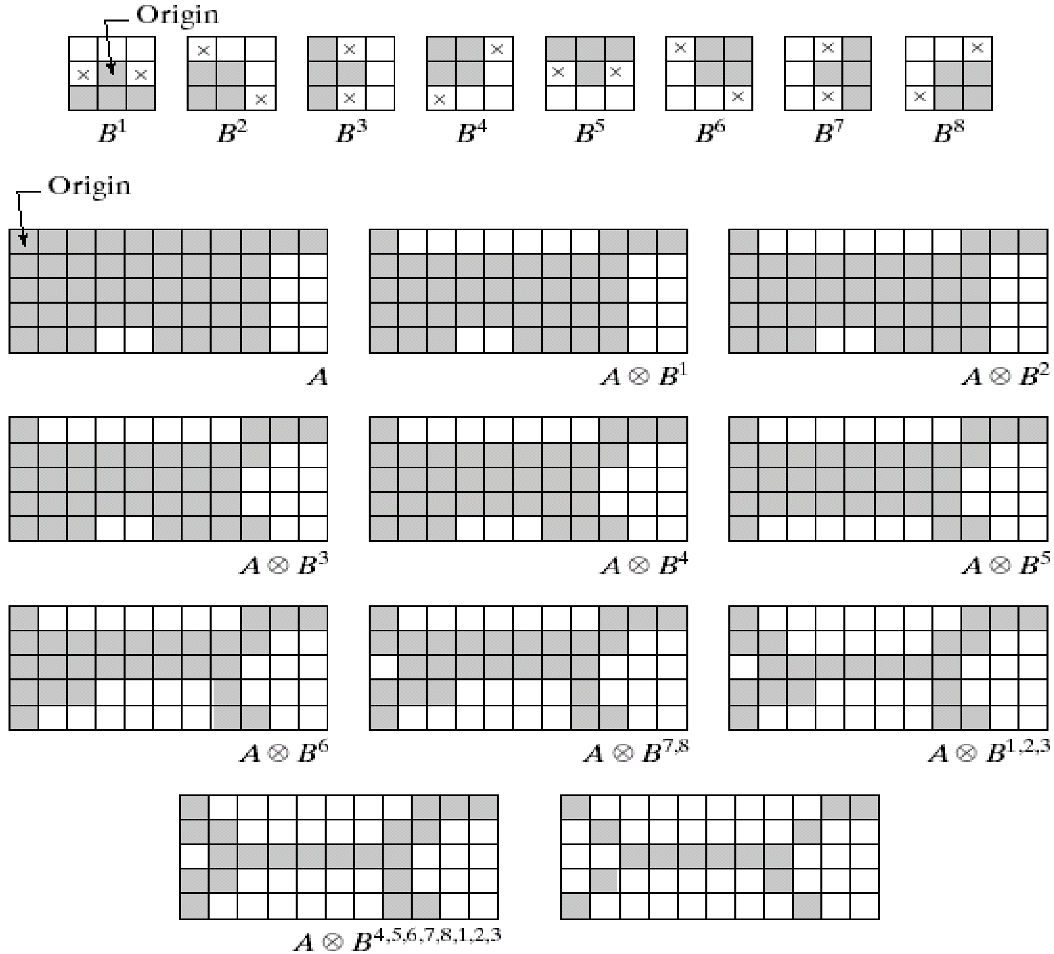
$$= A \cap (A \circledast B)^c$$

- A more useful expression for thinning A symmetrically is based on a sequence of structuring elements:

$$\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$$

$$A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2)\ldots) \otimes B^n)$$

Thinning



a
b c d
e f g
h i j
k l

FIGURE 9.21 (a) Sequence of rotated structuring elements used for thinning. (b) Set A. (c) Result of thinning with the first element. (d)–(i) Results of thinning with the next seven elements (there was no change between the seventh and eighth elements). (j) Result of using the first element again (there were no changes for the next two elements). (k) Result after convergence. (l) Conversion to m-connectivity.

Skeleton

- The skeleton of A is defined by terms of **erosions** and **openings**:

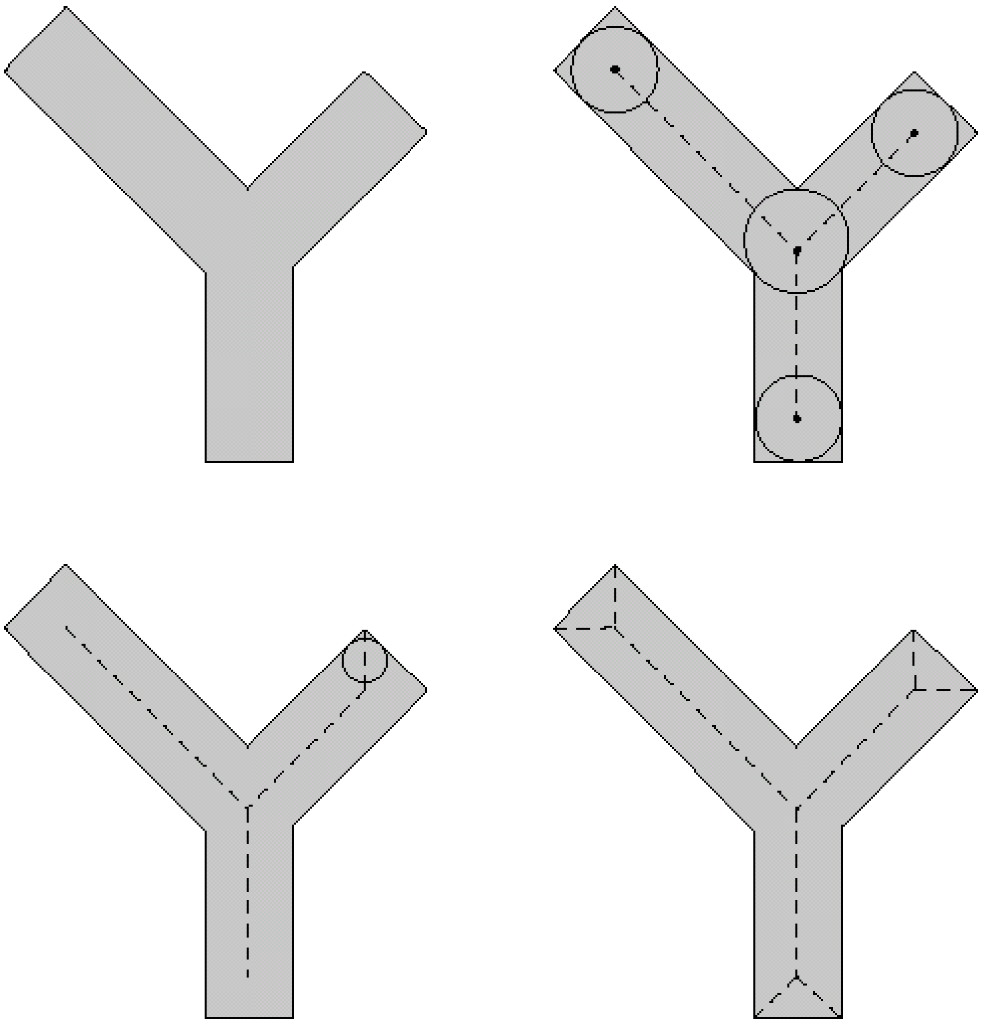
$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

- with $S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$
- Where B is the structuring element and $(A \ominus kB)$ indicates k successive erosions of A:

$$(A \ominus kB) = (\ldots((A \ominus B) \ominus B) \ominus \ldots) \ominus B$$

  - k times, and K is the last iterative step before A erodes to an empty set, in other words: $K = \max\{k \mid (A \ominus kB) \neq \emptyset\}$
- in conclusion S(A) can be obtained as the union of skeleton subsets Sk(A).

Skeletons



a	b
c	d

FIGURE 9.23 (a) Set A. (b) Various positions of maximum disks with centers on the skeleton of A. (c) Another maximum disk on a different segment of the skeleton of A. (d) Complete skeleton.

Skeleton

- A can be also be **reconstructed** from subsets Sk(A) by using the equation:

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$$

- Where $(S_k(A) \oplus kB)$ denotes k successive dilations of Sk(A) that is:

$$(S_k(A) \oplus kB) = ((\ldots((S_k(A) \oplus B) \oplus B) \oplus \ldots) \oplus B$$