Medical Image Processing

Module 6

CT Reconstruction

Syllabus

- Introduction, radon transform, algebraic reconstruction, some remarks on fourier transform and Filtering, filtered backprojection

INTRODUCTION

- Basic introduction to CT reconstruction.
- Describes the projection process of a parallel beam CT in terms of a mathematical function, the Radon Transform.
- Reconstruct the density distribution out of the projections first by an iterative solution of a big system of equations (algebraic reconstruction) and by filtered backprojection.
- Procedure to rotate images in an accurate way

INTRODUCTION

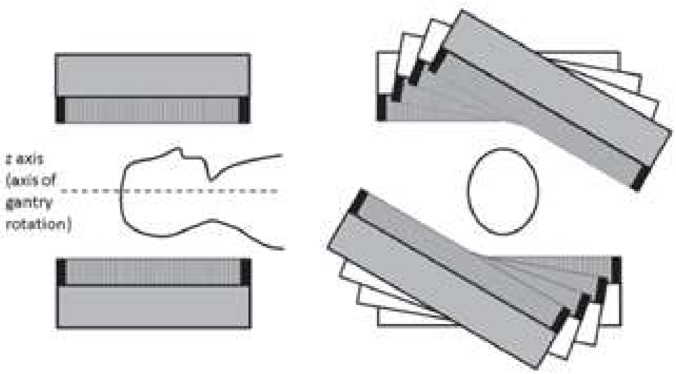
- Image reconstruction in CT is a mathematical process that generates tomographic images from X-ray projection data acquired at many different angles around the patient.
- Image reconstruction has fundamental impacts on image quality and therefore on radiation dose.
- For a given radiation dose it is desirable to reconstruct images with the lowest possible noise without sacrificing image accuracy and spatial resolution.
- Reconstructions that improve image quality can be translated into a reduction of radiation dose because images of the same quality can be reconstructed at lower dose.

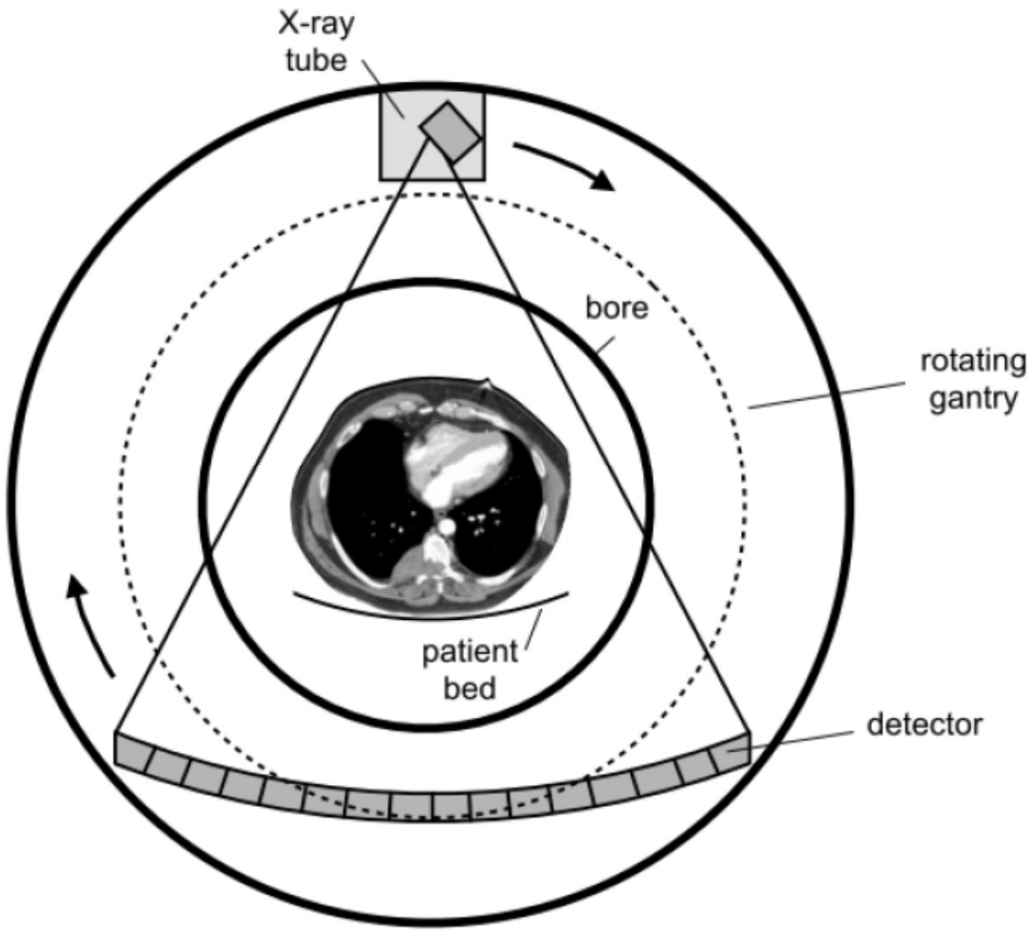
INTRODUCTION

- CT involves the exposure of the patient to x-ray radiation.
- This is associated with health risks (radiation-induced carcinogenesis) essentially proportional to the levels of radiation exposure.) 2% of cancers in the United States attributed to CT radiation.
- Radiation exposure can be directly reduced, this often leads to a lower SNR and/or lower image resolution) trade-o diagnostic quality vs. radiation dose.
- Another technique consists of sparse sampling (e.g., sparse-angle CT reconstruction)
- In some cases, only a \small" region-of-interest (ROI) needs to be reconstructed.

Computed Tomography acquisition

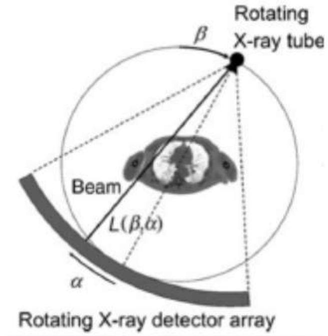
- Tomography: A series of planar images is acquired from different angles around the patent.





Computer Tomography (CT).

- Rotating X-ray tube + detector
- X-rays propagate through a x-section of patient



- Measure the exit beam <u>intensity</u> *integrated* along a <u>line</u> between X-ray source and detector

$$I_d = I_0 \exp\left[-\int_0^d \mu(s;\overline{E})\mathrm{d}s\right]$$

with μ linear attenuation coefficient as a function of the location s and the effective energy $\overline{E}$

- Basic measurement of CT: Line integral of the linear attenuation coefficient

Computed Tomography

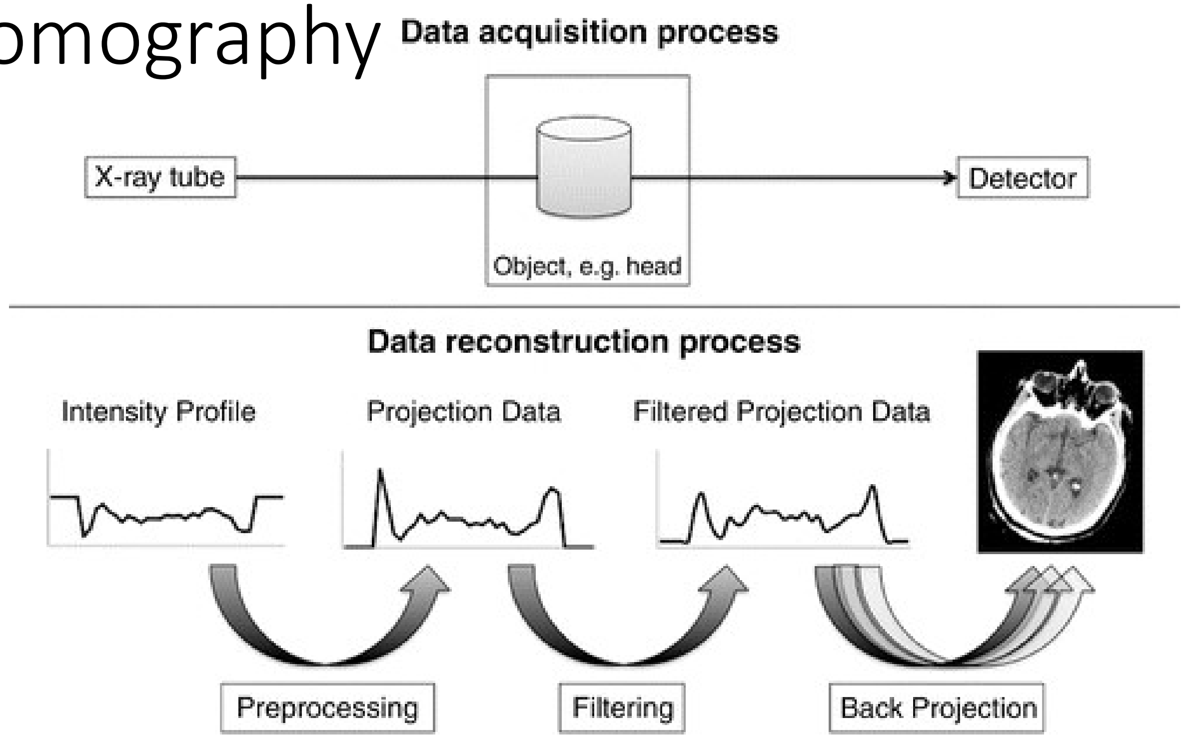


Figure : Simplified schematic of CT data reconstruction. Traditionally, several simplifications concerning the data acquisition process are made in the context of FBP: pencil-beam geometry of the x-ray, focal spot as an infinitely small point, intensity measured on a point located at the detector cell center. Regarding a single x-ray, photons with a known intensity are transmitted from the x-ray source through an object to the detector. According to the law of attenuation, the transmitted intensity decreases exponentially due to absorption within the object resulting in a lower measured intensity. Multiple x-rays result in the measurement of intensity profiles in the CT detector. By preprocessing, intensity values are transformed into attenuation values (projection data). Then, projection data are filtered using different reconstruction algorithms (kernels) to create specific image characteristics for soft-tissue or high-contrast visualization. Finally, the measured projection data are propagated into the image domain (back projection). Multiple projections are needed to solve the mathematical system with multiple equations and variables to generate the final CT image.

RADON TRANSFORM

- Classical x-ray imaging devices send x-ray beams through a body and display the mean density along the entire path of the ray. Density variations of soft tissues are rather small and are therefore difficult to see in conventional x-ray imaging.
- A mathematical description of the imaging process is given by the Radon transform.

RADON TRANSFORM

- Attenuation
- Attenuation of an x-ray when passing a volume element with attenuation coefficient μ and length ds is described by the Beer–Lambert law:

$$I = I_0 e^{-\mu \cdot ds}$$

- where I_0 and I are the intensities before and after the x-ray passed the volume element, respectively.
- Iterate Equation for different attenuation coefficients μi:

$$\begin{aligned} I &= I_0 e^{-\mu_1 \cdot ds} e^{-\mu_2 \cdot ds} \ldots e^{-\mu_n \cdot ds} \\ &= I_0 e^{-\sum_i \mu_i \cdot ds} \approx I_0 e^{-\int \mu(s)\, ds}. \end{aligned}$$

RADON TRANSFORM

- The integration is done along the ray. Out of the knowledge of the x-ray intensity before and after the x-ray passed the body we can therefore compute the integral of the attenuations

$$\int \mu(s)\,ds \approx \sum \mu_i \cdot ds = \log \frac{I_0}{I}$$

- CT-reconstruction means to derive the density (or attenuation) distribution μ out of the measured I_0/I.

Definition of the Radon transform in the plane

- To get a simple mathematical model for a CT we confine our description to a first generation CT-scanner that produces parallel projections through the object at different angles. The projections are restricted to a plane
- A slice through the human body can be interpreted as a two-dimensional distribution of attenuation coefficients $\mu(x, y)$.

RADON TRANSFORM

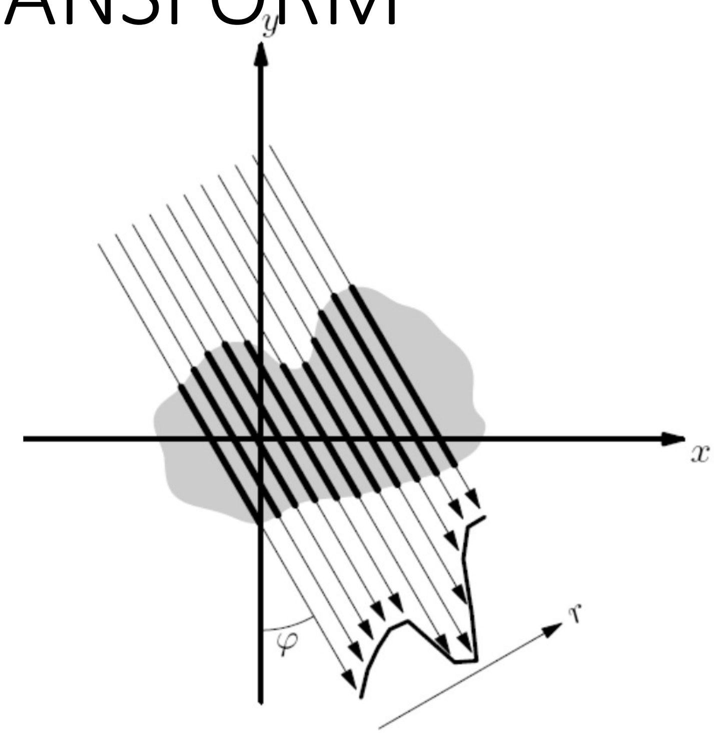


FIGURE: First generation CT scanners transmit x-rays using the same angle and different r, then change the angle. The lengths of the arrows in the figure correspond to the attenuation. The graph on the lower right therefore shows the attenuation as a function of r.

RADON TRANSFORM

- The x-rays sent through the slice have different angles and different distances from the origin. For line parametrization the Hesse normal form can be seen in Figure

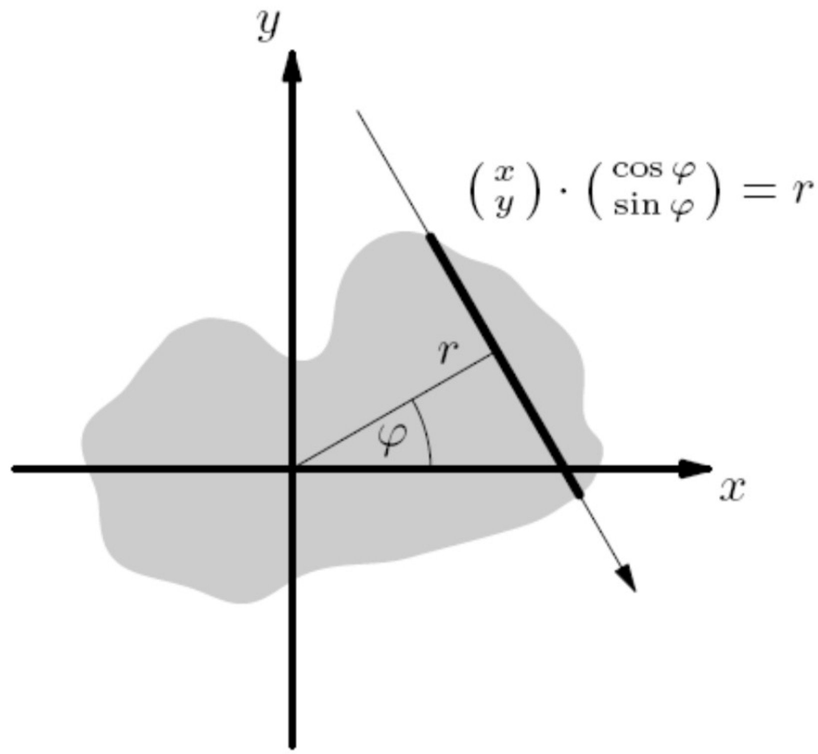


FIGURE : An x-ray beam goes through a gray body (a rather monotonic image). The normal vector of the line has an angle of ϕ to the x-axis and the orthogonal distance to the origin is r.
The equation of the line is x cosϕ + y sin ϕ = r.

RADON TRANSFORM

- The x-ray beam there is attenuated by every element of f(x, y) along the line with the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = x\cos\varphi + y\sin\varphi = r$$

If we think of this line as the rotation of the y-axis parallel line $\binom{r}{t}$, $t \in \mathbb{R}$ turned through φ degrees using a rotation matrix , $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, we have a parametrization of the line by multiplying matrix and vector:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} r\cos\varphi - t\sin\varphi \\ r\sin\varphi + t\cos\varphi \end{pmatrix}, \; t \in \mathbb{R}$$

RADON TRANSFORM

According to Equation the attenuation observed at the detector is given by the line integral

$$\int\limits_{\binom{x}{y}\cdot\binom{\cos\varphi}{\sin\varphi}=r} f(x,y)\,ds = \int\limits_{-\infty}^{\infty} f(r\cos\varphi - t\sin\varphi, r\sin\varphi + t\cos\varphi)\,dt,$$

using the parametrization from Equation Equation can be seen as a transformation from functions $f(x,y)$ defined in the plane $\mathbb{R}^2$ to other functions $R_f(\varphi,r)$ defined on $[0,2\pi]\times\mathbb{R}$:

$$f(x,y) \mapsto R_f(\varphi,r)$$

$$R_f(\varphi,r) := \int\limits_{\binom{x}{y}\cdot\binom{\cos\varphi}{\sin\varphi}=r} f(x,y)\,ds$$

This transformation is called the Radon transform

A CT measures the attenuation along many lines of different angles and radii and reconstructs the density distribution from them. The overall attenuation in the direction of a ray is the Radon transform

RADON TRANSFORM
Basic properties

Because of the linearity of integration (i.e., $\int f + \lambda g = \int f + \lambda \int g$), the Radon transform is linear as well:

$$R_{f+\lambda g} = R_f + \lambda R_g \quad \text{for } f \text{ and } g \text{ functions, and } \lambda \in \mathbb{R}.$$

Every image can be seen as a sum of images of single points in their respective gray values, we therefore take a look at the most elementary example, the Radon transformation of a point (x_0, y_0). As this would be zero everywhere, we should rather think of a small disc. Clearly the integral along lines that do not go through the point is zero and for the others it is more or less the same value, the attenuation caused by the point/disc. Now we have to find those parameters (φ, r) that define a line through (x_0, y_0). With the angle φ as the independent variable, the radial distance r is given by the line equation

$$r = x_0 \cos\varphi + y_0 \sin\varphi.$$

Therefore the Radon transform of a point/small disc is zero everywhere apart from the curve

RADON TRANSFORM
Basic properties

- The Radon transform of a point/small disc is zero everywhere apart from the curve, a trigonometric curve that can be seen in Figure.
- The shape of this curve also motivates the name sinogram for the Radon transform of an image.

FIGURE: The image of a disc and its Radon transform with appropriate intensity windowing. The second pair of images are a line and its Radon transform. The abscissa of the Radon transform is $r = 1, \ldots, 128$, and the ordinate is $\phi = 0, \ldots, 179$.

RADON TRANSFORM implementation

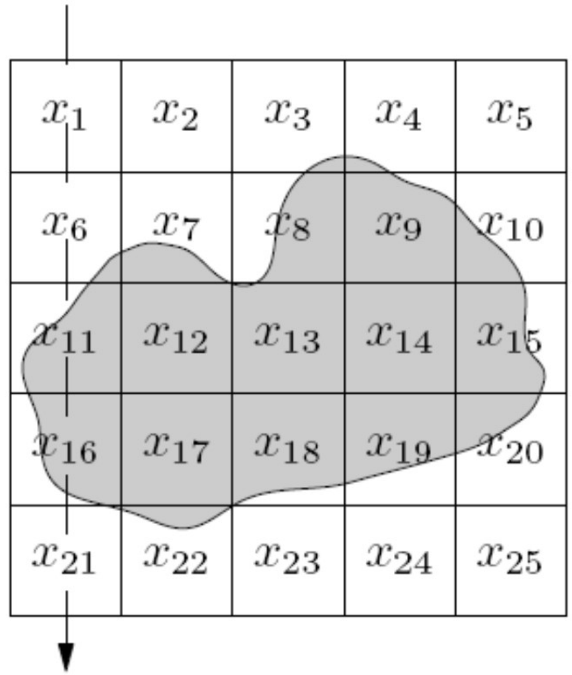


FIGURE : The area where the gray body lies is discretized, i.e., divided in 25 squares. We assume a constant density in each of the squares. The ray passing through the first column is solely attenuated by the elements $x_1, x_6, \ldots, x_{21}$. Therefore $R_f(0,0) = x_1 + x_6 + x_{11} + x_{16} + x_{21}$.

RADON TRANSFORM implementation

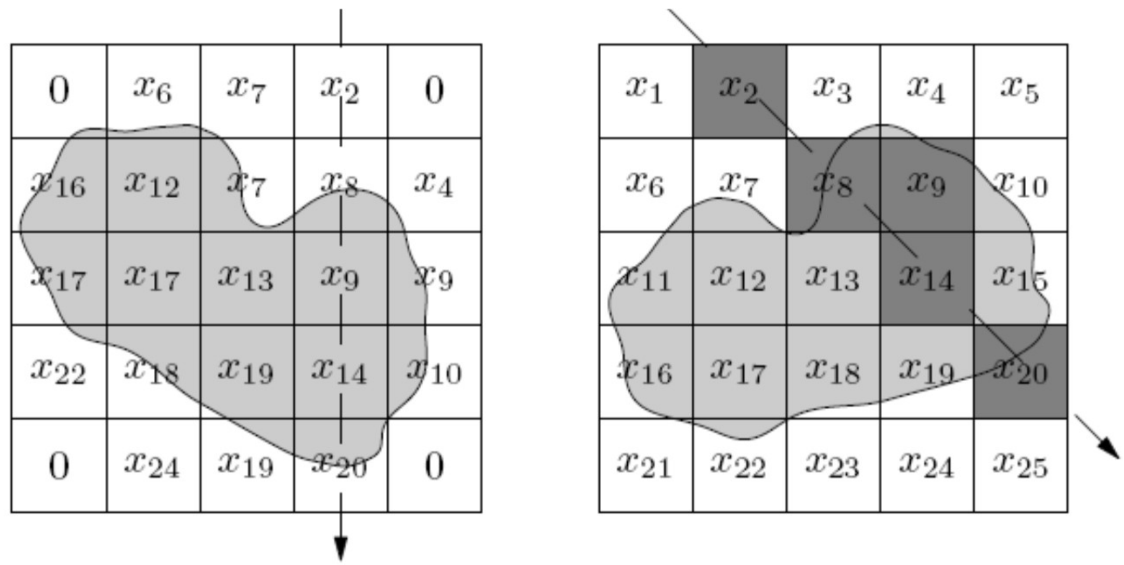


FIGURE : In the left image we see our test body rotated 45° clockwise. The matrix behind is rotated simultaneously and values outside the rotated image are set to zero. If we look at the squares in the fourth column we can see that they correspond to squares near a line with $\varphi = 45$ in the original image, as can be seen in the right image.

RADON TRANSFORM implementation

A typical slice image and its Radon transform

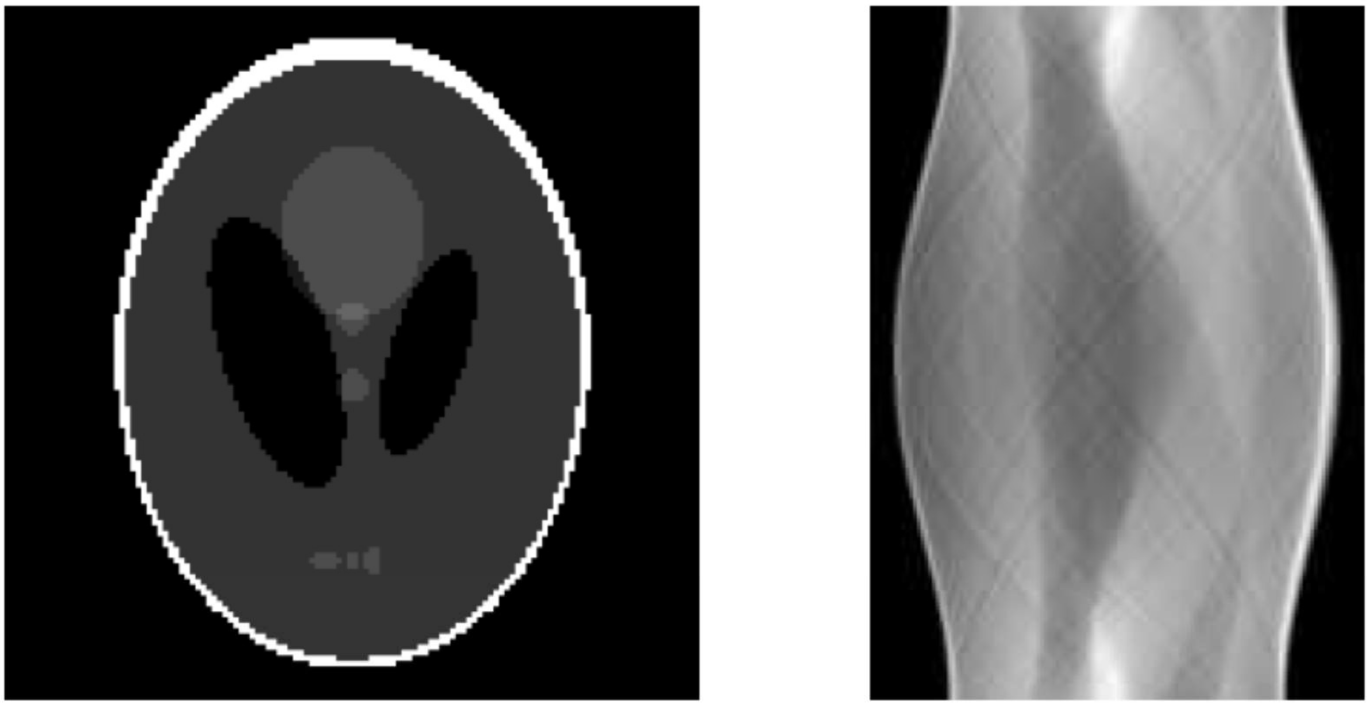
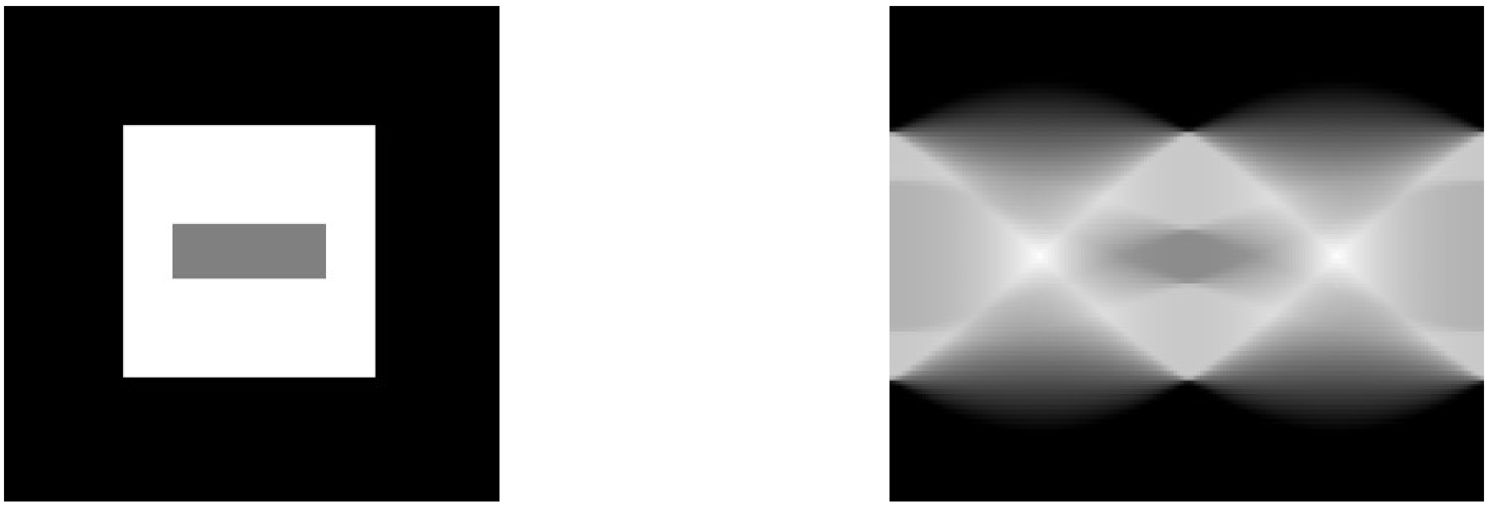


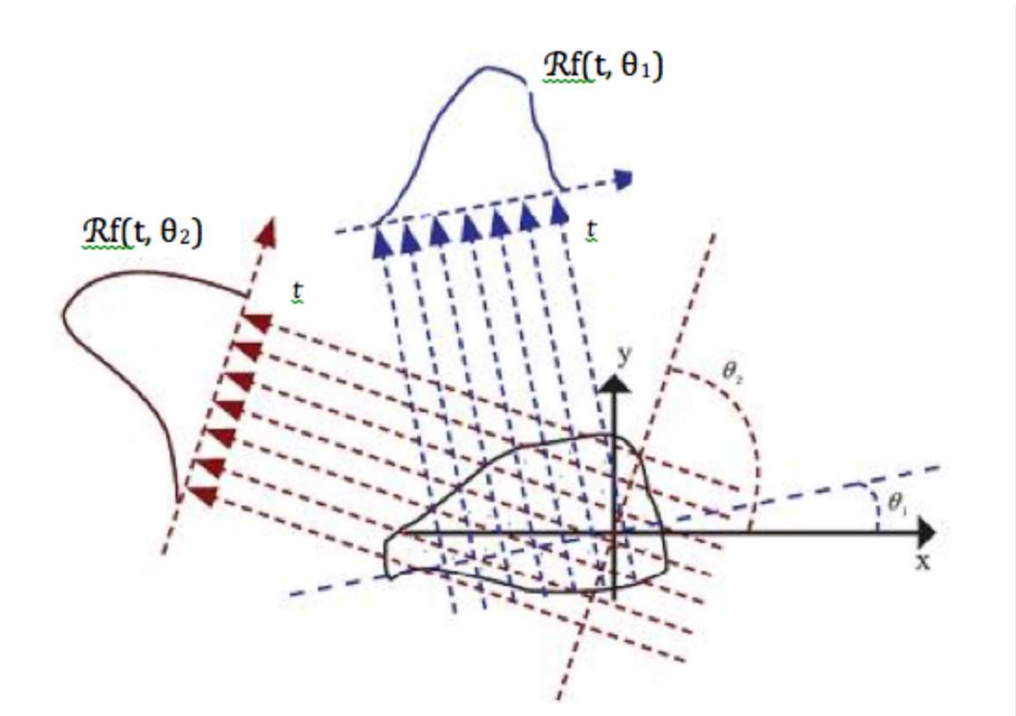
FIGURE : The Shepp and Logan head phantom and its (appropriately intensity windowed) Radon transform. The abscissa of the Radon transform is r, the ordinate is φ.

RADON TRANSFORM



A simple image (left) and the sinogram (right) produced by applying the Radon Transform

Radon Transform for Θ1 and Θ 2



ALGEBRAIC RECONSTRUCTION

- See the reconstruction of the original density distribution from the Radon transform can be interpreted as the solution of a large system of linear equations, and how this system can be solved.
- In the construction of the linear system we will use the same method we used for the implementation of the Radon transform above.

ALGEBRAIC RECONSTRUCTION

reconstruction task is the computation of the x_i, $i \in 1, \dots n^2$ out of the attenuations they cause to given x-rays, i.e., out of the Radon transform.

As we have seen above nearest neighbor interpolation guarantees our MATLAB implementation of the Radon transform to be a linear combination of the gray values x_i, $i \in 1, \dots n^2$, like in Equation 10.10 for `R(1,1)`,`R(1,2)`,`R(1,3)`, ...

By using 180 angle steps and n parallel beams we get $180 \times n$ linear equations in the unknowns x_i, $i \in 1, \dots n^2$ if we construct all the rays through the image in this way. If we put the whole image as well as the Radon transform in vectors, we can reformulate this using an $(180 \times n) \times (n \times n)$ matrix S such that

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}}_{S} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_{n^2} \end{pmatrix} = \begin{pmatrix} \texttt{R(1,1)} \\ \texttt{R(1,2)} \\ \dots \\ \texttt{R(1,n)} \\ \texttt{R(2,1)} \\ \dots \\ \texttt{R(180,n)} \end{pmatrix}. \tag{10.11}$$

ALGEBRAIC RECONSTRUCTION

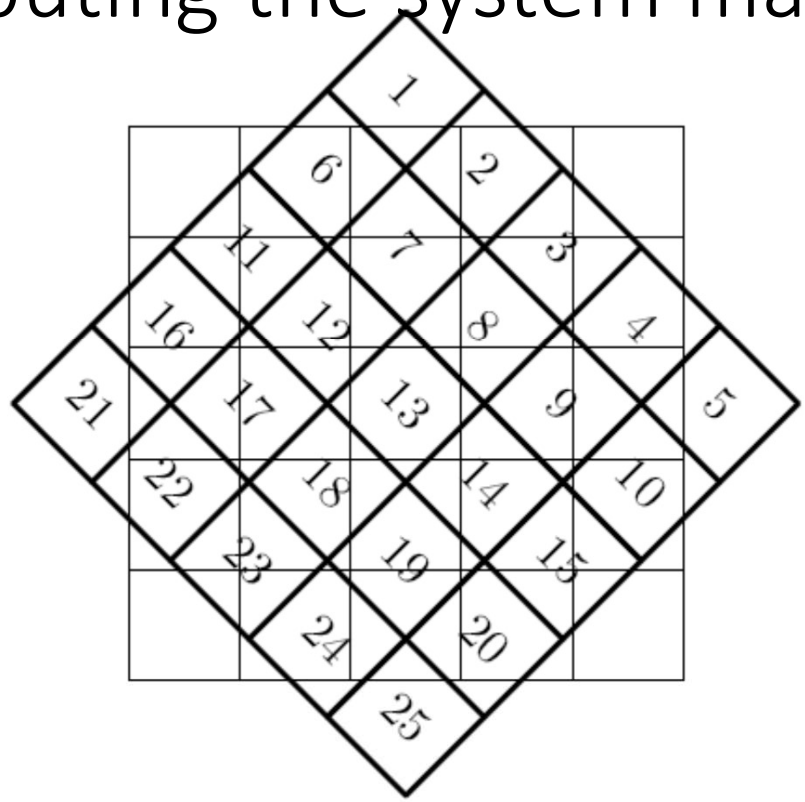
- The matrix S is called the System Matrix of the CT.
- The system matrix completely describes the image formation process of the scanner.
- Please note that the matrix starts with the sums of columns.
- By the definition of matrix multiplication it is clear that the elements in a line are the coefficients of the xi in the linear equation for R(i,j).
- A similar definition would have been possible even if we accounted for the way the beam crosses a square.
- Additional weighting factors w_{ij} would have been multiplied to the components of the matrix S.
- As the number of equations should be at least the number of unknowns, more angle steps have to be used for image sizes bigger than 180×180.

Computing the system matrix

- The system matrix could be constructed by finding nearest neighboring squares to a line crossing the image.
- This could be done by basic line drawing algorithms.
- However we will do this in the same way as we implemented the Radon transform above, by rotating the image instead of the line.
- In the columns of the rotated matrix we find field numbers that are near to lines with angle φ through the original matrix. The situation is illustrated for a line in Figure, using xi instead of numbers i.
- Summed up, the procedure to derive the system matrix consists of the following steps:

1. Generate a test matrix T filled with 1, . . . n2.
2. Rotate it by the angle −φ.
3. The non-zero numbers in column r are the pixels of the original matrix near to a line with angle φ and distance r. The corresponding element in the appropriate line of the system matrix S has to be increased by one.

Computing the system matrix



0	6	7	2	0
16	12	7	8	4
17	17	13	9	9
22	18	19	14	10
0	24	19	20	0

FIGURE A matrix filled with the numbers $1, \dots n^2$, rotated 45°clockwise. In the background a pattern for the rotated matrix can be seen; the right matrix shows the pattern filled with the nearest numbers of the rotated matrix.

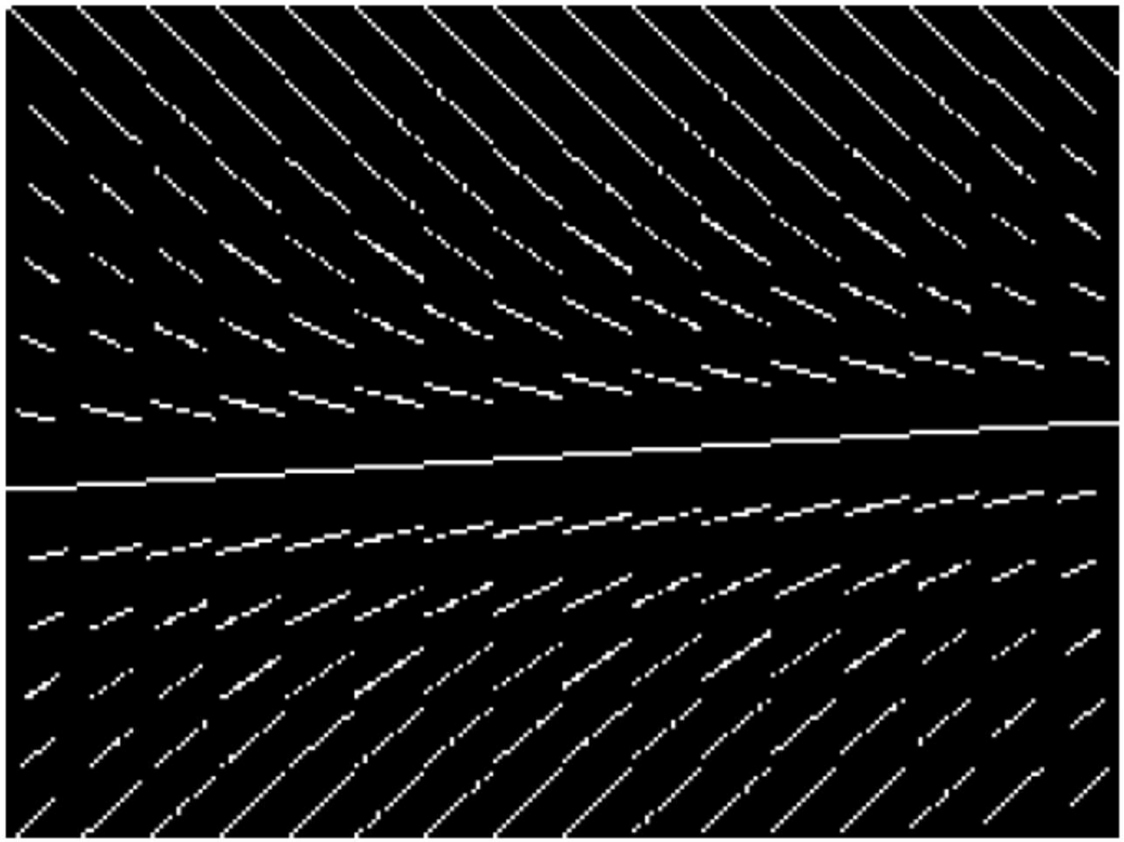


FIGURE An example for the shape of a $(16 \times 12) \times 16^2$ system matrix produced with the MATLAB code above. Black corresponds to zero, white is 1 or 2. The dominance of the black background shows that we are dealing with a sparse matrix.

FOURIER TRANSFORM AND FILTERING

- First we recall the definition of the Fourier transform for functions of one and two variables

$$\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{-\mathbf{i}kx} \, dx$$

$$\hat{f}(k,s) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x,y) e^{-\mathbf{i}\binom{x}{y}\cdot\binom{k}{s}} \, dx \, dy$$

- The Fourier transform of the disturbed function was set to zero at high frequencies and left unchanged for the others. This can also be done by a multiplication of the disturbed function's Fourier transform by a step function which is 0 at the frequencies we want to suppress, and 1 else. As this is the way the filter acts in the frequency domain we call the step function the frequency response of the filter.

FOURIER TRANSFORM AND FILTERING

- A linear filtering operations (convolutions) can be accomplished by a multiplication in Fourier space. A formal definition of the convolution of two functions avoiding the notation of a Fourier transform is given in Equation

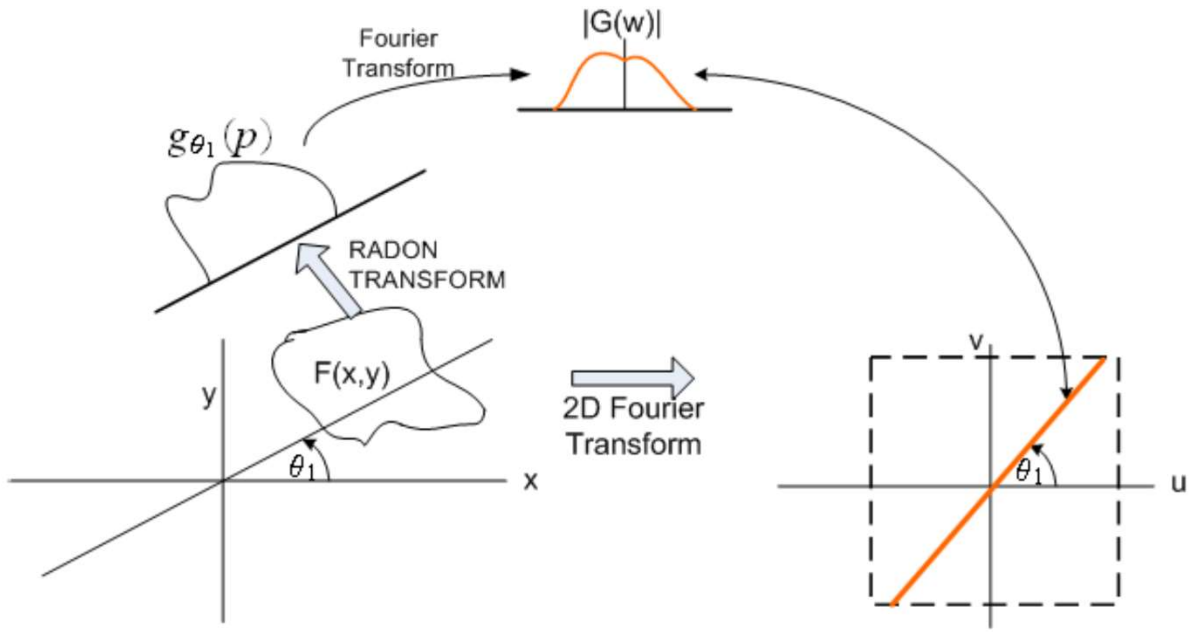
$$f \star g\,(x) := \int\limits_{-\infty}^{\infty} f(t)g(x-t)\; dt$$

$$\widehat{f \star g} = \sqrt{2\pi}\,\hat{f}\,\hat{g}$$

$$\begin{aligned}\sqrt{2\pi}\,\widehat{f \star g} &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(t)g(x-t)\; dt\; e^{-\mathbf{i}kx}\; dx = \int\limits_{-\infty}^{\infty} f(t) \int\limits_{-\infty}^{\infty} g(x-t)\; e^{-\mathbf{i}kx}\; dx\; dt \\ &= \int\limits_{-\infty}^{\infty} f(t) \int\limits_{-\infty}^{\infty} g(y)\; e^{-\mathbf{i}k(y+t)}\; dy\; dt = \int\limits_{-\infty}^{\infty} f(t)e^{-\mathbf{i}kt}\; dt \int\limits_{-\infty}^{\infty} g(y)\; e^{-\mathbf{i}ky}\; dy = 2\pi\,\hat{f}\hat{g}\end{aligned}$$

Central Slice Theorem

The central slice theorem (also known as the Fourier slice theorem or the slice projection theorem) is a link between the one and two dimensional Fourier transforms. The theorem tells us that the one dimensional Fourier transform of a projected function (the Radon transform) is equal to the two dimensional Fourier transform of the original function taken on the slice through the origin parallel to the line we projected our function on.



FILTERED BACKPROJECTION
Projection slice theorem

- From Fourier equation, consider s=0, $\hat{f}(k,s) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x,y) e^{-\mathbf{i} \binom{x}{y} \cdot \binom{k}{s}} \, dx \, dy$

$$\begin{aligned}
\hat{f}(k,0) &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x,y) e^{-\mathbf{i}xk} \, dx \, dy \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left[\int\limits_{-\infty}^{\infty} f(x,y) \, dy \right] e^{-\mathbf{i}xk} \, dx \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} R_f(0,x) e^{-\mathbf{i}xk} \, dx = \frac{1}{\sqrt{2\pi}} \widehat{R_f(0,\cdot)}(k)
\end{aligned}$$

where we used the fact that $R_f(0,x) = \int_{-\infty}^{\infty} f(x,y) \, dy$, and denote $\widehat{R_f(0,\cdot)}(k)$ for its Fourier transform. Summed up we have found:

$$\hat{f}(k,0) = \frac{1}{\sqrt{2\pi}} \widehat{R_f(0,\cdot)}(k)$$

Projection slice theorem

$$\hat{f}(k,0) = \frac{1}{\sqrt{2\pi}}\widehat{R_f(0,\cdot)}(k)$$

- Equation tells us that projecting (or summing up) a mass density function f along the y-axis (which is perpendicular to the x-axis) and then applying the Fourier transform is the same as taking the two dimensional Fourier transform and cutting a slice along the x-axis.
- The geometrical interpretation of Equation motivates the name projection slice theorem. It also holds for lines with angle φ other than zero.
- The procedure is illustrated in Figure

Projection slice theorem

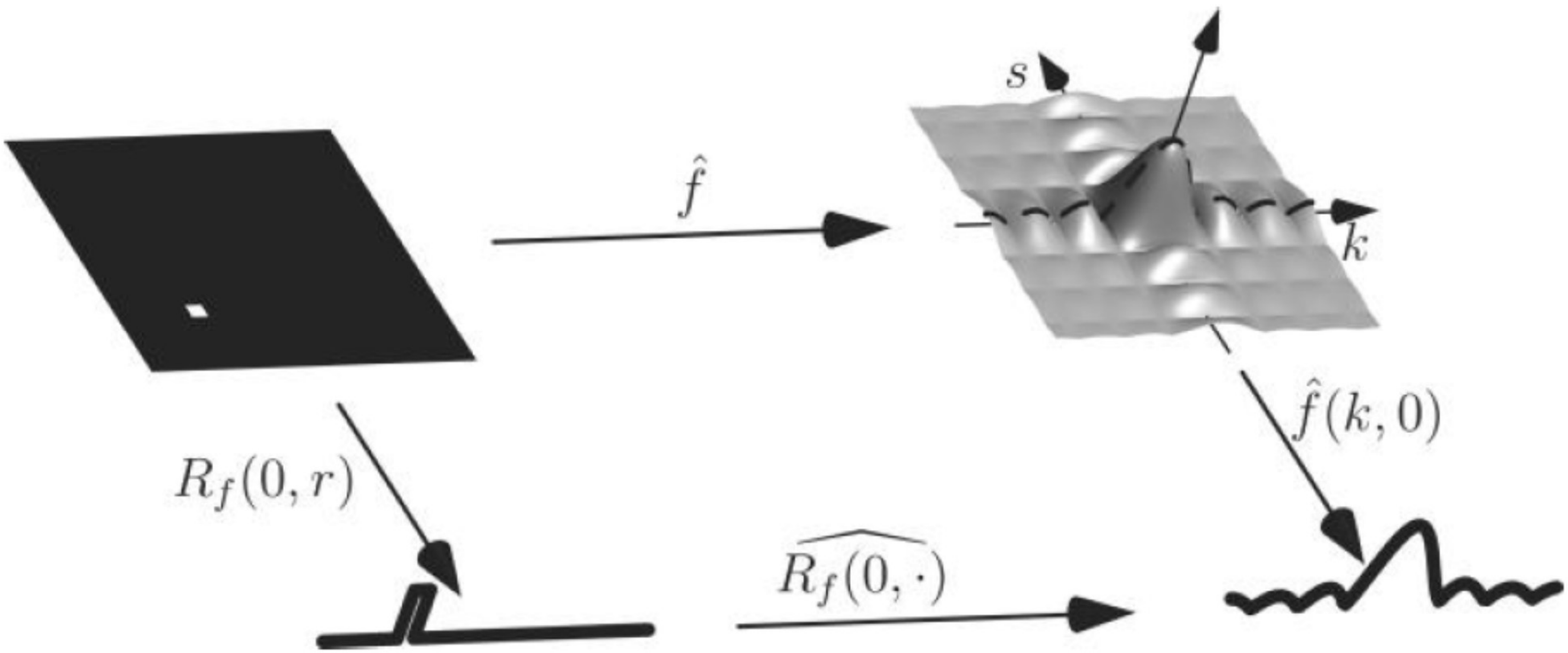


FIGURE: The projection slice theorem for ϕ = 0. Taking the two dimensional Fourier transform and subsequent taking the slice at s = 0 is the same as taking the Radon transform at ϕ = 0 and the one dimensional Fourier transform of the result. The image shows absolute values of the Fourier transforms.

Projection Slice Theorem 1

- Given a real-valued function defined on the plane, then

$$\hat{f}(\lambda\left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)) = \frac{1}{\sqrt{2\pi}}\widehat{R_f(\varphi,\cdot)}(\lambda)$$

- The proof is a straightforward integration using the substitution

$$\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right) = \left(\begin{smallmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{smallmatrix}\right)\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right)$$

  and the definition of the Radon transform

Projection Slice Theorem 1

- We start with the two dimensional Fourier transform along the line with angle φ

$$\hat{f}(\lambda\left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)e^{-\mathrm{i}\lambda\left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)}dx\ dy$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(\ldots)e^{-\mathrm{i}\lambda\left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)\cdot\left[a\left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)+b\left(\begin{smallmatrix}-\sin\varphi\\ \cos\varphi\end{smallmatrix}\right)\right]}da\ db$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\int\limits_{-\infty}^{\infty} f(\left(\begin{smallmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{smallmatrix}\right)\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right))db\right]e^{-\mathrm{i}\lambda a}da$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} R_f(\varphi,a)e^{-\mathrm{i}\lambda a}da = \frac{1}{\sqrt{2\pi}}\widehat{R_f(\varphi,\cdot)}(\lambda)$$

Projection Slice Theorem 1

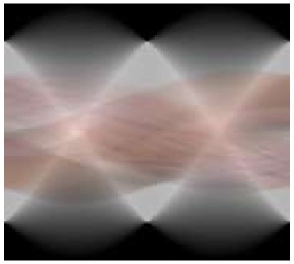
- The projection slice theorem provides a new way of reconstructing the function f from its Radon transform, often called direct Fourier method :

1. take one dimensional Fourier transforms of the given Radon transform Rf $(\varphi, \cdot)$, for a (hopefully large) number of angles φ.

2. take the inverse two dimensional Fourier transform of the above result.

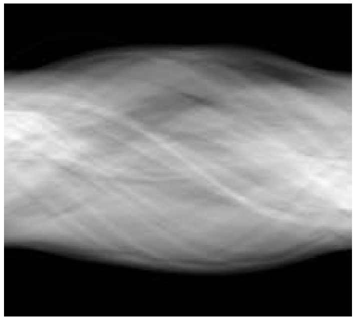
Inverse Radon Transform

- Inverse Radon uses robust numerical algorithms; hence, it can be used to reconstruct fairly complicated images. For example, the following is the Radon transform of an image showing the pancreas, liver, stomach and small intestine that was obtained using AnatomyData.



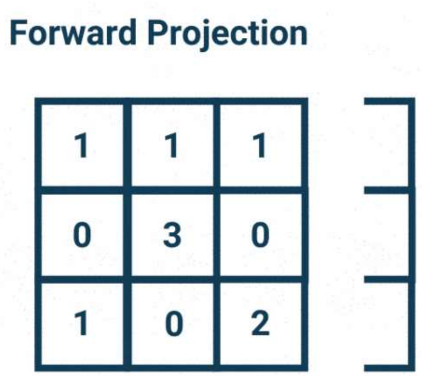
As seen here, InverseRadon gives a faithful reconstruction of the original image in an efficient manner.

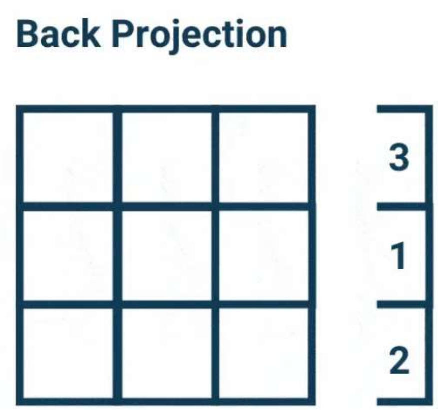
For example, the following shows the Radon image of a cross-section of the bile duct in a patient who is suspected of having cholangiocarcinoma (cancer of the bile duct).



Backprojection

- The backprojection operation is essentially trying to undo the forward projection operation. Since the forward projection operation mapped from the image into the detector space the backprojection operation maps from the detector back to the image.





Backprojection

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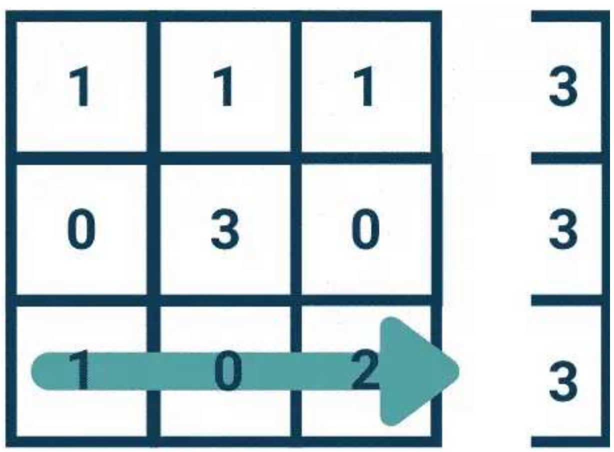
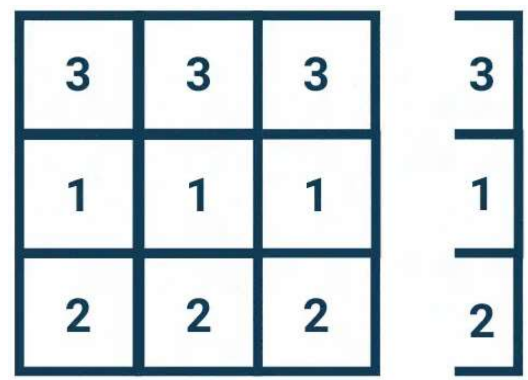
forward projection

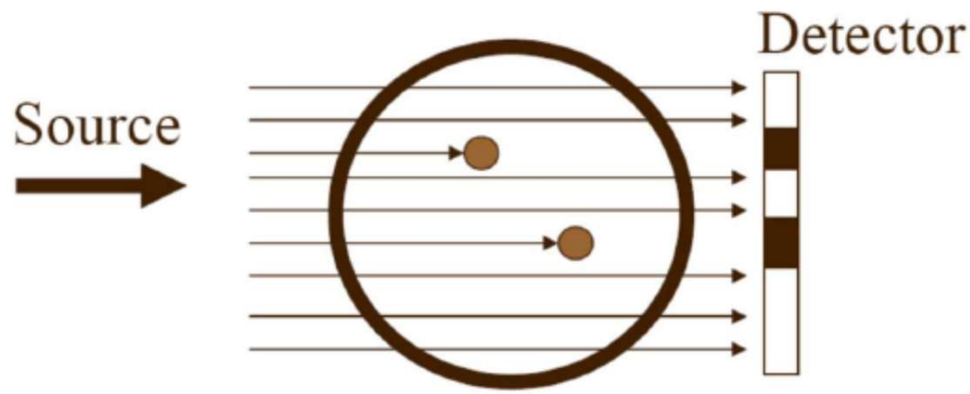


backprojection



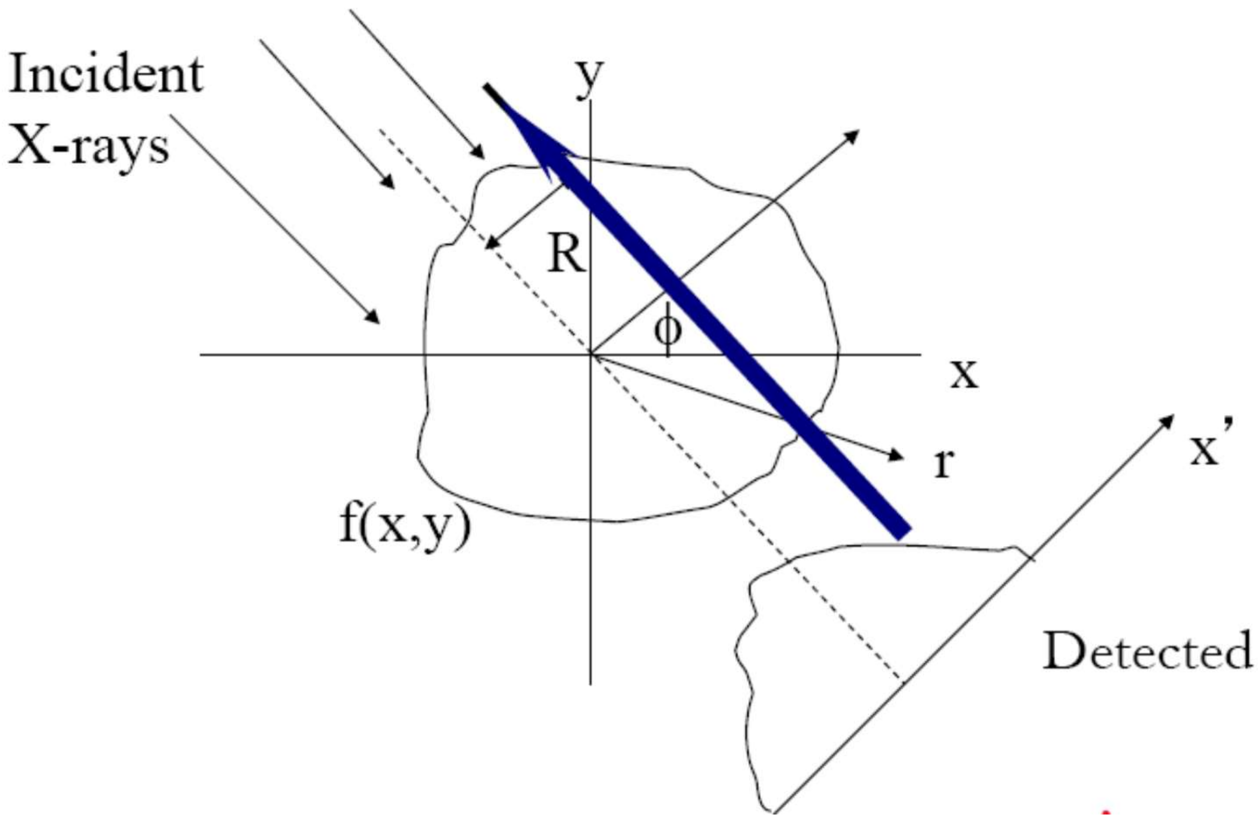
Geometrically, the backprojection operation simply propagates the measured sinogram back into the image space along the projection paths

Backprojection

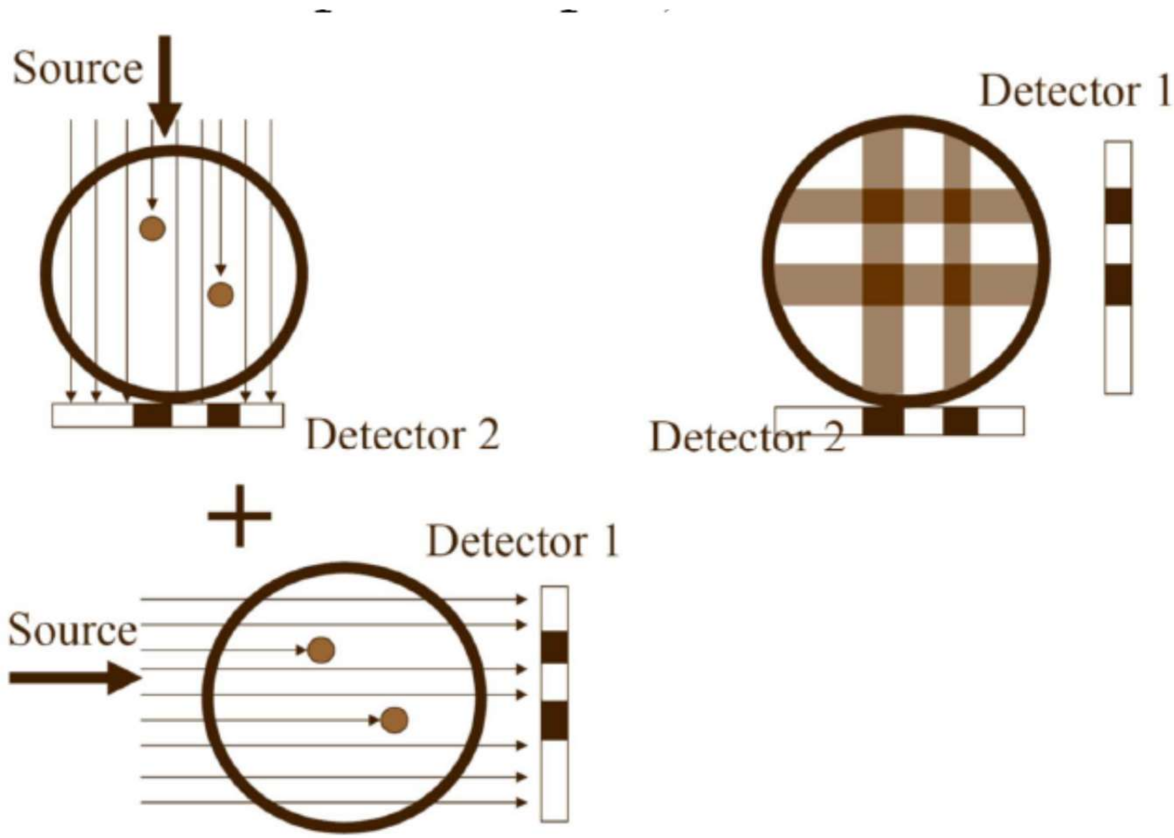


inverse = 1 back projection

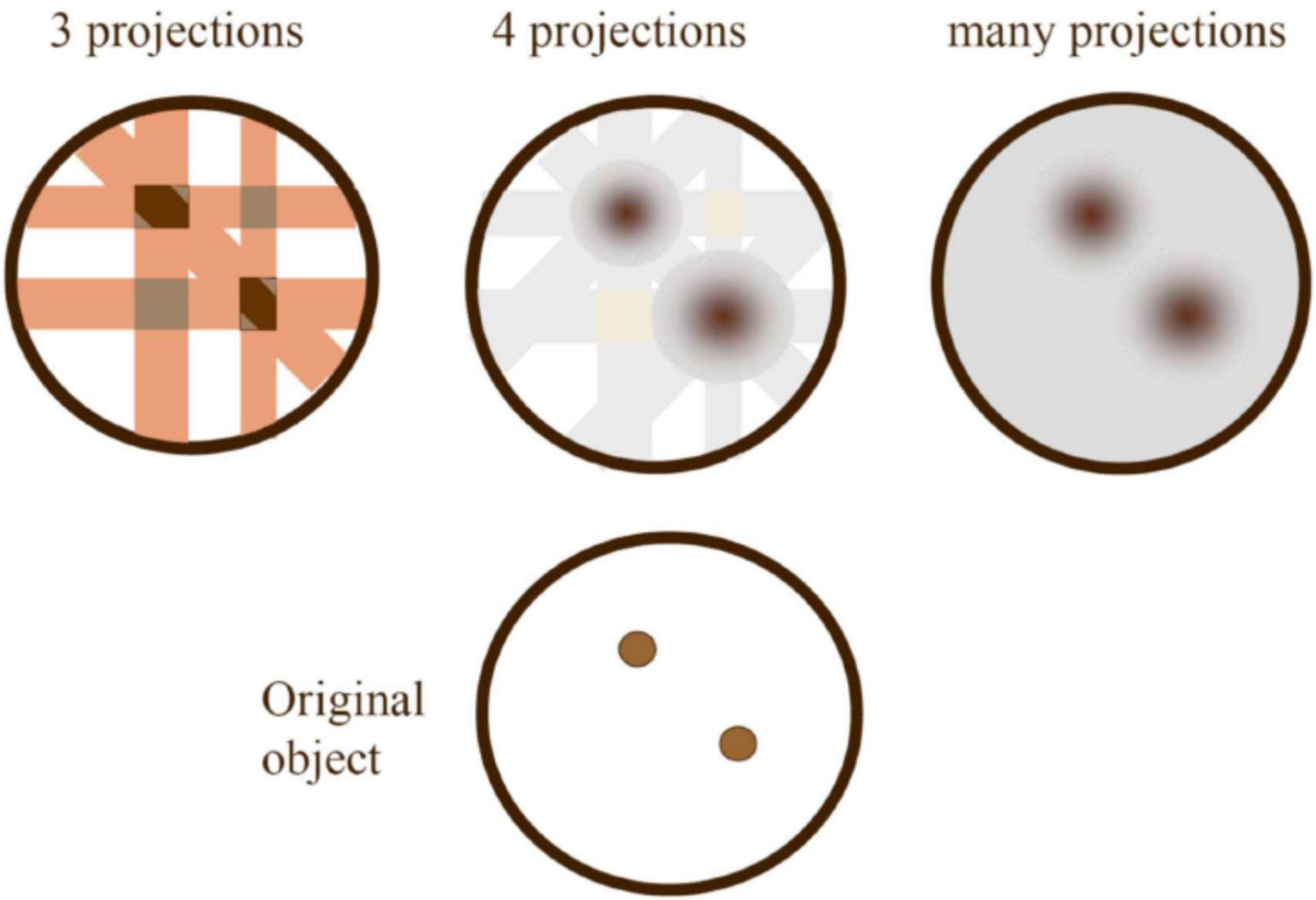
Backprojection



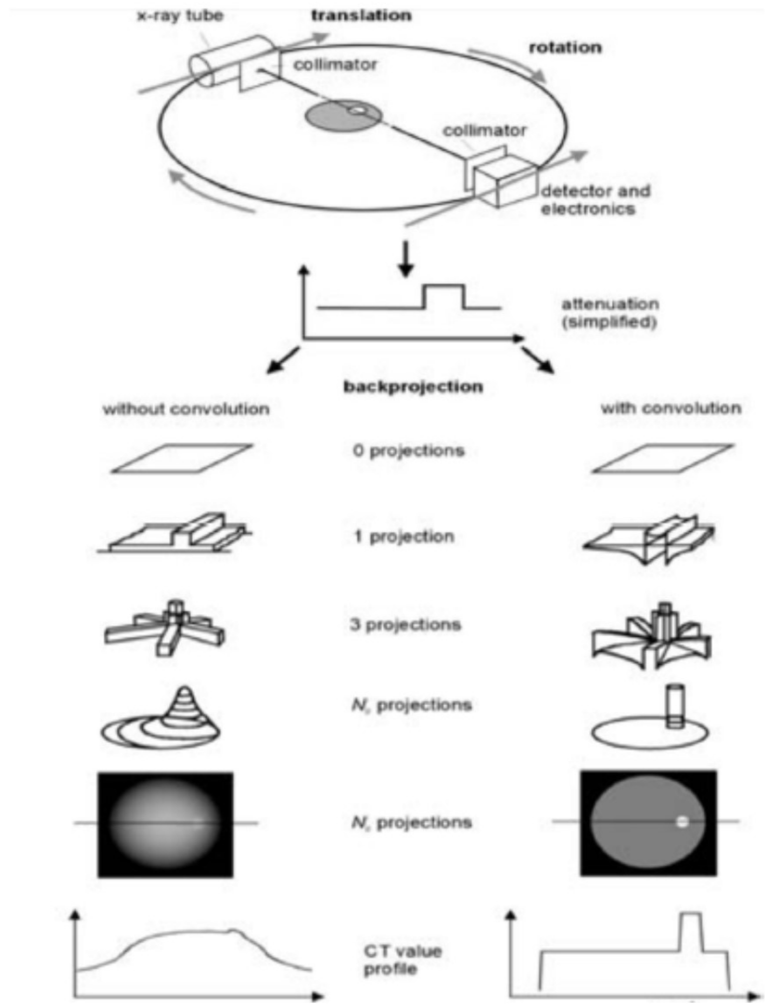
Backprojection



Backprojection



Backprojection



Filtered backprojection algorithm

Our starting point is the inverse two dimensional Fourier transform

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k,s) e^{\mathrm{i} \left(\begin{smallmatrix} k \\ s \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} \, dk \, ds \tag{10.19}$$

From the projection slice theorem, Equation 10.18, we know $\hat{f}$ at lines through the origin $r\left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix}\right)$. We therefore transform to a polar coordinate system (r,φ):

$$\begin{pmatrix} k \\ s \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}.$$

The determinant of the Jacobian matrix is

$$\det \frac{\partial(k,s)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial k}{\partial r} & \frac{\partial k}{\partial \varphi} \\ \frac{\partial s}{\partial r} & \frac{\partial s}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

Filtered backprojection algorithm

The inverse Fourier transform will therefore become

$$2\pi f(x,y) = \int_0^{2\pi}\int_0^{\infty} \hat{f}\left(r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right) e^{\mathbf{i}r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}}\; r\; dr\; d\varphi = \int_0^{\pi}\ldots + \int_{\pi}^{2\pi}\ldots$$

$$= \int_0^{\pi}\ldots + \int_0^{\pi}\int_0^{\infty} \hat{f}\left(r\begin{pmatrix}\cos(\varphi+\pi)\\ \sin(\varphi+\pi)\end{pmatrix}\right) e^{\mathbf{i}r\begin{pmatrix}\cos(\varphi+\pi)\\ \sin(\varphi+\pi)\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}}\; r\; dr\; d\varphi$$

$$= \int_0^{\pi}\ldots + \int_0^{\pi}\int_0^{\infty} \hat{f}\left(-r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right) e^{\mathbf{i}(-r)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}}\; r\; dr\; d\varphi$$

$$= \int_0^{\pi}\int_0^{\infty}\ldots + \int_0^{\pi}\int_{-\infty}^{0} \hat{f}\left(r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right) e^{\mathbf{i}r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}}\; (-r)\; dr\; d\varphi$$

$$= \int_0^{\pi}\int_{-\infty}^{\infty} \hat{f}\left(r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right) e^{\mathbf{i}r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}}\; |r|\; dr\; d\varphi$$

Filtered Back projection 1

- Let f(x, y) describe a function from the plane to the real numbers, and Rf (φ, r) its Radon transform, then

$$f(x,y) = \frac{1}{2\pi} \int_0^{\pi} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{R_f(\varphi,\cdot)}(r)\, |r|\, e^{\mathrm{i} r \binom{\cos\varphi}{\sin\varphi} \cdot \binom{x}{y}}\, dr \right] d\varphi$$

- In Equation the expression in squared brackets is the inverse Fourier transform of the function $\widehat{R_f(\varphi,\cdot)}(r)\,|r|$ evaluated at

$$t := \binom{\cos\varphi}{\sin\varphi} \cdot \binom{x}{y} = x\cos\varphi + y\sin\varphi$$

Filtered Back projection 1

- This is a filtering operation of the function Rf (φ, r) interpreted as a function of r (with a fixed angle φ) and a filter with a frequency response of |r|, which we will call ramp filter for obvious reasons (this filter is sometimes called the Ram-Lak filter)
- Toget the value of the original function f at the point (x, y) we have to derive Rf (φ, ·) ⋆ ramp for every φ, evaluate this at t = x cosφ + y sinφ, add them up (integrate them) and then divide the result by 2π:

$$f(x,y) \approx \frac{1}{2\pi} \sum_i (R_f(\varphi_i, \cdot) \star \mathrm{ramp})(t_i)$$

Filtered Backprojection 1

- All points on a certain line l1 with angle φ1 have clearly the same orthogonal distance t1 for this line, see Figure.
- Therefore all the points on the line with parameters (φ1, t1) have the summand (Rf (φ1, ·) ⋆ ramp)(t1) in common.

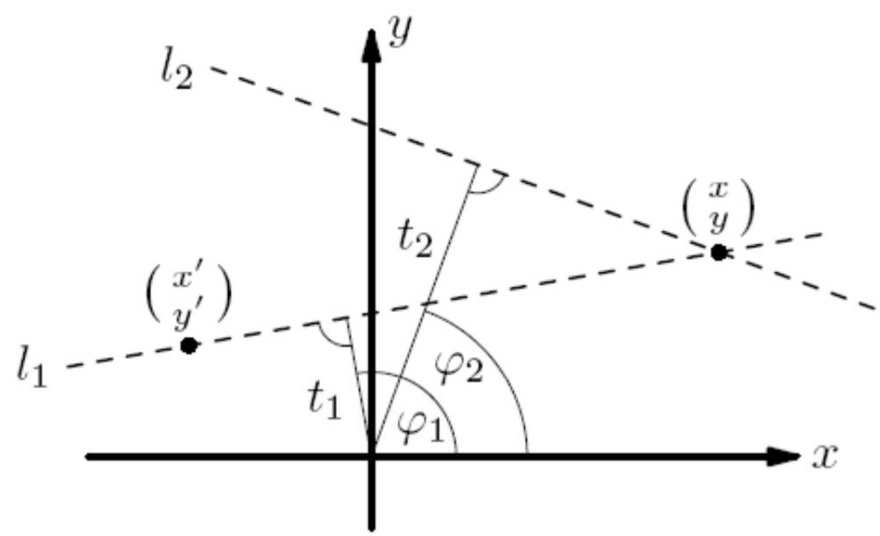


FIGURE To compute $f(x, y)$ one has to derive the orthogonal distance $t = x \cos\varphi + y \sin\varphi$ for all lines through the point (x, y). The value $(R_f(\varphi_1, \cdot) \star \mathrm{ramp})(t_1)$ can but used as a summand also for $f(x', y')$.

Filtered Backprojection 1

- A geometrical method to reconstruct f would be to smear the value $(Rf\ (\varphi 1, \cdot) \star \mathrm{ramp})(t)$ over the line with angle $\varphi 1$, to do this for all different radial distances t and then to proceed to $\varphi 2$ and smear $(Rf\ (\varphi 2, \cdot) \star \mathrm{ramp})(t)$ over the line with angle $\varphi 2$, and all different t values, and so on.

Filtered Backprojection 1

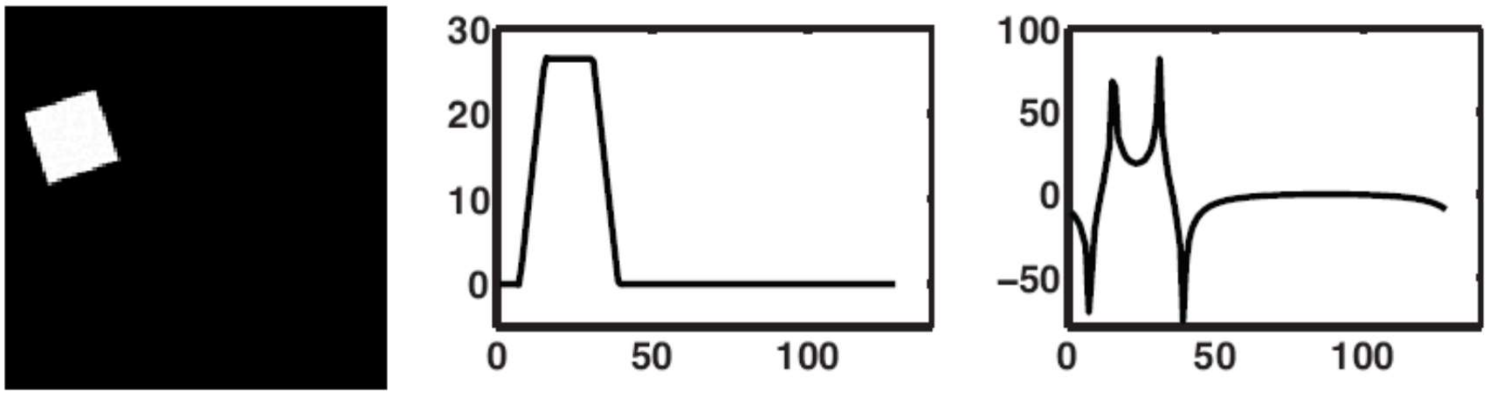


FIGURE The left image is a slightly rotated square, the middle image shows the Radon transform for $\varphi = 0$, i.e., $R_f(0, \cdot)$. In the right image we can see this function after a ramp filter was applied.

Filtered Backprojection 1

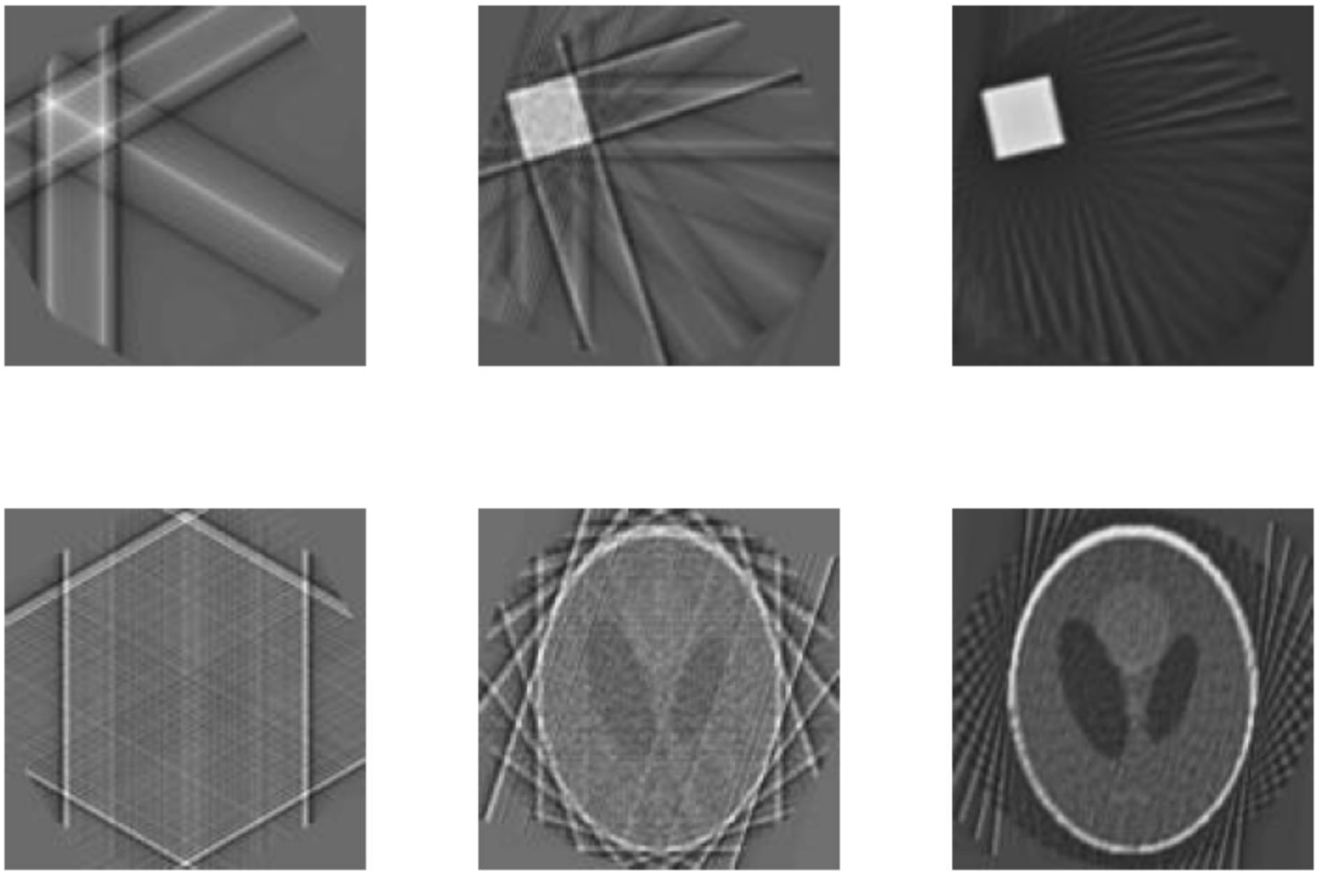


FIGURE : Filtered backprojection of the square from Figure 10.16 and a Shepp and Logan phantom with 3, 10 and 30 angle steps.

Filtered Backprojection 1

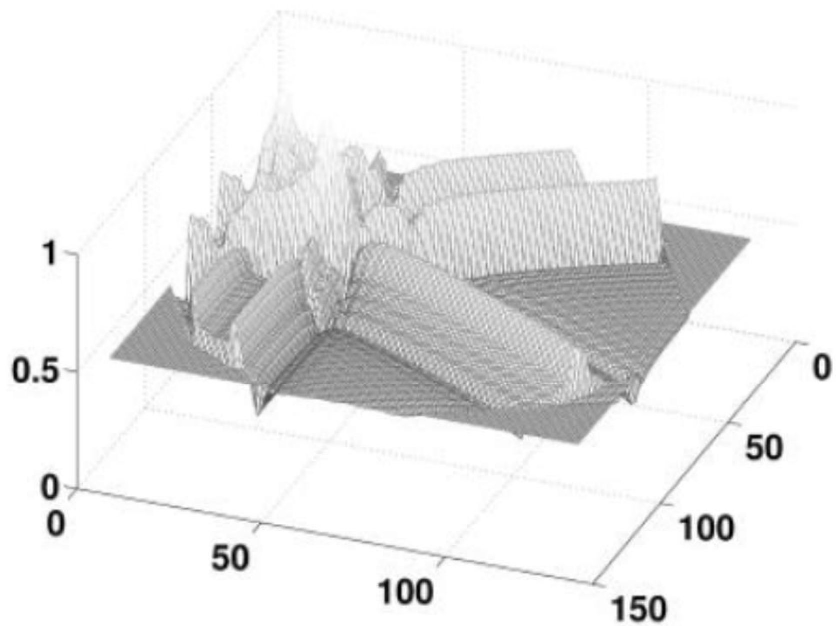
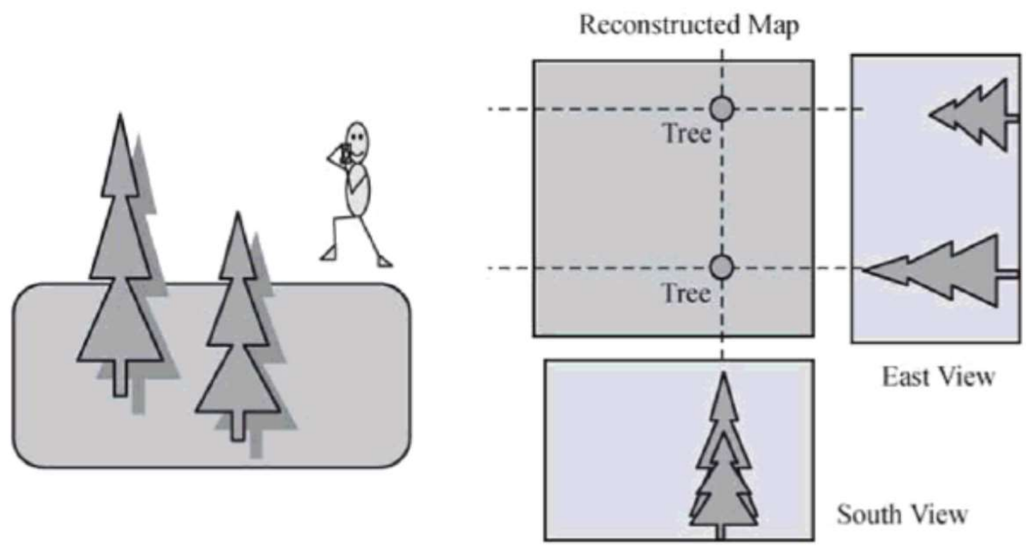


FIGURE : Mesh plot of the filtered backprojection of the square from Figure using three angle steps.

Basic Idea of Projection

Example: Photography

Two trees in a park, make 2 pictures from east and south, try to create a map of the park.



A photo is a projection of an object onto a plane

Basic Idea of Projection

Example: Another Photography

Other configuration: If you see two separate trees on both views, can you uniquely reconstruct the map of trees?
Here you *cannot* reconstruct the position and height of both trees.

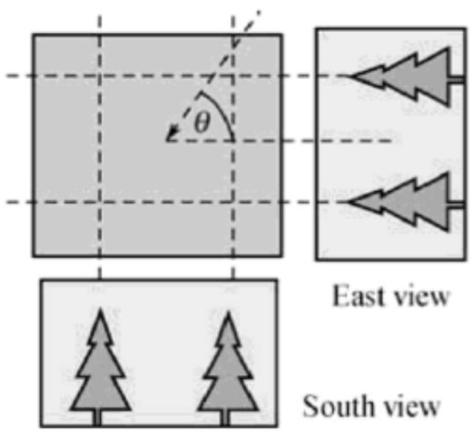


Figure: Two trees seen on *both* views

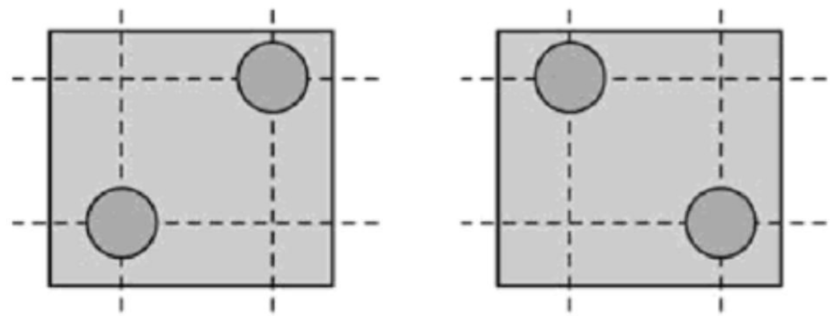


Figure: There are two solutions

Basic Idea of Projection

Example: Another Photography

Other configuration: If you see two separate trees on both views, can you uniquely reconstruct the map of trees?
Here you *cannot* reconstruct the position and height of both trees.

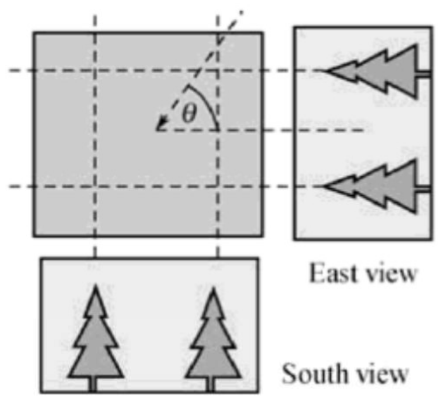


Figure: Two trees seen on *both* views

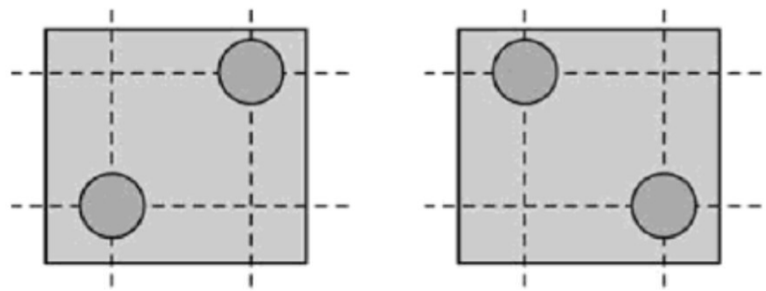
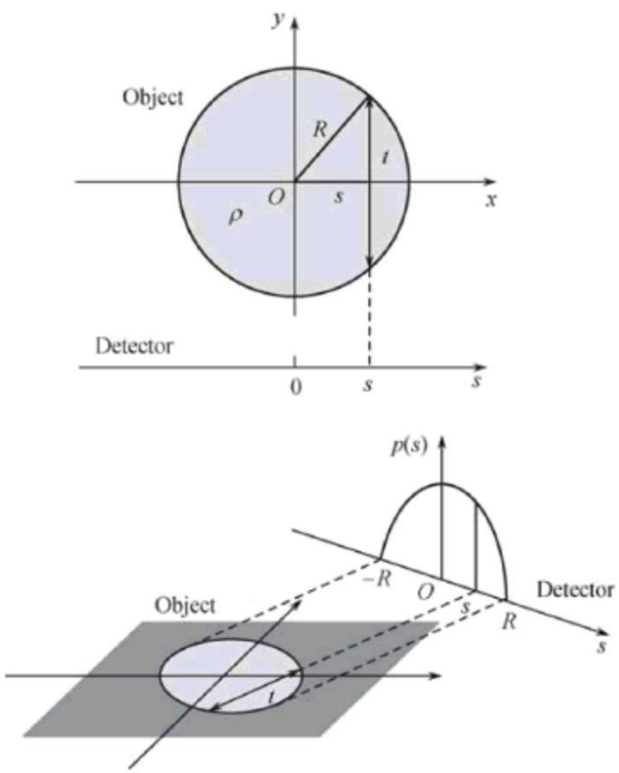


Figure: There are two solutions

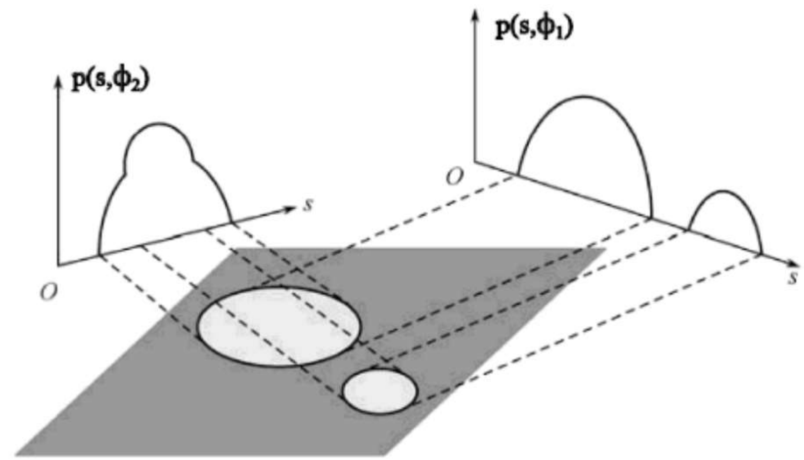
If we take another picture at 45°, we are able to solve the ambiguity.

Basic Idea: Projections.

- Before: photo, now: Projection is a line integral
- Projection $p(s, \phi)$ at angle ϕ, s is coordinate on detector



Projection $p(s)$ the same for any ϕ

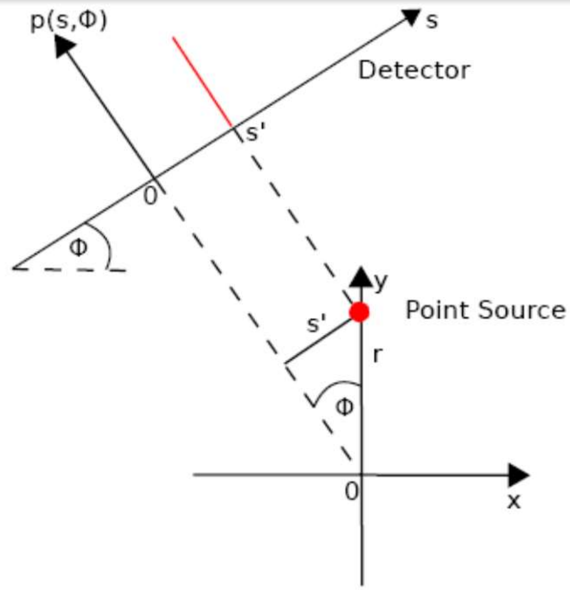


Projection $p(s, \phi)$ depends on orientation

Projections: Angle dependency.

Example: Point source on the y axis

Location s of the spike on the 1D detector: $s = r \sin \phi$.
The projection $p(s, \phi)$ in the s-ϕ-coordinate system is a sine function.



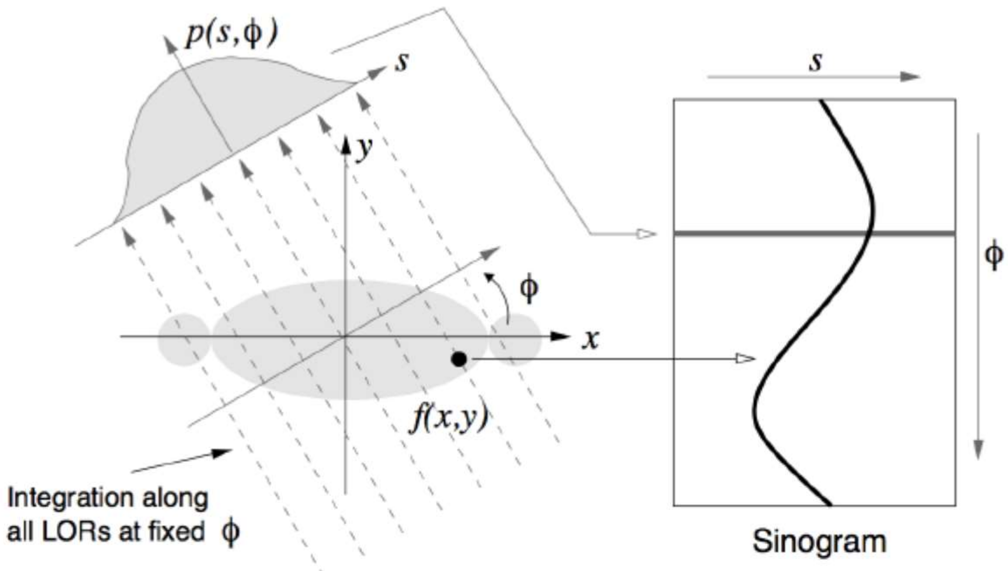
Sinogram

A sinogram is a representation of the projections on the s-ϕ plane.

Projections: Angle dependency.

Example: Point source on the y axis

Location s of the spike on the 1D detector: $s = r \sin \phi$.
The projection $p(s, \phi)$ in the s-ϕ-coordinate system is a sine function.

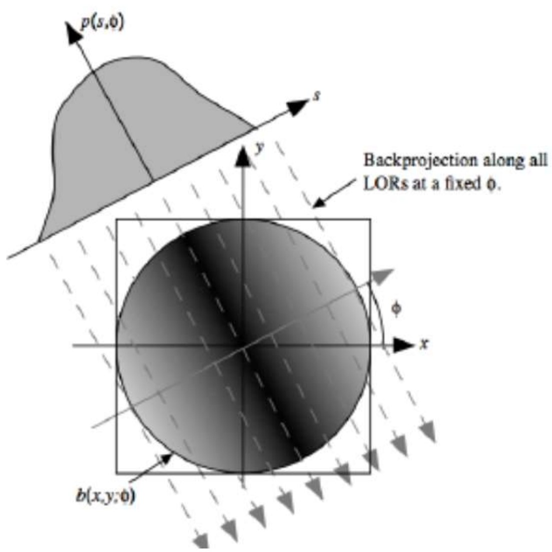


Sinogram

A sinogram is a representation of the projections on the s-ϕ plane.

Backprojection

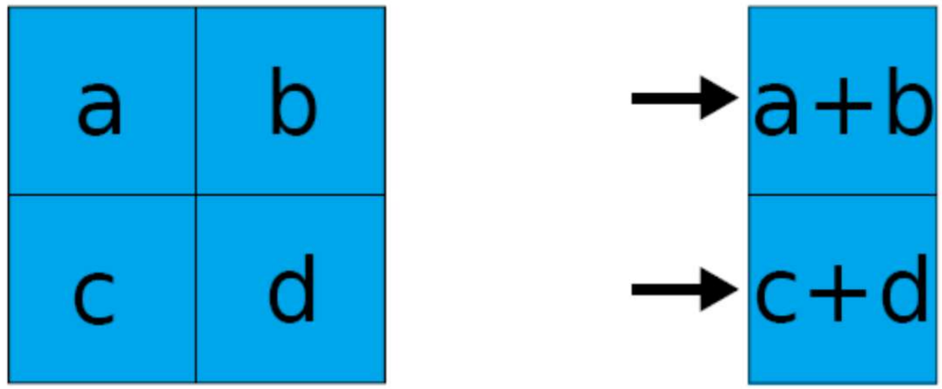
- Placing a value of $p(s, \phi)$ back into the position of the appropriate LOR
- But the knowledge of where the values came from was lost in the projection step
- The best we can do is place a constant value into all elements along the line



Lines of Response (LOR)

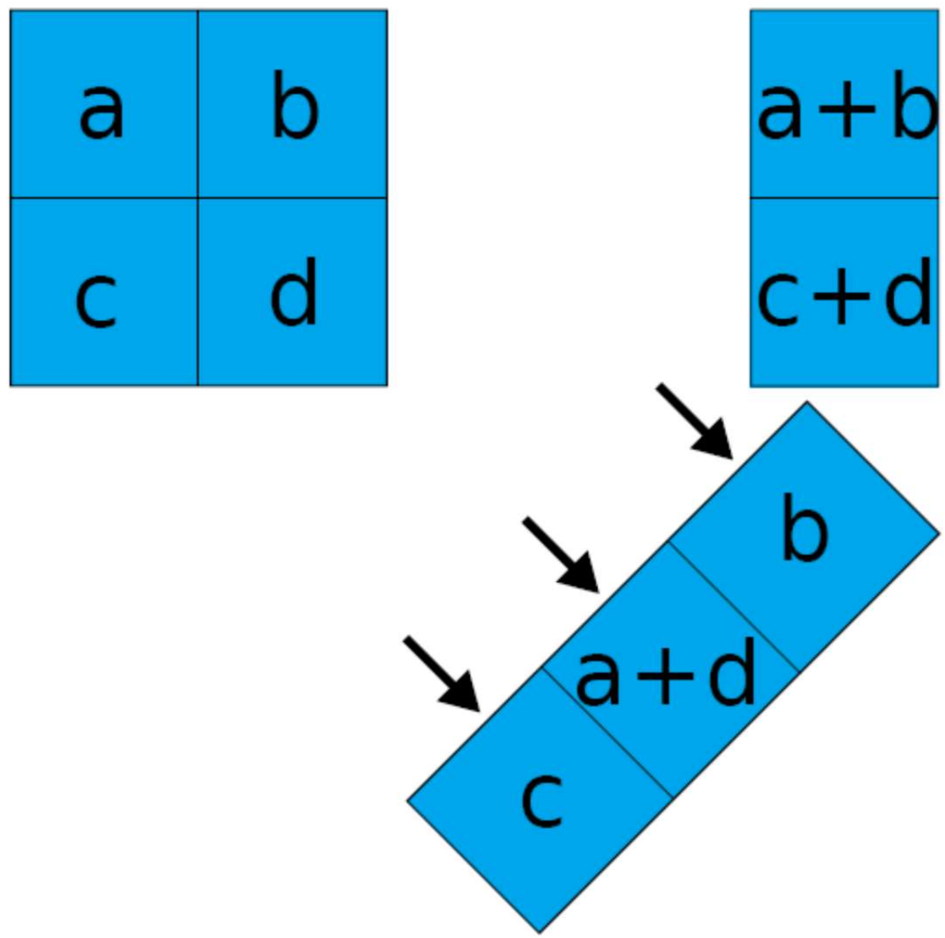
Backprojection Example.

1st projection



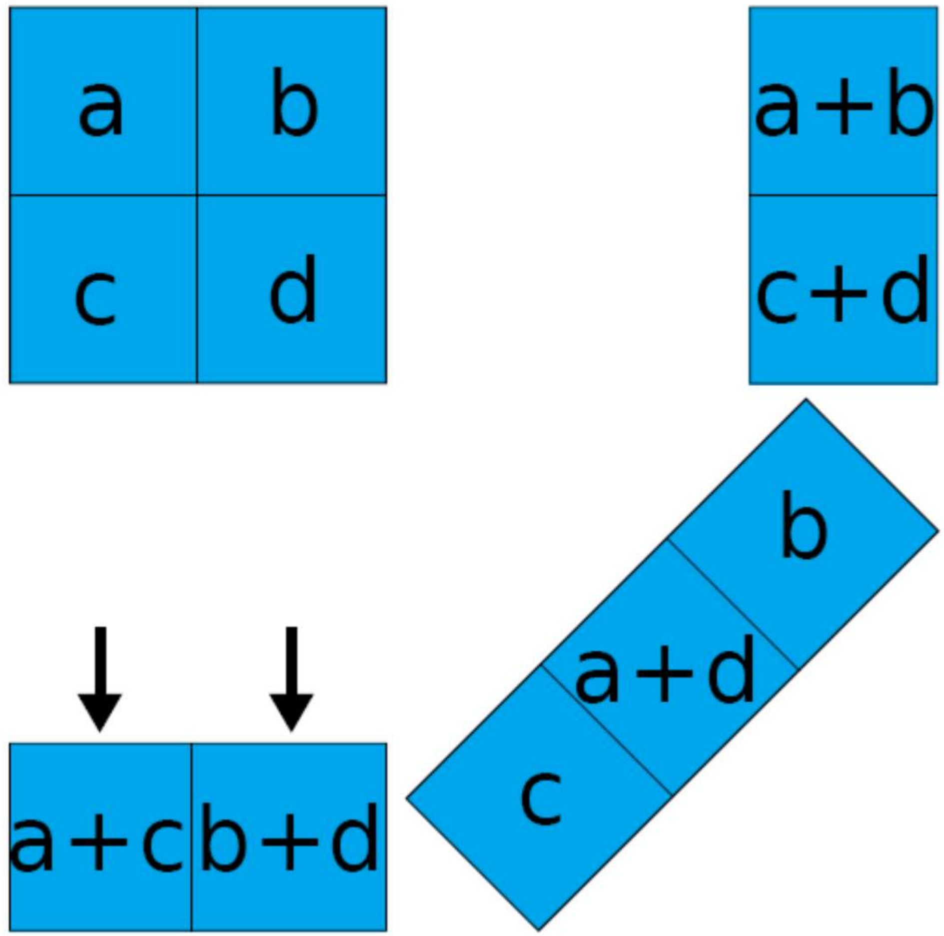
Backprojection Example.

2nd projection



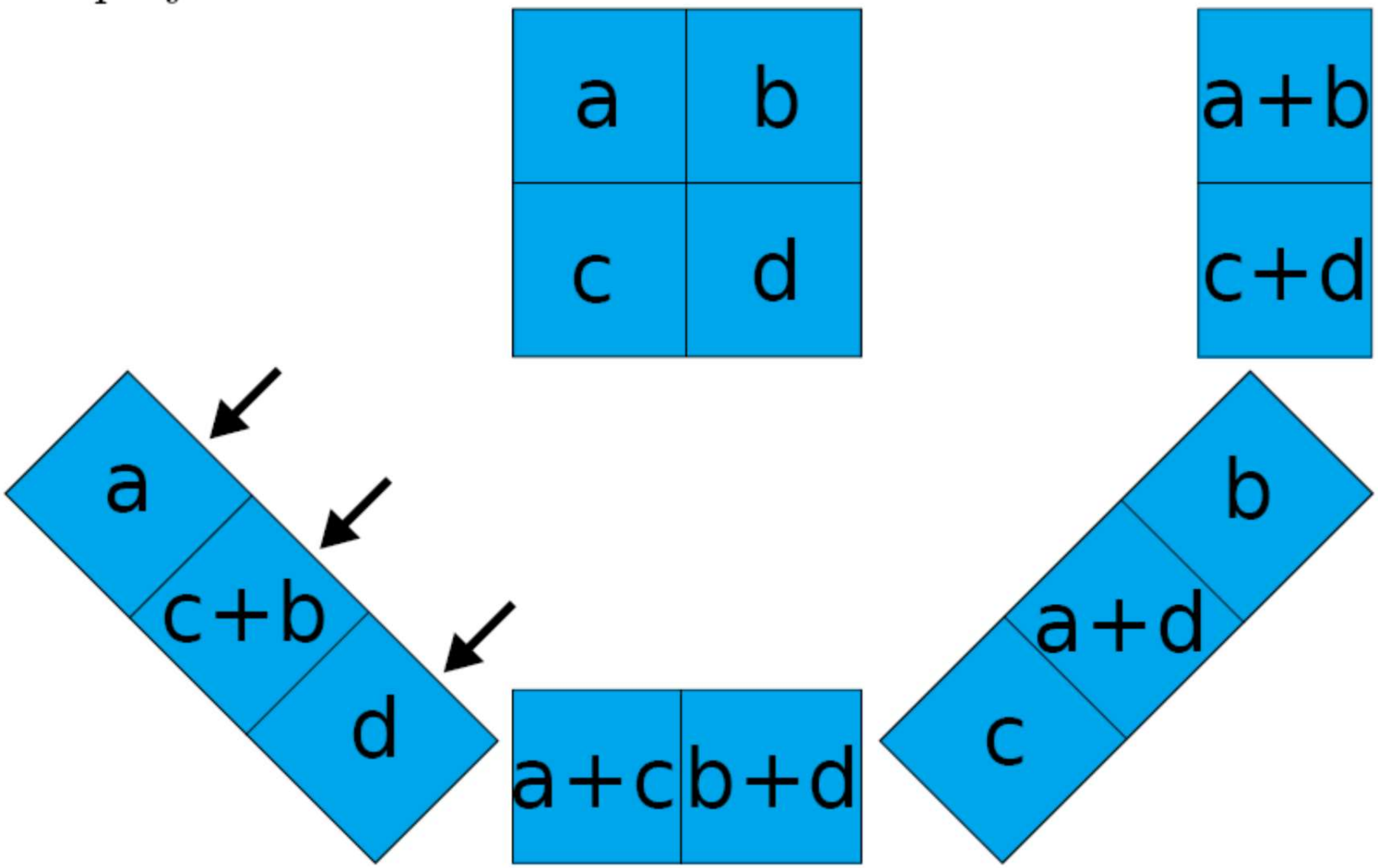
Backprojection Example.

3rd projection



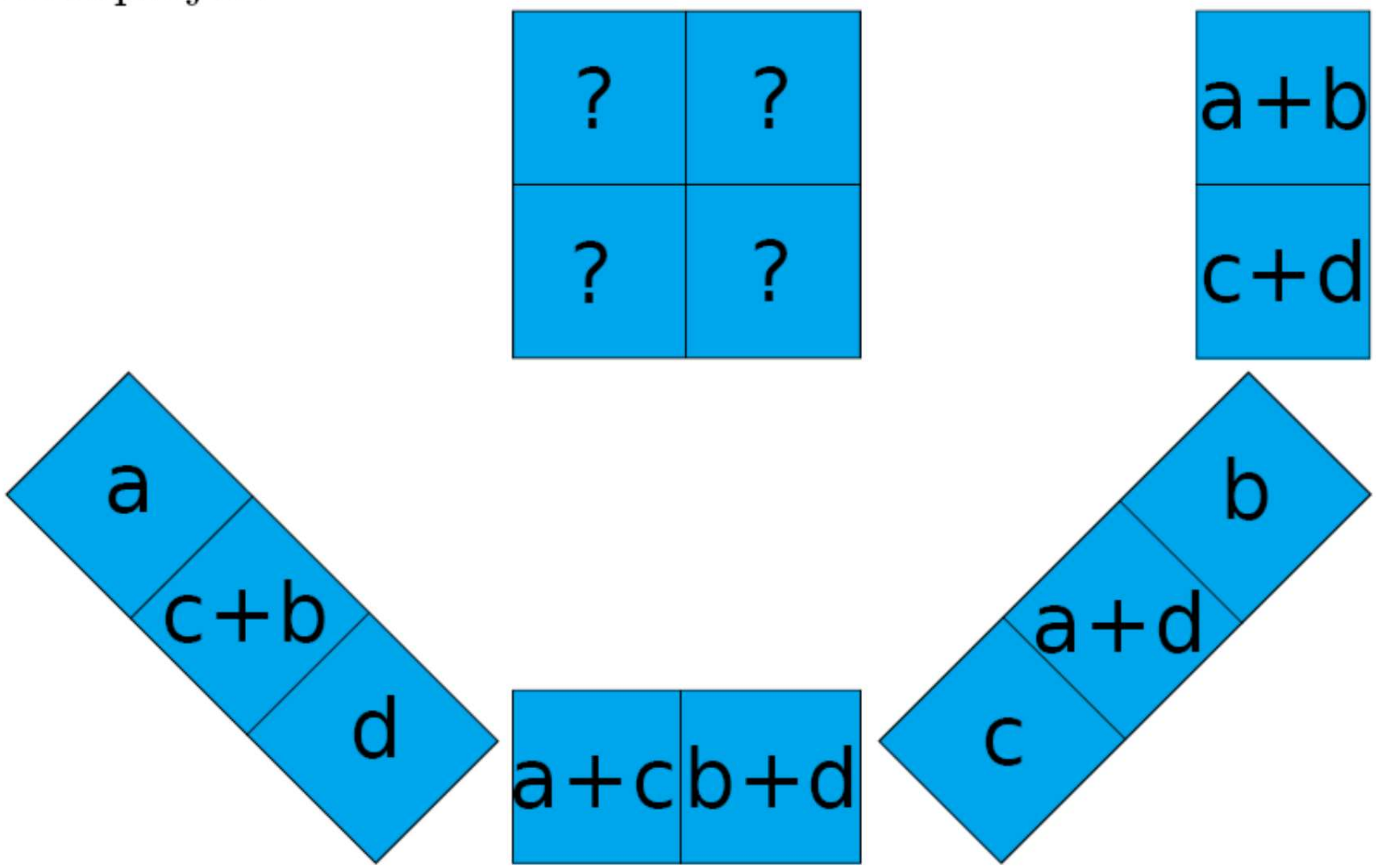
Backprojection Example.

4th projection



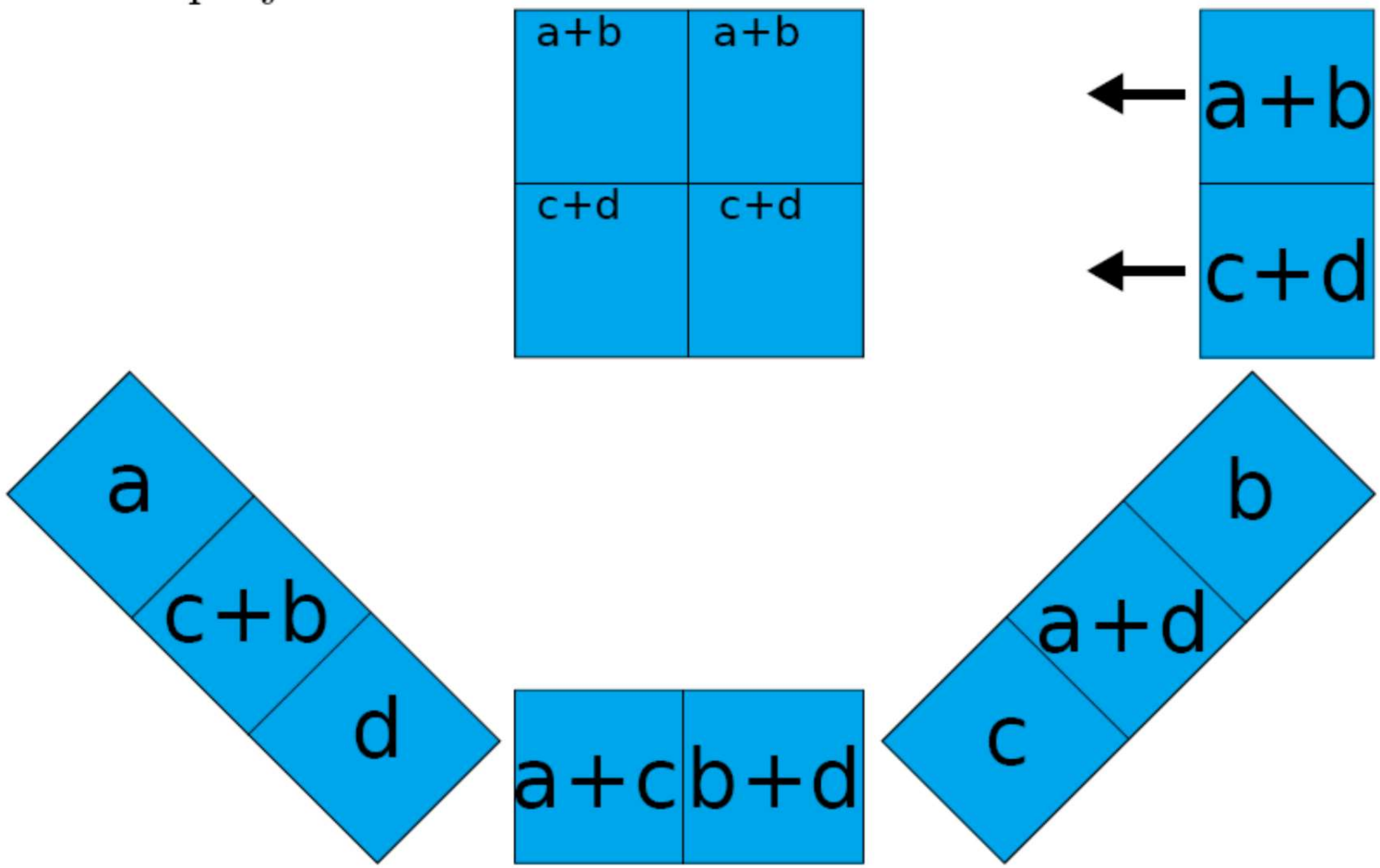
Backprojection Example.

Backproject



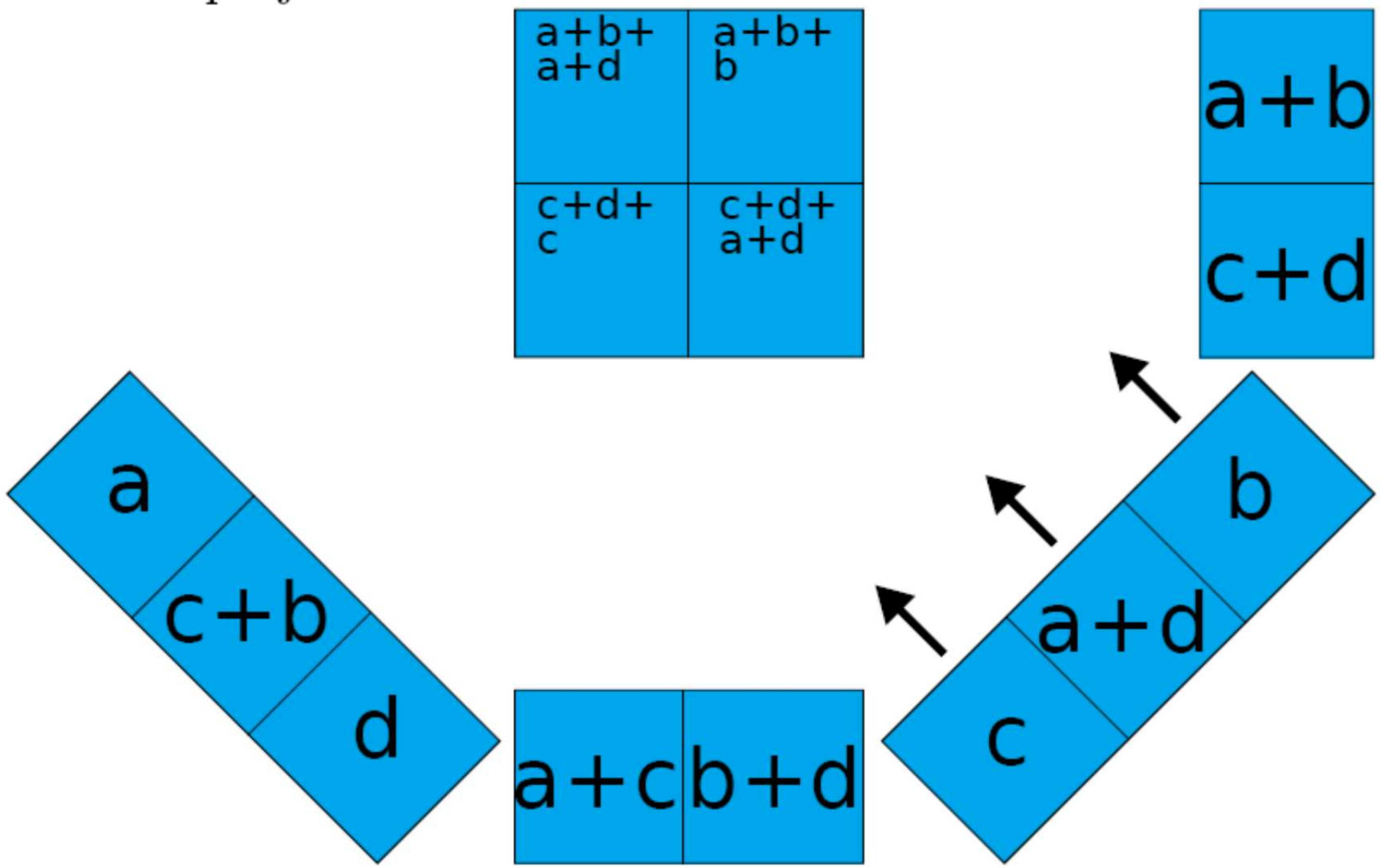
Backprojection Example.

1st backprojection



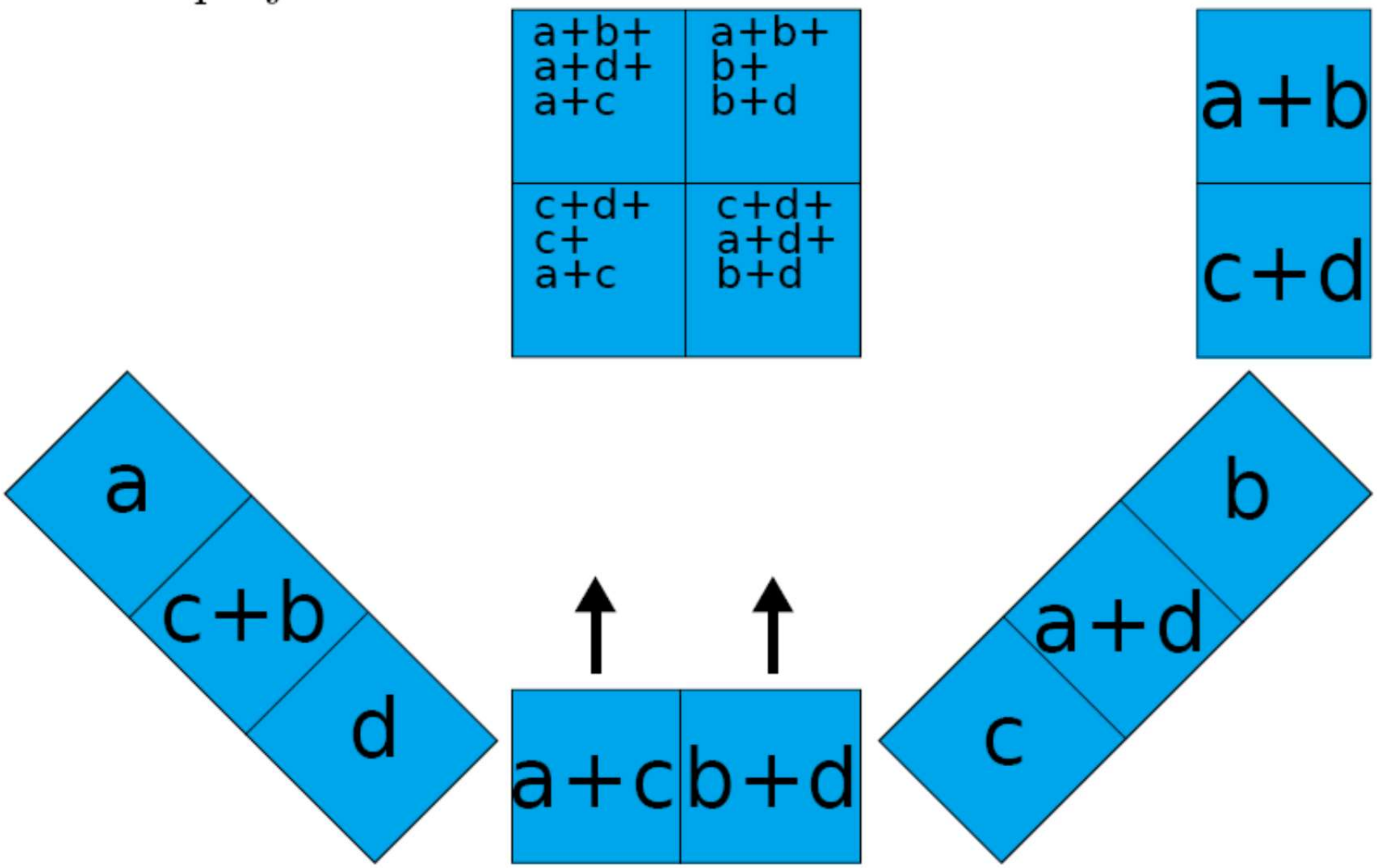
Backprojection Example.

2nd backprojection



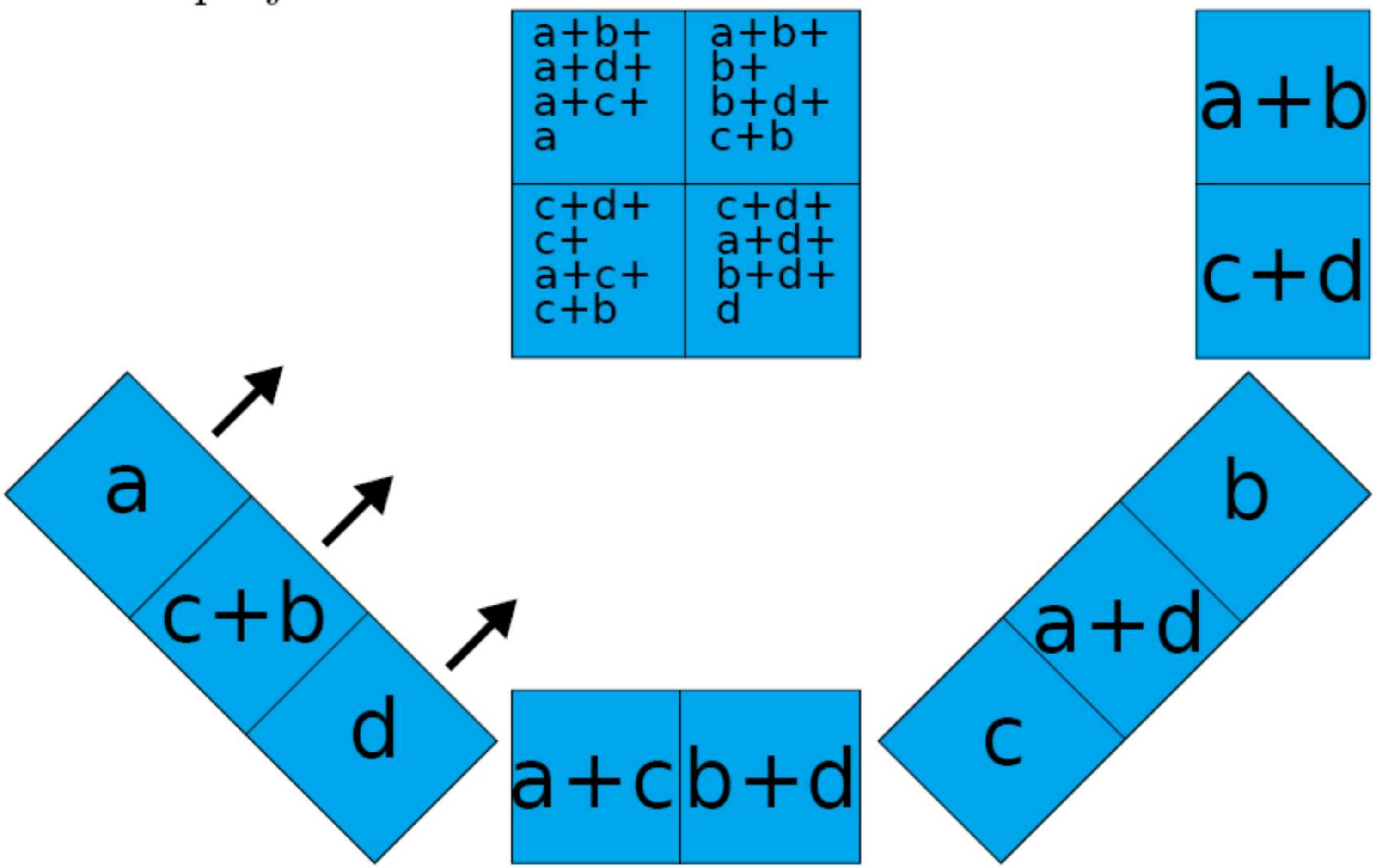
Backprojection Example.

3rd backprojection



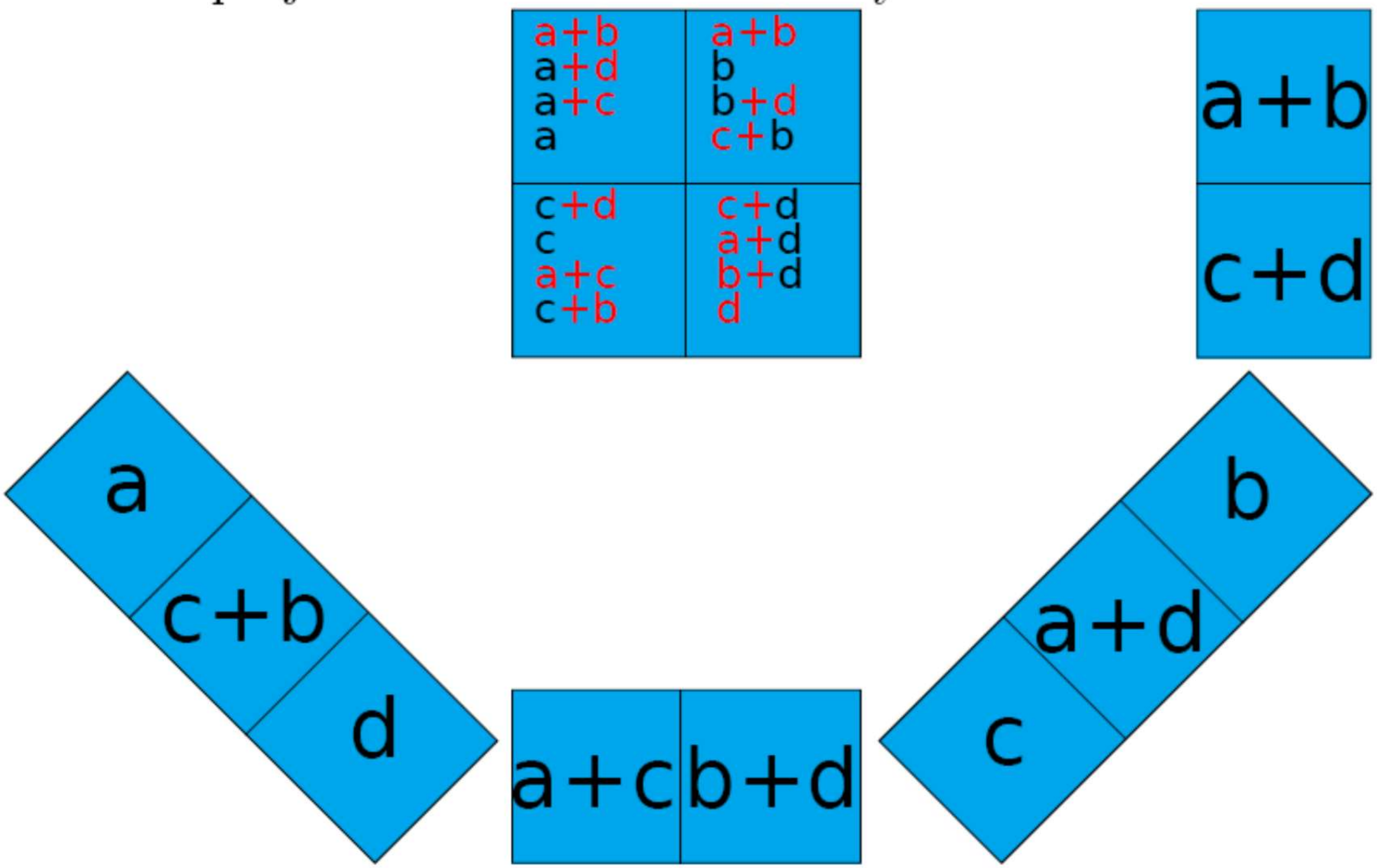
Backprojection Example.

4th backprojection



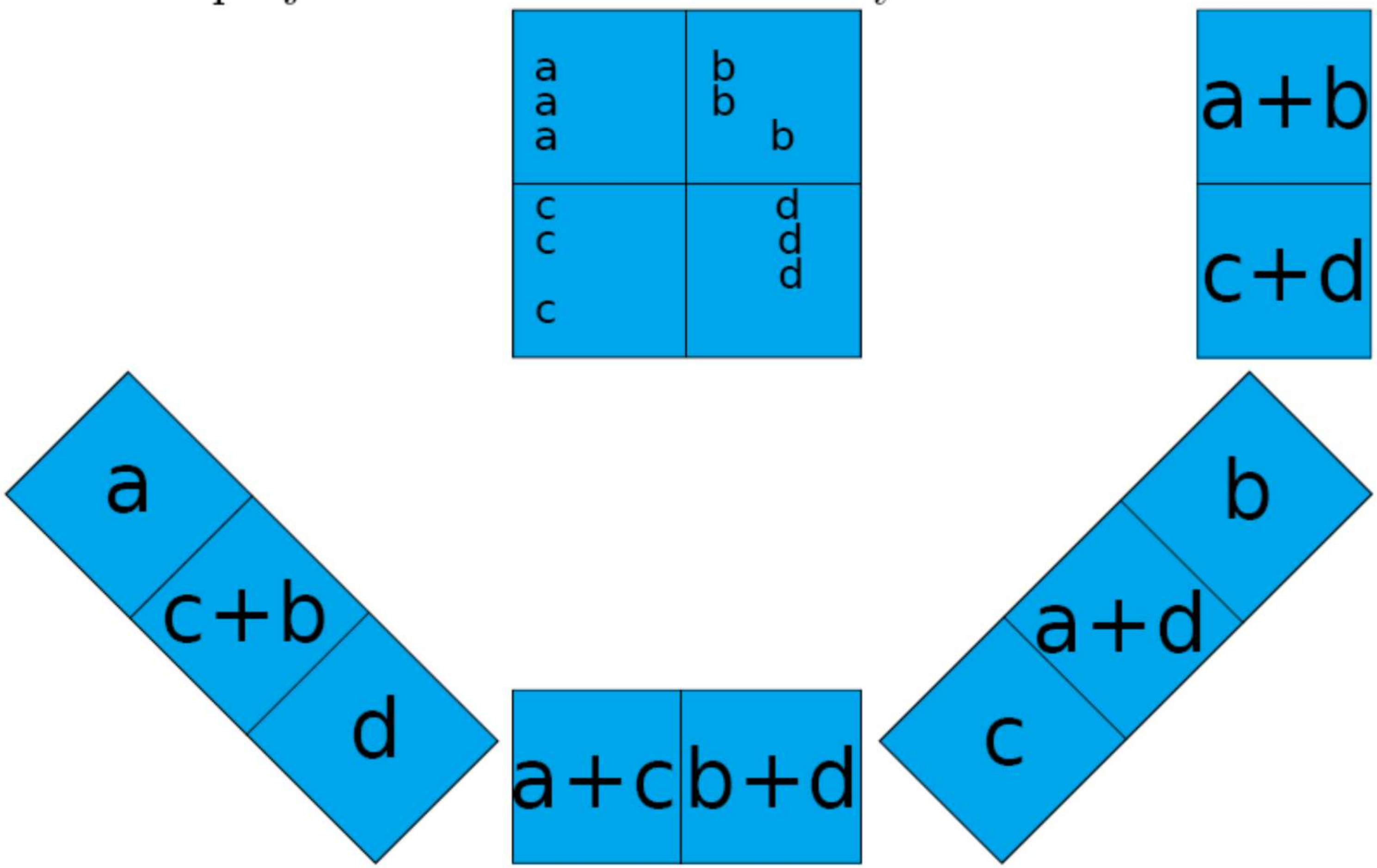
Backprojection Example.

Subtract projection sum from each entry



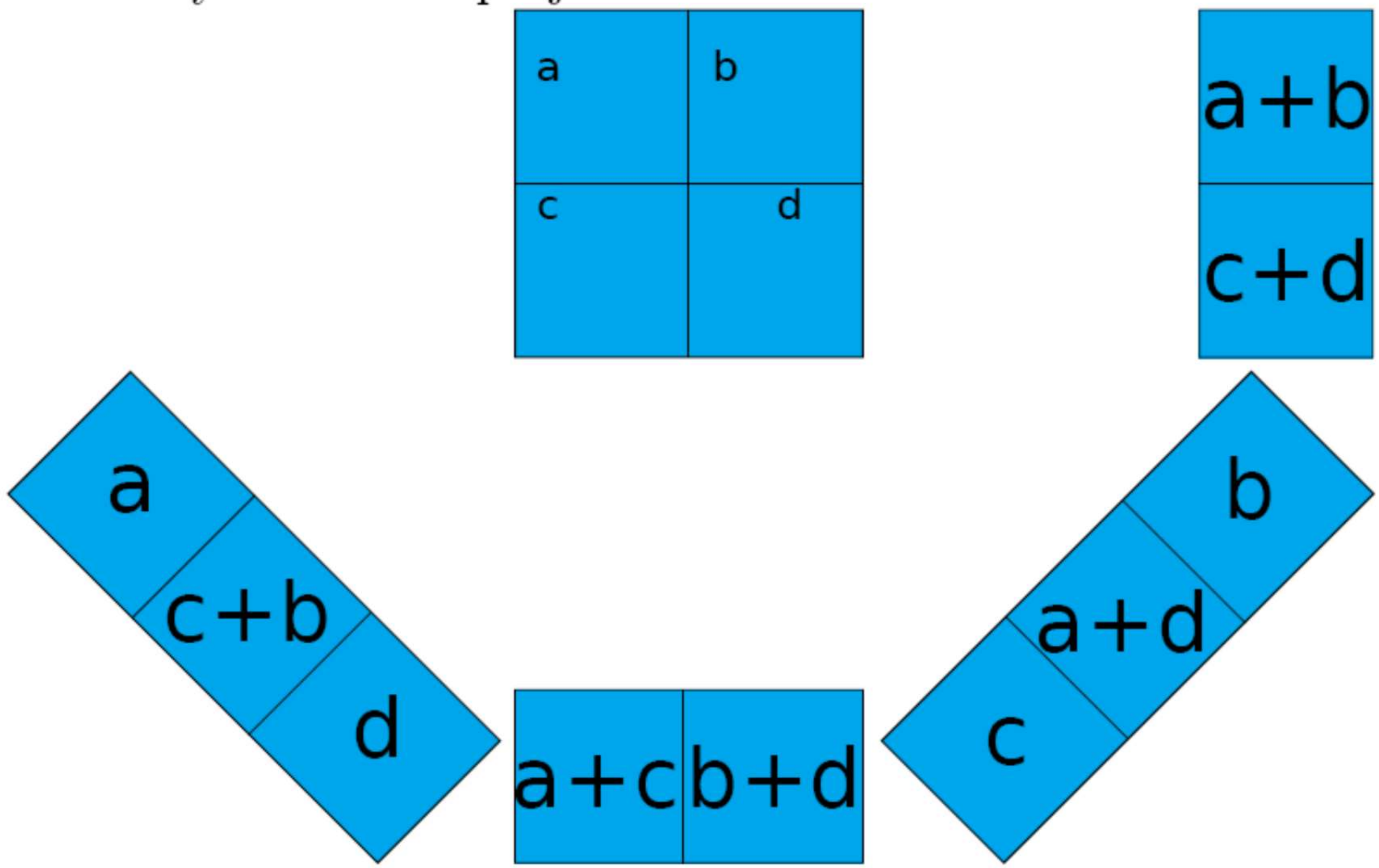
Backprojection Example.

Subtract projection sum from each entry

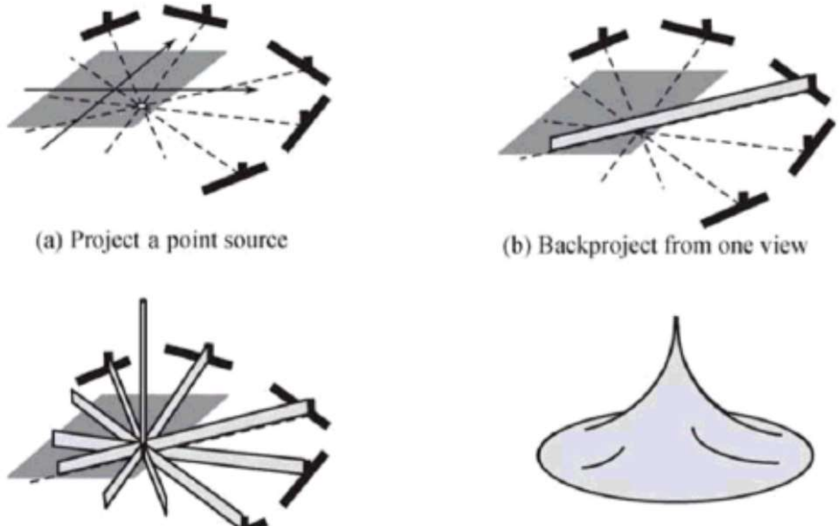


Backprojection Example.

Divide by number of projections $-1 = 3$



Backprojection Procedure.



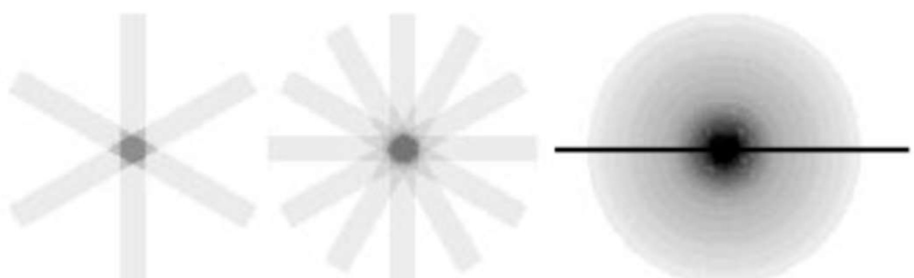
(a) Project a point source

(b) Backproject from one view

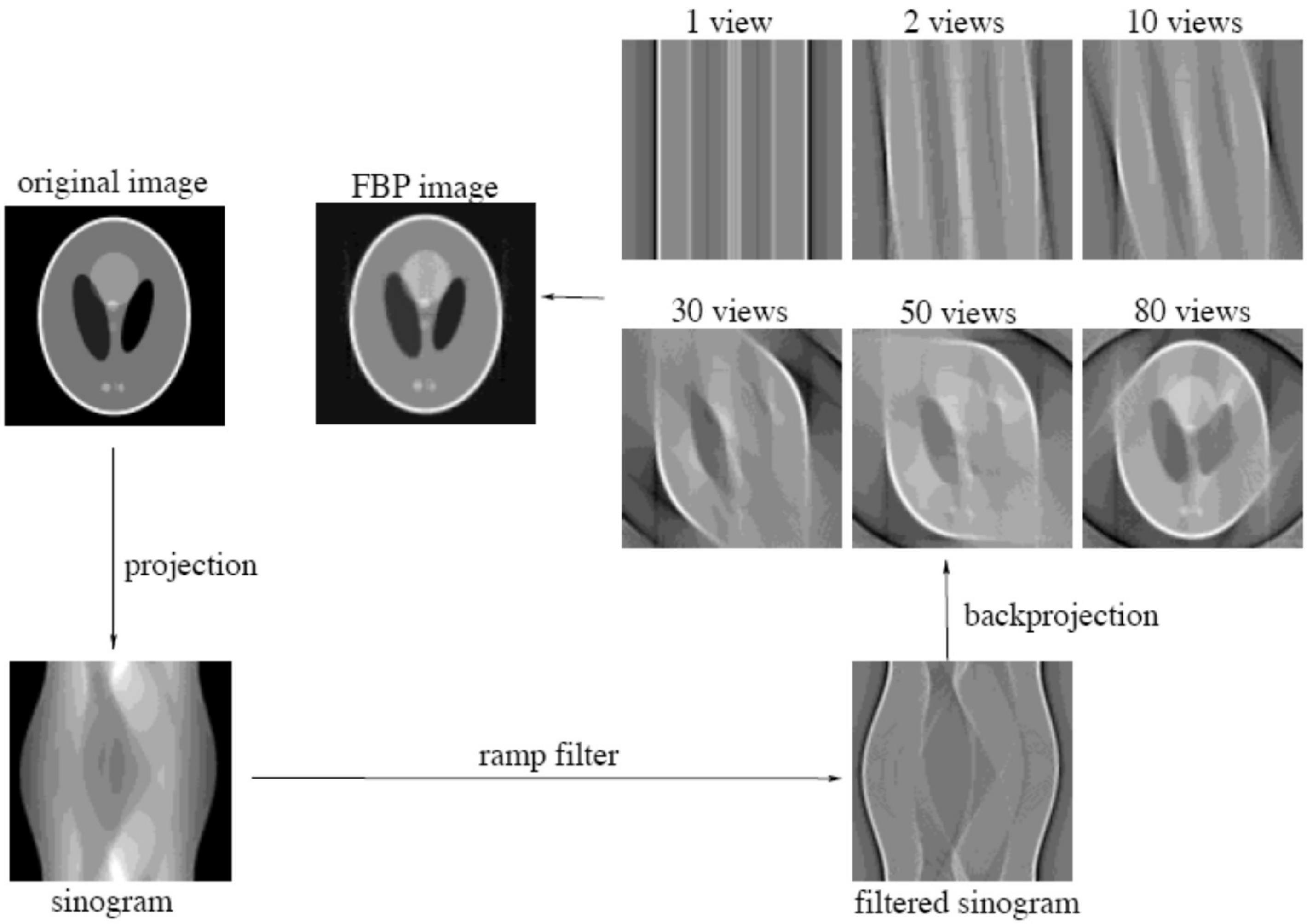
(c) Backproject from a few views

(d) Backproject from all views

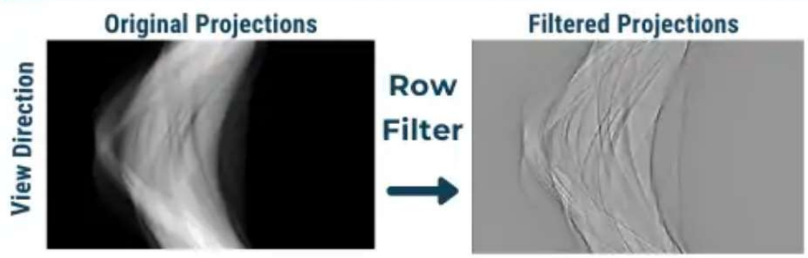
- 1 view: spike of intensity 1. This is sum of activity along projection path
- Re-distribute activity back to its original path
- Give equal activity everywhere along the line
- Many angles → Tall spike at the location of the point source
- (d) Ups...



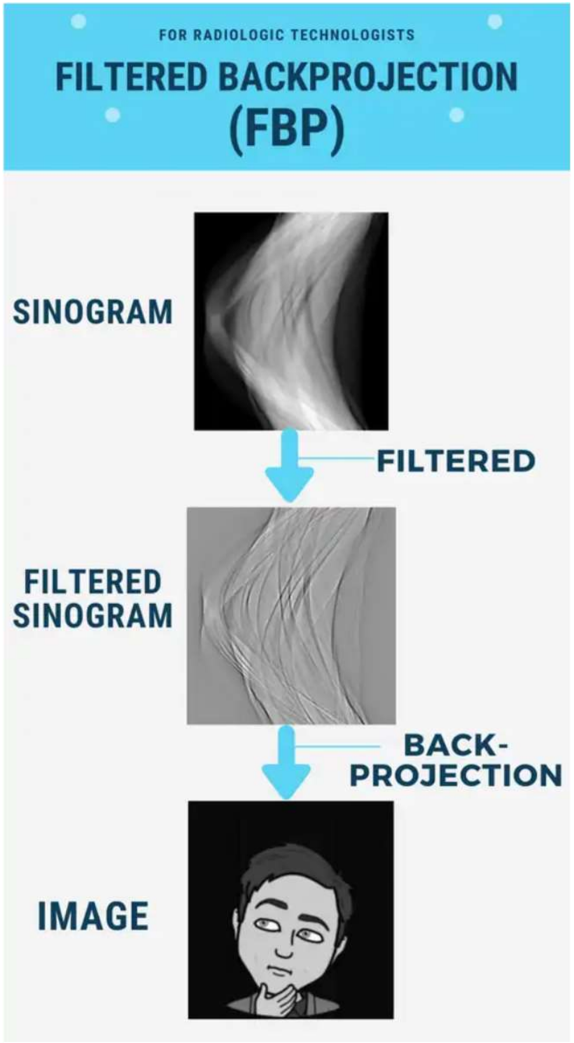
Filtered Backprojection.



SHARPENING FILTER



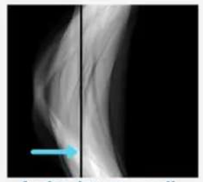
*THIS SHARPENING **F**ILTER IS THE **F** IN **F**BP.*


FOR RADIOLOGIC TECHNOLOGISTS
FILTERED BACKPROJECTION (FBP)
SINOGRAM
FILTERED
FILTERED SINOGRAM
BACK-PROJECTION
IMAGE

FOR RADIOLOGIC TECHNOLOGISTS

CT SINOGRAM ERRORS LEAD TO IMAGE ARTIFACTS

Sinogram w/ Bad Detector

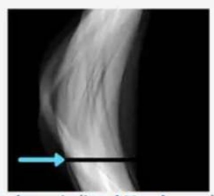


faulty detector cell

Ring (Arc) Artifacts

When the backprojection from one detector channel it causes a ring (arc) as the backprojection of that detector traces a circle.

Sinogram w/ Bad View



tube spit (i.e. kVp change)

Streaky Artifact

When the backprojection from one view is off it results in streaks from the source to the detector.

Thank you

Any Question????