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1. (a) $3^0 \; mod \; 7 = 1$
$3^1 \; mod \; 7 = 3$
$3^2 \; mod \; 7 = 2$
$3^3 \; mod \; 7 = 6$
$3^4 \; mod \; 7 = 4$
$3^5 \; mod \; 7 = 5$
$3^6 \; mod \; 7 = 1$

We see that the pattern repeats after 6 iterations. Therefore, we compute
$2015 \; mod \; 6 = 5$

and thus, we get the following:
$3^{2015} \equiv 3^5 \equiv 5 \; (mod \; 7)$
$\therefore 3^{2015} mod \; 7 = 5$

(b) Recall that quadratic formula for $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1, b = 2, c = 5$:

$$x = \frac{-2 \pm \sqrt{2^2 - 4\,(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

$\therefore x = -1 + 2i$ or $x = -1 - 2i$

(c) In matrix form, taking $\boldsymbol{A}x = \boldsymbol{b}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ 1 \end{bmatrix}$$

We will apply elementary matrix operations simultaneously to both matrices **A** and **b**, hence for the purpose of presentation, they will be written side-by-side as one augmented matrix. (Notation: R1 = Row 1, R2 = Row 2, R3 = Row 3)

$$\boldsymbol{A}|\boldsymbol{b} = \begin{bmatrix} 1 & -1 & 2 & 11 \\ 1 & 1 & 1 & 8 \\ 2 & -3 & 0 & 1 \end{bmatrix}$$

(1) Subtract R1 from R2 and place result in R2
(2) Subtract 2 x R1 from R3 and place result in R3

$$\begin{bmatrix} 1 & -1 & 2 & 11 \\ 1 & 1 & 1 & 8 \\ 2 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{Step\;(1)\;and\;(2)} \begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 2 & -1 & -3 \\ 0 & -1 & -4 & -21 \end{bmatrix}$$

(3) Add $\frac{1}{2}$R2 to R3 and place result in R3

$$\begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 2 & -1 & -3 \\ 0 & -1 & -4 & -21 \end{bmatrix} \xrightarrow{Step\;(3)} \begin{bmatrix} 1 & -1 & 2 & 11 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & -\frac{9}{2} & -\frac{45}{2} \end{bmatrix}$$

(4) Multiply R3 by $-\frac{2}{9}$

$$\begin{bmatrix}1 & -1 & 2 & 11\\ 0 & 2 & -1 & -3\\ 0 & 0 & -\dfrac{9}{2} & -\dfrac{45}{2}\end{bmatrix} \xrightarrow{Step\ (4)} \begin{bmatrix}1 & -1 & 2 & 11\\ 0 & 2 & -1 & -3\\ 0 & 0 & 1 & 5\end{bmatrix}$$

(5) Subtract 2 x R3 from R1 and place result in R1
(6) Add R3 to R2 and place result in R2
(7) Multiply R2 by $\frac{1}{2}$

$$\begin{bmatrix}1 & -1 & 2 & 11\\ 0 & 2 & -1 & -3\\ 0 & 0 & 1 & 5\end{bmatrix} \xrightarrow{Step\ (5),(6)\ and\ (7)} \begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 5\end{bmatrix}$$

(8) Add R2 to R1 and place result in R1

$$\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 5\end{bmatrix} \xrightarrow{Step\ (8)} \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 5\end{bmatrix}$$

Since we end up with an identity matrix for **A** after the above steps, we obtain a unique solution for the system, where
$x_1 = 2$
$x_2 = 1$
$x_3 = 5$

2 (a) Characteristic equation: $s^2 - 5s + 6 = 0$
Factorise: $(s-3)(s-2) = 0$
It is easy to see that the roots are $s = 3, s = 2$.

$\therefore a_n = \alpha(3)^n + \beta(2)^n$
Given that $a_1 = 8, a_2 = 20$, we obtain a system of 2 linear equations with 2 variables:

$$\begin{cases} 3\alpha + 2\beta = 8 \\ 9\alpha + 4\beta = 20 \end{cases}$$

Solving it yields $\alpha = \frac{4}{3}, \beta = 2$, giving us the following relation for the given recurrence relation:

$$a_n = \frac{4}{3}(3)^n + 2(2)^n$$

or simply:

$$a_n = 4(3)^{n-1} + (2)^{n+1}$$

(b) For one suit, the probably of choosing 5 cards from the same suit (taking note that as each successive card is chosen from the deck, it is not replaced):

$$P_1 = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} = \frac{33}{66640}$$

Recall that there are 4 suits in total:

$$\therefore P(flush) = 4 \times P_1 = \frac{33}{16660}$$

(c) Let P(n) be the proposition that $\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}\ for\ n \geq 1$

Base case: P(1)

$$LHS: \sum_{k=1}^{2^1} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

$$RHS: 1 + \frac{1}{2} = \frac{3}{2}$$

$$LHS \geq RHS \therefore P(1)\ is\ true.$$

Suppose P(s) is true for some $s \geq 1$, we need to show that P(s+1) is true, ie,

$$\sum_{k=1}^{2^{s+1}} \frac{1}{k} \geq 1 + \frac{s+1}{2}$$

We will take the LHS and attempt to "logically" simplify it. First we break the series into two parts:

$$\sum_{k=1}^{2^{s+1}} \frac{1}{k} = \sum_{k=1}^{2^s} \frac{1}{k} + \sum_{k=2^s+1}^{2^{s+1}} \frac{1}{k} \dots (1)$$

Digressing for a while. Suppose you have a series of fractions with ascending denominator and constant numerator, such as below:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{100}$$

We know that

$$\frac{1}{2} \geq \frac{1}{100}, \frac{1}{3} \geq \frac{1}{100} \cdots \frac{1}{99} > \frac{1}{100}$$

By that logic, we can deduce the following (LHS and RHS have the same number of terms):

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{100} \geq \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \cdots + \frac{1}{100} = \frac{99}{100}$$

Now we come back to (1) with the above logic in mind:

$$\sum_{k=1}^{2^{s+1}} \frac{1}{k} = \sum_{k=1}^{2^s} \frac{1}{k} + \sum_{k=2^s+1}^{2^{s+1}} \frac{1}{k}$$

$$\geq \sum_{k=1}^{2^s} \frac{1}{k} + \sum_{k=2^s+1}^{2^{s+1}} \frac{1}{2^{s+1}}$$

(we take the smallest fraction of the series and sum it up)

$$= \sum_{k=1}^{2^s} \frac{1}{k} + \frac{(2^{s+1} - (2^s + 1) + 1)}{2^{s+1}}$$

(Initial index = $2^s + 1$, Final index = 2^{s+1}, hence no. of elements = $(2^{s+1} - (2^s + 1) + 1)$)

$$= \sum_{k=1}^{2^s} \frac{1}{k} + \frac{2^{s+1} - 2^s}{2^{s+1}}$$

$$= \sum_{k=1}^{2^s} \frac{1}{k} + \frac{2 \times 2^s - 2^s}{2^{s+1}}$$

$$= \sum_{k=1}^{2^s} \frac{1}{k} + \frac{2^s}{2^{s+1}}$$
$$= \sum_{k=1}^{2^s} \frac{1}{k} + \frac{1}{2}$$

So far, we have that

$$\sum_{k=1}^{2^{s+1}} \frac{1}{k} \geq \sum_{k=1}^{2^s} \frac{1}{k} + \frac{1}{2}$$

Taking the assumption that P(s) is true, we have that

$$\sum_{k=1}^{2^s} \frac{1}{k} \geq 1 + \frac{s}{2}$$

and thus we can deduce that

$$\sum_{k=1}^{2^{s+1}} \frac{1}{k} \geq \sum_{k=1}^{2^s} \frac{1}{k} + \frac{1}{2} \geq \left(1 + \frac{s}{2}\right) + \frac{1}{2} = 1 + \frac{s+1}{2}$$

With that, we have proven $P(s) \rightarrow P(s+1)$ is true.

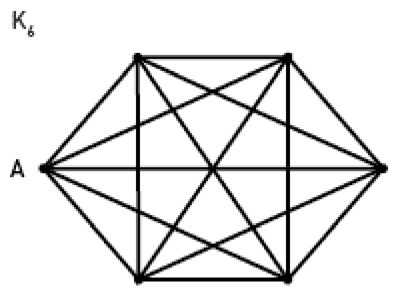
Since $P(1)$ is true, and that $P(s) \rightarrow P(s+1)$ is true, we have by mathematical induction that $\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}$ for all $n \geq 1$.

Q.E.D.

3 (a) $(A \cup B) - (A \cap B)$
$= (A \cup B) \cap \overline{(A \cap B)}$
$= (A \cup B) \cap (\overline{A} \cup \overline{B})$
$= \left[(A \cup B) \cap \overline{A}\right] \cup \left[(A \cup B) \cap \overline{B}\right]$
$= \left[\left(A \cap \overline{A}\right) \cup \left(B \cap \overline{A}\right)\right] \cup \left[\left(A \cap \overline{B}\right) \cup \left(B \cap \overline{B}\right)\right]$
$= [\phi \cup (B - A)] \cup [(A - B) \cup \phi]$
$= (B - A) \cup (A - B)$
$= (A - B) \cup (B - A)$

Q.E.D.

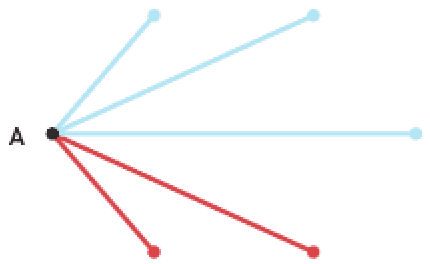
(b) We can symbolise the problem in terms of a connected graph with 6 nodes.



Each node represents one party member. We will then classify each edge in the graph (colour the edge) as either "friendship" or "stranger". If we can prove that there will always be at least one closed triangle subgraph with 3 edges of the same "colour", then we are done.

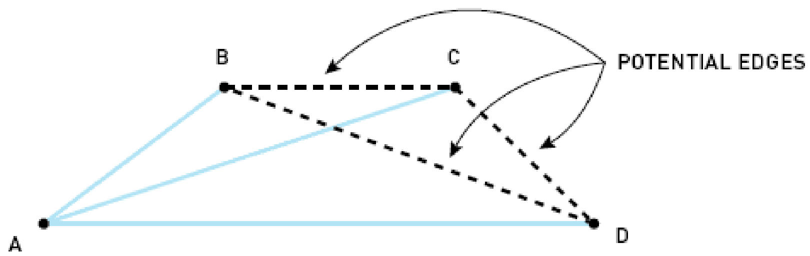
We first look at person A. Person A has connections to 5 other people, ie, his neighbours. These connections can either be “friends” or “strangers”. No matter how you classify these 5 connections, there will always be at least 3 “friend” connections or 3 “stranger” connections. (Pigeonhole principle: Classifying 5 edges into 2 groups implies that one group must have at least 3 edges in it).

A’S FIVE CONNECTIONS, ALSO KNOWN AS A’S NEIGHBORS



Now we just look at the 3 connections of the same type (in this case, the blue connection from the above diagram).

VERTICES B, C AND D WILL BE CONNECTED BY EITHER A RED EDGE OR A BLUE EDGE



Between A’s immediate neighbours, there will be connections as well; again, either blue “friendship” or red “stranger” connections. It is obvious that if edges BC, CD and BD are all of the same colour (blue or red), then we already have a **closed triangle subgraph**, and thus, mutual friendship/stranger between 3 people.

Suppose we had 1 blue edge, then whether it is on edge BC, CD or BD, it will still form a **closed triangle subgraph** with person A.

Suppose we had 2 blue edges instead, then it will definitely form at least one **closed triangle subgraph**.

Now, all we have left is to exchange blue “friendship” for red “stranger” in the above proof, and due to the rotational symmetry of the graph, we have shown that there has to be at least one **closed triangle subgraph** of the same coloured edges (at least 3 people who are mutual friends/strangers).

Q.E.D.

Adapted from:
https://www.learner.org/courses/mathilluminated/units/2/textbook/06.php

4 (a) If we choose x=2, and test whether xRx is true:

$$2^2 | 2 \rightarrow 4|2$$

But 4 does not divide 2, hence 2R2 is false (counter-example). It is not-reflexive, and hence cannot be a partial-order or an equivalence relation.

(b) Note: Composition of functions are associative, as a result of the associativity of relation compositions.

Let's first assume that the following exists
$(f \circ g)^{-1}$
With that, we have:
$[(f \circ g) \circ (f \circ g)^{-1}](x) = x \ldots (1)$

Now, since f and g are invertible, we can perform the following:
$x = (f \circ f^{-1})(x)$
$= (f \circ (g \circ g^{-1}) \circ f^{-1})(x)$
$= (f \circ g \circ g^{-1} \circ f^{-1})(x)$
$= [(f \circ g) \circ (g^{-1} \circ f^{-1})](x) \ldots (2)$ (recall the note above)

We then compare results from (1) and (2)
$[(f \circ g) \circ (f \circ g)^{-1}](x) = [(f \circ g) \circ (g^{-1} \circ f^{-1})](x)$
By inspection, we see that:
$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Adapted from: http://math.stackexchange.com/questions/2349/how-to-prove-f-circ-g-1-g-1-circ-f-1