Exercise 2. Solutions

1.

Find the base with the largest absolute value!

(1) $$a_n = \frac{n\cdot 5^n + (-6)^{n+1}}{(-6)^n + 2^{n-1}} = \frac{n\cdot 5^n + (-6)(-6)^n}{(-6)^n + \frac{1}{2}\cdot 2^n} =$$

$$= \frac{n\cdot\left(-\frac{5}{6}\right)^n + (-6)}{1 + \frac{1}{2}\cdot\left(-\frac{2}{6}\right)^n} \to \frac{-6}{1} = -6 \text{ as } n\to\infty.$$

We simplify by $(-6)^n$.

Use that $q^n \to 0$ as $n\to\infty$ if $|q|<1$.

(2) $$a_n = \frac{\overbrace{n^3+2n+1}^{=p(n)}}{\underbrace{(3n+1)^3}_{=q(n)}}$$

Ratio of two polynomials!

$\deg(p) = 3 = 3 = \deg(q) \Rightarrow$

$\Rightarrow (a_n)$ is convergent!

the limit is the ratio of the leading coefficients: $\frac{1}{27}$.

(3) $$a_n = \underbrace{\frac{(n+1)3^n + n^2}{5^n + 3^{n+1}}}_{\text{Type like (1).}} = \underbrace{\frac{(n+1)\cdot\left(\frac{3}{5}\right)^n + n^2\cdot\left(\frac{1}{5}\right)^n}{1 + 3\cdot\left(\frac{3}{5}\right)^n}} \to 0 \text{ as } n\to\infty.$$

Use that $p(n)\cdot q^n \to 0$ as $n\to\infty$ if p is a polynomial and $|q|<1$.

(4) $$a_n = \left(\frac{2n+1}{2n+10}\right)^{3n-1} = \left(\frac{2n+10-10+1}{2n+10}\right)^{3n-1} = \left(1+\frac{-9}{2n+10}\right)^{3n-1}$$

$$= \Bigg[\underbrace{\left(1+\frac{-9}{2n+10}\right)^{2n+10}}_{\to e^{-9}}\Bigg]^{\frac{3n-1}{2n+10}\ \to\ \frac{3}{2}} \Rightarrow a_n \to \left(e^{-9}\right)^{\frac{3}{2}} \text{ as } n\to\infty.$$

(2)

(5) $a_n = \dfrac{n^2+4}{n^4-16}$ $\quad \begin{matrix} =p(n) \\ =q(n) \end{matrix}$ $\qquad \deg(p) = 2 < 4 = \deg(q) \Rightarrow$

$\Rightarrow (a_n)$ is a nullsequence!

(6)

$$a_n = \sqrt{2n^2+2} - \sqrt{2n^2+1} = \frac{\overbrace{\left(\sqrt{2n^2+2}-\sqrt{2n^2+1}\right)\left(\sqrt{2n^2+2}+\sqrt{2n^2+1}\right)}^{(a-b)(a+b)=a^2-b^2}}{\sqrt{2n^2+2}+\sqrt{2n^2+1}}$$

$$= \frac{2n^2+2-(2n^2+1)}{\sqrt{2n^2+2}+\sqrt{2n^2+1}} = \frac{\overset{\deg=0}{1}}{\underset{\deg=1}{\sqrt{2n^2+2}}+\underset{\deg=1}{\sqrt{2n^2+1}}} \Rightarrow$$

$\Rightarrow (a_n)$ is a nullsequence!

(7)

$$a_n = \sqrt{n^3+1} - n = \ldots$$

Use the identity $(a-b)(a^2+ab+b^2) = a^3-b^3$!!

Solution: $\lim\limits_{n\to\infty} a_n = 0$.

(8) $a_n = \left(\dfrac{n+1}{n-3}\right)^{2n+1}$ Solve it as in problem (4)!

Solution: $\lim\limits_{n\to\infty} a_n = e^8$.

(9) $$a_n = \frac{(-1)^n + 25\cdot 5^n}{5^n + \left(-\frac{1}{2}\right)\cdot(-2)^n} = \frac{\left(-\frac{1}{5}\right)^n + 25}{1+\left(-\frac{1}{2}\right)\cdot\left(-\frac{2}{5}\right)^n} \to 25$$ as $n\to\infty$.

(10) $a_n = \dfrac{n^3+2n^2-4}{n^2+2n^3+4}$ $\qquad \left.\begin{matrix}\deg(p)=3\\ \deg(q)=3\end{matrix}\right\} \Rightarrow \lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$.

(11) $a_n = n - \sqrt{n^2+1}$ Solution:

Solve it as in problem (6). $\lim_{n\to\infty} a_n = 0$.

(12)

$$a_n = \left(\frac{3n+4}{3n+3}\right)^{n+1} = \left(\frac{3n+3-3+4}{3n+3}\right)^{n+1} = \left(1 + \frac{1}{3n+3}\right)^{n+1}$$

$$= \left[\left(1 + \frac{1}{3n+3}\right)^{3n+3}\right]^{\frac{n+1}{3n+3}} \longrightarrow e^{1/3} \text{ as } n\to\infty.$$

(13)

$$a_n = \frac{n^4+2n+3}{5^{n+1}+2^n} \Bigg\} \; a_n = \frac{\text{polynomial}}{\text{exponential function}} \Rightarrow$$

$\Rightarrow a_n \to 0$ as $n \to \infty$.

(14)

$$a_n = \frac{(n+1)^8 \cdot 7^n + n^2+1}{(-8)^{n+1} + n^3(-7)^{n+1}} = \frac{(n+1)^8 \cdot \left(-\frac{7}{8}\right)^n + (n^2+1)\left(-\frac{1}{8}\right)^n}{-8 + (-7)\cdot n^3 \cdot \left(\frac{7}{8}\right)^n}$$

$\Rightarrow a_n \to 0$ as $n \to \infty$.

(15)

$a_n = \sqrt[n]{2^n+3^n}$ The limit is the largest basis.

- $a_n = \sqrt[n]{2^n+3^n} \geq \sqrt[n]{3^n} = 3 \to 3$ as $n\to\infty$
- $a_n = \sqrt[n]{2^n+3^n} \leq \sqrt[n]{3^n+3^n} = \sqrt[n]{2\cdot 3^n} = \sqrt[n]{2}\cdot 3 \to 3$ as $n\to\infty$.

$\Rightarrow a_n \to 3$ as $n\to\infty$.

(16) $a_n = \sqrt[n]{4^{n+1} + 5^n}$

- lower estimation:

$$a_n = \sqrt[n]{4^{n+1} + 5^n} \geq \sqrt[n]{5^n} = 5 \to 5 \text{ as } n \to \infty.$$

- upper estimation:

$$a_n = \sqrt[n]{4^{n+1} + 5^n} \leq \sqrt[n]{5^{n+1} + 5^n} = \sqrt[n]{6 \cdot 5^n} =$$

$$= \sqrt[n]{6} \cdot 5 \to 5 \text{ as } n \to \infty.$$

$\Rightarrow$

$\Rightarrow$ In view of Squeeze theorem, the limit of (a_n) is 5.

(17) $a_n = \dfrac{(2n^2+4n)^2}{4n^4+5n+1}$ $\Big\}$ $\deg(p) = 4$, $\deg(q) = 4$ $\Rightarrow$ (a_n) is convergent!

The limit is 1.

(18) $$a_n = \left(\frac{n-4}{n+4}\right)^n = \left[\underbrace{\left(1 + \frac{-8}{n+4}\right)^{n+4}}_{\to e^{-8}}\right]^{\frac{n}{n+4} \to 1} \Rightarrow$$

$\Rightarrow$ $a_n \to e^{-8}$ as $n \to \infty$.