Ex. 1: (1) $f(x) = (x+2)^2$ $\qquad f'(x) = 2(x+2)\cdot\underbrace{(x+2)'}_{=1+0} = 2x+4 \qquad f''(x) = (f'(x))' = (2x+4)' = 2$

(2) $f(x) = x^3 + 5^x \cdot \cos(x) \qquad f'(x) = 3x^2 + 5^x \cdot \ln(5)\cdot\cos(x) + 5^x\cdot(-\sin(x)) =$

$$= 3x^2 + 5^x\left(\ln(5)\cdot\cos(x) - \sin(x)\right)$$

$$f''(x) = 6x + 5^x \cdot \ln(5)\left(\ln(5)\cdot\cos(x) - \sin(x)\right) + 5^x\left(\ln(5)(-\sin(x)) - \cos(x)\right)$$

(3)

$$f(x) = (x^2+1)^{\cos(x)} = \left(e^{\ln(x^2+1)}\right)^{\cos(x)} = e^{\ln(x^2+1)\cdot\cos(x)}$$

$$f'(x) = e^{\ln(x^2+1)\cdot\cos x} \cdot \left(\ln(x^2+1)\cdot\cos(x)\right)' = e^{\ln(x^2+1)\cdot\cos(x)} \cdot \Big(\frac{1}{x^2+1}\cdot\overbrace{(2x+0)}^{(x^2+1)'}\cdot\cos(x)$$

$$+ \ln(x^2+1)\cdot(-\sin(x))\Big) = e^{\ln(x^2+1)\cdot\cos(x)}\left(\frac{2x\cdot\cos(x)}{x^2+1} - \ln(x^2+1)\cdot\sin(x)\right)$$

(4) $f(x) = \ln(x^3+3x^2+3x+1)$

$$f'(x) = \frac{1}{x^3+3x^2+3x+1}\cdot(x^3+3x^2+3x+1)' = \frac{3x^2+6x+3}{x^3+3x^2+3x+1}$$

$$f''(x) = (f'(x))' = \frac{(6x+6)(x^3+3x^2+3x+1) - (3x^2+6x+3)^2}{(x^3+3x^2+3x+1)^2}$$

$$(3x^2+6x+3)' = 6x+6$$

$$(x^3+3x^2+3x+1)' = 3x^2+6x+3$$

(5)

$$f(x) = \tan(x^2) \qquad f'(x) = \frac{1}{\cos^2(x^2)}\cdot 2x = \frac{2x}{\cos^2(x^2)}$$

$$f''(x) = \frac{2\cdot\cos^2(x^2) - 2x\cdot\overbrace{2\cdot\cos(x^2)\cdot(-\sin(x^2))\cdot 2x}^{=(\cos^2(x^2))'}}{\cos^4(x^2)}$$

(6) $f(x) = \frac{x^2}{\log_5(x)}$ $\qquad f'(x) = \frac{2x\cdot\log_5(x) - x^2\cdot\frac{1}{x\cdot\ln(5)}}{(\log_5(x))^2}$

(7) $f(x) = \cos(\sin(x)+1)$

$f'(x) = -\sin(\sin(x)+1)\cdot\overbrace{\cos(x)}^{(\sin(x)+1)'}$

$f''(x) = \overbrace{-\cos(\sin(x)+1)\cdot\cos(x)}^{(-\sin(\sin(x)+1))'}\cdot\cos(x) + (-\sin(\sin(x)+1))\cdot\overbrace{(-\sin(x))}^{\cos(x)'}$

(8) $f(x) = \underbrace{\sqrt{x}}_{=x^{\frac{1}{2}}}\cdot\ln(x)$ $\qquad f'(x) = \frac{1}{2\sqrt{x}}\cdot\ln(x) + \sqrt{x}\cdot\frac{1}{x} = \frac{\ln(x)}{2\sqrt{x}} + \underbrace{\frac{1}{\sqrt{x}}}_{x^{-\frac{1}{2}}}$

$f''(x) = \frac{\frac{1}{x}\cdot 2\sqrt{x} - \ln(x)\cdot 2\cdot\frac{1}{2\sqrt{x}}}{(2\sqrt{x})^2} + \left(-\frac{1}{2}\right)\cdot x^{-\frac{3}{2}}$

(9) $f(x) = \frac{x^2+4x+4}{(x+2)^3} = \frac{(x+2)^2}{(x+2)^3} = \frac{1}{x+2} = (x+2)^{-1}$

$f'(x) = (-1)(x+2)^{-2}\cdot(x+2)' = (-1)\cdot(x+2)^{-2} = \frac{-1}{(x+2)^2}$

$f''(x) = (-1)\cdot(-2)\cdot(x+2)^{-3} = 2\cdot(x+2)^{-3} = \frac{2}{(x+2)^3}$

(10) $f(x) = (x^3+1)^{x^3+1} = \left(e^{\ln(x^3+1)}\right)^{x^3+1} = e^{(x^3+1)\cdot\ln(x^3+1)}$

$f'(x) = e^{(x^3+1)\cdot\ln(x^3+1)}\cdot\left((x^3+1)\cdot\ln(x^3+1)\right)' =$

$= e^{(x^3+1)\cdot\ln(x^3+1)}\left(\underbrace{3x^2}_{(x^3+1)'}\cdot\ln(x^3+1) + (x^3+1)\cdot\underbrace{\frac{1}{x^3+1}\cdot\overbrace{3x^2}^{(x^3+1)'}}_{(\ln(x^3+1))'}\right)$

(11) $f(x) = 3^{x+1}\cdot\sinh(x^4+2x)$

$f'(x) = 3^{x+1}\cdot\ln(3)\cdot\underbrace{(x+1)'}_{=1}\cdot\sinh(x^4+2x) + 3^{x+1}\cdot\cosh(x^4+2x)\cdot\overbrace{(4x^3+2)}^{(x^4+2x)'}$

(12) $f(x) = x^x = \left(e^{\ln(x)}\right)^x = e^{x\cdot\ln(x)}$

$$f'(x) = e^{x\cdot\ln(x)}\cdot(x\cdot\ln(x))' = e^{x\ln(x)}\left(\underbrace{1}_{x'}\cdot\ln(x) + x\cdot\underbrace{\frac{1}{x}}_{(\ln(x))'}\right) = e^{x\cdot\ln(x)}(\ln(x)+1)$$

$$f''(x) = \left(e^{x\cdot\ln(x)}\right)'\cdot(\ln(x)+1) + e^{x\ln(x)}\cdot(\ln(x)+1)' =$$

$$= e^{x\cdot\ln(x)}\cdot(\ln(x)+1)^2 + e^{x\ln(x)}\cdot\frac{1}{x}$$

(13) $f(x) = \arctan(x)$ $\qquad f'(x) = \dfrac{1}{1+x^2}$

$$f''(x) = \left((1+x^2)^{-1}\right)' = (-1)\cdot(1+x^2)^{-2}\cdot\underbrace{2x}_{(1+x^2)'} = \frac{-2x}{(1+x^2)^2}$$ or

$$f''(x) = \frac{1}{1+x^2} = \frac{\underbrace{0}_{1'}\cdot(1+x^2) - 1\cdot\underbrace{2x}_{(1+x^2)'}}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

(14) $f(x) = \cosh(3x+1)$ $\qquad f'(x) = \sinh(3x+1)\cdot\underbrace{3}_{(3x+1)'}$

$$f''(x) = 3\cdot\cosh(3x+1)\cdot\underbrace{3}_{(3x+1)'} = 9\cdot\cosh(3x+1)$$

(15) $f(x) = \dfrac{4^x\cdot(x+2)^3}{\ln^x x} = \dfrac{4^x\cdot(x+2)^3}{e^{\ln(\ln(x))\cdot x}}$

$$(\ln(x))^x = \left(e^{\ln(\ln(x))}\right)^x = e^{\ln(\ln(x))\cdot x}$$

$$\left(4^x\cdot(x+2)^3\right)' = \underbrace{4^x\cdot\ln(4)}_{(4^x)'}\cdot(x+2)^3 + 4^x\cdot\overbrace{3(x+2)^2\cdot\underbrace{1}_{(x+2)'}}^{((x+2)^3)'} = 4^x\cdot(x+2)^2\left(\ln(4)\cdot(x+3)+3\right)$$

$$\left(e^{\ln(\ln(x))\cdot x}\right)' = e^{\ln(\ln(x))x}\left(\underbrace{\frac{1}{\ln(x)}\cdot\frac{1}{x}}_{(\ln(\ln(x)))'}\cdot x + \ln(\ln(x))\cdot\underbrace{1}_{x'}\right) =$$

$$= \underbrace{e^{\ln(\ln(x))\cdot x}}_{(\ln(x))^x}\left(\frac{1}{\ln(x)} + \ln(\ln(x))\right)$$

$$f'(x) = \frac{\left(4^x\cdot\ln(4)\cdot(x+2)^3 + 4^x\cdot 3\cdot(x+2)^2\right)\cdot(\ln(x))^x - 4^x\cdot(x+2)^3\cdot(\ln(x))^x\cdot\left(\frac{1}{\ln(x)} + \ln(\ln(x))\right)}{(\ln(x))^{2x}}$$

<u>Ex. 4.</u> (1) $f(x) = 2^x \quad f'(x) = 2^x \cdot \ln(2),\ f''(x) = 2^x (\ln(2))^2,\ f'''(x) = 2^x (\ln(2))^3,$

$$\dots \Rightarrow f^{(n)}(x) = 2^x \cdot (\ln(2))^n$$

(2) $f(x) = \cos(x),\ f'(x) = -\sin(x),\ f''(x) = -\cos(x),\ f''' = \sin(x),$

$$f^{IV}(x) = f(x) = \cos(x) \Rightarrow$$

$$f^{(n)}(x) = \begin{cases} \cos(x) & \text{if } n = 4k \\ -\sin(x) & \text{if } n = 4k+1 \\ -\cos(x) & \text{if } n = 4k+2 \\ \sin(x) & \text{if } n = 4k+3 \end{cases} \qquad (k \in \mathbb{N})$$

(3) $f(x) = \dfrac{1}{2x+1} = (2x+1)^{-1} \quad f'(x) = (-1) \cdot (2x+1)^{-2} \cdot \overbrace{2}^{(2x+1)'} = (-2) \cdot (2x+1)^{-2}$

$$f''(x) = (-1)(-2)(2x+1)^{-3} \cdot \underbrace{2 \cdot 2}_{2^2},\quad f'''(x) = (-1)(-2)(-3) \cdot (2x+1)^{-4} \cdot 2^3, \dots$$

$$\Rightarrow f^{(n)}(x) = (-1)^n \cdot n!\,(2x+1)^{-(n+1)} \cdot 2^n$$

(4) $f(x) = \sin(3x),\ f'(x) = \cos(3x) \cdot 3,\ f''(x) = -\sin(3x) \cdot 3^2,$

$$f'''(x) = -\cos(3x) \cdot 3^3,\ f^{IV}(x) = \sin(3x) \cdot 3^4 = f(x) \cdot 3^4$$

$$\Rightarrow \quad f^{(n)}(x) = \begin{cases} \sin(3x) \cdot 3^n & \text{if } n = 4k \\ \cos(3x) \cdot 3^n & \text{if } n = 4k+1 \\ -\sin(3x) \cdot 3^n & \text{if } n = 4k+2 \\ -\cos(3x) \cdot 3^n & \text{if } n = 4k+3 \end{cases}$$