University of Debrecen, Faculty of Informatics

Logic in computer science

Zero-order logic
(propositional logic)
syntax

September 22, 2022

The zero-order language of classical logic

An ordered triple $\mathcal{L}^{(0)} = \langle \mathbf{LC}, \mathbf{Con}, \mathbf{Form} \rangle$ defines the language of the zero-order (propositional) logic, where

1. $\mathbf{LC} = \{ \neg, \supset, \wedge, \vee, \equiv, (,) \}$ is the set of *logical constants*,
2. $\mathbf{Con} \neq \varnothing$; is the set of *non-logical constants* such that
$$\mathbf{LC} \cap \mathbf{Con} = \varnothing;$$
3. $\mathbf{Form}$ is the set of *formulae*.

The logical and non-logical constants contains the letters of the alphabet, while the formulas as sequences of letters are the words of the zero-order language.

Zero-order formulas

The words of the zero-order language are the members of the **Form** set, and we call them *zero-order formulas*. The **Form** set is inductively defined as follows:

- $\mathbf{Con} \subseteq \mathbf{Form}$ (if $\mathbf{A} \in \mathbf{Con}$ then $\mathbf{A}$ *atomic*)
- if $\mathbf{B} \in \mathbf{Form}$ then $\neg\mathbf{B} \in \mathbf{Form}$.
- if $\mathbf{B} \in \mathbf{Form}$ and $\mathbf{C} \in \mathbf{Form}$, then
  - $(\mathbf{B} \wedge \mathbf{C}) \in \mathbf{Form}$,
  - $(\mathbf{B} \vee \mathbf{C}) \in \mathbf{Form}$,
  - $(\mathbf{B} \supset \mathbf{C}) \in \mathbf{Form}$,
  - $(\mathbf{B} \equiv \mathbf{C}) \in \mathbf{Form}$.

Direct subformula

If $\mathbf{A} \in \mathbf{Con}$ then

- the $\mathbf{A}$ atomic formula has no direct subformula;

if $\mathbf{B} \in \mathbf{Form}$ and $\mathbf{C} \in \mathbf{Form}$ are arbitrary formulas, then

- the formula $\neg\mathbf{B}$ has one direct subformula: the formula $\mathbf{B}$;
- the formulae $(\mathbf{B} \wedge \mathbf{C})$, $(\mathbf{B} \vee \mathbf{C})$, $(\mathbf{B} \supset \mathbf{C})$ and $(\mathbf{B} \equiv \mathbf{C})$ have two direct subformulae: the formula $\mathbf{B}$ and the formula $\mathbf{C}$.

Set of subformulae

The *set of subformulae* of a formula $\mathbf{A} \in \mathbf{Form}$ is set of formulae denoted by $SF(\mathbf{A})$ and is defined inductively as follows:

1. $\mathbf{A} \in SF(\mathbf{A})$;
2. if $\mathbf{B} \in SF(\mathbf{A})$ and $\mathbf{C}$ is a direct subformula of $\mathbf{B}$, then $\mathbf{C} \in SF(\mathbf{A})$.

Note that SF is a function such that:

$$SF : \mathbf{Form} \to 2^{\mathrm{Form}} \setminus \{\varnothing\}$$

Construction tree

The construction tree of a formula $\mathbf{A} \in \mathbf{From}$ is the finite ordered binary tree defined as follows:

- their nodes are formulae,
- their root is labelled by the formula $\mathbf{A}$,
- a node labelled by $\neg\mathbf{B}$ has only one child labelled by $\mathbf{B}$,
- a node labelled by $(\mathbf{B} \wedge \mathbf{C})$, $(\mathbf{B} \vee \mathbf{C})$, $(\mathbf{B} \supset \mathbf{C})$ or $(\mathbf{B} \equiv \mathbf{C})$ has exactly two children labelled by $\mathbf{B}$ és $\mathbf{C}$,
- the leaves labelled by atomic formulae.

Logical degree

The *logical degree* of a formula is a nonnegative integer defined as follows:

$$\ell : \mathbf{Form} \to \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$$

If $\mathbf{A} \in \mathbf{Con}$ then

- the logical degree of an atomic formula $\mathbf{A}$ is 0
  $\ell(\mathbf{A}) \;\rightleftharpoons\; \mathbf{0}$;

if $\mathbf{B} \in \mathbf{Form}$ and $\mathbf{C} \in \mathbf{Form}$ are arbitrary formulas,

- $\ell(\ \neg\mathbf{B}\ ) \;\rightleftharpoons\; \mathbf{1} + \ell(\mathbf{B})$;
- $\ell(\ (\mathbf{B} \circ \mathbf{C})\ ) \;\rightleftharpoons\; \mathbf{1} + \ell(\mathbf{B}) + \ell(\mathbf{C})$ where $\circ \in \{\wedge, \vee, \supset, \equiv\}$.

Scope of a connective

For a connective $\neg$, $\wedge$, $\vee$, $\supset$ or $\equiv$ appearing in a non-atomic formula **A**, the *scope* is the formula

- with the smallest logical degree

such that

- it is a subformula of **A**, and
- it contains the connective.

The *main logical connective* of a formula is the connective whose scope equals the formula itself.

Abbreviations by the omission of parentheses

The precedence of connectives from the strongest to the weakest is:

$$\neg, \wedge, \vee, \supset, \equiv.$$

Let $\oplus \in \{\wedge, \vee, \supset, \equiv\}$ and $\oslash \in \{\wedge, \vee, \supset, \equiv\}$ two connective. Then

- in any subformula in the form $(\mathbf{A} \oslash (\mathbf{B} \oplus \mathbf{C}))$, the inner pair of parentheses is omittable if $\oplus$ is not weaker than $\oslash$.
- in any subformula in the form $((\mathbf{B} \oplus \mathbf{C}) \oslash \mathbf{A})$ the inner pair of parentheses is omittable if $\oplus$ is stronger than $\oslash$;
- in any formula in the form $(\mathbf{A} \oslash \mathbf{B})$ the (outmost) pair of parentheses is omittable.

Abbreviations by the omission of parentheses

Because the strongest connective is the negation sign ($\neg$), the parentheses immediately following it are surely not omittable.

The parentheses omission abbreviations define a relation between formulas and their abbreviations which is

- injective (because there is no two different formula with equal abbreviated form), and
- not functional (because no restriction requires eliminating all the parentheses).