University of Debrecen, Faculty of Informatics

Logic in computer science

Zero-order logic
(propositional logic)
semantics

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Truth values

The set $\{0, 1\}$ is the *set of truth values* in the classical meaning, where

- 0 represents *false*,
- 1 represents *true*.

The $\varrho : Con \to \{0, 1\}$ mapping (function) is called an *interpretation* of the zero-order language $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$.

$$\varrho \in \{0, 1\}^{Con}$$

Remark

The ϱ interpretation determines the truth value for the non-logical constants of the language.

Instead of using the above-defined function, an interpretation of the classical zero-order language is sometimes given as a subset of non-logical constants Con.

$$\{ \ a \mid a \in Con \text{ and } \varrho(a) = 1 \ \}$$

Here the interpretation as a set contains those non-logical constants which said to be true.

Truth value of a formula

The *truth value of a formula* $\mathcal{F}$ with respect to a given interpretation ϱ is denoted by $|\,\mathcal{F}\,|_\varrho$ and is defined recursively as follows:

- if $A \in Con$ then $|\,A\,|_\varrho \rightleftharpoons \varrho(A)$,
- if $B \in Form$ and $C \in Form$ then

$$
\begin{aligned}
|\,\neg B\,|_\varrho &\rightleftharpoons 1 - |\,B\,|_\varrho \\
|\,(B \wedge C)\,|_\varrho &\rightleftharpoons \min(\,|\,B\,|_\varrho, |\,C\,|_\varrho\,) \\
|\,(B \vee C)\,|_\varrho &\rightleftharpoons \max(\,|\,B\,|_\varrho, |\,C\,|_\varrho\,) \\
|\,(B \supset C)\,|_\varrho &\rightleftharpoons \max(\,1 - |\,B\,|_\varrho, |\,C\,|_\varrho\,) \\
|\,(B \equiv C)\,|_\varrho &\rightleftharpoons \begin{cases} 1 & \text{if } |\,B\,|_\varrho = |\,C\,|_\varrho, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Truth table of connectives

$\lvert B \rvert_\varrho$	$\lvert C \rvert_\varrho$	$\lvert \neg B \rvert_\varrho$	$\lvert (B \wedge C) \rvert_\varrho$	$\lvert (B \vee C) \rvert_\varrho$	$\lvert (B \supset C) \rvert_\varrho$	$\lvert (B \equiv C) \rvert_\varrho$
0	0	1	0	0	1	1
0	1		0	1	1	0
1	0	0	0	1	0	0
1	1		1	1	1	1

Model

Let $\Gamma \subseteq Form$ be an arbitrary set of formulae of a zero-order language $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$.

- The interpretation ϱ is the *model* of the set of fromulae Γ if $|\,A\,|_{\varrho} = 1$ for all $A \in \Gamma$.

Let $A \in Form$ be a formula fo a zero-order language $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$.

- The interpretation ϱ is the *model* of the formula A if the ϱ is the model of the singleton set $\{A\}$.

Satisfiability (SAT)

Let $\Gamma \subseteq Form$ be an arbitrary set of formulae of a $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$ zero-order language.

- The Γ set of formulae is *satisfiable*, if it has a model.
- The Γ set of formulae is *unsatisfiable*, if it is not satisfiable.

Let $A \in Form$ be an arbitrary formula of a $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$ zero-order language.

- The formula A is *satisfiable* if the $\{A\}$ singleton set is satisfiable.
- The formula A is *unsatisfiable* if the $\{A\}$ singleton set is unsatisfiable.

Semantic equivalence

Let $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$ be a zero-order language, $\Gamma \subseteq Form$ be an arbitrary set of formulae, and $A \in Form$ be an arbitrary formula of the language.

- The formula A is the *semantic consequence* of Γ if $\Gamma \cup \{\neg A\}$ is unsatisfiable.

  (Notation: $\Gamma \models A$)

- The formula A is *valid* (*tautology*) if $\emptyset \models A$.

  (Notation: $\models A$)

Logical equivalence

Let $\mathcal{L}^{(0)} = \langle LC, Con, Form \rangle$ be a zero-order language and $A \in Form$ and $B \in Form$ be two formulas of $\mathcal{L}^{(0)}$.

The formula A and formula B are *logically equivalent*, if

$$A \models B \quad \text{and} \quad B \models A.$$

(Notation: $A \Leftrightarrow B$)