Readme File for Github

Ljuboslav Boskic

May 24, 2019

Theory

Koopman operator theory is an alternative formulation of dynamical system theory which provides a versatile framework for data-driven methods of high-dimensional nonlinear systems. The theory originated in the 1930s through the work done by Koopman and Von Neumann.[1] Work done in the previous few years has proven the spectral decomposition[2][3] , introducing the idea of Koopman modes. This theory led to data-driven methods to approximate the Koopman operator spectrum and modes.

In a discrete time setting if:

$$x' = T(x) \tag{1}$$

is a discrete time dynamical system where $x \in \mathcal{M}$ and $T : \mathcal{M} \to \mathcal{M}$, the the associated Koopman operator U is defined by:

$$Uf(x) = f \circ T(x) \tag{2}$$

We call $\phi : \mathcal{M} \to \mathcal{C}$ an eigenfunction of U associated with $\lambda \in \mathcal{C}$ then,

$$U\phi = \lambda\phi \tag{3}$$

And in continous time,

$$U^t\phi = e^{\lambda t}\phi \tag{4}$$

The eigenfunctions and eigenvalues of the Koopman operator have lots of information about the dynamics. In the previous years a proof of the Koopman Mode Decomposition (KMD) is another outcome of this theory.

The Koopman spectrum consists only of eigenvalues, the evolution of observables can be expanded in terms of Koopman eigenfunction denoted as ϕ_j where $(j = 0, 1...)$ and Koopman eigenvalues λ_j where $(j = 0, 1, ...)$.
The evolution of f is given by:

$$U^n f(x_o) = f \circ T^n(x_o) = \sum_{j=1}^{\infty} v_j \phi_j(x_o) \lambda_j^n \tag{5}$$

In this decomposition v_j are the Koopman modes associated with the pair (λ_j, ϕ_j). These modes correspond to components of the physical field characterized by exponential growth and possible oscillations in time.

In recent years, various methods have been made to compute spectral properties (λ, ϕ, v) from data sets. Data-driven algorithms have been created and utilize data or measurements to approximate the KMD of the system. With this analysis in hand, one is able to identify stability or instability of the modes present within the dynamics. A large fraction of these algorithms are known as Dynamic Mode Decomposition (DMD).

[1]BO Koopman and J v Neumann. "Dynamical systems of continuous spectra". In: *Proceedings of the National Academy of Sciences of the United States of America* 18.3 (1932), p. 255, p. 1.

[2]Igor Mezić and Andrzej Banaszuk. "Comparison of systems with complex behavior". In: *Physica D: Nonlinear Phenomena* 197.1 (2004), pp. 101 –133. ISSN: 0167-2789. DOI: `https://doi.org/10.1016/j.physd.2004.06.015`. URL: `http://www.sciencedirect.com/science/article/pii/S0167278904002507`, p. 2.

[3]Igor Mezić. "Spectral Properties of Dynamical Systems, Model Reduction and Decompositions". In: *Nonlinear Dynamics* 41.1 (2005), pp. 309–325, p. 3.

Applications of DMD

The Koopman operator is an infinite-dimensional, linear operator that acts on a Hilbert space of functions called the space of observables. The eigenvalues and eigenfunctions of this linear operator are capable of capturing key dynamics charactersticis of a linear or nonlinear dynamical system. Additionally, the Koopman modes, corresponding to a particular choice of observable function, allow one to reconstruct and forecast (predict) the observed quantity. Togethere these three values of Koopman eigenvalues, eigenfunctions, and modes yield the Koopman mode decomposition (KMD) of an arbitrary observable.[4] There has been numerous work in control[5],[6] For error approximation of time shifted delays the following will be used:

$$Error = \frac{\|AX - Y\|_F}{\|Y\|_F} \tag{6}$$

Temperature Data[7]

Temperature readings are collected at five locations within a Laboratory at UCSB and an outdoor reading from the Santa Barbara Municipal Airport ($\sim$ 2 miles away). In addition, there is humidity, light, pressure, and noise measurements for the indoor sensors. Measurements from these sensors (along with outdoor temperature) taken every 5 minutes over the course of 5 days. Looking at just the sensors we can see the indoor and outdoor temperature in figure (1).

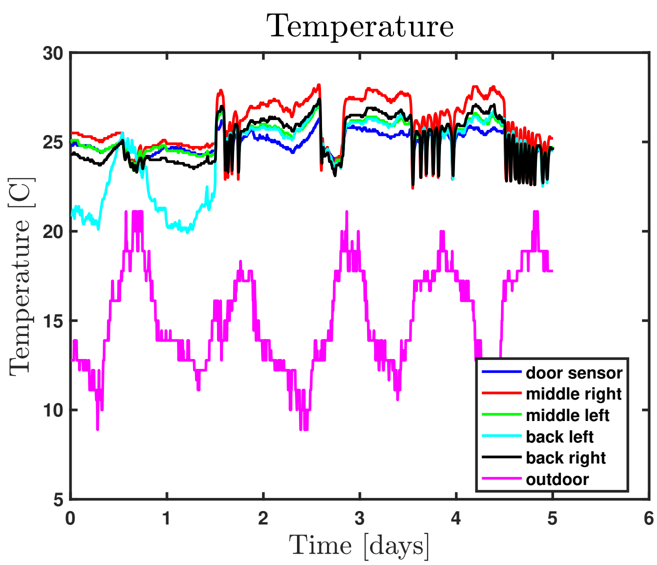

Figure 1: Indoor-Outdoor Temperature for May 6^{th} 0:00 to May 11^{th} 0:00 in laboratory area.

In the following figures we have several plots representing Koopman spectral quantities related to our chosen obsevables. In following figures we use 300 time shifted observables for each of our six

[4]Hassan Arbabi and Igor Mezić. "Study of dynamics in post-transient flows using Koopman mode decomposition". In: *Physical Review Fluids* 2.12 (2017), p. 124402, p. 4.

[5]Hassan Arbabi, Milan Korda, and Igor Mezic. "A data-driven Koopman model predictive control framework for nonlinear flows". In: *arXiv preprint arXiv:1804.05291* (2018), p. 5.

[6]Milan Korda and Igor Mezić. "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control". In: *Automatica* 93 (2018), pp. 149 –160. ISSN: 0005-1098. DOI: `https://doi.org/10.1016/j.automatica.2018.03.046`. URL: `http://www.sciencedirect.com/science/article/pii/S000510981830133X`, p. 6.

[7]Ljuboslav Boskic. "Application of Koopman Mode Analysis in Residential Environments". MA thesis. University of Califorina Santa Barbara, June 2019, p. 7.

temperature sensors.

1. The top left plot of figure 2 contains the frequency of each DMD eigenvalue with its power. We can notice bumps in the spectrum which would call for further analysis in order to get a better understanding of the dynamics going on at that particular frequency. In particular we can see a first bump around 15. The top right plots shows us the growth rate of each DMD eigenvalue with it's power, showing us most of our modes are decaying.

2. In figure 3 shows us the magnitude and phase of the six higher power modes with frequency near 1. Frequency in these plots is seen as $1.036(day)^{-1}$ which corresponds to once every day so this is the daily mode. Here we can look at the magnitude plot showing us that there is no significant dominance of any sensor in the laboratory, seen by the color bar. At this 1 per day frequency we see most of the dynamics that is going on is mostly due to the outdoors. This is also seen at the phase plot which shows that all the indoor sensors are in phase while the outdoors is out of phase by a value of π due to the color bar going from $-\pi$ to π.

3. In figure 4 also shows us the magnitude and phase of the six higher power modes but with frequency near 15. Again frequency is terms of inverse days, so this frequency corresponds to 15 times per day or about every 1.5 hours. **The interesting thing that can be seen in the magnitude is that now the middle right sensor is the most dominant in this system, this means there is something interesting going on in the dynamics of this sensor.** Further analysis of the phase shows us that the middle right sensor is pi/pi or 1 radian out of phase with the rest of the sensors. **We have found the heating and cooling control in the laboratory to be every 1.5 hours, purely from a data-driven approach.**

4. Error approximation of the optimal amount of time shifted observables was done by using equation (6), illustrated in figure 5.

   The analysis using Koopman modes gives us a remarkable insight into the thermal dynamics of indoor spaces. Namely, the distinction between the zones affected strongly by the outside conditions, near the window, are evident. In addition, there are modes that clearly indicate dynamics of the controllers. In this way, the external influences are separated from control dynamics and the refinement of the analysis using the model in the previous section is possible. *For use in residential buildings, such understanding of thermal behavior is of essence in design and control and we have provided the tools for it.*

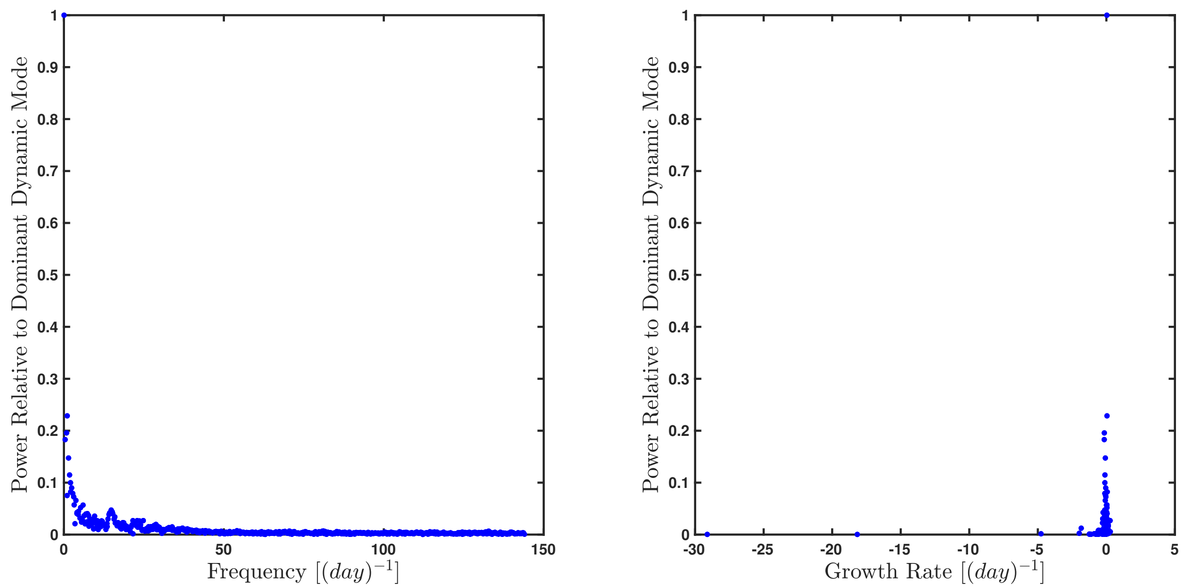

Figure 2:

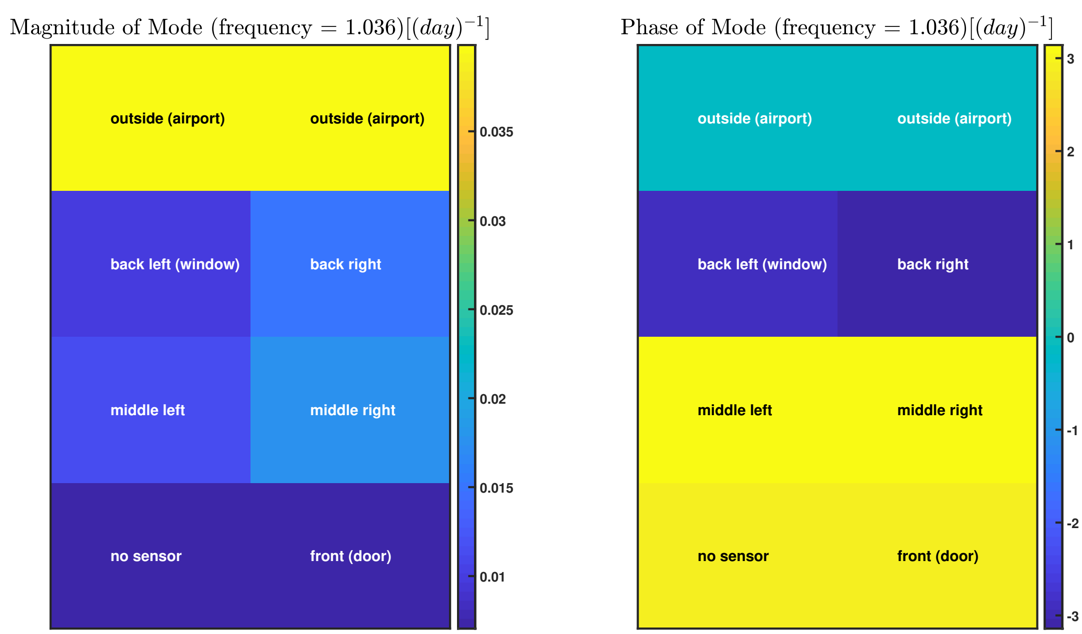

Figure 3:

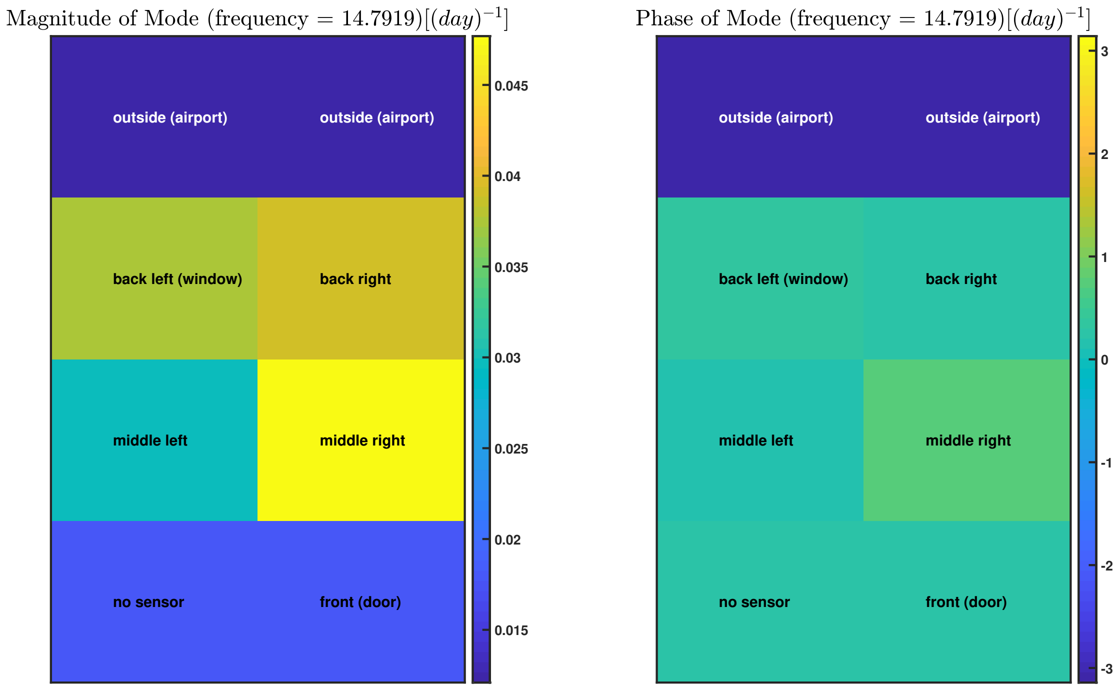

Figure 4:

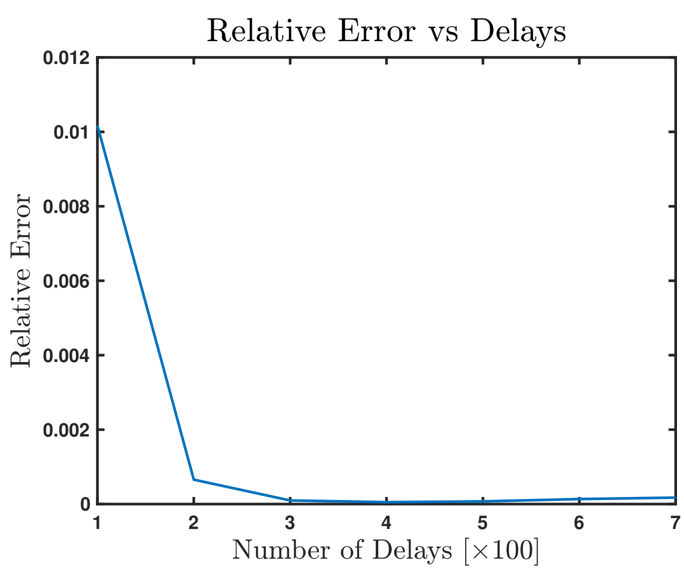

Figure 5:

1 Companion Matrix DMD[8,9]

We will call D as the data matrix.

1) Define $X = [D_o, D_1, \cdots, D_{m-1}]$

2) Compute c_j values:

$$X^{\dagger} D_m = \begin{bmatrix} c_o \\ c_1 \\ . \\ . \\ . \\ c_{m-2} \end{bmatrix}$$

Where $X^{\dagger}$ is the pseudoinverse of X.

3) Form the companion matrix:

$$C := \begin{bmatrix} 0 & 0 & ... & 0 & c_o \\ 1 & 0 & ... & 0 & c_1 \\ 0 & 1 & .. & .0 & c_2 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & c_{m-1} \end{bmatrix}$$

4) Get the eigenvalue/vectors from C, let (λ_j, w_j) be the eigenvalue-vector pair.

λ_j are the dynamic eigenvalues
v_j are the dynamic modes

$$v_j = X w_j$$

5) Compute Modes:

$$\Phi_j = Y V \Sigma^{-1} v_j$$

2 Schmid DMD[10]

We will call D as the data matrix.

1) Form $X = [D_o, D_1, \cdots, D_{m-1}]$ and $Y = [D_1, D_2, \cdots, D_m]$

---

[8] Y. Susuki and I. Mezić. "Nonlinear Koopman Modes and Power System Stability Assessment Without Models". In: *IEEE Transactions on Power Systems* 29.2 (2014), pp. 899–907. ISSN: 0885-8950. DOI: 10.1109/TPWRS.2013.2287235, p. 8.

[9] Clarence W Rowley et al. "Spectral analysis of nonlinear flows". In: *Journal of fluid mechanics* 641 (2009), pp. 115–127, p. 9.

[10] PETER J. SCHMID. "Dynamic mode decomposition of numerical and experimental data". In: *Journal of Fluid Mechanics* 656 (2010), 5–28. DOI: 10.1017/S0022112010001217, p. 10.

2) Compute the singular value decomposition (SVD) of X:

$$svd(X) = U\Sigma V^*$$

3) Create $\tilde{A}$ matrix:

$$\tilde{A} = U^* Y V \Sigma^{-1}$$

4) Get the eigenvalue/vectors from $\tilde{A}w = w\lambda$. Let (λ_j, w_j) be the eigenvalue-vector pair.

λ_js are the dynamic eigenvalues
v_js are the dynamic modes

$$v_j = Uw_j$$

5) Compute Modes:

$$\Phi_j = YV\Sigma^{-1}v_j$$

3 Exact DMD[11]

We will call D as the data matrix.

1) Form $X = [D_o, D_1, \cdots, D_{m-1}]$ and $Y = [D_1, D_2, \cdots, D_m]$

2) Compute the singular value decomposition (SVD) of X:

$$svd(X) = U\Sigma V^*$$

3) Create $\tilde{A}$ matrix:

$$\tilde{A} = U^* Y V \Sigma^{-1}$$

4) Get the eigenvalue/vectors from $\tilde{A}w = w\lambda$. Let (λ_j, w_j) be the eigenvalue-vector pair.

λ_js are the dynamic eigenvalues
v_js are the dynamic modes

[11] Jonathan H Tu et al. "On dynamic mode decomposition: theory and applications". In: *arXiv preprint arXiv:1312.0041* (2013), p. 11.

Exact DMD Mode can be given by:

$$v_j = \frac{1}{\lambda_j}(YV\Sigma^{-1}w_j)$$

An alternative giving the projected dynamic modes is given by:

$$v_j = Uw_j$$

5) Compute Modes:

$$\Phi_j = YV\Sigma^{-1}v_j$$

4 Power Spectrum DMD[12]

We will call D as the data matrix.

1) Form $X = [D_o, D_1, \cdots, D_{m-1}]$ and $Y = [D_1, D_2, \cdots, D_m]$

2) Compute the singular value decomposition (SVD) of X:

$$svd(X) = U\Sigma V^*$$

3) Create $\tilde{A}$ matrix:

$$\tilde{A} = \Sigma^{-\frac{1}{2}}U^*YV\Sigma^{-\frac{1}{2}}$$

4) Get the eigenvalue/vectors from $\tilde{A}w = w\lambda$. Let (λ_j, w_j) be the eigenvalue-vector pair.

λ_js are the dynamic eigenvalues
v_js are the dynamic modes

$$v_j = \Sigma^{-\frac{1}{2}}Uw_j$$

5) Compute Modes:

$$\Phi_j = YV\Sigma^{-1}v_j$$

[12] Bingni W. Brunton et al. "Extracting spatial–temporal coherent patterns in large-scale neural recordings using dynamic mode decomposition". In: *Journal of Neuroscience Methods* 258 (2016), pp. 1 –15, p. 12.

5 Hankel DMD[13]

We will call $D_{m\times n}$ as the data matrix, being a long matrix such that $m >> n$ i.e $[6x1441]$

1) Form Hankel matrix H with a delay embedding d

$$H = \begin{bmatrix} D_o & D_1 & D_2 & \cdots & D_{(n-1)-d} \\ D_1 & D_2 & D_3 & \cdots & D_{(n-2)-d} \\ D_2 & \cdots & \cdots & \cdots & \vdots \\ \vdots & & & & \\ D_{(d+1)\times m} & \cdots & \cdots & \cdots & D_{((d+1)\times m),(n-(d+1))} \end{bmatrix}$$

2) Form data matrix X and Y:

$$X = [H]$$

$$Y = [UH]$$

H is the Hankel matrix defined above and UH is the same matrix but shifted one step forward in time. 2) Compute the singular value decomposition (SVD) of X:

$$svd(X) = U\Sigma V^*$$

3) Create $\tilde{A}$ matrix:

$$\tilde{A} = U^* Y V \Sigma^{-1}$$

4) Get the eigenvalue/vectors from $\tilde{A}w = w\lambda$. Let (λ_j, w_j) be the eigenvalue-vector pair.

λ_js are the dynamic eigenvalues
v_js are the dynamic modes

$$v_j = Uw_j$$

5) Compute Modes:

$$\Phi_j = YV\Sigma^{-1}v_j$$

[13] Arbabi and Mezić, "Study of dynamics in post-transient flows using Koopman mode decomposition", op. cit., p. 13.