

Difference Equations

in many fields such as ecology, economics, and engineering, a need arises to model mathematically a dynamic system that changes over time. Several features of the system are each measured at discrete time intervals, producing a sequence of vectors x_0, x_1, x_2, \dots . The entries in x_k provide information about the state of the system at the time of the k th measurements

if there is a matrix A such that $x_1 = Ax_0, x_2 = Ax_1$, and in general

$$x_{k+1} = Ax_k \text{ for } k = 0, 1, 2, \dots \quad (5)$$

then (5) is called a **linear difference equation** (or *recurrence relation*). Given such an equation, one can compute x_1, x_2 and so on, provided x_0 is known.

A subject of interest to demographers is the movement of populations or groups of people from one region to another. The simple model here considers the changes in the population of a certain city and its surrounding suburbs over a period of years.

Fix an initial year—say, 2000—and denote the populations of the city and suburbs that year by r_0 and s_0 , respectively. Let \mathbf{x}_0 be the population vector

$$\mathbf{x}_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} \quad \begin{array}{l} \text{City population, 2000} \\ \text{Suburban population, 2000} \end{array}$$

For 2001 and subsequent years, denote the populations of the city and suburbs by the vectors

$$\mathbf{x}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} r_3 \\ s_3 \end{bmatrix}, \dots$$

Our goal is to describe mathematically how these vectors might be related.

Suppose demographic studies show that each year about 5% of the city's population moves to the suburbs (and 95% remains in the city), while 3% of the suburban population moves to the city (and 97% remains in the suburbs). See Fig. 2.



FIGURE 2 Annual percentage migration between city and suburbs.

After 1 year, the original r_0 persons in the city are now distributed between city and suburbs as

$$\begin{bmatrix} .95r_0 \\ .05r_0 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} \begin{array}{l} \text{Remain in city} \\ \text{Move to suburbs} \end{array} \quad (6)$$

The s_0 persons in the suburbs in 2000 are distributed 1 year later as

$$s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} \begin{array}{l} \text{Move to city} \\ \text{Remain in suburbs} \end{array} \quad (7)$$

The vectors in (6) and (7) account for all of the population in 2001.³ Thus

$$\begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} + s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

That is,

$$\mathbf{x}_1 = M\mathbf{x}_0 \quad (8)$$

where M is the **migration matrix** determined by the following table:

From:		
City	Suburbs	To:
$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$		City
		Suburbs

Equation (8) describes how the population changes from 2000 to 2001. If the migration percentages remain constant, then the change from 2001 to 2002 is given by

$$\mathbf{x}_2 = M\mathbf{x}_1$$

and similarly for 2002 to 2003 and subsequent years. In general,

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \dots \quad (9)$$

The sequence of vectors $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots\}$ describes the population of the city/suburban region over a period of years.

EXAMPLE 3 Compute the population of the region just described for the years 2001 and 2002, given that the population in 2000 was 600,000 in the city and 400,000 in the suburbs.

SOLUTION The initial population in 2000 is $\mathbf{x}_0 = \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix}$. For 2001,

$$\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} = \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix}$$

For 2002,

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix} = \begin{bmatrix} 565,440 \\ 434,560 \end{bmatrix} \quad \blacksquare$$

The model for population movement in (9) is *linear* because the correspondence $\mathbf{x}_k \mapsto \mathbf{x}_{k+1}$ is a linear transformation. The linearity depends on two facts: the number of people who chose to move from one area to another is *proportional* to the number of people in that area, as shown in (6) and (7), and the cumulative effect of these choices is found by *adding* the movement of people from the different areas.
