

THEOREM 7

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then for each \mathbf{x} in V , there exists a unique set of scalars c_1, \dots, c_n such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n \quad (1)$$

A Graphical interpretation of Coordinates

EXAMPLE 3 In crystallography, the description of a crystal lattice is aided by choosing a basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for \mathbb{R}^3 that corresponds to three adjacent edges of one “unit cell” of the crystal. An entire lattice is constructed by stacking together many copies of one cell. There are fourteen basic types of unit cells; three are displayed in Fig. 3.²

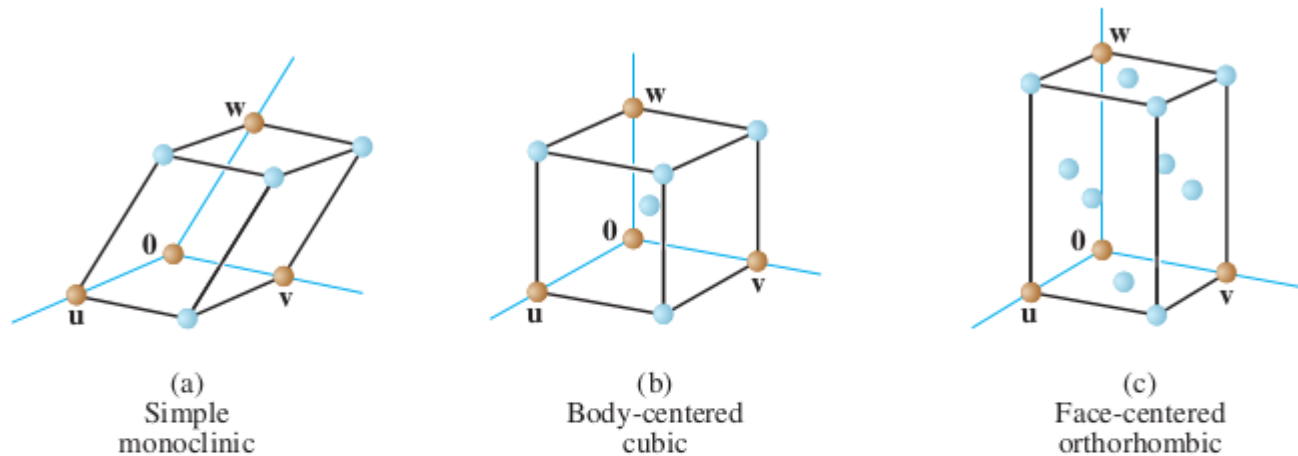


FIGURE 3 Examples of unit cells.

The coordinates of atoms within the crystal are given relative to the basis for the lattice. For instance,

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

identifies the top face-centered atom in the cell in Fig. 3(c). ■

When a basis \mathcal{B} for \mathbb{R}^n is fixed, the \mathcal{B} -coordinate vector of a specified \mathbf{x} is easily found, as in the next example.

EXAMPLE 4 Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of \mathbf{x} relative to \mathcal{B} .

SOLUTION The \mathcal{B} -coordinates c_1, c_2 of \mathbf{x} satisfy

$$\underset{\mathbf{b}_1}{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \underset{\mathbf{b}_2}{c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \underset{\mathbf{x}}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}$$

or

$$\underset{\mathbf{b}_1}{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}} \underset{\mathbf{b}_2}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\mathbf{x}}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}} \quad (3)$$

This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left. In any case, the solution is $c_1 = 3$, $c_2 = 2$. Thus $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$, and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



See Fig. 4

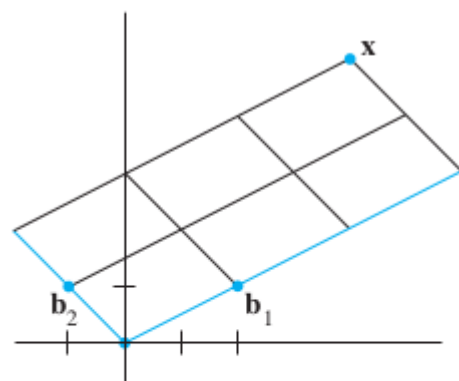


FIGURE 4

FIGURE 4

The \mathcal{B} -coordinate vector of \mathbf{x} is
 $(3, 2)$.

The matrix in (3) changes the \mathcal{B} -coordinates of a vector \mathbf{x} into the standard coordinates for \mathbf{x} . An analogous change of coordinates can be carried out in \mathbb{R}^n for a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$$

Then the vector equation

$$\mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_n \mathbf{b}_n$$

is equivalent to

$$\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}} \quad (4)$$

We call $P_{\mathcal{B}}$ the **change-of-coordinates matrix** from \mathcal{B} to the standard basis in \mathbb{R}^n . Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into \mathbf{x} . The change-of-coordinates equation (4) is important and will be needed at several points in Chapters 5 and 7.

Since the columns of $P_{\mathcal{B}}$ form a basis for \mathbb{R}^n , $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem). Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts \mathbf{x} into its \mathcal{B} -coordinate vector:

$$P_{\mathcal{B}}^{-1} \mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced here by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping mentioned earlier. Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from \mathbb{R}^n onto \mathbb{R}^n , by the Invertible Matrix Theorem. (See

The Coordinate Mapping

Choosing a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for a vector space V introduces a coordinate system in V . The coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ connects the possibly unfamiliar space V to the familiar space \mathbb{R}^n . See Fig. 5. Points in V can now be identified by their new “names.”

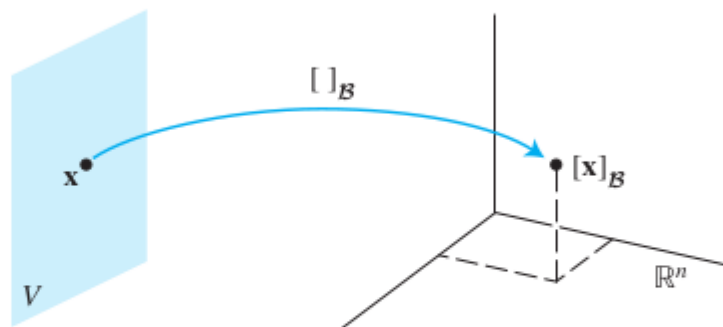


FIGURE 5 The coordinate mapping from V onto \mathbb{R}^n .

THEOREM 8

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto \mathbb{R}^n .

The coordinate mapping in Theorem 8 is an important example of an *isomorphism* from V onto \mathbb{R}^n . In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W (*iso* from the Greek for “the same,” and *morph* from the Greek for “form” or “structure”). The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces. *Every vector space calculation in V is accurately reproduced in W , and vice versa.* In particular, any real vector space with a basis of n vectors is indistinguishable from \mathbb{R}^n . See Exercises 25 and 26.

EXAMPLE 7 Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$$

and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then \mathcal{B} is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if \mathbf{x} is in H , and if it is, find the coordinate vector of \mathbf{x} relative to \mathcal{B} .

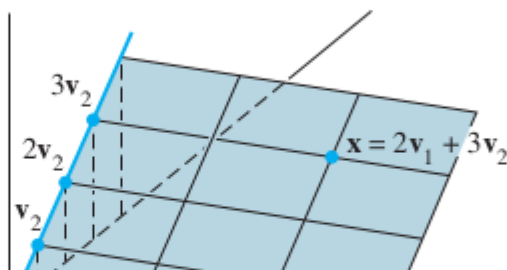
SOLUTION If \mathbf{x} is in H , then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

The scalars c_1 and c_2 , if they exist, are the \mathcal{B} -coordinates of \mathbf{x} . Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $c_1 = 2$, $c_2 = 3$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The coordinate system on H determined by \mathcal{B} is shown in Fig. 7. ■



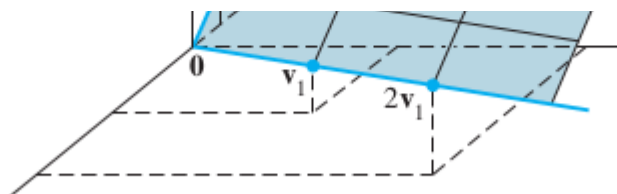


FIGURE 7 A coordinate system on a plane H in \mathbb{R}^3 .
