#linear_algebra

The Null space of a matrix

in applications of liner algebrasubspaces R^n usually arise in one of two ways:

1. as the set of all solutions to a system of homogeneous linear equations
2. as the set of all linear combinations of certain specified vectos.

- When Nul A contains nonzero vectors,the number of vectors in the spanning set for Nul A equals the number of free variables in the equation Ax = 0.

The columns space of a matrix

DEFINITION

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\,]$, then

$$\text{Col } A = \text{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

Since Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a subspace, by Theorem 1, the next theorem follows from the definition of Col A and the fact that the columns of A are in $\mathbb{R}^m$.

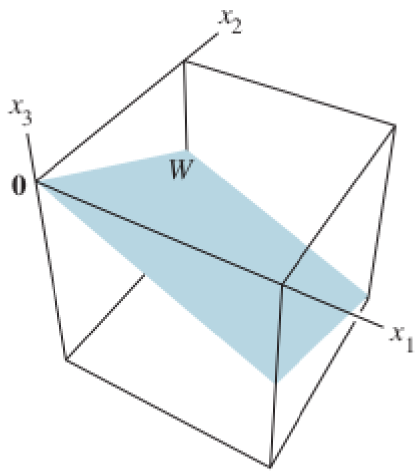
THEOREM 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Note that a typical vector in Col A can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

The notation $A\mathbf{x}$ for vectors in Col A also shows that Col A is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. We will return to this point of view at the end of the section.



EXAMPLE 4 Find a matrix A such that $W = \text{Col}\,A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

SOLUTION First, write W as a set of linear combinations.

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Second, use the vectors in the spanning set as the columns of A. Let $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Then $W = \text{Col}\,A$, as desired. ■

Recall from Theorem 4 in Section 1.4 that the columns of A span $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$. We can restate this fact as follows:

> The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

a. If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

b. If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

SOLUTION

a. The columns of A each have three entries, so Col A is a subspace of $\mathbb{R}^k$, where $k = 3$.

b. A vector $\mathbf{x}$ such that $A\mathbf{x}$ is defined must have four entries, so Nul A is a subspace of $\mathbb{R}^k$, where $k = 4$. ■

When a matrix is not square, as in Example 5, the vectors in Nul A and Col A live in entirely different "universes." For example, no linear combination of vectors in $\mathbb{R}^3$ can produce a vector in $\mathbb{R}^4$. When A is square, Nul A and Col A do have the zero vector in common, and in special cases it is possible that some nonzero vectors belong to both Nul A and Col A.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

DEFINITION

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W). The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. If T happens to arise as a matrix transformation—say, $T(\mathbf{x}) = A\mathbf{x}$

for some matrix A—then the kernel and the range of T are just the null space and the column space of A, as defined earlier.

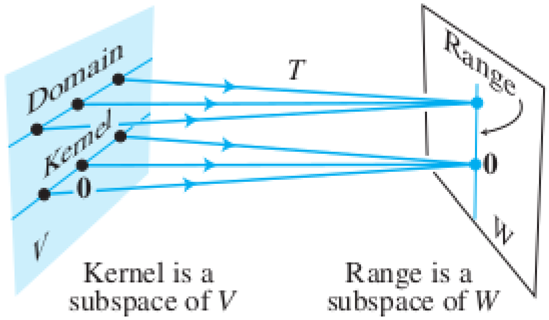


FIGURE 2 Subspaces associated with a linear transformation.

in applications, a subspace usually arises as either the kernel or the range of an appropriate linear transformation.For instance,the set of all solutions of a homogenous linear differential equation turns out to be the kernel of a linear transformation.Typically,such a linear transformation is described in terms of one or more derivatives of a fucntion.

Examples

EXAMPLE 8 (Calculus required) Let V be the vector space of all real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$. Let W be the vector space $C[a, b]$ of all continuous functions on $[a, b]$, and let $D : V \to W$ be the transformation that changes f in V into its derivative f'. In calculus, two simple differentiation rules are

$$D(f + g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f)$$

That is, D is a linear transformation. It can be shown that the kernel of D is the set of constant functions on $[a, b]$ and the range of D is the set W of all continuous functions on $[a, b]$. ■

EXAMPLE 9 (Calculus required) The differential equation

$$y'' + \omega^2 y = 0 \tag{4}$$

where ω is a constant, is used to describe a variety of physical systems, such as the vibration of a weighted spring, the movement of a pendulum, and the voltage in an inductance-capacitance electrical circuit. The set of solutions of (4) is precisely the kernel of the linear transformation that maps a function $y = f(t)$ into the function $f''(t) + \omega^2 f(t)$. Finding an explicit description of this vector space is a problem in differential equations. The solution set turns out to be the space described in Exercise 19 in Section 4.1. ■