now that powerful computers are widely available,more and more scientific and engineering problems are being treated in a way that uses discrete,or digital,data rather than continuous data.
Difference equations are often the appropriate too to analyze such data.
Even when a differential equation is used to model a continuous process,a numerical solution is often produced from a related difference equation.

Discrete-Time signals

The vector space $\mathbb{S}$ of discrete-time signals was introduced in Section 4.1. A **signal** in $\mathbb{S}$ is a function defined only on the integers and is visualized as a sequence of numbers, say, $\{y_k\}$. Figure 1 shows three typical signals whose general terms are $(.7)^k$, 1^k, and $(-1)^k$, respectively.

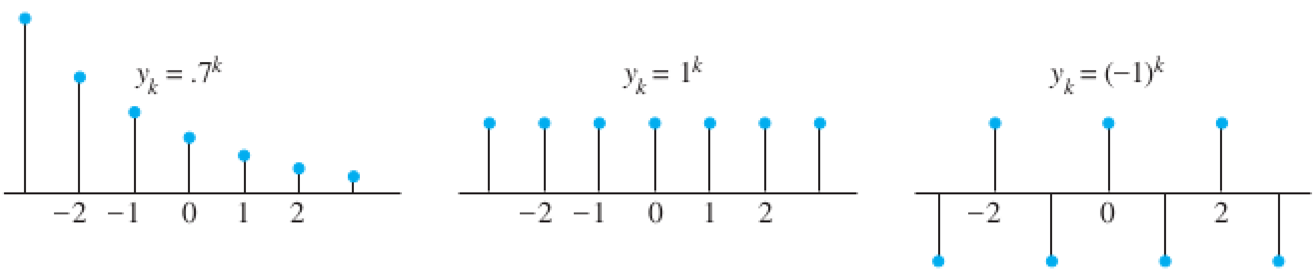


FIGURE 1 Three signals in $\mathbb{S}$.

Digital signals obviously arise in electrical and control signals engineering,but discrete-data sequences are also generated in biology,physics,economics,demography and many other areas.
wherever a process is measured,or sampled,at discrete time intervals.When a process begins at a specific time,it is sometimes convenient to write a signal as a sequence of the form{y0,y1,y2,...} The terms yk for

$k<0$ either are assumed to be zero or are simply omitted.

EXAMPLE 1 The crystal-clear sounds from a compact disc player are produced from music that has been sampled at the rate of 44,100 times per second. See Fig. 2. At each measurement, the amplitude of the music signal is recorded as a number, say, y_k. The original music is composed of many different sounds of varying frequencies, yet the sequence $\{y_k\}$ contains enough information to reproduce all the frequencies in the sound up to about 20,000 cycles per second, higher than the human ear can sense. ■

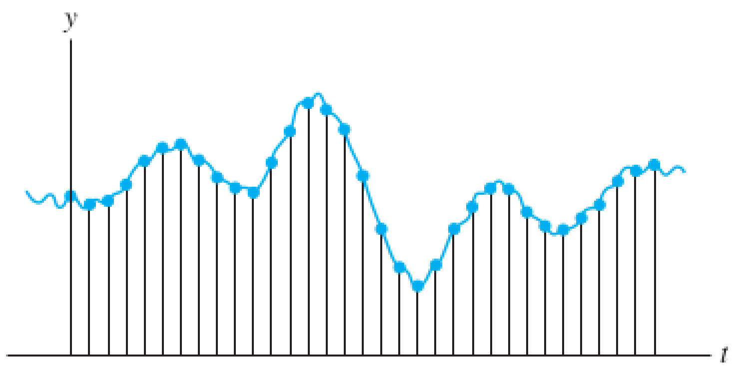


FIGURE 2 Sampled data from a music signal.

linear independence in the space S of Signals

To simplify notation, we consider a set of only three signals in $\mathbb{S}$, say, $\{u_k\}$, $\{v_k\}$, and $\{w_k\}$. They are linearly independent precisely when the equation

$$c_1 u_k + c_2 v_k + c_3 w_k = 0 \quad \text{for all } k \tag{1}$$

implies that $c_1 = c_2 = c_3 = 0$. The phrase "for all k" means for all integers—positive, negative, and zero. One could also consider signals that start with $k = 0$, for example, in which case, "for all k" would mean for all integers $k \geq 0$.

Suppose c_1, c_2, c_3 satisfy (1). Then equation (1) holds for any three consecutive values of k, say, k, $k+1$, and $k+2$. Thus (1) implies that

$$c_1 u_{k+1} + c_2 v_{k+1} + c_3 w_{k+1} = 0 \quad \text{for all } k$$

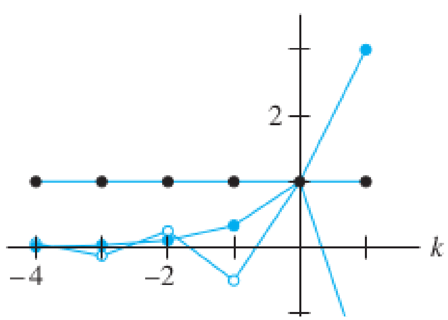
and

$$c_1 u_{k+2} + c_2 v_{k+2} + c_3 w_{k+2} = 0 \quad \text{for all } k$$

Hence c_1, c_2, c_3 satisfy

$$\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for all } k \tag{2}$$

The coefficient matrix in this system is called the **Casorati matrix** of the signals, and the determinant of the matrix is called the **Casoratian** of $\{u_k\}$, $\{v_k\}$, and $\{w_k\}$. If the Casorati matrix is invertible for at least one value of k, then (2) will imply that $c_1 = c_2 = c_3 = 0$, which will prove that the three signals are linearly independent.

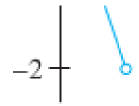


EXAMPLE 2 Verify that 1^k, $(-2)^k$, and 3^k are linearly independent signals.

SOLUTION The Casorati matrix is

$$\begin{bmatrix} 1^k & (-2)^k & 3^k \\ 1^{k+1} & (-2)^{k+1} & 3^{k+1} \\ 1^{k+2} & (-2)^{k+2} & 3^{k+2} \end{bmatrix}$$

Row operations can show fairly easily that this matrix is always invertible. However, it is faster to substitute a value for k—say, $k = 0$—and row reduce the numerical matrix:



The signals 1^k, $(-2)^k$, and 3^k.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 10 \end{bmatrix}$$

The Casorati matrix is invertible for $k = 0$. So 1^k, $(-2)^k$, and 3^k are linearly independent. ■

Linear difference equations

Given scalars $a_0, \ldots, a_n$, with a_0 and a_n nonzero, and given a signal $\{z_k\}$, the equation

$$a_0 y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k \quad \text{for all } k \qquad (3)$$

is called a linear difference equation(or linear recurrence relation) of order n.For simplicity,a0 is often taken equal to 1.If {zk} if the zero sequence,the equation is homogeneous,otherwise,teh equation is nonhomogeneous.

EXAMPLE 3 In digital signal processing, a difference equation such as (3) describes a **linear filter**, and $a_0, \ldots, a_n$ are called the **filter coefficients**. If $\{y_k\}$ is treated as the input and $\{z_k\}$ as the output, then the solutions of the associated homogeneous equation are the signals that are filtered *out* and transformed into the zero signal. Let us feed two different signals into the filter

$$.35y_{k+2} + .5y_{k+1} + .35y_k = z_k$$

Here .35 is an abbreviation for $\sqrt{2}/4$. The first signal is created by sampling the continuous signal $y = \cos(\pi t/4)$ at integer values of t, as in Fig. 3(a). The discrete signal is

$$\{y_k\} = \{\ldots, \cos(0), \cos(\pi/4), \cos(2\pi/4), \cos(3\pi/4), \ldots\}$$

For simplicity, write $\pm.7$ in place of $\pm\sqrt{2}/2$, so that

$$\{y_k\} = \{\ldots, \underset{\substack{\uparrow \\ k=0}}{1}, .7, 0, -.7, -1, -.7, 0, .7, 1, .7, 0, \ldots\}$$

Table 1 shows a calculation of the output sequence $\{z_k\}$, where .35(.7) is an abbreviation for $(\sqrt{2}/4)(\sqrt{2}/2) = .25$. The output is $\{y_k\}$, shifted by one term.

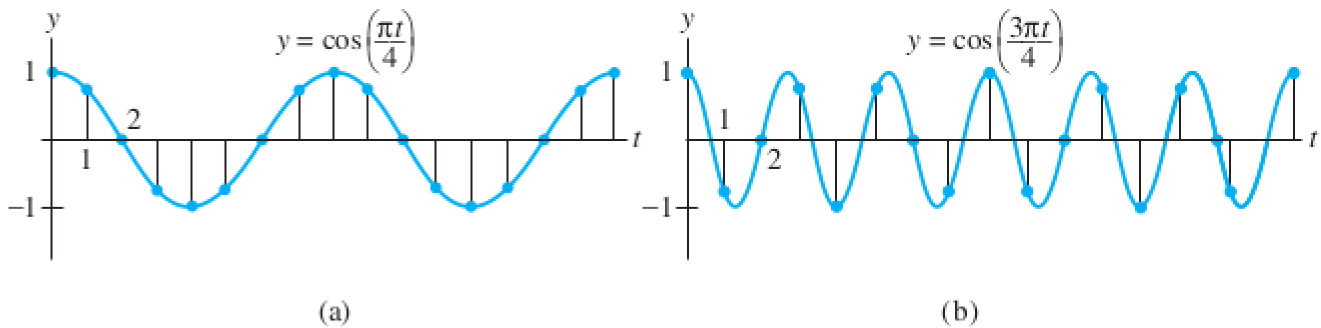


FIGURE 3 Discrete signals with different frequencies.

TABLE 1 Computing the Output of a Filter

k	y_k	y_{k+1}	y_{k+2}	$.35y_k$	$+ .5y_{k+1}$	$+ .35y_{k+2}$	$= z_k$
0	1	.7	0	.35(1)	+ .5(.7)	+ .35(0)	= .7
1	.7	0	−.7	.35(.7)	+ .5(0)	+ .35(−.7)	= 0
2	0	−.7	−1	.35(0)	+ .5(−.7)	+ .35(−1)	= −.7
3	−.7	−1	−.7	.35(−.7)	+ .5(−1)	+ .35(−.7)	= −1
4	−1	−.7	0	.35(−1)	+ .5(−.7)	+ .35(0)	= −.7
5	−.7	0	.7	.35(−.7)	+ .5(0)	+ .35(.7)	= 0
⋮	⋮						⋮

A different input signal is produced from the higher frequency signal $y = \cos(3\pi t/4)$, shown in Fig. 3(b). Sampling at the same rate as before produces a new input sequence:

$$\{w_k\} = \{\ldots, \underset{\substack{\uparrow \\ k=0}}{1}, -.7, 0, .7, -1, .7, 0, -.7, 1, -.7, 0, \ldots\}$$

When $\{w_k\}$ is fed into the filter, the output is the zero sequence. The filter, called a *low-pass filter*, lets $\{y_k\}$ pass through, but stops the higher frequency $\{w_k\}$. ■

In many applications, a sequence $\{z_k\}$ is specified for the right side of a difference equation (3), and a $\{y_k\}$ that satisfies (3) is called a **solution** of the equation. The next example shows how to find solutions for a homogeneous equation.

examples

EXAMPLE 4 Solutions of a homogeneous difference equation often have the form $y_k = r^k$ for some r. Find some solutions of the equation

$$y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0 \quad \text{for all } k \tag{4}$$

SOLUTION Substitute r^k for y_k in the equation and factor the left side:

$$r^{k+3} - 2r^{k+2} - 5r^{k+1} + 6r^k = 0 \tag{5}$$

$$r^k(r^3 - 2r^2 - 5r + 6) = 0$$

$$r^k(r-1)(r+2)(r-3) = 0 \tag{6}$$

Since (5) is equivalent to (6), r^k satisfies the difference equation (4) if and only if r satisfies (6). Thus 1^k, $(-2)^k$, and 3^k are all solutions of (4). For instance, to verify that 3^k is a solution of (4), compute

$$\begin{aligned} &3^{k+3} - 2 \cdot 3^{k+2} - 5 \cdot 3^{k+1} + 6 \cdot 3^k \\ &= 3^k(27 - 18 - 15 + 6) = 0 \quad \text{for all } k \end{aligned}$$

■

In general, a nonzero signal r^k satisfies the homogeneous difference equation

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = 0 \quad \text{for all } k$$

if and only if r is a root of the **auxiliary equation**

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

We will not consider the case in which r is a repeated root of the auxiliary equation. When the auxiliary equation has a *complex root*, the difference equation has solutions of the form $s^k \cos k\omega$ and $s^k \sin k\omega$, for constants s and ω. This happened in Example 3.

Solution sets of linear difference equations

Given $a_1, \dots, a_n$, consider the mapping $T : \mathbb{S} \to \mathbb{S}$ that transforms a signal $\{y_k\}$ into a signal $\{w_k\}$ given by

$$w_k = y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k$$

It is readily checked that T is a *linear* transformation. This implies that the solution set of the homogeneous equation

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = 0 \quad \text{for all } k$$

is the kernel of T (the set of signals that T maps into the zero signal), and hence the solution set is a *subspace* of $\mathbb{S}$. Any linear combination of solutions is again a solution.

The next theorem, a simple but basic result, will lead to more information about the solution sets of difference equations.

THEOREM 16

If $a_n \neq 0$ and if $\{z_k\}$ is given, the equation

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k \quad \text{for all } k \tag{7}$$

has a unique solution whenever $y_0, \dots, y_{n-1}$ are specified.

THEOREM 17

The set H of all solutions of the nth-order homogeneous linear difference equation

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = 0 \quad \text{for all } k \tag{10}$$

is an n-dimensional vector space.

EXAMPLE 5 Find a basis for the set of all solutions to the difference equation

$$y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0 \quad \text{for all } k$$

SOLUTION Our work in linear algebra really pays off now! We know from Examples 2 and 4 that 1^k, $(-2)^k$, and 3^k are linearly independent solutions. In general, it can be difficult to verify directly that a set of signals *spans* the solution space. But that is no problem here because of two key theorems—Theorem 17, which shows that the solution space is exactly three-dimensional, and the Basis Theorem in Section 4.5, which says that a linearly independent set of n vectors in an n-dimensional space is automatically a basis. So 1^k, $(-2)^k$, and 3^k form a basis for the solution space. ■

The standard way to describe the "general solution" of the difference equation (10) is to exhibit a basis for the subspace of all solutions. Such a basis is usually called a **fundamental set of solutions** of (10). In practice, if you can find n linearly independent signals that satisfy (10), they will automatically span the n-dimensional solution space, as explained in Example 5.

Nonhomogeneous equations

The general solution of the nonhomogeneous difference equation

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k \quad \text{for all } k \tag{11}$$

can be written as one particular solution of (11) plus an arbitrary linear combination of a fundamental set of solutions of the corresponding homogeneous equation (10). This fact is analogous to the result in Section 1.5 showing that the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ are parallel. Both results have the same explanation: The mapping $\mathbf{x} \mapsto A\mathbf{x}$ is linear, and the mapping that transforms the signal $\{y_k\}$ into the signal $\{z_k\}$ in (11) is linear. See Exercise 35.

EXAMPLE 6 Verify that the signal $y_k = k^2$ satisfies the difference equation

$$y_{k+2} - 4y_{k+1} + 3y_k = -4k \quad \text{for all } k \tag{12}$$

Then find a description of all solutions of this equation.

SOLUTION Substitute k^2 for y_k on the left side of (12):

$$\begin{aligned}(k+2)^2 - 4(k+1)^2 + 3k^2 \\ &= (k^2 + 4k + 4) - 4(k^2 + 2k + 1) + 3k^2 \\ &= -4k\end{aligned}$$

So k^2 is indeed a solution of (12). The next step is to solve the homogeneous equation

$$y_{k+2} - 4y_{k+1} + 3y_k = 0 \tag{13}$$

The auxiliary equation is

$$r^2 - 4r + 3 = (r-1)(r-3) = 0$$

The roots are $r = 1, 3$. So two solutions of the homogeneous difference equation are 1^k and 3^k. They are obviously not multiples of each other, so they are linearly independent signals. By Theorem 17, the solution space is two-dimensional, so 3^k and 1^k form a basis for the set of solutions of equation (13). Translating that set by a particular solution of the nonhomogeneous equation (12), we obtain the general solution of (12):

$$k^2 + c_1 1^k + c_2 3^k, \quad \text{or} \quad k^2 + c_1 + c_2 3^k$$

Figure 4 gives a geometric visualization of the two solution sets. Each point in the figure corresponds to one signal in $\mathbb{S}$. ■

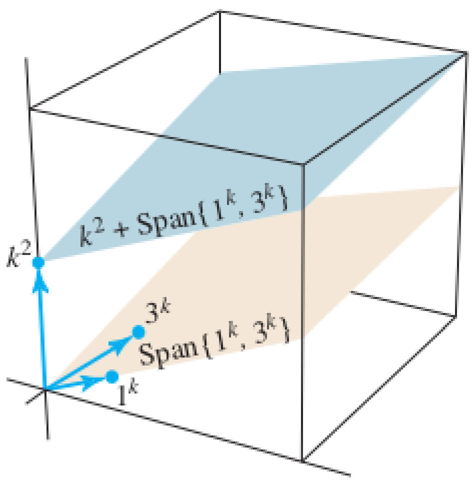


FIGURE 4
Solution sets of difference equations (12) and (13).

Reduction to systems of first-order equations

A modern way to study a homogeneous nth-order linear difference equation is to replace it by an equivalent system of first-order difference equations, written in the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for all } k$$

where the vectors $\mathbf{x}_k$ are in $\mathbb{R}^n$ and A is an $n \times n$ matrix.

it's a vector-valued difference equation.

EXAMPLE 7 Write the following difference equation as a first-order system:

$$y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0 \quad \text{for all } k$$

SOLUTION For each k, set

$$\mathbf{x}_k = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$$

The difference equation says that $y_{k+3} = -6y_k + 5y_{k+1} + 2y_{k+2}$, so

$$\mathbf{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix} = \begin{bmatrix} 0 + y_{k+1} + 0 \\ 0 + 0 + y_{k+2} \\ -6y_k + 5y_{k+1} + 2y_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$$

That is,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for all } k, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \qquad \blacksquare$$