#linear_algebra

THEOREM 15

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\,\mathbf{x}\,]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\,\mathbf{x}\,]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.[2] Figure 2 illustrates the change-of-coordinates equation (4).

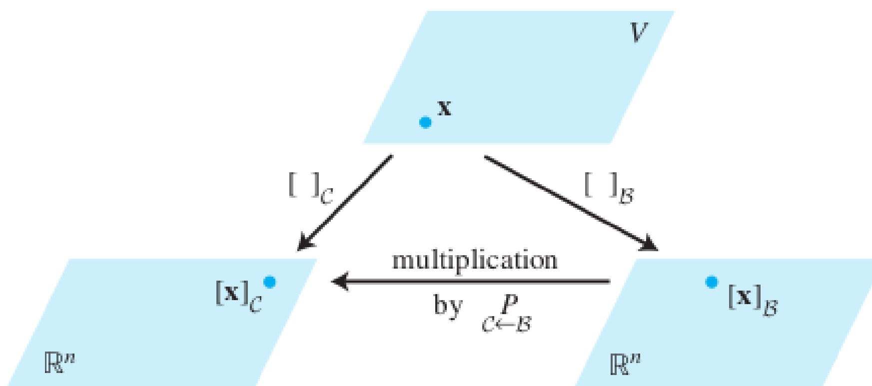

FIGURE 2 Two coordinate systems for V.

$$(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

EXAMPLE 2 Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

SOLUTION The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\,\mathbf{b}_1\,]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\,\mathbf{b}_2\,]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

To solve both systems simultaneously, augment the coefficient matrix with $\mathbf{b}_1$ *and* $\mathbf{b}_2$, and row reduce:

$$\left[\begin{array}{cc:cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right] \tag{7}$$

Thus

$$[\,\mathbf{b}_1\,]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \text{and} \quad [\,\mathbf{b}_2\,]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

The desired change-of-coordinates matrix is therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & [\,\mathbf{b}_2\,]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

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Observe that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Example 2 already appeared in (7). This is not surprising because the first column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ results from row reducing $[\,\mathbf{c}_1 \quad \mathbf{c}_2 \;\vdots\; \mathbf{b}_1\,]$ to $[\,I \;\vdots\; [\,\mathbf{b}_1\,]_{\mathcal{C}}\,]$, and similarly for the second column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Thus

$$\boxed{[\,\mathbf{c}_1 \quad \mathbf{c}_2 \;\vdots\; \mathbf{b}_1 \quad \mathbf{b}_2\,] \sim [\,I \;\vdots\; \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,]}$$

An analogous procedure works for finding the change-of-coordinates matrix between any two bases in $\mathbb{R}^n$.

EXAMPLE 3 Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

SOLUTION

a. Notice that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is needed rather than $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, and compute

$$[\,\mathbf{b}_1 \quad \mathbf{b}_2 \;\vdots\; \mathbf{c}_1 \quad \mathbf{c}_2\,] = \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

So

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

b. By part (a) and property (6) above (with $\mathcal{B}$ and $\mathcal{C}$ interchanged),

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = (\underset{\mathcal{B}\leftarrow\mathcal{C}}{P})^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix} \quad \blacksquare$$