#linear_algebra

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

In words,a subspace is closed under addition and scalar multiplication

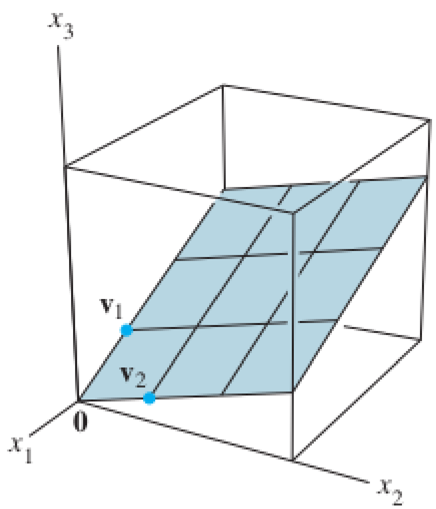
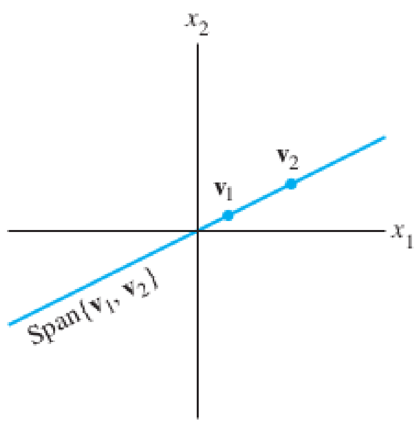


FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.



$\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2 = k\mathbf{v}_1$.

EXAMPLE 2 A line L *not* through the origin is *not* a subspace, because it does not contain the origin, as required. Also, Fig. 2 shows that L is not closed under addition or scalar multiplication. ■

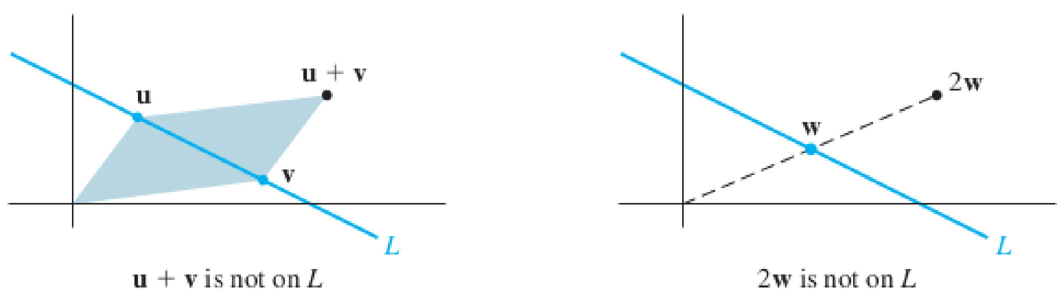


FIGURE 2

EXAMPLE 3 For $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a subspace of $\mathbb{R}^n$. The verification of this statement is similar to the argument given in Example 1. We shall now refer to Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ as **the subspace spanned** (or **generated**) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$. ■

Note that $\mathbb{R}^n$ is a subspace of itself because it has the three properties required for a subspace. Another special subspace is the set consisting of only the zero vector in $\mathbb{R}^n$. This set, called the **zero subspace**, also satisfies the conditions for a subspace.

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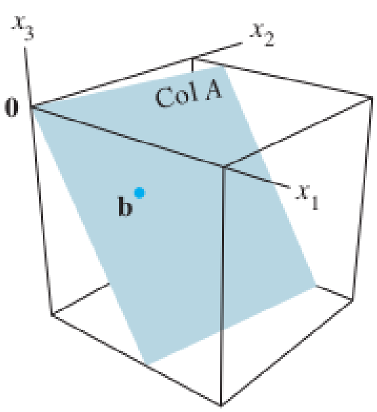
column space and Null space of a matrix

Subspaces of $\mathbb{R}^n$ usually occur in applications and theory in one of two ways. In both cases, the subspace can be related to a matrix.

DEFINITION

> The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

If $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$, with the columns in $\mathbb{R}^m$, then Col A is the same as Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Example 4 shows that the **column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$**. Note that Col A equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, Col A is only part of $\mathbb{R}^m$.



EXAMPLE 4 Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether $\mathbf{b}$ is in the column space of A.

SOLUTION The vector $\mathbf{b}$ is a linear combination of the columns of A if and only if $\mathbf{b}$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$, that is, if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Row reducing the augmented matrix $[\,A \ \ \mathbf{b}\,]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we conclude that $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{b}$ is in Col A. ■

THEOREM 12

> The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Basis for Subspace

Becasue a subspace typically contains an infinite number of vectors,some problems involving a subspace are handled best be working with a small finite set of vectors that span the subspace..The smaller the set ,the better.It can be shown that the smallest possible spanning set must be linearly independent.

A *basis* for a subspace H of R^n is a linearly independent set in H that spans R^n

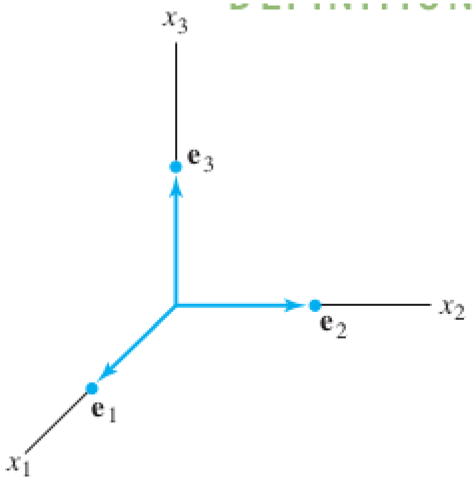


FIGURE 3

The standard basis for $\mathbb{R}^3$.

EXAMPLE 6 Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

SOLUTION First, write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \tag{1}$$

Equation (1) shows that Nul A coincides with the set of all linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. That is, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ generates Nul A. In fact, this construction of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ automatically makes them linearly independent, because equation (1) shows that $\mathbf{0} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$ only if the weights x_2, x_4, and x_5 are all zero. (Examine entries 2, 4, and 5 in the vector $x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$.) So $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a *basis* for Nul A. ■

Theorem 13:

The pivot columns of a matrix A form a basis for the column space of A

Warning: Be careful to use *pivot columns of A itself* for the basis of Col A. The columns of an echelon form B are often not in the column space of A. (For instance, in Examples 7 and 8, the columns of B all have zeros in their last entries and cannot generate the columns of A.)