fact

the homogeneous equation Ax=0 has a nontrivial solution if and only if the equation has at least one free variable.

EXAMPLE 2 A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0 \tag{1}$$

SOLUTION There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = .3x_2 + .2x_3$, with x_2 and x_3 free. As a vector, the general solution is
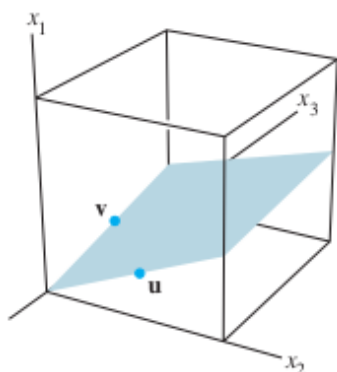
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} \quad \text{(with } x_2, x_3 \text{ free)} \tag{2}$$

FIGURE 2

This calculation shows that every solution of (1) is a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, shown in (2). That is, the solution set is Span $\{\mathbf{u}, \mathbf{v}\}$. Since neither $\mathbf{u}$ nor $\mathbf{v}$ is a scalar multiple of the other, the solution set is a plane through the origin. See Fig. 2. ■

Examples 1 and 2, along with the exercises, illustrate the fact that the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can always be expressed explicitly as Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for suitable vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$. If the only solution is the zero vector, then the solution set is Span $\{\mathbf{0}\}$. If the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable, the solution set is a line through the origin, as in Fig. 1. A plane through the origin, as in Fig. 2, provides a good mental image for the solution set of $A\mathbf{x} = \mathbf{0}$ when there are two or more free variables. Note, however, that a similar figure can be used to visualize Span $\{\mathbf{u}, \mathbf{v}\}$ even when $\mathbf{u}$ and $\mathbf{v}$ do not arise as solutions of $A\mathbf{x} = \mathbf{0}$. See Fig. 11 in Section 1.3.

Parametric Vector Form

The original equation (1) for the plane in Example 2 is an *implicit* description of the plane. Solving this equation amounts to finding an *explicit* description of the plane as the set spanned by $\mathbf{u}$ and $\mathbf{v}$. Equation (2) is called a **parametric vector equation** of the plane. Sometimes such an equation is written as

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})$$

to emphasize that the parameters vary over all real numbers. In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line. Whenever a solution set is described explicitly with vectors as in Examples 1 and 2, we say that the solution is in **parametric vector form**.

Solutions of Nonhomogeneous systems

when a nonhomongeneous linear has many solutions ,the general solution can be written in parametric vector form as one vector plus an arbitray linear combination of vectors that satisfy the corresponding homogeneous system.

Theorem 6:
Suppose the equation Ax=b is consistent for some given b,and let P be a solution.Then the solution set of Ax

$= b$ is the set of all vectors of the form $W = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$

Theorem 6 says that if $Ax = b$ has a solution, then the solution set is obtained by translating the solution set of $Ax = 0$, using any particular solution p of $Ax = b$ for the translation. Figure 6 illustrates the case in which there are two free variables. Even when $n > 3$, our mental image of the solution set of a consistent system $Ax = b$ (with $b \neq 0$) is either a single nonzero point or a line or plane not passing through the origin.
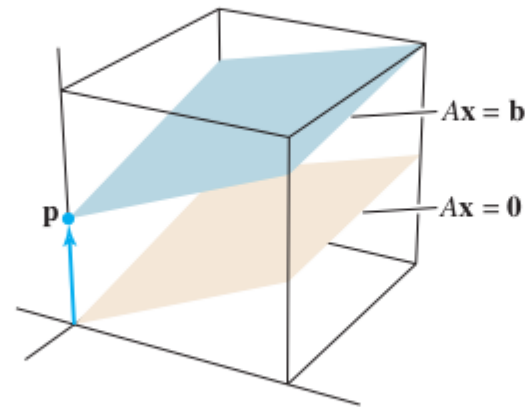

FIGURE 6 Parallel solution sets of $Ax = b$ and $Ax = 0$.

Warning: Theorem 6 and Fig. 6 apply only to an equation $Ax = b$ that has at least one nonzero solution p. When $Ax = b$ has no solution, the solution set is empty.

•