

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

- a. A is an invertible matrix.
- b. A is row equivalent to the $n \times n$ identity matrix.
- c. A has n pivot positions.
- d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- e. The columns of A form a linearly independent set.
- f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
- h. The columns of A span \mathbb{R}^n .
- i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- j. There is an $n \times n$ matrix C such that $CA = I$.
- k. There is an $n \times n$ matrix D such that $AD = I$.
- l. A^T is an invertible matrix.

Note:

the invertible theorem applies only to square matrices.

Invertible Linear Transformations

Recall from Section 2.1 that matrix multiplication corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}Ax = x$ can be viewed as a statement about linear transformations. See Fig. 2.

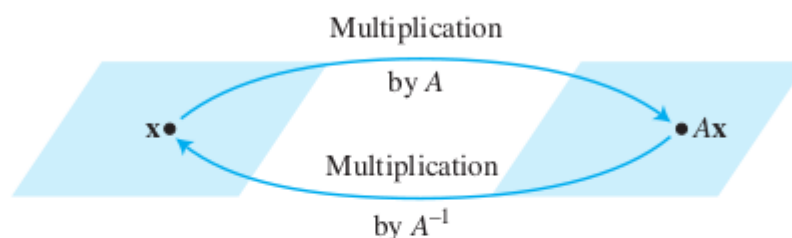


FIGURE 2 A^{-1} transforms Ax back to x .

A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (1)$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (2)$$

The next theorem shows that if such an S exists, it is unique and must be a linear transformation. We call S the **inverse** of T and write it as T^{-1} .

THEOREM 9

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T . Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

PROOF Suppose that T is invertible. Then (2) shows that T is onto \mathbb{R}^n , for if \mathbf{b} is in \mathbb{R}^n and $\mathbf{x} = S(\mathbf{b})$, then $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$, so each \mathbf{b} is in the range of T . Thus A is

Let \mathbf{x} and $\mathbf{y} = S(\mathbf{x})$, then $T(\mathbf{x}) = T(S(\mathbf{x})) = \mathbf{x}$, so every \mathbf{x} is in the range of T . Thus T is invertible, by the Invertible Matrix Theorem, statement (i).

Conversely, suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation, and S obviously satisfies (1) and (2). For instance,

$$S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$$

Thus T is invertible. The proof that S is unique is outlined in Exercise 38. ■

EXAMPLE 2 What can you say about a one-to-one linear transformation T from \mathbb{R}^n into \mathbb{R}^n ?

SOLUTION The columns of the standard matrix A of T are linearly independent (by Theorem 12 in Section 1.9). So A is invertible, by the Invertible Matrix Theorem, and T maps \mathbb{R}^n onto \mathbb{R}^n . Also, T is invertible, by Theorem 9. ■

in practical work, you might occasionally encounter a "nearly singular" or **ill-conditioned** matrix--an invertible matrix that can become singular if some of its entries are changed ever so slightly. In this case, row reduction may produce fewer than n pivot positions, as a result of roundoff error. Also, roundoff error can sometimes make a singular matrix appear to be invertible.