#linear_algebra

linear algebra played an essential role in Nobel prize-winning work of wassily leontief,as mentioned at the beginning.

The economic model described is the basis for more elaborate models used in many parts of the world.

Suppose a nation's economy is divided into n sectors that produce goods or services, and let $\mathbf{x}$ be a **production vector** in $\mathbb{R}^n$ that lists the output of each sector for one year. Also, suppose another part of the economy (called the *open sector*) does not produce goods or services but only consumes them, and let $\mathbf{d}$ be a **final demand vector** (or **bill of final demands)** that lists the values of the goods and services demanded from the various sectors by the nonproductive part of the economy. The vector $\mathbf{d}$ can represent consumer demand, government consumption, surplus production, exports, or other external demands.

As the various sectors produce goods to meet consumer demand, the producers themselves create additional **intermediate demand** for goods they need as inputs for their own production. The interrelations between the sectors are very complex, and the connection between the final demand and the production is unclear. Leontief asked if there is a production level $\mathbf{x}$ such that the amounts produced (or "supplied") will exactly balance the total demand for that production, so that

$$\left\{\begin{matrix}\text{amount}\\ \text{produced}\\ \mathbf{x}\end{matrix}\right\} = \left\{\begin{matrix}\text{intermediate}\\ \text{demand}\end{matrix}\right\} + \left\{\begin{matrix}\text{final}\\ \text{demand}\\ \mathbf{d}\end{matrix}\right\} \tag{1}$$

The basic assumption of Leontief's input–output model is that for each sector, there is a **unit consumption vector** in $\mathbb{R}^n$ that lists the inputs needed *per unit of output* of the sector. All input and output units are measured in millions of dollars, rather than in quantities such as tons or bushels. (Prices of goods and services are held constant.)

As a simple example, suppose the economy consists of three sectors—manufacturing, agriculture, and services—with unit consumption vectors $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$, as shown in the table that follows.

	Inputs Consumed per Unit of Output		
Purchased from:	**Manufacturing**	**Agriculture**	**Services**

Manufacturing	.50	.40	.20
Agriculture	.20	.30	.10
Services	.10	.10	.30
	$\uparrow$ $\mathbf{c}_1$	$\uparrow$ $\mathbf{c}_2$	$\uparrow$ $\mathbf{c}_3$

EXAMPLE 1 What amounts will be consumed by the manufacturing sector if it decides to produce 100 units?

SOLUTION Compute

$$100\mathbf{c}_1 = 100\begin{bmatrix} .50 \\ .20 \\ .10 \end{bmatrix} = \begin{bmatrix} 50 \\ 20 \\ 10 \end{bmatrix}$$

To produce 100 units, manufacturing will order (i.e., "demand") and consume 50 units from other parts of the manufacturing sector, 20 units from agriculture, and 10 units from services. ■

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If manufacturing decides to produce x_1 units of output, then $x_1\mathbf{c}_1$ represents the *intermediate demands* of manufacturing, because the amounts in $x_1\mathbf{c}_1$ will be consumed in the process of creating the x_1 units of output. Likewise, if x_2 and x_3 denote the planned outputs of the agriculture and services sectors, $x_2\mathbf{c}_2$ and $x_3\mathbf{c}_3$ list their corresponding intermediate demands. The total intermediate demand from all three sectors is given by

$$\begin{aligned}\{\text{intermediate demand}\} &= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 \\ &= C\mathbf{x} \qquad (2)\end{aligned}$$

where C is the **consumption matrix** $[\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3\,]$, namely,

$$C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix} \qquad (3)$$

Equations (1) and (2) yield Leontief's model.

THE LEONTIEF INPUT–OUTPUT MODEL, OR PRODUCTION EQUATION

$$\underset{\text{Amount produced}}{\mathbf{x}} = \underset{\text{Intermediate demand}}{C\mathbf{x}} + \underset{\text{Final demand}}{\mathbf{d}} \qquad (4)$$

Equation (4) may also be written as $I\mathbf{x} - C\mathbf{x} = \mathbf{d}$, or

$$(I - C)\mathbf{x} = \mathbf{d} \qquad (5)$$

If the matrix $I - C$ is invertible, then we can apply Theorem 5 in Section 2.2, with A replaced by $(I - C)$, and from the equation $(I - C)\mathbf{x} = \mathbf{d}$ obtain $\mathbf{x} = (I - C)^{-1}\mathbf{d}$. The theorem below shows that in most practical cases, $I - C$ *is* invertible and the production vector $\mathbf{x}$ is economically feasible, in the sense that the entries in $\mathbf{x}$ are nonnegative.

In the theorem, the term **column sum** denotes the sum of the entries in a column of a matrix. Under ordinary circumstances, the column sums of a consumption matrix are less than 1 because a sector should require less than one unit's worth of inputs to produce one unit of output.

THEOREM 11

Let C be the consumption matrix for an economy, and let $\mathbf{d}$ be the final demand. If C and $\mathbf{d}$ have nonnegative entries and if each column sum of C is less than 1, then $(I - C)^{-1}$ exists and the production vector

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

has nonnegative entries and is the unique solution of

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

A Formula for $(I - C)^{-1}$

Imagine that the demand represented by $\mathbf{d}$ is presented to the various industries at the beginning of the year, and the industries respond by setting their production levels at $\mathbf{x} = \mathbf{d}$, which will exactly meet the final demand. As the industries prepare to produce $\mathbf{d}$, they send out orders for their raw materials and other inputs. This creates an intermediate demand of $C\mathbf{d}$ for inputs.

To meet the additional demand of $C\mathbf{d}$, the industries will need as additional inputs the amounts in $C(C\mathbf{d}) = C^2\mathbf{d}$. Of course, this creates a second round of intermediate demand, and when the industries decide to produce even more to meet this new demand, they create a third round of demand, namely, $C(C^2\mathbf{d}) = C^3\mathbf{d}$. And so it goes.

Theoretically, this process could continue indefinitely, although in real life it would not take place in such a rigid sequence of events. We can diagram this hypothetical situation as follows:

	Demand That Must Be Met	**Inputs Needed to Meet This Demand**
Final demand	$\mathbf{d}$	$C\mathbf{d}$
Intermediate demand		
1st round	$C\mathbf{d}$	$C(C\mathbf{d}) = C^2\mathbf{d}$
2nd round	$C^2\mathbf{d}$	$C(C^2\mathbf{d}) = C^3\mathbf{d}$
3rd round	$C^3\mathbf{d}$	$C(C^3\mathbf{d}) = C^4\mathbf{d}$
	$\vdots$	$\vdots$

The production level $\mathbf{x}$ that will meet all of this demand is

$$\mathbf{x} = \mathbf{d} + C\mathbf{d} + C^2\mathbf{d} + C^3\mathbf{d} + \cdots$$

$$= (I + C + C^2 + C^3 + \cdots)\mathbf{d} \qquad (6)$$

$$= (I + C + C^2 + C^3 + \cdots)\mathbf{d} \tag{6}$$

To make sense of equation (6), consider the following algebraic identity:

$$(I - C)(I + C + C^2 + \cdots + C^m) = I - C^{m+1} \tag{7}$$

It can be shown that if the column sums in C are all strictly less than 1, then $I - C$ is invertible, C^m approaches the zero matrix as m gets arbitrarily large, and $I - C^{m+1} \to I$. (This fact is analogous to the fact that if a positive number t is less than 1, then $t^m \to 0$ as m increases.) Using equation (7), write

$$(I - C)^{-1} \approx I + C + C^2 + C^3 + \cdots + C^m \quad \text{when the column sums of } C \text{ are less than 1.} \tag{8}$$

The approximation in (8) means that the right side can be made as close to $(I - C)^{-1}$ as desired by taking m sufficiently large.

In actual input–output models, powers of the consumption matrix approach the zero matrix rather quickly. So (8) really provides a practical way to compute $(I - C)^{-1}$. Likewise, for any $\mathbf{d}$, the vectors $C^m\mathbf{d}$ approach the zero vector quickly, and (6) is a practical way to solve $(I - C)\mathbf{x} = \mathbf{d}$. If the entries in C and $\mathbf{d}$ are nonnegative, then (6) shows that the entries in $\mathbf{x}$ are nonnegative, too.

The Economic Importance of Entries in $(I - C)^{-1}$

The entries in $(I - C)^{-1}$ are significant because they can be used to predict how the production $\mathbf{x}$ will have to change when the final demand $\mathbf{d}$ changes. In fact, the entries in column j of $(I - C)^{-1}$ are the *increased* amounts the various sectors will have to produce in order to satisfy *an increase of 1 unit* in the final demand for output from sector j. See Exercise 8.

NUMERICAL NOTE

In any applied problem (not just in economics), an equation $A\mathbf{x} = \mathbf{b}$ can always be written as $(I - C)\mathbf{x} = \mathbf{b}$, with $C = I - A$. If the system is large and *sparse* (with mostly zero entries), it can happen that the column sums of the absolute values in C are less than 1. In this case, $C^m \to 0$. If C^m approaches zero quickly enough, (6) and (8) will provide practical formulas for solving $A\mathbf{x} = \mathbf{b}$ and finding A^{-1}.