#linear_algebra

The secret of determinants lies in how they change when row operations are performed.

Theorem 3

Row operations.

Let A be a square matrix.

1. if a multiple of one row of A is added to another row to product a matrix B,then det B = det A.
2. if two rows of A are interchanged to product B,then det B = -det A

3. if one row of A is multiplied by k to produce B,then det B =k.detA.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U = 0$

FIGURE 1

Typical echelon forms of square matrices.

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. (This is always possible. See the row reduction algorithm in Section 1.2.) If there are r interchanges, then Theorem 3 shows that

$$\det A = (-1)^r \det U$$

Since U is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries $u_{11}, \dots, u_{nn}$. If A is invertible, the entries u_{ii} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's). Otherwise, at least u_{nn} is zero, and the product $u_{11} \cdots u_{nn}$ is zero. See Fig. 1. Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases} \tag{1}$$

It is interesting to note that although the echelon form U described above is not unique (because it is not completely row reduced), and the pivots are not unique, the *product* of the pivots *is* unique, except for a possible minus sign.

Formula (1) not only gives a concrete interpretation of $\det A$ but also proves the main theorem of this section:

THEOREM 4

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4 adds the statement "$\det A \neq 0$" to the Invertible Matrix Theorem. A useful corollary is that $\det A = 0$ when the columns of A are linearly dependent. Also, $\det A = 0$ when the *rows* of A are linearly dependent. (Rows of A are columns of A^T, and linearly dependent columns of A^T make A^T singular. When A^T is singular, so is A, by the Invertible Matrix Theorem.) In practice, linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

Columns Operations

Theorem5

If A is an n *n matrix,then det* A^T *= det A .*

Theorem6

If A and B ar n nmatrices,then detAB = (detA)(detB).

however det(A+B) is not equal to detA + detB.

A linearity property of the determinant function.

for an n * n matrix A,we can consider det A as a function of the n column vectors in A.