

## Exercise 1 - Univariate Gaussians (Hellinger distance).

Solution: by definition, the distance is given by:

$$\begin{aligned}
 d(\pi_1, \pi_2)^2 &= \frac{1}{2} \int (\sqrt{\pi_1(x)} - \sqrt{\pi_2(x)})^2 dx \\
 &= \frac{1}{2} \int (\pi_1(x) - 2\sqrt{\pi_1(x)\pi_2(x)} + \pi_2(x)) dx \\
 &= \frac{1}{2} \left( \int \pi_1(x) dx - 2 \int \sqrt{\pi_1(x)\pi_2(x)} dx + \int \pi_2(x) dx \right) \\
 &= \frac{1}{2} (2 - 2 \int \sqrt{\pi_1(x)\pi_2(x)} dx) = 1 - \int \sqrt{\pi_1(x)\pi_2(x)} dx = 1 - Z_{(\pi_1, \pi_2)} \quad ①
 \end{aligned}$$

let's calculate the integral:

$$\begin{aligned}
 Z_{(\pi_1, \pi_2)} &= \int \sqrt{\pi_1(x)\pi_2(x)} dx = \int \sqrt{\frac{1}{\delta_1 \sqrt{2\pi}} \cdot e^{-\frac{x^2 + \bar{x}_1^2 - 2x\bar{x}_1}{2\delta_1^2}} \cdot \frac{1}{\delta_2 \sqrt{2\pi}} \cdot e^{-\frac{x^2 + \bar{x}_2^2 - 2x\bar{x}_2}{2\delta_2^2}}} dx \\
 &= \int \sqrt{\frac{1}{\delta_1 \sqrt{2\pi}} \cdot \frac{1}{\delta_2 \sqrt{2\pi}} \cdot e^{-\frac{x^2 \delta_1^2 + \delta_2^2 \bar{x}_1^2 - 2x\bar{x}_1 \delta_1^2 + \delta_2^2 + x^2 \delta_2^2 + \bar{x}_2^2 \delta_1^2 - 2x\bar{x}_2 \delta_1^2}{2\delta_1^2 \delta_2^2}}} dx \\
 &= \int \sqrt{\frac{1}{\sqrt{\delta_1^2 2\pi}} \cdot \frac{1}{\sqrt{\delta_2^2 2\pi}} \cdot \exp\left(-\frac{x^2 (\delta_1^2 + \delta_2^2) - 2x(\bar{x}_1 \delta_1^2 + \bar{x}_2 \delta_1^2) + (\delta_1^2 \bar{x}_1^2 + \delta_2^2 \bar{x}_2^2)}{2\delta_1^2 \delta_2^2}\right)} dx \\
 &= \int \sqrt{\frac{1}{\sqrt{\delta_1^2 2\pi}} \cdot \frac{1}{\sqrt{\delta_2^2 2\pi}} \cdot \exp\left(-\frac{\delta_1^2 + \delta_2^2}{2\delta_1^2 \delta_2^2} \cdot [x^2 - 2x \frac{\bar{x}_1 \delta_1^2 + \bar{x}_2 \delta_1^2}{\delta_1^2 + \delta_2^2} + \frac{\bar{x}_1^2 \delta_2^2 + \bar{x}_2^2 \delta_1^2}{\delta_1^2 + \delta_2^2}]\right)} dx \\
 &= \frac{1}{\sqrt{\delta_1 \delta_2}} \cdot \frac{1}{\sqrt{2\pi}} \int \exp\left(-\frac{\delta_1^2 + \delta_2^2}{4\delta_1^2 \delta_2^2} \left[x^2 - 2x \frac{\bar{x}_1 \delta_1^2 + \bar{x}_2 \delta_1^2}{\delta_1^2 + \delta_2^2} + \frac{(\bar{x}_1 \delta_1^2 + \bar{x}_2 \delta_1^2)^2}{\delta_1^2 + \delta_2^2} - \frac{(\bar{x}_1 \delta_2^2 + \bar{x}_2 \delta_1^2)^2}{\delta_1^2 + \delta_2^2} + \frac{\bar{x}_1^2 \delta_2^2 + \bar{x}_2^2 \delta_1^2}{\delta_1^2 + \delta_2^2}\right]\right) dx \\
 &= \frac{1}{\sqrt{\delta_1 \delta_2}} \cdot \sqrt{\frac{2\delta_1^2 \delta_2^2}{\delta_1^2 + \delta_2^2}} \cdot \sqrt{\frac{\delta_1^2 + \delta_2^2}{2\delta_1^2 \delta_2^2}} \cdot \frac{1}{\sqrt{2\pi}} \left\{ \exp\left(\frac{1}{2} \cdot \frac{\delta_1^2 + \delta_2^2}{2\delta_1^2 \delta_2^2} \left[x - \frac{\bar{x}_1 \delta_1^2 + \bar{x}_2 \delta_1^2}{\delta_1^2 + \delta_2^2}\right]^2\right) \right. \\
 &\quad \left. + \frac{\delta_1^2 + \delta_2^2}{4\delta_1^2 \delta_2^2} \left[ -\frac{\bar{x}_1 \delta_2^2 + \bar{x}_2 \delta_1^2}{\delta_1^2 + \delta_2^2} \right] \right\} dx \quad ② Y
 \end{aligned}$$

let's simplify ②Y:

$$\begin{aligned}
 Y &= -\frac{\delta_1^2 + \delta_2^2}{4\delta_1^2 \delta_2^2} \left[ -\frac{\bar{x}_1 \delta_2^2 + \bar{x}_2 \delta_1^2}{\delta_1^2 + \delta_2^2} + \frac{\bar{x}_1^2 \delta_2^2 + \bar{x}_2^2 \delta_1^2}{\delta_1^2 + \delta_2^2} \right] = \frac{1}{4\delta_1^2 \delta_2^2} \left[ \frac{(\bar{x}_1 \delta_2^2 + \bar{x}_2 \delta_1^2)^2 - (\delta_1^2 + \delta_2^2)(\bar{x}_1^2 \delta_2^2 + \bar{x}_2^2 \delta_1^2)}{\delta_1^2 + \delta_2^2} \right] \\
 &= \frac{1}{4\delta_1^2 \delta_2^2} \left[ \frac{\bar{x}_1^2 \delta_2^4 + \bar{x}_2^2 \delta_1^4 + 2\bar{x}_1 \delta_2^2 \bar{x}_2 \delta_1^2 - \delta_1^2 \bar{x}_1^2 \delta_2^2 - \delta_2^2 \bar{x}_1^2 - \delta_1^4 \bar{x}_2^2 - \delta_2^2 \bar{x}_2^2 \delta_1^2}{\delta_1^2 + \delta_2^2} \right]
 \end{aligned}$$

$$Y = \frac{1}{4\delta_1^2 \delta_2^2} \left[ \frac{2\bar{x}_1 \delta_1^2 \bar{x}_2 \delta_2^2 - \delta_1^2 \bar{x}_1^2 \bar{x}_2^2 - \delta_2^2 \bar{x}_1^2 \bar{x}_2^2}{\delta_1^2 + \delta_2^2} \right] = -\frac{1}{4\delta_1^2 \delta_2^2} \left[ \frac{(\delta_1 \bar{x}_1 \delta_2 - \delta_2 \bar{x}_2 \delta_1)^2}{\delta_1^2 + \delta_2^2} \right]$$

$$= -\frac{1}{4\delta_1^2 \cdot \delta_2^2} \left[ \frac{(\delta_1 \delta_2)^2 \cdot (\bar{x}_1 - \bar{x}_2)^2}{\delta_1^2 + \delta_2^2} \right] = -\frac{1}{4} \left[ \frac{(\bar{x}_1 - \bar{x}_2)^2}{\delta_1^2 + \delta_2^2} \right]$$

we plug that in  $Z$  and get:

$$\begin{aligned} Z_{(\bar{x}_1, \bar{x}_2)} &= \frac{1}{\sqrt{\delta_1 \delta_2}} \cdot \sqrt{\frac{2\delta_1^2 \cdot \delta_2^2}{(\delta_1^2 + \delta_2^2)}} \cdot \sqrt{\frac{(\delta_1^2 + \delta_2^2)}{2\delta_1^2 \cdot \delta_2^2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int \exp\left(-\frac{1}{2} \cdot \frac{\delta_1^2 + \delta_2^2}{2\delta_1^2 \cdot \delta_2^2} \left[x - \frac{\bar{x}_1 \delta_2^2 + \bar{x}_2 \delta_1^2}{\delta_1^2 + \delta_2^2}\right]^2\right. \\ &\quad \left.- \frac{1}{4} \cdot \frac{(\bar{x}_1 - \bar{x}_2)^2}{\delta_1^2 + \delta_2^2}\right) dx \\ &= \sqrt{\frac{2\delta_1 \cdot \delta_2}{\delta_1^2 + \delta_2^2}} \cdot -\frac{1}{4} \cdot \frac{(\bar{x}_1 - \bar{x}_2)^2}{\delta_1^2 + \delta_2^2} \end{aligned}$$

we plug that in ① and get:

$$\begin{aligned} d_{\text{Hell}}^{(\bar{x}_1, \bar{x}_2)^2} &= 1 - Z_{(\bar{x}_1, \bar{x}_2)} \\ &= 1 - \sqrt{\frac{2\delta_1 \delta_2}{(\delta_1^2 + \delta_2^2)}} \cdot -\frac{1}{4} \cdot \frac{(\bar{x}_1 - \bar{x}_2)^2}{\delta_1^2 + \delta_2^2} \\ &= 1 - \sqrt{\frac{2\delta_1 \delta_2}{\delta_1^2 + \delta_2^2}} \cdot \left[ e^{-\frac{1}{2} \cdot \frac{(\bar{x}_1 - \bar{x}_2)^2}{\delta_1^2 + \delta_2^2}} \right]^2 \end{aligned}$$

$$= 1 - \sqrt{\frac{2\delta_1 \delta_2}{\delta_1^2 + \delta_2^2}} \cdot \sqrt{e^{-\frac{(\bar{x}_1 - \bar{x}_2)^2}{2(\delta_1^2 + \delta_2^2)}}}$$

$$\Rightarrow d_{\text{Hell}}^{(\bar{x}_1, \bar{x}_2)^2} = 1 - \sqrt{e^{-\frac{(\bar{x}_1 - \bar{x}_2)^2}{2(\delta_1^2 + \delta_2^2)}} \cdot \frac{2\delta_1 \delta_2}{\delta_1^2 + \delta_2^2}}$$

### Exercise 2:

We have the forward model:  $y = \alpha + \varepsilon$ , i.e.:  $h(\alpha) = \alpha$   
 where prior pdf:  $\pi_0(\alpha) \sim \text{Normal}(0, \sigma^2)$  and  
 $\varepsilon \sim \text{Normal}(0, 1)$

According to the Bayes' theorem:

$$\pi_x(x | y_{\text{obs}}) = \frac{\pi_y(y_{\text{obs}} | x) \pi_0(x)}{\pi_y(y_{\text{obs}})} \quad \text{where } y_{\text{obs}} = y_i$$

$$\begin{aligned} \Rightarrow \pi_i &= \pi(x | y_i) = \frac{\pi_\varepsilon(y_i - x) \pi_0(x)}{\int_{\mathbb{R}} \pi_\varepsilon(y_i - x) \pi_0(x) dx} \\ &\propto \exp\left(-\frac{1}{2}(y_i - x)^2\right) \exp\left(-\frac{x^2}{2\sigma^2}\right) \\ &= \exp\left(-\frac{1}{2}\left(y_i^2 - 2xy_i + x^2 + \frac{x^2}{\sigma^2}\right)\right) \\ &= \exp\left(-\frac{1}{2}\left(x^2\left(\frac{\sigma^2+1}{\sigma^2}\right) - 2xy_i + y_i^2\right)\right) \\ &= \exp\left(-\frac{1}{2 \cdot \frac{\sigma^2}{\sigma^2+1}} \left(x^2 - 2xy_i \frac{\sigma^2}{\sigma^2+1} + y_i^2 \cdot \frac{\sigma^2}{\sigma^2+1}\right)\right) \\ &= \exp\left(-\frac{1}{2 \cdot \frac{\sigma^2}{\sigma^2+1}} \left(\left(x - \frac{\sigma^2}{\sigma^2+1} y_i\right)^2 - \frac{\sigma^4}{(\sigma^2+1)^2} y_i^2 + y_i^2 \cdot \frac{\sigma^2}{\sigma^2+1}\right)\right) \\ &= \exp\left(-\frac{1}{2 \cdot \frac{\sigma^2}{\sigma^2+1}} \left(x - \frac{\sigma^2}{\sigma^2+1} y_i\right)^2\right) \underbrace{\exp\left(\frac{\sigma^2}{2(\sigma^2+1)} y_i^2 - \frac{1}{2} y_i^2\right)}_{\text{belongs to normalization term.}} \\ &\propto \exp\left(-\frac{1}{2 \cdot \frac{\sigma^2}{\sigma^2+1}} \left(x - \frac{\sigma^2}{\sigma^2+1} y_i^2\right)\right) \end{aligned}$$

Hence the posterior is Gaussian with mean  $\left(\frac{\sigma^2}{\sigma^2+1} y_i^2\right)$  and

the variance  $\left(\frac{\sigma^2}{\sigma^2+1}\right)$

~~mean~~

~~$\sigma^2$~~

Now we have two posterior Gaussian densities  $N(\bar{x}_1, \delta_1^2)$  and  $N(\bar{x}_2, \delta_2^2)$

$$\text{where } \bar{x}_1 = \frac{\delta^2}{\delta^2 + 1} y_1, \quad \bar{x}_2 = \frac{\delta^2}{\delta^2 + 1} y_2$$

$$\text{and } \delta_1^2 = \delta_2^2 = \frac{\delta^2}{\delta^2 + 1}.$$

Applied the formula of Hellinger distance from Exercise 1 we obtain:

$$d_{\text{Hell}}^2(\pi_1, \pi_2) = 1 - \sqrt{\exp\left(-\frac{\left(\frac{\delta^2}{\delta^2+1} y_1 - \frac{\delta^2}{\delta^2+1} y_2\right)^2}{2 \cdot 2 \cdot \frac{\delta^2}{\delta^2+1}}\right)} \cdot \frac{2 \delta^2}{2 \delta^2}$$

$$= 1 - \sqrt{\exp\left(-\left(\frac{\delta^2}{\delta^2+1}\right)^2 \frac{(y_1 - y_2)^2}{4 \cdot \frac{\delta^2}{\delta^2+1}}\right)}$$

$$\exp(x) \stackrel{\beta}{\approx}$$

$$= \exp\left(-\frac{1}{4} \cdot \frac{\delta^2}{\delta^2+1} \cdot (y_1 - y_2)^2\right) \stackrel{\leftarrow}{=} \exp(x) \approx$$

$$= 1 - \exp\left(-\frac{1}{8} \cdot \frac{\delta^2}{\delta^2+1} (y_1 - y_2)^2\right) \stackrel{\leftarrow}{=} \exp(1-x)$$

$$= \exp(x\beta) \leq \frac{1}{8} \cdot \frac{\delta^2}{\delta^2+1} (y_1 - y_2)^2 \quad (\text{Using } 1 - \exp(-x) \leq x \text{ } \textcircled{*})$$

$$\Rightarrow d_{\text{Hell}}(\pi_1, \pi_2) \leq \sqrt{\frac{1}{8} \cdot \frac{\delta^2}{\delta^2+1} |y_1 - y_2|^2}$$

$$= \frac{\delta}{2\sqrt{2}(\delta^2+1)} |y_1 - y_2|$$

$$\sum \frac{x}{x \leq 1}$$

$$|y_1 - y_2| \leq 2$$

$$\text{Thus } c = \frac{\delta}{2\sqrt{2}(\delta^2+1)}$$

r

Claim  $\textcircled{*}$   $1 - \exp(-x) \leq x \Leftrightarrow x - 1 + \exp(-x) \geq 0$ .

Proof: We have  $(x-1)$  convex and  $\exp(-x)$  strictly convex  
(cause  $g''(x) = \exp(-x) > 0$ )

$\Rightarrow x - 1 + \exp(-x)$  strictly convex.

$\Rightarrow x - 1 + \exp(-x)$  has unique minimum.

We can take derivative wrt  $x$ :

$$\frac{\partial(x - 1 + \exp(-x))}{\partial x} = 1 - \exp(-x) \stackrel{!}{=} 0 \Rightarrow x^* = 0.$$

$$\Rightarrow x - 1 + \exp(-x) \geq x^* - 1 + \exp(-x^*) = 0 - 1 + 1 = 0 \quad \square$$

### Exercise 3:

Markov transition kernel  $\pi(z'|z)$  with two PIs  
 $\pi_1$  &  $\pi_2$

$$\hat{\pi}_1(z) = \int_{\mathbb{R}} \pi(z'|z) \pi_1(z') dz'$$

$$\hat{\pi}_2(z) = \int_{\mathbb{R}} \pi(z'|z) \pi_2(z') dz'$$

Show that  $d_{TV}(\hat{\pi}_1, \hat{\pi}_2) \leq d_{TV}(\pi_1, \pi_2)$

$$d_{TV}(\pi_1, \pi_2) = \frac{1}{2} \sup_{|g|_p \leq 1} |\pi_2(g) - \pi_1(g)|$$

$$= \frac{1}{2} \int_{\mathbb{R}} |\pi_2(z) - \pi_1(z)| dz$$

$$d_{TV}(\hat{\pi}_1, \hat{\pi}_2) = \frac{1}{2} \int_{\mathbb{R}} |\hat{\pi}_2(z) - \hat{\pi}_1(z)| dz$$

$$= \frac{1}{2} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \pi(z'|z) \pi_2(z') dz' dz - \int_{\mathbb{R}} \int_{\mathbb{R}} \pi(z'|z) \pi_1(z') dz' dz \right|$$

*Jensen  
ineq.*

$$\leq \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |\pi(z'|z) \pi_2(z') - \pi(z'|z) \pi_1(z')| dz' dz \quad \text{⑤}$$

$$= \cancel{\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |\pi(z'|z)(\pi_2(z') - \pi_1(z'))| dz' dz}$$

$$\leq \underbrace{\int_{\mathbb{R}} |\pi(z'|z)| dz}_{=} \cdot \frac{1}{2} \underbrace{\int_{\mathbb{R}} |\pi_2(z) - \pi_1(z)| dz}_{=} = d_{TV}(\pi_1, \pi_2)$$

$$\Rightarrow d_{TV}(\hat{\pi}_1, \hat{\pi}_2) \leq d_{TV}(\pi_1, \pi_2)$$

$$\textcircled{1} = \iint_{\mathbb{R} \times \mathbb{R}} |\pi(z'|z) (\pi_1(z) - \pi_2(z))| dz dz'$$

$$= \iint_{\mathbb{R} \times \mathbb{R}} \pi(z'|z) |\pi_1(z) - \pi_2(z)| dz dz'$$

" ↑  
 $|\pi(z'|z)|$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \pi(z'|z) dz | \pi_1(z) - \pi_2(z) | dz$$

$\int_{\mathbb{R}}$  ↓  
 $= 1$  b.c.  
integral pdf

$$= \int_{\mathbb{R}} |\pi_1(z) - \pi_2(z)| dz = d_{TV}(\pi_1, \pi_2)$$

How to prove it starting from the def.  
of TV on the sheet

$$\textcircled{1} \|f\|_\infty \leq 1 \quad g(z) := \int f(z') \pi(z'|z) dz$$

$\leadsto \|g\|_\infty \leq 1$

$$\textcircled{2} \quad d_{TV}(\hat{\pi}_1, \hat{\pi}_2) = \sup_{\|g\|_\infty \leq 1} |\hat{\pi}_1[g] - \hat{\pi}_2[g]|$$

$$= \sup_{g_g, \|g\|_\infty \leq 1} |\pi_1[g] - \pi_2[g]| \leq \sup_{\|g\|_\infty \leq 1} |\pi_1[g] - \pi_2[g]|$$

$$P_{ij} = P[X_i | X_j]$$

$$\forall \alpha \in [0, 1]$$

Ex we can calc.

$$P[Z_0 \leq \alpha] \quad \forall \alpha$$

$$Z_0 \sim \pi_0$$

$$P[\gamma(z_0) \leq \alpha] = P[Z_0 \leq \alpha] \quad \forall \alpha$$

calc