Exercise 1. Determine the ANOVA decomposition for

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$

under the uniform measure on $[0,1] \times [0,1]$ and compute the associated variances σ_1^2, σ_2^2, and σ_{12}^2. Which terms in the ANOVA decomposition contribute most significantly to the total variance σ^2?

We seek a decomposition of $f(x_1,x_2)$ of the form

$$f(x_1,x_2) = \underbrace{f_0}_{1} + \underbrace{f_1(x_1)}_{2} + \underbrace{f_2(x_2)}_{3} + \underbrace{f_{12}(x_1,x_2)}_{4}$$

$$\begin{aligned}
f_0 = E[f(X)] &= \int_{[0,1]^2} f(x_1,x_2)\,dx_1\,dx_2 \\
&= \int_0^1 \left[\int_0^1 12x_1 + 6x_2 - 6x_1x_2 \; dx_1\right]^{(*)} dx_2 \\
&= \int_0^1 \Big[6x_1^2 + 6x_1x_2 - 3x_1^2x_2\Big]_0^{1 = x_1} dx_2 \\
&= \int_0^1 (6 + 6x_2 - 3x_2)\; dx_2 \\
&= \int_0^1 (3x_2 + 6)^{(*)}\, dx_2 \\
&= \left[\tfrac{3}{2}x_2^2 + 6x_2\right]_0^1 \\
1 \quad &= \frac{15}{2}
\end{aligned}$$

$$\begin{aligned}
f_1(x_1) &= \int_{[0,1]} f(x)\,dx_2 - \overset{1}{f_0} \\
&= \int_0^1 12x_1 + 6x_2 - 6x_1x_2 \; dx_2 - \frac{15}{2} \\
&= \Big[12x_1x_2 + 3x_2^2 - 3x_1x_2^2\Big]_0^{1 = x_2} - \frac{15}{2} \\
&= (12x_1 + 3 - 3x_1) - \frac{15}{2} \\
2 \quad &= 9x_1 - \frac{9}{2}
\end{aligned}$$

$$\begin{aligned}
f_2(x_2) &= \int_{[0,1]} f(x)\,dx_1 - \overset{1}{f_0} \\
&= \left[\int_0^1 12x_1 + 6x_2 - 6x_1x_2 \; dx_1\right]^{(*)} - \frac{15}{2} \\
&= (3x_2 + 6) - \frac{15}{2} \\
3 \quad &= 3x_2 - \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\underbrace{f_{12}(x_1,x_2)}_{4} &= f(x_1,x_2) - 1 - 2 - 3 \\
&= (12x_1 + 6x_2 - 6x_1x_2) - \left(\frac{15}{2}\right) - \left(9x_1 - \frac{9}{2}\right) - \left(3x_2 - \frac{3}{2}\right) \\
&= 12x_1 - 9x_1 + 6x_2 - 3x_2 - 6x_1x_2 - \frac{15}{2} + \frac{9}{2} + \frac{3}{2} \\
4 \quad &= 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}
\end{aligned}$$

$$\sigma_1^2 = Var(f_1(X_1))$$
$$= var(9X_1 - \tfrac{9}{2}) = var(9X_1) = 81\, \underbrace{Var(X_1)}_{=\frac{1}{12}}$$
$$= 81 \cdot \tfrac{1}{12} = \tfrac{27}{4} = \sigma_1^2$$

(since we know that $X_1 \sim U_{[0,1]}$ then, $Var(X_1) = \frac{(1-0)^2}{12} = \frac{1}{12}$)

$$\sigma_2^2 = Var(f_2(X_2))$$
$$= Var(3X_2 - \tfrac{3}{2}) = var(3X_2) = 9\, \underbrace{Var(X_2)}_{=\frac{1}{12}}$$
$$= 9 \cdot \tfrac{1}{12} = \tfrac{3}{4} = \sigma_2^2$$

$$\sigma_{12}^2 = Var(f_{12}(X_1, X_2))$$
$$= var(3X_1 + 3X_2 - 6X_1X_2 - \tfrac{3}{2})$$
$$= var(3X_1 + 3X_2 - 6X_1X_2)$$
$$= Var(3X_1 + 3X_2) + Var(6X_1X_2) - 2Cov(3X_1 + 3X_2, 6X_1X_2)$$

(since X_1 and X_2 are independent)

$$= Var(3X_1) + var(3X_2) + \underbrace{2Cov(3X_1, 3X_2)}_{=0} + Var(6X_1X_2) - 2Cov(3X_1 + 3X_2, 6X_1X_2)$$
$$= 9var(X_1) + 9var(X_2) + 36var(X_1X_2) - 36Cov(X_1 + X_2, X_1X_2)$$
$$= 9var(X_1) + 9var(X_2) + 36var(X_1X_2) - 36Cov(X_1, X_1X_2) - 36Cov(X_2, X_1X_2)$$

$var(X_1X_2)$:
$$= E[X_1^2X_2^2] - (E[X_1X_2])^2$$
$$= \int_0^1 x_1^2 x_2^2\, dx_1 dx_2 - (E[X_1]E[X_2])^2$$

(since X_1 and X_2 are independent)

$$= \int_0^1 \left[\tfrac{1}{3}x_1^3 x_2^2\right]_0^{1 = x_1} dx_2 - \left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right)^2$$
$$= \int_0^1 \tfrac{1}{3}x_2^2\, dx_2 - \left(\tfrac{1}{4}\right)^2$$
$$= \left[\tfrac{1}{9}x_2^3\right]_0^1 - \tfrac{1}{16}$$
$$= \tfrac{1}{9} - \tfrac{1}{16}$$
$$= \tfrac{7}{144}$$

$Cov(X_1, X_1X_2) = Cov(X_2, X_1X_2)$:
$$= E[X_1X_1X_2] - E[X_1]E[X_1X_2]$$

(since independent)

$$= \int_0^1 x_1^2 x_2\, dx_1 dx_2 - \tfrac{1}{2}E[X_1]E[X_2]$$
$$= \int_0^1 \left[\tfrac{1}{3}x_1^3 x_2\right]_0^{1 = x_1} dx_2 - \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}$$
$$= \int_0^1 \tfrac{1}{3}x_2\, dx_2 - \tfrac{1}{8}$$
$$= \left[\tfrac{1}{6}x_2^2\right]_0^1 - \tfrac{1}{8}$$
$$= \tfrac{1}{6} - \tfrac{1}{8}$$
$$= \tfrac{1}{24}$$

$$= 9\cdot\tfrac{1}{12} + 9\cdot\tfrac{1}{12} + 36\cdot\tfrac{7}{144} - 36\cdot\tfrac{1}{24} - 36\cdot\tfrac{1}{24}$$
$$= 9\cdot\tfrac{1}{6} + \tfrac{7}{4} - \tfrac{36}{12}$$
$$= \tfrac{3}{2} + \tfrac{7}{4} - 3$$
$$= \frac{6+7-12}{4}$$
$$\sigma_{12}^2 = \tfrac{1}{4}$$

To verify the ANOVA decomposition of $f(x_1, x_2)$

$$\sigma^2 = Var(f(X_1X_2)) =$$
$$= Var(12X_1 + 6X_2 - 6X_1X_2)$$
$$= Var(12X_1 + 6X_2) + Var(6X_1X_2) - 2Cov(12X_1 + 6X_2, 6X_1X_2)$$
$$= Var(12X_1) + Var(6X_2) + Var(6X_1X_2) - 2Cov(12X_1, 6X_1X_2) - 2Cov(6X_2, 6X_1X_2)$$
$$= 144\,Var(X_1) + 36\,Var(X_2) + 36\,Var(X_1X_2) - 144\,Cov(X_1, X_1X_2) - 72\,Cov(X_2, X_1X_2)$$
$$= 144 \cdot \frac{1}{12} + 36 \cdot \frac{1}{12} + 36 \cdot \frac{7}{144} - 144 \cdot \frac{1}{24} - 72 \cdot \frac{1}{24}$$
$$= 12 + 3 + \frac{7}{4} - 6 - 3$$
$$= 6 + \frac{7}{4}$$
$$= \frac{31}{4}$$

$=$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_{12}^2 = \underbrace{\frac{27}{4}}_{\sigma_1^2} + \underbrace{\frac{3}{4}}_{\sigma_2^2} + \underbrace{\frac{1}{4}}_{\sigma_{12}^2} = \frac{31}{4}$$

σ_1^2 contributes most significantly to the total variance σ^2!

Exercise 2. Consider a sequence $\{X_i\}_{i\geq 1}$ of independent and identically distributed univariate Gaussian random variables with mean zero and variance $\sigma^2 = 1$. Apply the central limit theorem to the induced sequence of random variables

$$Y_M = M^{-1/2} \sum_{i=1}^{M} (X_i^2 - \mathbb{E}[X_i^2]),$$

in order to determine the PDF π_{Y_M} for Y_M as $M \to \infty$.

$X_i \sim N(0,1)$

$\Rightarrow X_i^2 \sim \chi_1^2$

X_i's are independent $i = 1, 2, \dots, M$

$$\underbrace{\sum_{i=1}^{M} X_i^2}_{:= Z_M} = \chi_M^2$$ (Chi-squared distribution with M degree of freedom)

By Central Limit theorem

$$\frac{Z_M - E[Z_M]}{\sqrt{Var(Z_M)}} \xrightarrow{m\to\infty} N(0,1) \quad (*)$$

Since we have

$$\begin{aligned} E[Z_M] &= E[\sum_{i=1}^{M} X_i^2] \\ &= E[X_1^2 + X_2^2 + \cdots X_M^2] \\ &= \underbrace{E[X_1^2] + E[X_2^2] + \cdots E[X_M^2]}_{\text{M terms and all } E[X_i] \text{ are same.}} \quad \text{(since linearity of expectation)} \\ &= M \underbrace{E[X_1^2]}_{=1} \end{aligned}$$

$$\begin{aligned} Var(Z_M) &= Var(\sum_{i=1}^{M} X_i^2) \\ &= Var(X_1^2 + X_2^2 + \cdots + X_M^2) \quad \text{(since } X_i\text{'s are independent)} \\ &= \underbrace{Var(X_1^2) + Var(X_2^2) + \cdots Var(X_M^2)}_{\text{M terms and all } Var(X_i) \text{ are same.}} \\ &= M\, Var(X_1^2) \\ &= M(E[X_1^4] - (E[X_1^2])^2) \\ &= M(3 - 1^2) \\ &= 2M \end{aligned}$$

We can reformulate $(*)$ above

$$\frac{\sum_{i=1}^{M} X_i^2 - M E[X_i^2]}{\sqrt{2M}} = \frac{\sum_{i=1}^{M} (X_i^2 - E[X_i^2])}{\sqrt{2M}} \xrightarrow{M\to\infty} N(0,1)$$

We want to determine the PDF π_{Y_M} for Y_M as $M \to \infty$ where

$$Y_M = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} (X_i^2 - E[X_i^2])$$
$$= \sqrt{2} \underbrace{\left[\frac{1}{\sqrt{2M}} \sum_{i=1}^{M} (X_i^2 - E[X_i^2]) \right]}$$

We found this term $\xrightarrow{M\to\infty} N(0,1)$
which we will define as $Z_{M\to\infty}$

$$\Rightarrow Y_{M\to\infty} = \sqrt{2}\, Z_{M\to\infty} \quad \text{where } Z_{M\to\infty} \sim N(0,1)$$

We know that affine transformation of Gaussian distributed R.V is also Gaussian distributed R.V with a different mean and variance. Therefore, it is sufficient to find the mean and variance of $Y_{M\to\infty}$.

$$E[Y_{M\to\infty}] = E[\sqrt{2}\, Z_{M\to\infty}] = \sqrt{2}\, \underbrace{E[Z_{M\to\infty}]}_{=0} = \sqrt{2}\cdot 0 = 0$$

(by linearity)

$$Var(Y_{M\to\infty}) = Var(\sqrt{2}\, Y_{M\to\infty}) = 2\, \underbrace{Var(Y_{M\to\infty})}_{=1} = 2\cdot 1 = 2$$

Hence, $Y_{M\to\infty} \sim N(0,2)$ where the PDF π_{Y_M} is

$$\pi_{Y_M}(y) = \frac{1}{\sqrt{2\pi 2^2}} \exp\left[-\frac{1}{2} \frac{y^2}{2^2} \right]$$
$$= \frac{1}{\sqrt{8\pi}} \exp\left[-\frac{y^2}{8} \right]$$

In [1]:
```python
import numpy as np
```

Exercise 3. Implement Algorithm 3.17 from the book. The input parameters are the integers M, L, and a set of weights $w_i \geq 0$, $i = 1, \ldots, M$, with $\sum_{i=1}^{M} w_i = 1$. The output of the algorithm are M integers $\bar{\xi}_i \geq 0$ which satisfy $\sum_{i=1}^{M} \bar{\xi}_i = L$. Verify your algorithm by checking that $\bar{\xi}_i/L \approx w_i$ for $L \gg M$.

Algorithm 3.27 (Multinomial samples) The integer-valued variable $\bar{\xi}_i$, $i = 1, \ldots, M$, is set equal to zero initially.

For $l = 1, \ldots, L$ do:

(i) Draw a number $u \in [0, 1]$ from the uniform distribution $\mathrm{U}[0, 1]$.

(ii) Determine the integer $i^* \in \{1, \ldots, M\}$ which satisfies

$$i^* = \arg\min_{i \geq 1} \sum_{j=1}^{i} w_j \geq u.$$

(iii) Increment $\bar{\xi}_{i^*}$ by one.

In [2]:
```python
def multinomialSamples (M, L, w_arr):
    eta_bar = np.zeros(M)
    # immediately drawing a list with uniform samples of size L
    u_list = np.random.uniform(low=0.0, high=1.0, size=L)

    for current in range(L):
        u = u_list[current]
        w_sum = w_arr[0]

        # determine i_star
        i_star = 0
        while w_sum < u:
            i_star += 1
            w_sum += w_arr[i_star]

        # increment eta_bar[i_star] to get the right distribution
        eta_bar[i_star] += 1
    return eta_bar
```

In [3]:
```python
L = 10000
w_arr = np.asarray([0.2, 0.1, 0.05, 0.25, 0.33, 0.07])
M = len(w_arr)

eta_bar = multinomialSamples(M, L, w_arr)

print("Compare the two arrays for verification:")
print(w_arr)
print(eta_bar/L)
```

```
Compare the two arrays for verification:
[0.2  0.1  0.05 0.25 0.33 0.07]
[0.2061 0.0974 0.0453 0.2488 0.3318 0.0706]
```

Exercise 4. Go through Appendix 3.5 which explains the convergence of multinomial resampling using the concept of random probability measures and their distance. The final bound, that is $4/L^{1/2}$, on page 94 is not optimal. Can you improve it?

$$d(\mu_M^{\Omega}, \nu_M^{\Omega}) = \sup_{|f|\le 1} \sqrt{E\left[\left(\frac{1}{L}\sum_{l=1}^{L} f(X_l) - \bar{f}_M\right)^2\right]}$$

$$= \frac{1}{L^{1/2}} \sup_{|f|\le 1} \sqrt{E\left[\frac{1}{L}\sum_{l=1}^{L} (f(X_l) - \bar{f}_M)^2\right]}$$

$$= \frac{1}{L^{1/2}} \sup_{|f|\le 1} \sqrt{\frac{1}{L}\sum_{l=1}^{L} E\left[f(X_l)^2 - 2f(X_l)\bar{f}_M + \bar{f}_M^2\right]}$$

$$= \frac{1}{L^{1/2}} \sup_{|f|\le 1} \sqrt{\frac{1}{L}\sum_{l=1}^{L} \left(E[f(X_l)^2] - 2\bar{f}_M \underbrace{E[f(X_l)]}_{=\bar{f}_M} + \bar{f}_M^2\right)}$$

$$= \frac{1}{L^{1/2}} \sqrt{\frac{1}{L}\sum_{l=1}^{L} \sup_{|f|\le 1} \underbrace{\left(E[f(X_l)^2] - \bar{f}_M^2\right)}}$$

Since $0 \le f(X_l)^2 \le 1$ and $0 \le \overline{f_M^2} \le 1$
under the assumption $|f(x)| \le 1$ for all $x \in \mathbb{R}^{N_x}$

$$\le \frac{1}{L^{1/2}} \sqrt{\frac{1}{L} \underbrace{\sum_{l=1}^{L} (1)}_{=L}}$$

$$= \frac{1}{L^{1/2}}$$