Exercise 1

$\mathbb{E}[(\hat{X}-X)^2] = \mathbb{E}[(aY+b-X)^2] =$

$\mathbb{E}[(ahX+a\Xi+b-X)^2] = \mathbb{E}[(X(ah-1)+a\Xi+b)^2]$

$= \underbrace{\mathbb{E}[X^2]}_{=Var[X]+\mathbb{E}[X]^2 = \sigma^2+\bar{x}^2}(ah-1)^2 + a^2\underbrace{\mathbb{E}[\Xi^2]}_{=r^2} + b^2$

$+ 2(ah-1)a\underbrace{\mathbb{E}[X\Xi]}_{=\mathbb{E}[X]\mathbb{E}[\Xi]=0} + 2(ah-1)b\underbrace{\mathbb{E}[X]}_{=\bar{x}} + 2ab\underbrace{\mathbb{E}[\Xi]}_{=0}$

since indep.   since mean is 0

$= (ah-1)^2(\sigma^2+\bar{x}^2) + a^2r^2 + b^2 + 2(ah-1)b\bar{x} = \ell(a,b)$

To minimize we set the gradient to 0

$$\frac{\partial \ell}{\partial b} = 2b + 2(ah-1)\bar{x} \stackrel{!}{=} 0$$

$$\Rightarrow b = (1-ah)\bar{x}$$

Plug this in $\ell \Rightarrow$

$$\ell(a) = (ah-1)^2\sigma^2 + a^2r^2$$

$$= a^2(h^2\sigma^2+r^2) - 2ah\sigma^2 + \sigma^2$$

Setting derivative to 0 gives us

$$a^* = \frac{h\sigma^2}{h^2\sigma^2+r^2}$$

$$\Rightarrow b^* = \frac{r^2}{h^2\sigma^2+r^2}\bar{x}$$

Since ℓ is a quadratic polynomial in a, b with positive coefficients before a^2 and b^2 we know it has only one critical point which is its global minimum

$\Rightarrow a^*, b^*$ minimize ℓ

Exercise 21

$c = \pi(x_1)$

$$\pi(x_1,x_2) = \frac{1}{Z}\exp(-x_1^2 - x_2^2 - x_1^2 x_2^2)$$

$$\pi(x_2 \mid x_1) = \frac{\pi(x_1,x_2)}{c}, \quad c := \int_{-\infty}^{\infty} \pi(x_1,x_2)\,dx_2$$

$$\frac{c}{\frac{1}{Z}e^{-x_1^2}} = \int_{-\infty}^{\infty} e^{-x_2^2(1+x_1^2)}\,dx_2 = \frac{\sqrt{\pi}}{\sqrt{1+x_1^2}} \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{1+x_1^2}{\pi}}\, e^{-x_2^2(1+x_1^2)}}_{\text{pdf of } \mathcal{N}\left(0, \frac{1}{2(1+x_1^2)}\right) \text{ integrates to } 1}\,dx_2$$

$$= \frac{\sqrt{\pi}}{\sqrt{1+x_1^2}}$$

$$\Rightarrow \quad c = \frac{1}{Z} e^{-x_1^2} \frac{\sqrt{\pi}}{\sqrt{1+x_1^2}}, \quad \pi(x_2 \mid x_1) = \sqrt{\frac{1+x_1^2}{\pi}}\, e^{-x_2^2(1+x_1^2)} = * = \mathcal{N}\Big(0, \underbrace{\frac{1}{2(1+x_1^2)}}_{\sigma^2}\Big)$$

$$\boxed{e^{-x_2^2/2\sigma^2} = e^{-x_2^2/\left(2\cdot\frac{1}{2(1+x_1^2)}\right)} = *}$$

Shorter way: One can also use that $\pi(x_2 \mid x_1)$ is a pdf which is proportional to $e^{-x_2^2(1+x_1^2)}$. Since it has to integrate to 1, the constant in front is clear.

$$\mathbb{E}[X_1 X_2^2 \mid X_1 = x_1] = x_1 \underbrace{\mathbb{E}[X_2^2 \mid X_1 = x_1]}_{\text{Variance of } \mathcal{N}\left(0, \frac{1}{2(1+x_1^2)}\right)}$$

$$= x_1 \frac{1}{2(1+x_1^2)}$$

E3)

From the script, Ex. 2.18 we have

$$P^{-1} = \frac{1}{\sigma_c^2}\begin{pmatrix} 1 & -\left(\frac{\sigma_{12}}{\sigma_{22}}\right)^2 \\ \left(\frac{\sigma_{12}}{\sigma_{22}}\right)^2 & \left(\frac{\sigma_{11}}{\sigma_{22}}\right)^2 \end{pmatrix} \quad \text{with} \quad \sigma_c^2 = \sigma_{11}^2 - \sigma_{12}^2\sigma_{22}^{-2}\sigma_{21}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 a y_1 + x_1 b y_2 + x_2 c y_1 + x_2 d y_2$$

Let $\bar{x}_c = \bar{x}_1 + \sigma_{12}^2\sigma_{22}^{-2}(x_2 - \bar{x}_2)$

We treat the exponent seperately and multiply it with σ_c^2:

$$\sigma_c^2 (z-\bar{z})^t P^{-1}(z-\bar{z}) = (x_1-\bar{x}_1)^2 + \left(\frac{\sigma_{11}}{\sigma_{22}}\right)^2 (x_2-\bar{x}_2)^2 + 2\left(\frac{\sigma_{12}}{\sigma_{22}}\right)^2 (x_1-\bar{x}_1)(x_2-\bar{x}_2) =: (*)$$

We want to show that $(*) = (\#)$ where

$$(\#) = (x_1 - \bar{x}_c)^2 + \frac{\sigma_c^2}{\sigma_{22}^2}(x_2-\bar{x}_2)^2$$

$$\frac{\sigma_c^2}{\sigma_{22}} = \frac{\sigma_{11}^2}{\sigma_{22}^2} - \frac{\sigma_{12}^4}{\sigma_{22}^4}$$

$$(x_1-\bar{x}_c)^2 = (x_1 - \bar{x}_1 - \sigma_{12}\sigma_{22}^{-2}(x_2-\bar{x}_2))$$

$$(\#) = (x_1-\bar{x}_1)^2 + 2\frac{\sigma_{12}^2}{\sigma_{22}^2}(x_1-\bar{x}_1)(x_2-\bar{x}_2) + \left(\cancel{\frac{\sigma_{12}^4}{\sigma_{22}^4}} + \frac{\sigma_{11}^2}{\sigma_{22}^2} - \cancel{\frac{\sigma_{12}^4}{\sigma_{22}^4}}\right)(x_2-\bar{x}_2)^2$$

We see that $(*) = (\#)$

Now for the factor before the $e^{-\text{"}-}$

$$\frac{1}{2\pi |P|^{1/2}} = \frac{1}{2\pi}\frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}$$

$$= \frac{1}{2\pi}\frac{1}{\sigma_c^2\sigma_{22}^2} = \frac{1}{\sqrt{2\pi}\,\sigma_c^2}\;\frac{1}{\sqrt{2\pi}\,\sigma_{22}^2}$$

Density of $\pi(x_1) = \int \pi(x_1, x_2)\, dx_2$

$$\frac{1}{\sqrt{2\pi\sigma_c^2}} \exp\left(-\frac{1}{2\sigma_c^2}(x_1 - \bar{x}_1)^2\right)$$ since the density

$$\frac{1}{\sqrt{2\pi\sigma_{22}^2}} \exp\left(-\frac{1}{2\sigma_{22}^2}(x_2 - \bar{x}_2)^2\right) = N(x_2, \sigma_{22}^2)$$

integrates to 1

$$\pi(x_2 | x_1) = \frac{\pi(x_1, x_2)}{\pi(x_1)} = \frac{1}{\sqrt{2\pi\sigma_{22}^2}}$$

We now factored $\pi(x_1, x_2) = \pi(x_1 | x_2)\, \pi(x_2)$

For the formulas for $\pi(x_1)$, $\pi(x_2 | x_1)$ we would have to switch things around:

$$\pi(x_1) = \frac{1}{\sqrt{2\pi\sigma_{11}^2}} \exp\left(-\frac{1}{2\sigma_{11}^2}(x_1 - \bar{x}_1)^2\right)$$

$$\pi(x_2 | x_1) = \frac{1}{\sqrt{2\pi\sigma_d^2}} \exp\left(-\frac{1}{2\sigma_d^2}(x_2 - \bar{x}_d)^2\right)$$

with
$$\sigma_d^2 = \sigma_{22}^2 - \sigma_{12}^4 \sigma_{11}^{-2}$$
$$\bar{x}_d = \bar{x}_2 + \frac{\sigma_{12}^2}{\sigma_{11}^2}(x_1 - \bar{x}_1).$$

Exercise 4

$$d^2_{Hell}(p,q) = 1 - \int \sqrt{pq}\, dx$$

$$= \underbrace{\int p\, dx}_{\text{density integrates to } 1} - \int \sqrt{\frac{q}{p}}\, p\, dx$$

$$= \int p \left(1 - \sqrt{\frac{q}{p}}\right) dx$$

$$\underset{(*)}{\leq} -\int p \log\left(\sqrt{\frac{q}{p}}\right) dx$$

$$= -\frac{1}{2} \int p \log\left(\frac{q}{p}\right) dx$$

$$= KL(p|q)$$

(*) Claim: $1 - x \leq -\log x \iff x - 1 - \log x \geq 0$

Pf: $x - 1$ convex, $-\log x$ strictly convex

$\Rightarrow x - 1 + (-\log x)$ strictly convex

$\Rightarrow x - 1 - \log x$ has unique minimum

Take derivative w.r.t. x

$$\nabla_x\, x - 1 - \log x = 1 - \frac{1}{x} \overset{!}{=} 0$$

$$\Rightarrow x^* = 1$$

$$\Rightarrow x - 1 - \log x \geq x^* - 1 - \log x^* = 1 - 1 - 0 = 0$$

$\square$