Exercise 1. Consider the following discrete time deterministic dynamical system

$$z^{n+1} = \nu z^n(1-z^n) \tag{1}$$

with $\nu = 4$ and $z^n \in (0,1)\ \forall\ n \in \mathbb{N}$. Its invariant density is given by

$$\frac{1}{\pi\sqrt{z(1-z)}}. \quad = * \tag{2}$$

a) Show analytically that (2) is invariant under the mapping (1).
Hint: obtain an integral equation for the evolution of the density π^n by starting from $\mathbb{P}(z^{n+1} \leq z)$.

We want to obtain the marginal density π^{n+1} and show that it is invariant, i.e. $\pi^{n+1} = \pi^n$.

$$\begin{aligned}
\underset{\text{CDF of } z^{n+1}}{\overset{\shortparallel}{F_{z^{n+1}}(z)}} &= \mathbb{P}(4z^n(1-z^n) < z) \\
&= \mathbb{P}\left(z^n - (z^n)^2 < \cancel{\tfrac{4}{z}}\ \tfrac{z}{4}\right) \\
&= \mathbb{P}\left((z^n)^2 - z^n + \tfrac{4}{z} > 0\right) \\
&= \mathbb{P}\left(z^n < \tfrac{1-\sqrt{1-z}}{2}\right) + \mathbb{P}\left(z^n > \tfrac{1+\sqrt{1-z}}{2}\right) \\
&= \mathbb{P}\left(z^n < \tfrac{1-\sqrt{1-z}}{2}\right) + \left(1 - \mathbb{P}\left(z^n < \tfrac{1+\sqrt{1-z}}{2}\right)\right) \\
&= \left[\frac{2}{\pi}\arcsin\left(\sqrt{\frac{1-\sqrt{1-z}}{2}}\right) + 1 - \frac{2}{\pi}\arcsin\left(\sqrt{\frac{1+\sqrt{1-z}}{2}}\right)\right] \circledast
\end{aligned}$$

$$\frac{d}{dz}\int_0^{\frac{1-\sqrt{1-z}}{2}} * \, dz$$

Since
$$\begin{aligned}
&\int \frac{1}{\pi\sqrt{z(1-z)}}\,dz \\
&= \frac{1}{\pi}\int \frac{1}{\sqrt{1-z}\,\sqrt{z}}\,dz \\
&= \frac{2}{\pi}\int \frac{1}{\sqrt{1-x^2}}\,dx \\
\text{(standard integral)} &= \frac{2}{\pi}\cdot\arcsin(x) \\
&= \frac{2}{\pi}\cdot\arcsin(\sqrt{z})
\end{aligned}$$

(change of variables $x = \sqrt{z}$, $\frac{dx}{dz} = \frac{1}{2\sqrt{z}}$, $dx = \frac{1}{2\sqrt{z}}dz$)

μ_1, μ_2 and you want to show

$\mu_1 = \mu_2$

then you do that by showing

$\mu_1(A) = \mu_2(A)$ for all subsets A

In 1-d it is enough to show that

$$\mu_1((-\infty, x]) = \mu_2((-\infty, x])$$

$$\overset{\shortparallel}{F_1(x)} \qquad\qquad \overset{\shortparallel}{F_2(x)}$$

CDF $\qquad\qquad$ CDF

$$\pi^{n+1}(z) = F_{z^{n+1}}{}'(z) = (\circledast)'$$

PDF of z^{n+1}

$$= \frac{2}{\pi}\left(\frac{1}{\sqrt{1-\left(\frac{1}{2}-\frac{\sqrt{1-z}}{2}\right)}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}-\frac{\sqrt{1-z}}{2}}}\cdot\left(-\frac{1}{2}\cdot\frac{1}{2\sqrt{1-z}}\cdot(-1)\right)\right.$$

$$\left.-\frac{1}{\sqrt{1-\left(\frac{1}{2}+\frac{\sqrt{1-z}}{2}\right)}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}+\frac{\sqrt{1-z}}{2}}}\cdot\left(\frac{1}{2}\cdot\frac{1}{2\sqrt{1-z}}\cdot(-1)\right)\right)$$

$$= \frac{1}{4\pi}\left(\frac{1}{\sqrt{\frac{1}{2}+\frac{\sqrt{1-z}}{2}}\cdot\sqrt{\frac{1}{2}-\frac{\sqrt{1-z}}{2}}\cdot\sqrt{1-z}} + \frac{}{\sqrt{\frac{1}{2}-\frac{\sqrt{1-z}}{2}}\cdot\sqrt{\frac{1}{2}+\frac{\sqrt{1-z}}{2}}\cdot\sqrt{1-z}}\right.$$

$$= \frac{1}{4\pi}\left(\frac{2}{\sqrt{\frac{1}{4}-\left(\frac{1-z}{4}\right)}\cdot\sqrt{1-z}}\right)$$

$$= \frac{1}{\cancel{4}\pi}\left(\frac{\cancel{2}}{\frac{\sqrt{z}}{\cancel{2}}\cdot\sqrt{1-z}}\right)$$

$$= \frac{1}{\pi\sqrt{z(1-z)}} = \pi^{n}(z)$$

Hence, $\pi^{n+1}(z) = \pi^{n}(z)$, $\frac{1}{\pi\sqrt{z(1-z)}}$ is invariant under the mapping $z^{n+1} = 4z(1-z)$.

In [1]:

```python
import numpy as np
from matplotlib import pyplot as plt
%matplotlib inline
import seaborn as sns
sns.set_style('darkgrid')
#sns.set(rc={'figure.figsize':(4,4)})
```

b) *Simulation exercise:* Consider z^0 as being uniformly distributed on $(0,1)$. Write a code to determine how this uniform distribution evolves under (1). Produce histograms after $5, 10, 100$ iterations and overlay the invariant density (2).

We start from U[0,1] and see how this density evolves.

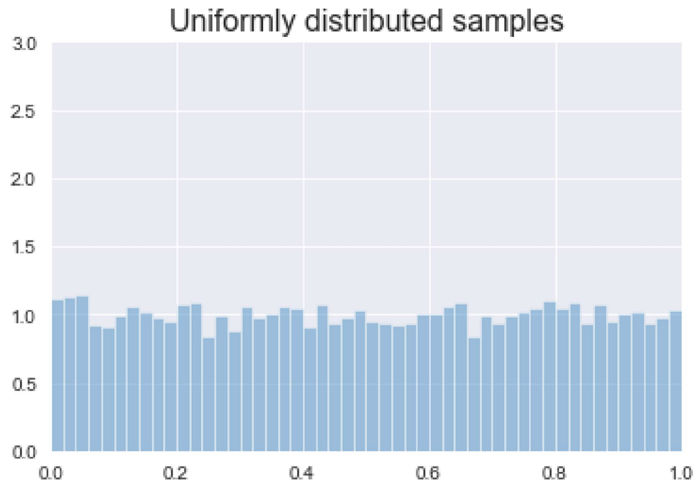
In [2]:

```python
samples = np.random.uniform(low=0, high=1, size=10000)

fig, ax = plt.subplots()
ax.set(xlim=(0,1), ylim=(0,3))
sns.distplot(samples, bins=50, ax=ax)
ax.get_lines()[0].remove()
plt.title("Uniformly distributed samples", fontsize=16)
```

Out[2]:

```
Text(0.5, 1.0, 'Uniformly distributed samples')
```



Deterministic mapping function

$$z^{n+1} = \nu z^n (1 - z^n) \tag{1}$$

In [3]:

```python
def DeterministicDynamicalSystem (samples_mapped, iterations):
    for i in range(iterations):
        samples_mapped = 4*samples_mapped*(1-samples_mapped)
    return samples_mapped
```

Inveriant density function

$$\frac{1}{\pi\sqrt{z(1-z)}}. \tag{2}$$

In [4]:

```python
def InvariantDensity(z):
    z = 1/(np.pi*np.sqrt(z*(1-z)))
    return z
invariant_density = InvariantDensity(np.linspace(0.001, 0.999, 10000))
```

function for plotting histogram

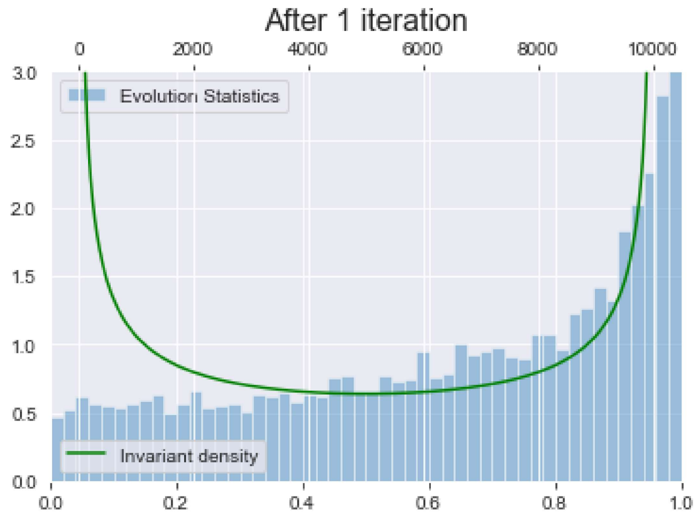
In [5]:

```python
def PlotHistogram(samples_iter, title):
    fig, ax = plt.subplots()
    ax.set(xlim=(0,1), ylim=(0,3))
    sns.distplot(samples_iter, bins=50, ax=ax)
    ax.get_lines()[0].remove()
    ax2 = ax.twiny()
    ax2.plot(invariant_density,color='green')
    ax2.legend(labels=['Invariant density'])
    ax.legend(labels=['Evolution Statistics'])
    plt.title(title, fontsize=16)
```

Now we will see the evolution of the initial density.

In [6]:

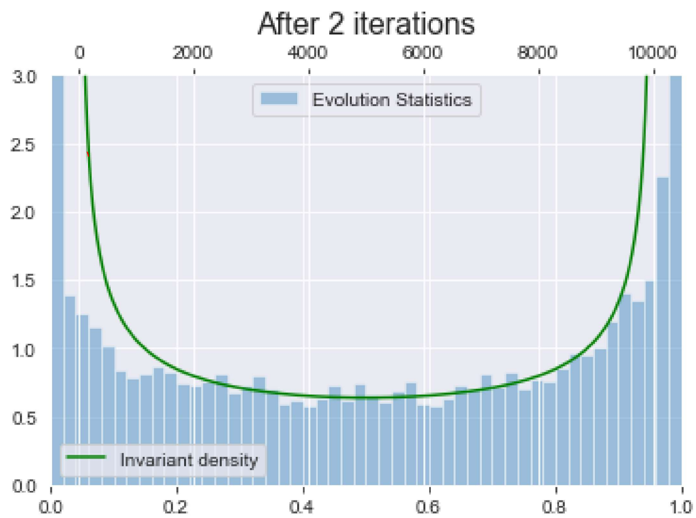
```python
samples_iter1 = DeterministicDynamicalSystem (samples, 1)
PlotHistogram(samples_iter1, "After 1 iteration")
```



In [7]:

```python
samples_iter2 = DeterministicDynamicalSystem (samples, 2)
PlotHistogram(samples_iter2, "After 2 iterations")
```

"geometric ergodicity"

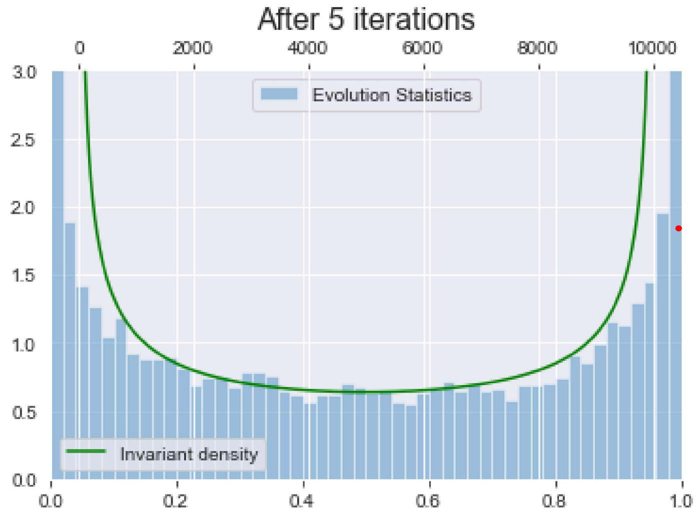


Unif Distr

Even after 2 iterations, it already approachs to invariant density!

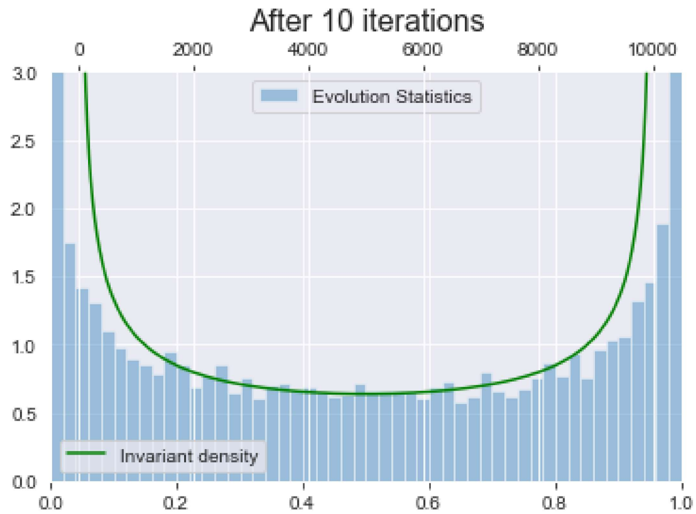
In [8]:

```
samples_iter5 = DeterministicDynamicalSystem (samples, 5)
PlotHistogram(samples_iter5, "After 5 iterations")
```



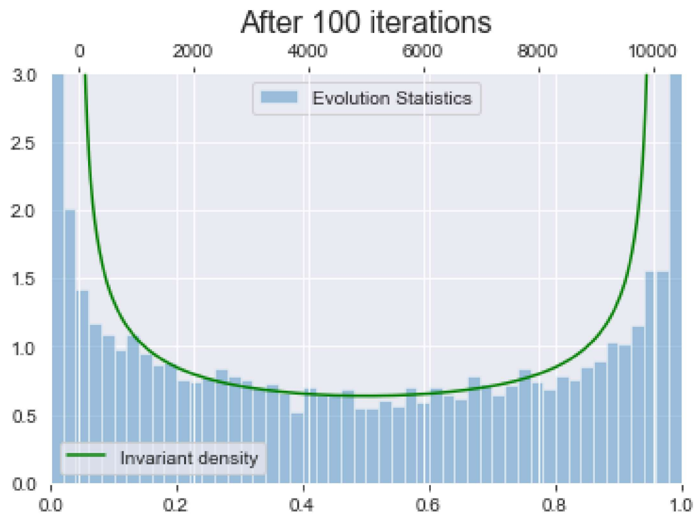
In [9]:

```
samples_iter10 = DeterministicDynamicalSystem (samples, 10)
PlotHistogram(samples_iter10, "After 10 iterations")
```



In [10]:

```
samples_iter100 = DeterministicDynamicalSystem (samples, 100)
PlotHistogram(samples_iter100, "After 100 iterations")
```



Exercise 2. The weather in a simple earth system has 4 possible states: sun, rain, snow, hail. We want to use a discrete time Markov chain to model the daily weather of this system. The weather does not change from one day to the next with probability $1-3\alpha$. If the weather does change, then all other 3 states are equally likely. We have that $0<\alpha<\frac{1}{3}$.

(a) Show that $\mathrm{P}^n_{i,i}=\frac{1}{4}+\frac{3}{4}(1-4\alpha)^n\ \forall\ i=\{1,2,3,4\}$ where n corresponds to the nth time index. Then obtain an expression for $\mathrm{P}^n_{i,j}$ for $i\neq j$.

(Approach using induction)

Since the other three $(P^n)_{ij}$ for $i\neq j$ take the same values in each column,

$$(P^n)_{ij}=\left(1-\left(\tfrac{1}{4}+\tfrac{3}{4}(1-4\alpha)^n\right)\right)/3$$
$$=\tfrac{1}{4}-\tfrac{1}{4}(1-4\alpha)^n$$

1A (what you want show)

Claim: $(P^n)_{ii}=\frac{1}{4}\left(1+3(1-4\alpha)^n\right)$ and $(P^n)_{ij}=\frac{1}{4}\left(1-(1-4\alpha)^n\right)$ for $i\neq j,\ i,j\in\{1,2,3,4\}$ $C(n)$

Proof by induction:

1B $(n=1)$

* Base case: $(P^1)_{ii}=\frac{1}{4}+\frac{3}{4}-3\alpha=1-3\alpha$
$(P^1)_{ij}=\frac{1}{4}-\frac{1}{4}+\alpha=\alpha$ $\Leftrightarrow C(n)$

IS $(n\to n+1)$

* Inductive step: Show that for any $n\geq 1$ with $C(n)\Rightarrow C(n+1)$

$$(P^{n+1})_{ii}=\sum_{k=1}^{4}(P)_{ik}(P^n)_{ki}$$
$$\overset{C(n)}{=}P_{ii}\cdot(P^n)_{ii}+3\cdot P_{il}\cdot(P^n)_{il}\quad\text{for } l\neq i$$
$$\overset{C(n)}{=}(1-3\alpha)\left(\tfrac{1}{4}+\tfrac{3}{4}(1-4\alpha)^n\right)+3\cdot\alpha\cdot\left(\tfrac{1}{4}-\tfrac{1}{4}(1-4\alpha)^n\right)$$
$$=\tfrac{1}{4}\left(1+3(1-4\alpha)^n-\cancel{3\alpha}-9\alpha(1-4\alpha)^n+\cancel{3\alpha}-3\alpha(1-4\alpha)^n\right)$$
$$=\tfrac{1}{4}\left(1+(1-4\alpha)^n(3-12\alpha)\right)$$
$$=\tfrac{1}{4}\left(1+3\cdot(1-4\alpha)^{n+1}\right)$$
$$=\underline{\tfrac{1}{4}+\tfrac{3}{4}(1-4\alpha)^{n+1}}$$

$$(P^{n+1})_{ij}=\sum_{k=1}^{4}(P)_{ik}(P^n)_{kj}$$
$$\overset{C(n)}{=}(P)_{ii}(P^n)_{ij}+(P)_{ij}(P^n)_{jj}+2\cdot(P)_{il}(P^n)_{lj}\quad\text{for } l\neq i,j$$
$$\overset{C(n)}{=}(1-3\alpha)\cdot\tfrac{1}{4}\left(1-(1-4\alpha)^n\right)+\alpha\cdot\tfrac{1}{4}\left(1+3(1-4\alpha)^n\right)+2\cdot\alpha\cdot\tfrac{1}{4}\left(1-(1-4\alpha)^n\right)$$
$$=\tfrac{1}{4}\left(1-(1-4\alpha)^n-\cancel{3\alpha}+3\alpha(1-4\alpha)^n+\cancel{\alpha}+3\alpha(1-4\alpha)^n+\cancel{2\alpha}-2\alpha(1-4\alpha)^n\right)$$
$$=\tfrac{1}{4}\left(1-(1-4\alpha)^n(1-4\alpha)\right)$$
$$=\underline{\tfrac{1}{4}\left(1-(1-4\alpha)^{n+1}\right)}$$

By sym. it has to hold that all P^{n+1}_{ij} are equal $i\neq j$

$\to P^{n+1}_{ij}=\frac{1}{3}(1-P^{n}_{ii})$

(alternative approach using diagonalization)

$$P = \begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix}$$

Since P is a real symmetric matrix, P can be diagnalized, i.e.

$P = Q\Lambda Q^T$ where Λ is $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ $\quad \Lambda \in \mathbb{R}^{4\times4}$

and Q is orthogonal. $\quad Q \in \mathbb{R}^{4\times4}$

Since P is irreducible, by Perron-frobenius theorem,
We know that only one eigenvalue of P is 1. $\Rightarrow \lambda_1 = 1$
Since the sum of the all eigenvalue of P is equal to its trace,

$$1 + \lambda_2 + \lambda_3 + \lambda_4 = 4 - 12\alpha \quad \Rightarrow \lambda_2 + \lambda_3 + \lambda_4 = 3 - 12\alpha$$

Since the converge rate P_{ij} for $i \neq j$ are the same, $\lambda_2 = \lambda_3 = \lambda_4 = 1 - 4\alpha$.

Eigenvalues and Eigenvectors of P:

$$\underset{\lambda^1}{1}\ \underset{V^1}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \qquad \underset{\lambda^2 \text{ with multiplicity } 3}{(1-4\alpha)}\ \underset{V^2}{\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}}, \underset{V^3}{\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}}, \underset{V^3}{\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}}$$

Since $\|V^1\| = \|V^2\| = \|V^3\| = \|V^4\| = \sqrt{4} = 2$,

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$P = Q\Lambda Q^T = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1-4\alpha & & \\ & & 1-4\alpha & \\ & & & 1-4\alpha \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$P^n = Q\Lambda^n Q^T = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & & & \\ & (1-4\alpha)^n & & \\ & & (1-4\alpha)^n & \\ & & & (1-4\alpha)^n \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & (1-4\alpha)^n & (1-4\alpha)^n & (1-4\alpha)^n \\ 1 & -(1-4\alpha)^n & (1-4\alpha)^n & -(1-4\alpha)^n \\ 1 & (1-4\alpha)^n & -(1-4\alpha)^n & (1-4\alpha)^n \\ 1 & -(1-4\alpha)^n & -(1-4\alpha)^n & (1-4\alpha)^n \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$P^n = \frac{1}{4}\begin{bmatrix} 1+3(1-4\alpha)^n & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n \\ 1-(1-4\alpha)^n & 1+3(1-4\alpha)^n & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n \\ 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 1+3(1-4\alpha)^n & 1-(1-4\alpha)^n \\ 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 1+3(1-4\alpha)^n \end{bmatrix}$$

Hence, $P^n_{ii} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$ for $\forall i = \{1, 2, 3, 4\}$

and $P^n_{ij} = \frac{1}{4} + \frac{1}{4}(1-4\alpha)^n$ for $i \neq j$ #

(b) What is the long-run proportion of time the chain is in each state?

From $0<\alpha<\frac{1}{3}$, it follows that $|1-4\alpha|<1$.

As $n\to\infty$, $(1-4\alpha)^n \to 0$.

Hence, $P_{ii}^n = \frac{1}{4}$ and $P_{ij}^n = \frac{1}{4}$ for $i\neq j$ and $i,j\in\{1,2,3,4\}$

and the limiting distribution is $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$.

2

→ chain conv. to unif distr

+ unif. row is its stat.

(either by using that this is implied by conv., or directly showing $P\begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix} = \begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix}$)

1

$\alpha \neq 0, 1$

Exercise 3. Consider the following 2-state discrete time Markov chain taking values $\{1, -1\}$ with $\pi_Z^0 = \pi_Z^*$ (invariant distribution) with transition probability matrix

$$\mathrm{P} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}$$

i) Show analytically that the (unnormalised) autocorrelation at lag τ, $\tau > 0$ defined as

$$\underline{\mathbb{E}[Z^n Z^{n+\tau}]}$$

$$P\pi_Z^* = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}\begin{bmatrix} \pi_{Z_1}^* \\ \pi_{Z_2}^* \end{bmatrix} = \begin{bmatrix} \alpha\pi_{Z_1}^* + (1-\alpha)\pi_{Z_2}^* \\ (1-\alpha)\pi_{Z_1}^* + \alpha\pi_{Z_2}^* \end{bmatrix} = \begin{bmatrix} \pi_{Z_1}^* \\ \pi_{Z_2}^* \end{bmatrix}$$

$\Rightarrow$ ... simple computation ...

$\Rightarrow \pi_{Z_1}^* = \pi_{Z_2}^* = 1/2$

claim: $P^n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{(2\alpha-1)^n}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Leftrightarrow C(n)$

Proof by induction: Show the claim for $n \geq 1$ with $C(n) \Rightarrow C(n+1)$

$$P^1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{(2\alpha-1)^1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \begin{pmatrix} \alpha-1/2 & -\alpha+1/2 \\ -\alpha+1/2 & \alpha-1/2 \end{pmatrix} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$$

$$P^{n+1} = P \cdot \underset{C(n)}{P^n} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{(2\alpha-1)^n}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} \alpha+(1-\alpha) & \alpha+(1-\alpha) \\ (1-\alpha)+\alpha & (1-\alpha)+\alpha \end{pmatrix} + \frac{(2\alpha-1)^n}{2}\begin{pmatrix} \alpha-(1-\alpha) & -\alpha+(1-\alpha) \\ (1-\alpha)-\alpha & -(1-\alpha)+\alpha \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{(2\alpha-1)^n}{2}\begin{pmatrix} 2\alpha-1 & -2\alpha+1 \\ -2\alpha+1 & 2\alpha-1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{(2\alpha-1)^{n+1}}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\mathbb{E}[Z^n Z^{n+\tau}] = 1 \cdot 1 \cdot \mathbb{P}(Z^n = 1, Z^{n+\tau} = 1) + 1 \cdot (-1) \cdot \mathbb{P}(Z^n = 1, Z^{n+\tau} = -1)$$
$$+ (-1) \cdot 1 \cdot \mathbb{P}(Z^n = -1, Z^{n+\tau} = 1) + (-1) \cdot (-1) \cdot \mathbb{P}(Z^n = -1, Z^{n+\tau} = -1)$$

Def. of Exp. value

$$= \mathbb{P}(Z^{n+\tau} = 1 \mid Z^n = 1)\,\mathbb{P}(Z^n = 1) - \mathbb{P}(Z^{n+\tau} = -1 \mid Z^n = 1)\,\mathbb{P}(Z^n = 1)$$
$$- \mathbb{P}(Z^{n+\tau} = 1 \mid Z^n = -1)\,\mathbb{P}(Z^n = -1) + \mathbb{P}(Z^{n+\tau} = -1 \mid Z^{n+\tau} = -1)\,\mathbb{P}(Z^n = -1)$$

$$= (P^\tau)_{22} \cdot \frac{1}{2} - (P^\tau)_{12} \cdot \frac{1}{2} - (P^\tau)_{21} \cdot \frac{1}{2} + (P^\tau)_{11} \cdot \frac{1}{2}$$

$$P = \begin{array}{cc} -1 & 1 \\ \begin{bmatrix} P_{11} & P_{21} \\ P_{21} & P_{22} \end{bmatrix} \end{array} \begin{array}{c} -1 \\ 1 \end{array}$$

p^n is in stat. distr. $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$$= \frac{1}{2}\left(\cancel{\frac{1}{2}} + \frac{(2\alpha-1)^\tau}{2} - \cancel{\frac{1}{2}} + \frac{(2\alpha-1)^\tau}{2} - \cancel{\frac{1}{2}} + \frac{(2\alpha-1)^\tau}{2} + \cancel{\frac{1}{2}} + \frac{(2\alpha-1)^\tau}{2}\right)$$

$$= \frac{1}{2}\,\frac{4(2\alpha-1)^\tau}{2}$$

$$= (2\alpha-1)^\tau \checkmark$$

ii) What happens to the autocorrelation function as the lag τ increases? Explain which property of the Markov chain leads to this behaviour.

$$\begin{aligned} Cov[Z^n, Z^{n+\tau}] &= \mathbb{E}[Z^n Z^{n+\tau}] - \mathbb{E}[Z^n]\mathbb{E}[Z^{n+\tau}] \\ &= \mathbb{E}[Z^n Z^{n+\tau}] \\ &= (2\alpha-1)^\tau \end{aligned}$$

* Case $0<\alpha<1$, then $(2\alpha-1)^\tau \to 0$ as $\alpha \to \infty$, which means that Z^n and $Z^{n+\tau}$ becomes uncorrelated.

The Markov chain with $0<\alpha<1$ is irreducible and aperiodic. $\Rightarrow$ ergodic

* Case $\alpha=0$, then the autocorrelation function oscillates between -1 and 1.

The Markov chain with $\alpha=0$, i.e. $P=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has one communicating class with periodicity 2. (Irreducible and Periodic)

* Case $\alpha=1$, then $(2\alpha-1)^\tau$ stays 1, which means that Z^n and $Z^{n+\tau}$ are completely positively correlated.

The Markov chain with $\alpha=1$, i.e. $P=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ consists of two self-transition communicating classes (reducible).

There are no unique stationary (invariant) distribution in this case.