Exercise Sheet 6 Group H 4.6.2020

Exercise 1- Univariate Gaussians (Hellinger distance)

Solution: by definition, the distance is given by:

$$d_{Hell}(\pi_1, \pi_2)^2 = \frac{1}{2}\int (\sqrt{\pi_1(x)} - \sqrt{\pi_2(x)})^2\, dx$$

$$= \frac{1}{2}\int (\pi_1(x) - 2\sqrt{\pi_1(x)\cdot\pi_2(x)} + \pi_2(x)\, dx$$

$$= \frac{1}{2}(\int \pi_1(x)\,dx - 2\int\sqrt{\pi_1(x)\,\pi_2(x)}\,dx + \int \pi_2(x)\,dx)$$

$$= \frac{1}{2}(2 - 2\int\sqrt{\pi_1(x)\,\pi_2(x)}\,dx) = 1 - \int\sqrt{\pi_1(x)\cdot\pi_2(x)}\,dx = 1 - Z_{(\pi_1,\pi_2)} \quad (1)$$

let's calculate the integral:

$$Z_{(\pi_1,\pi_2)} = \int\sqrt{\pi_1(x)\,\pi_2(x)}\,dx = \int\sqrt{\frac{1}{\delta_1\sqrt{2\pi}}\, e^{-\frac{x^2+\bar{x}_1^2-2x\bar{x}_1}{2\delta_1^2}} \cdot \frac{1}{\delta_2\sqrt{2\pi}}\cdot e^{-\frac{x^2+\bar{x}_2^2-2x\bar{x}_2}{2\delta_2^2}}}\,dx$$

$$= \int\sqrt{\frac{1}{\delta_1\sqrt{2\pi}}\cdot\frac{1}{\delta_2\sqrt{2\pi}}\cdot e^{-\frac{x^2\delta_2^2+\delta_2^2\bar{x}_1^2-2x\bar{x}_1\delta_2^2+x^2\delta_1^2+\bar{x}_2^2\delta_1^2-2x\bar{x}_2\delta_1^2}{2\delta_1^2\delta_2^2}}}\,dx$$

$$= \int\sqrt{\frac{1}{\sqrt{\delta_1^2 2\pi}}\cdot\frac{1}{\sqrt{\delta_2^2 2\pi}}\cdot \exp(-\frac{x^2(\delta_1^2+\delta_2^2)-2x(\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2)+(\delta_2^2\bar{x}_1^2+\delta_1^2\bar{x}_2^2)}{2\delta_1^2\delta_2^2})}\,dx$$

$$= \int\sqrt{\frac{1}{\sqrt{\delta_1^2 2\pi}}\cdot\frac{1}{\sqrt{\delta_2^2 2\pi}}\cdot \exp(-\frac{\delta_1^2+\delta_2^2}{2\delta_1^2\delta_2^2}\cdot[x^2-2x\frac{\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2}{\delta_1^2+\delta_2^2}+\frac{\bar{x}_1^2\delta_2^2+\bar{x}_2^2\delta_1^2}{\delta_1^2+\delta_2^2}])}\,dx$$

$$= \frac{1}{\sqrt{\delta_1\cdot\delta_2}}\cdot\frac{1}{\sqrt{2\pi}}\int \exp(-\frac{\delta_1^2+\delta_2^2}{4\delta_1^2\delta_2^2}[x^2-2x\frac{\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2}{\delta_1^2+\delta_2^2}+\frac{(\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2)^2}{\delta_1^2+\delta_2^2}-\frac{(\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2)^2}{\delta_1^2+\delta_2^2}+\frac{\bar{x}_1^2\delta_2^2+\bar{x}_2^2\delta_1^2}{\delta_1^2+\delta_2^2}])\,dx$$

$$= \frac{1}{\sqrt{\delta_1\delta_2}}\cdot\sqrt{\frac{2\delta_1^2\delta_2^2}{\delta_1^2+\delta_2^2}}\cdot\sqrt{\frac{\delta_1^2+\delta_2^2}{2\delta_1^2\cdot\delta_2^2}}\;\frac{1}{\sqrt{2\pi}}\int \exp(-\frac{1}{2}\cdot\frac{\delta_1^2+\delta_2^2}{2\delta_1^2\cdot\delta_2^2}[x-\frac{\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2}{\delta_1^2+\delta_2^2}]^2 - \frac{\delta_1^2+\delta_2^2}{4\delta_1^2\delta_2^2}[-\frac{\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2}{\delta_1^2+\delta_2^2} + \frac{\bar{x}_1^2\delta_2^2+\bar{x}_2^2\delta_1^2}{\delta_1^2+\delta_2^2}])\,dx$$

(2) Y

let's simplify (2)Y:

$$Y = -\frac{\delta_1^2+\delta_2^2}{4\delta_1^2\delta_2^2}[-\frac{\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2}{\delta_1^2+\delta_2^2}+\frac{\bar{x}_1^2\delta_2^2+\bar{x}_2^2\delta_1^2}{\delta_1^2+\delta_2^2}] = \frac{1}{4\delta_1^2\delta_2^2}[\frac{(\bar{x}_1\delta_2^2+\bar{x}_2\delta_1^2)^2-(\delta_1^2+\delta_2^2)(\bar{x}_1^2\delta_2^2+\bar{x}_2^2\delta_1^2)}{\delta_1^2+\delta_2^2}]$$

$$= \frac{1}{4\delta_1^2\delta_2^2}[\frac{\bar{x}_1^2\delta_2^4+\bar{x}_2^2\delta_1^4+2\bar{x}_1\delta_2^2\bar{x}_2\delta_1^2-\delta_1^2\bar{x}_1^2\delta_2^2-\delta_2^4\bar{x}_1^2-\delta_1^4\bar{x}_2^2-\delta_2^2\bar{x}_2^2\delta_1^2}{\delta_1^2+\delta_2^2}]$$

$$Y = \frac{1}{4\delta_1^2\delta_2^2}\left[\frac{2\bar{x}_1\delta_2^2\bar{x}_2\delta_1^2 - \delta_1^2\bar{x}_1^2\delta_2^2 - \delta_2^2\bar{x}_2^2\delta_1^2}{\delta_1^2+\delta_2^2}\right] = \frac{-1}{4\delta_1^2\delta_2^2}\left[\frac{(\delta_1\bar{x}_1\delta_2 - \delta_2\bar{x}_2\delta_1)^2}{\delta_1^2+\delta_2^2}\right]$$

$$= \frac{-1}{4\delta_1^2 \cdot \delta_2^2}\left[\frac{(\delta_1\delta_2)^2 \cdot (\bar{x}_1-\bar{x}_2)^2}{\delta_1^2+\delta_2^2}\right] = -\frac{1}{4}\left[\frac{(\bar{x}_1-\bar{x}_2)^2}{\delta_1^2+\delta_2^2}\right]$$

we plug that in Z and get:

$$Z_{(\pi_1,\pi_2)} = \frac{1}{\sqrt{\delta_1\delta_2}} \cdot \sqrt{\frac{2\delta_1^2 \cdot \delta_2^2}{(\delta_1^2+\delta_2^2)}} \cdot \sqrt{\frac{(\delta_1^2+\delta_2^2)}{2\delta_1^2 \cdot \delta_2^2}}\frac{1}{\sqrt{2\pi}} \cdot \int \exp\left(-\frac{1}{2} \cdot \frac{\delta_1^2+\delta_2^2}{2\delta_1^2\delta_2^2}\left[x - \frac{\bar{x}_1\delta_2^2+\bar{x}_2\cdot\delta_1^2}{\delta_1^2+\delta_2^2}\right]^2 - \frac{1}{4}\cdot\frac{(\bar{x}_1-\bar{x}_2)^2}{\delta_1^2+\delta_2^2}\right) \cdot dx$$

$$= \sqrt{\frac{2\delta_1 \cdot \delta_2}{\delta_1^2+\delta_2^2}} \cdot e^{-\frac{1}{4}\cdot\frac{(\bar{x}_1-\bar{x}_2)^2}{\delta_1^2+\delta_2^2}}$$

we plug that in ① and get:

$$d_{Hell}(\pi_1,\pi_2)^2 = 1 - Z_{(\pi_1,\pi_2)}$$

$$= 1 - \sqrt{\frac{2\delta_1\delta_2}{(\delta_1^2+\delta_2^2)}} \cdot e^{-\frac{1}{4}\cdot\frac{(\bar{x}_1-\bar{x}_2)^2}{\delta_1^2+\delta_2^2}}$$

$$= 1 - \sqrt{\frac{2\delta_1\delta_2}{\delta_1^2+\delta_2^2}} \cdot \left[e^{-\frac{1}{2}\cdot\frac{(\bar{x}_1-\bar{x}_2)^2}{\delta_1^2+\delta_2^2}}\right]^{\frac{1}{2}}$$

$$= 1 - \sqrt{\frac{2\delta_1\delta_2}{\delta_1^2+\delta_2^2}} \cdot \sqrt{e^{-\frac{(\bar{x}_1-\bar{x}_2)^2}{2(\delta_1^2+\delta_2^2)}}}$$

$$\Rightarrow \quad d_{Hell}(\pi_1,\pi_2)^2 = 1 - \sqrt{e^{-\frac{(\bar{x}_1-\bar{x}_2)^2}{2(\delta_1^2+\delta_2^2)}} \cdot \frac{2\delta_1\cdot\delta_2}{\delta_1^2+\delta_2^2}}$$

Exercise 2:

We have the forward model: $y = x + \varepsilon$, i.e: $h(x) = x$
where prior pdf: $\pi_0(x) \sim \text{Normal}(0, \sigma'^2)$ and
$\varepsilon \sim \text{Normal}(0,1)$

According to the Bayes' theorem:

$$\pi_X(x \mid y_{obs}) = \frac{\pi_Y(y_{obs} \mid x)\,\pi_0(x)}{\pi_Y(y_{obs})} \quad \text{where } y_{obs} = y_i$$

$$\Rightarrow \pi_i = \pi(x \mid y_i) = \frac{\pi_\varepsilon(y_i - x)\,\pi_0(x)}{\int_{\mathbb{R}} \pi_\varepsilon(y_i - x)\,\pi_0(x)\,dx}$$

$$\propto \exp\left(-\frac{1}{2}(y_i - x)^2\right) \exp\left(-\frac{x^2}{2\sigma'^2}\right)$$

$$= \exp\left(-\frac{1}{2}\left(y_i^2 - 2xy_i + x^2 + \frac{x^2}{\sigma'^2}\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left(x^2\left(\frac{\sigma'^2+1}{\sigma'^2}\right) - 2xy_i + y_i^2\right)\right)$$

$$= \exp\left(-\frac{1}{2 \cdot \frac{\sigma'^2}{\sigma'^2+1}}\left(x^2 - 2xy_i\frac{\sigma'^2}{\sigma'^2+1} + y_i^2 \cdot \frac{\sigma'^2}{\sigma'^2+1}\right)\right)$$

$$= \exp\left(-\frac{1}{2 \cdot \frac{\sigma'^2}{\sigma'^2+1}}\left(\left(x - \frac{\sigma'^2}{\sigma'^2+1}y_i\right)^2 - \frac{\sigma'^4}{(\sigma'^2+1)^2}y_i^2 + y_i^2 \cdot \frac{\sigma'^2}{\sigma'^2+1}\right)\right)$$

$$= \exp\left(-\frac{1}{2 \cdot \frac{\sigma'^2}{\sigma'^2+1}}\left(x - \frac{\sigma'^2}{\sigma'^2+1}y_i\right)^2\right) \underbrace{\exp\left(\frac{\sigma'^2}{2(\sigma'^2+1)}y_i^2 - \frac{1}{2}y_i^2\right)}_{\text{belongs to normalization term.}}$$

$$\propto \exp\left(-\frac{1}{2\,\frac{\sigma'^2}{\sigma'^2+1}}\left(x - \frac{\sigma'^2}{\sigma'^2+1}y_i^2\right)\right)$$

Hence the posterior is Gaussian with mean $\left(\frac{\sigma'^2}{\sigma'^2+1}y_i^2\right)$ and the variance $\left(\frac{\sigma'^2}{\sigma'^2+1}\right)$

Now we have two posterior Gaussian densities $N(\bar{x}_1, \sigma_1'^2)$ and $N(\bar{x}_2, \sigma_2'^2)$

where $\bar{x}_1 = \frac{\sigma'^2}{\sigma'^2+1} y_1$, $\bar{x}_2 = \frac{\sigma'^2}{\sigma'^2+1} y_2$

and $\sigma_1'^2 = \sigma_2'^2 = \frac{\sigma'^2}{\sigma'^2+1}$.

Applied the formula of Hellinger distance from Exercise 1 we obtain:

$$d^2_{Hell}(\pi_1, \pi_2) = 1 - \sqrt{\exp\left(-\frac{\left(\frac{\sigma'^2}{\sigma'^2+1} y_1 - \frac{\sigma'^2}{\sigma'^2+1} y_2\right)^2}{2 \cdot 2 \cdot \frac{\sigma'^2}{\sigma'^2+1}}\right) \cdot \frac{2\sigma_1'^2}{2\sigma_1'^2}}$$

$$= 1 - \sqrt{\exp -\left(\frac{\sigma'^2}{\sigma'^2+1}\right)^2 \frac{(y_1-y_2)^2}{4 \cdot \frac{\sigma'^2}{\sigma'^2+1}}}$$

$$= 1 - \sqrt{\exp\left(-\frac{1}{4}\frac{\sigma'^2}{\sigma'^2+1} \cdot (y_1-y_2)^2\right)}$$

$$= 1 - \exp\left(-\frac{1}{8} \cdot \frac{\sigma'^2}{\sigma'^2+1} (y_1-y_2)^2\right)$$

$$\leq \frac{1}{8} \cdot \frac{\sigma'^2}{\sigma'^2+1} (y_1-y_2)^2 \quad (\text{Using } 1-\exp(-x) \leq x \ \circledast)$$

$$\Rightarrow d_{Hell}(\pi_1, \pi_2) \leq \sqrt{\frac{1}{8}\frac{\sigma'^2}{\sigma'^2+1}|y_1-y_2|^2}$$

$$= \frac{\sigma'}{2\sqrt{2}(\sigma'^2+1)} |y_1-y_2|$$

Thus $c = \frac{\sigma'}{2\sqrt{2}(\sigma'^2+1)}$

Claim $\circledast$ $1-\exp(-x) \leq x \iff x - 1 + \exp(-x) \geq 0$.

Proof: We have $(x-1)$ convex and $\exp(-x)$ strictly convex (cause $g''(x) = \exp(-x) > 0$)

$\Rightarrow x - 1 + \exp(-x)$ strictly convex.

$\Rightarrow x - 1 + \exp(-x)$ has unique minimum.

We can take derivative wrt x:

$$\frac{\partial(x-1+\exp(-x))}{\partial x} = 1 - \exp(-x) \overset{!}{=} 0 \Rightarrow x^* = 0.$$

$$\Rightarrow x - 1 + \exp(-x) \geq x^* - 1 + \exp(-x^*) = 0 - 1 + 1 = 0 \qquad \square$$

Exercise 3:

Markov transition kernel $\pi(z'|z)$ with two PDF π_1 & π_2

$$\hat{\pi}_1(z') = \int_{\mathbb{R}} \pi(z'|z)\, \pi_1(z)\, dz$$
$$\hat{\pi}_2(z') = \int_{\mathbb{R}} \pi(z'|z)\, \pi_2(z)\, dz$$

Show that $d_{TV}(\hat{\pi}_1, \hat{\pi}_2) \leq d_{TV}(\pi_1, \pi_2)$

$$d_{TV}(\pi_1, \pi_2) = \frac{1}{2} \sup_{\|z\|_\infty \leq 1} |\pi_2(z) - \pi_1(z)|$$
$$= \frac{1}{2} \int_{\mathbb{R}} |\pi_2(z) - \pi_1(z)|\, dz$$

$$d_{TV}(\hat{\pi}_1, \hat{\pi}_2) = \frac{1}{2} \int_{\mathbb{R}} |\hat{\pi}_2(z) - \hat{\pi}_1(z)|\, dz$$
$$= \frac{1}{2} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \pi(z'|z)\, \pi_2(z)\, dz - \int_{\mathbb{R}} \pi(z'|z\; \pi_2(z) dz \right|$$
$$\leq \frac{1}{2} \int_{\mathbb{R}}\int_{\mathbb{R}} \left| \pi(z'|z)\, \pi_2(z) - \pi(z'|z)\, \pi_2(z) \right| dz\, dz$$
$$= \frac{1}{2} \int_{\mathbb{R}}\int_{\mathbb{R}} \left| \pi(z'|z)(\pi_2(z) - \pi_1(z)) \right| dz\, dz$$
$$\leq \underbrace{\int_{\mathbb{R}} \left| \pi(z'|z)\, dz \right|}_{=1} \cdot \underbrace{\frac{1}{2} \int_{\mathbb{R}} |\pi_2(z) - \pi_1(z)|\, dz}_{= d_{TV}(\pi_1, \pi_2)}$$

$$\Rightarrow d_{TV}(\hat{\pi}_1, \hat{\pi}_2) \leq d_{TV}(\pi_1, \pi_2)$$