set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with. QED

The main result of this subsection, that all of the bases in a finite-dimensional vector space have the same number of elements, is the single most important result in this book. As Example 2.12 shows, it describes what vector spaces and subspaces there can be.

One immediate consequence brings us back to when we considered the two things that could be meant by the term 'minimal spanning set'. At that point we defined 'minimal' as linearly independent but we noted that another reasonable interpretation of the term is that a spanning set is 'minimal' when it has the fewest number of elements of any set with the same span. Now that we have shown that all bases have the same number of elements, we know that the two senses of 'minimal' are equivalent.

Exercises

Assume that all spaces are finite-dimensional unless otherwise stated.

✓ 2.16 Find a basis for, and the dimension of, $\mathcal{P}_2$.

2.17 Find a basis for, and the dimension of, the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

✓ 2.18 Find a basis for, and the dimension of, each space.

(a) $\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x - w + z = 0\}$

(b) the set of 5×5 matrices whose only nonzero entries are on the diagonal (e.g., in entry $1,1$ and $2,2$, etc.)

(c) $\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + a_1 = 0 \text{ and } a_2 - 2a_3 = 0\} \subseteq \mathcal{P}_3$

2.19 Find a basis for, and the dimension of, $\mathcal{M}_{2\times 2}$, the vector space of 2×2 matrices.

2.20 Find the dimension of the vector space of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

subject to each condition.

(a) $a, b, c, d \in \mathbb{R}$

(b) $a - b + 2c = 0$ and $d \in \mathbb{R}$

(c) $a + b + c = 0$, $a + b - c = 0$, and $d \in \mathbb{R}$

✓ 2.21 Find the dimension of this subspace of $\mathbb{R}^2$.

$$S = \{\begin{pmatrix} a + b \\ a + c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

✓ 2.22 Find the dimension of each.

(a) The space of cubic polynomials $p(x)$ such that $p(7) = 0$

(b) The space of cubic polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$

(d) The space of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$

1.30 We've seen that the result of reordering a basis can be another basis. Must it be?

1.31 Can a basis contain a zero vector?

✓ 1.32 Let $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ be a basis for a vector space.

(a) Show that $\langle c_1\vec{\beta}_1, c_2\vec{\beta}_2, c_3\vec{\beta}_3 \rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. What happens when at least one c_i is 0?

(b) Prove that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle$ is a basis where $\vec{\alpha}_i = \vec{\beta}_1 + \vec{\beta}_i$.

1.33 Find one vector $\vec{v}$ that will make each into a basis for the space.

(a) $\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v} \rangle$ in $\mathbb{R}^2$ (b) $\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v} \rangle$ in $\mathbb{R}^3$ (c) $\langle x, 1 + x^2, \vec{v} \rangle$ in $\mathcal{P}_2$

✓ 1.34 Where $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis, show that in this equation

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

each of the c_i's is zero. Generalize.

1.35 A basis contains some of the vectors from a vector space; can it contain them all?

1.36 Theorem 1.12 shows that, with respect to a basis, every linear combination is unique. If a subset is not a basis, can linear combinations be not unique? If so, must they be?

✓ 1.37 A square matrix is *symmetric* if for all indices i and j, entry i, j equals entry j, i.

(a) Find a basis for the vector space of symmetric 2×2 matrices.

(b) Find a basis for the space of symmetric 3×3 matrices.

(c) Find a basis for the space of symmetric $n\times n$ matrices.

1.38 We can show that every basis for $\mathbb{R}^3$ contains the same number of vectors.

(a) Show that no linearly independent subset of $\mathbb{R}^3$ contains more than three vectors.

(b) Show that no spanning subset of $\mathbb{R}^3$ contains fewer than three vectors. *Hint:* recall how to calculate the span of a set and show that this method cannot yield all of $\mathbb{R}^3$ when we apply it to fewer than three vectors.

1.39 One of the exercises in the Subspaces subsection shows that the set

$$\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \}$$

is a vector space under these operations.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx - r + 1 \\ ry \\ rz \end{pmatrix}$$

Find a basis.

(a) $\langle\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\rangle$ (b) $\langle\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}\rangle$ (c) $\langle\begin{pmatrix}0\\2\\-1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\5\\0\end{pmatrix}\rangle$

(d) $\langle\begin{pmatrix}0\\2\\-1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\3\\0\end{pmatrix}\rangle$

✓ 1.20 Represent the vector with respect to the basis.

(a) $\begin{pmatrix}1\\2\end{pmatrix}$, $B=\langle\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\rangle\subseteq\mathbb{R}^2$

(b) x^2+x^3, $D=\langle 1,1+x,1+x+x^2,1+x+x^2+x^3\rangle\subseteq\mathcal{P}_3$

(c) $\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$, $\mathcal{E}_4\subseteq\mathbb{R}^4$

1.21 Represent the vector with respect to each of the two bases.

$$\vec{v}=\begin{pmatrix}3\\-1\end{pmatrix}\quad B_1=\langle\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\rangle,\ B_2=\langle\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\rangle$$

1.22 Find a basis for $\mathcal{P}_2$, the space of all quadratic polynomials. Must any such basis contain a polynomial of each degree: degree zero, degree one, and degree two?

1.23 Find a basis for the solution set of this system.

$$\begin{aligned}x_1-4x_2+3x_3-\ x_4&=0\\2x_1-8x_2+6x_3-2x_4&=0\end{aligned}$$

✓ 1.24 Find a basis for $\mathcal{M}_{2\times2}$, the space of 2×2 matrices.

✓ 1.25 Find a basis for each.

(a) The subspace $\{\,a_2x^2+a_1x+a_0\mid a_2-2a_1=a_0\,\}$ of $\mathcal{P}_2$

(b) The space of three-wide row vectors whose first and second components add to zero

(c) This subspace of the 2×2 matrices

$$\{\begin{pmatrix}a&b\\0&c\end{pmatrix}\mid c-2b=0\}$$

1.26 Find a basis for each space, and verify that it is a basis.

(a) The subspace $M=\{a+bx+cx^2+dx^3\mid a-2b+c-d=0\}$ of $\mathcal{P}_3$.

(b) This subspace of $\mathcal{M}_{2\times2}$.

$$W=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\mid a-c=0\}$$

1.27 Check Example 1.6.

✓ 1.28 Find the span of each set and then find a basis for that span.

(a) $\{1+x,1+2x\}$ in $\mathcal{P}_2$ (b) $\{2-2x,3+4x^2\}$ in $\mathcal{P}_2$

✓ 1.29 Find a basis for each of these subspaces of the space $\mathcal{P}_3$ of cubic polynomials.

(a) The subspace of cubic polynomials $p(x)$ such that $p(7)=0$

(b) The subspace of polynomials $p(x)$ such that $p(7)=0$ and $p(5)=0$

(c) The subspace of polynomials $p(x)$ such that $p(7)=0$, $p(5)=0$, and $p(3)=0$

1.15 Remark Definition 1.1 requires that a basis be a sequence because without that we couldn't write these coordinates in a fixed order.

1.16 Example In $\mathbb{R}^2$, where $\vec{v} = \binom{3}{2}$, to find the coordinates of that vector with respect to the basis

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

we solve

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

and get that $c_1 = 3$ and $c_2 = -1/2$.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ -1/2 \end{pmatrix}$$

1.17 Remark This use of column notation and the term 'coordinate' has both a disadvantage and an advantage. The disadvantage is that representations look like vectors from $\mathbb{R}^n$, which can be confusing when the vector space is $\mathbb{R}^n$, as in the prior example. We must infer the intent from the context. For example, the phrase 'in $\mathbb{R}^2$, where $\vec{v} = \binom{3}{2}$' refers to the plane vector that, when in canonical position, ends at $(3, 2)$. And in the end of that example, although we've omitted a subscript B from the column, that the right side is a representation is clear from the context.

The advantage of the notation and the term is that they generalize the familiar case: in $\mathbb{R}^n$ and with respect to the standard basis $\mathcal{E}_n$, the vector starting at the origin and ending at $(v_1, \ldots, v_n)$ has this representation.

$$\mathrm{Rep}_{\mathcal{E}_n}(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{\mathcal{E}_n}$$

Our main use of representations will come later but the definition appears here because the fact that every vector is a linear combination of basis vectors in a unique way is a crucial property of bases, and also to help make a point. For calculation of coordinates among other things, we shall restrict our attention to spaces with bases having only finitely many elements. That will start in the next subsection.

Exercises

✓ 1.18 Decide if each is a basis for $\mathcal{P}_2$.
 (a) $\langle x^2 - x + 1, 2x + 1, 2x - 1 \rangle$ (b) $\langle x + x^2, x - x^2 \rangle$

✓ 1.19 Decide if each is a basis for $\mathbb{R}^3$.

1.30 In any vector space V, the empty set is linearly independent. What about all of V?

1.31 Show that if $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent then so are all of its proper subsets: $\{\vec{x}, \vec{y}\}$, $\{\vec{x}, \vec{z}\}$, $\{\vec{y}, \vec{z}\}$, $\{\vec{x}\}$,$\{\vec{y}\}$, $\{\vec{z}\}$, and $\{\}$. Is that 'only if' also?

1.32 (a) Show that this

$$S = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \}$$

is a linearly independent subset of $\mathbb{R}^3$.

(b) Show that

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

is in the span of S by finding c_1 and c_2 giving a linear relationship.

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Show that the pair c_1, c_2 is unique.

(c) Assume that S is a subset of a vector space and that $\vec{v}$ is in $[S]$, so that $\vec{v}$ is a linear combination of vectors from S. Prove that if S is linearly independent then a linear combination of vectors from S adding to $\vec{v}$ is unique (that is, unique up to reordering and adding or taking away terms of the form $0 \cdot \vec{s}$). Thus S as a spanning set is minimal in this strong sense: each vector in $[S]$ is a combination of elements of S a minimum number of times—only once.

(d) Prove that it can happen when S is not linearly independent that distinct linear combinations sum to the same vector.

1.33 Prove that a polynomial gives rise to the zero function if and only if it is the zero polynomial. (*Comment.* This question is not a Linear Algebra matter but we often use the result. A polynomial gives rise to a function in the natural way: $x \mapsto c_n x^n + \cdots + c_1 x + c_0$.)

1.34 Return to Section 1.2 and redefine point, line, plane, and other linear surfaces to avoid degenerate cases.

1.35 (a) Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

(b) Is this true for any set of five? Any set of three?

(c) What is the most number of elements that a linearly independent subset of $\mathbb{R}^2$ can have?

1.36 Is there a set of four vectors in $\mathbb{R}^3$ such that any three form a linearly independent set?

1.37 Must every linearly dependent set have a subset that is dependent and a subset that is independent?

1.38 In $\mathbb{R}^4$ what is the biggest linearly independent set you can find? The smallest? The biggest linearly dependent set? The smallest? ('Biggest' and 'smallest' mean that there are no supersets or subsets with the same property.)

(a) $\{\begin{pmatrix}1\\-3\\5\end{pmatrix},\begin{pmatrix}2\\2\\4\end{pmatrix},\begin{pmatrix}4\\-4\\14\end{pmatrix}\}$ (b) $\{\begin{pmatrix}1\\7\\7\end{pmatrix},\begin{pmatrix}2\\7\\7\end{pmatrix},\begin{pmatrix}3\\7\\7\end{pmatrix}\}$ (c) $\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\0\\4\end{pmatrix}\}$

(d) $\{\begin{pmatrix}9\\9\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}3\\5\\-4\end{pmatrix},\begin{pmatrix}12\\12\\-1\end{pmatrix}\}$

✓ 1.21 Which of these subsets of $\mathcal{P}_3$ are linearly dependent and which are independent?

(a) $\{3-x+9x^2, 5-6x+3x^2, 1+1x-5x^2\}$
(b) $\{-x^2, 1+4x^2\}$
(c) $\{2+x+7x^2, 3-x+2x^2, 4-3x^2\}$
(d) $\{8+3x+3x^2, x+2x^2, 2+2x+2x^2, 8-2x+5x^2\}$

1.22 Determine if each set is linearly independent in the natural space.

(a) $\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}-1\\1\\0\end{pmatrix}\}$ (b) $\{(1\ \ 3\ \ 1),(-1\ \ 4\ \ 3),(-1\ \ 11\ \ 7)\}$

(c) $\{\begin{pmatrix}5&4\\1&2\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\-1&4\end{pmatrix}\}$

✓ 1.23 Prove that each set $\{f,g\}$ is linearly independent in the vector space of all functions from $\mathbb{R}^+$ to $\mathbb{R}$.

(a) $f(x)=x$ and $g(x)=1/x$
(b) $f(x)=\cos(x)$ and $g(x)=\sin(x)$
(c) $f(x)=e^x$ and $g(x)=\ln(x)$

✓ 1.24 Which of these subsets of the space of real-valued functions of one real variable is linearly dependent and which is linearly independent? (We have abbreviated some constant functions; e.g., in the first item, the '2' stands for the constant function $f(x)=2$.)

(a) $\{2, 4\sin^2(x), \cos^2(x)\}$ (b) $\{1, \sin(x), \sin(2x)\}$ (c) $\{x, \cos(x)\}$
(d) $\{(1+x)^2, x^2+2x, 3\}$ (e) $\{\cos(2x), \sin^2(x), \cos^2(x)\}$ (f) $\{0, x, x^2\}$

1.25 Does the equation $\sin^2(x)/\cos^2(x)=\tan^2(x)$ show that this set of functions $\{\sin^2(x), \cos^2(x), \tan^2(x)\}$ is a linearly dependent subset of the set of all real-valued functions with domain the interval $(-\pi/2..\pi/2)$ of real numbers between $-\pi/2$ and $\pi/2)$?

1.26 Is the xy-plane subset of the vector space $\mathbb{R}^3$ linearly independent?

✓ 1.27 Show that the nonzero rows of an echelon form matrix form a linearly independent set.

1.28 (a) Show that if the set $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent then so is the set $\{\vec{u},\vec{u}+\vec{v},\vec{u}+\vec{v}+\vec{w}\}$.

(b) What is the relationship between the linear independence or dependence of $\{\vec{u},\vec{v},\vec{w}\}$ and the independence or dependence of $\{\vec{u}-\vec{v},\vec{v}-\vec{w},\vec{w}-\vec{u}\}$?

1.29 Example 1.10 shows that the empty set is linearly independent.

(a) When is a one-element set linearly independent?
(b) How about a set with two elements?

1.18 Corollary A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s}_i$ is a linear combination of the vectors $\vec{s}_1, \ldots, \vec{s}_{i-1}$ listed before it.

Proof Consider $S_0 = \{\}$, $S_1 = \{\vec{s}_1\}$, $S_2 = \{\vec{s}_1, \vec{s}_2\}$, etc. Some index $i \geqslant 1$ is the first one with $S_{i-1} \cup \{\vec{s}_i\}$ linearly dependent, and there $\vec{s}_i \in [S_{i-1}]$. QED

The proof of Corollary 1.16 describes producing a linearly independent set by shrinking, by taking subsets. And the proof of Corollary 1.18 describes finding a linearly dependent set by taking supersets. We finish this subsection by considering how linear independence and dependence interact with the subset relation between sets.

1.19 Lemma Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

Proof Both are clear. QED

Restated, subset preserves independence and superset preserves dependence.

Those are two of the four possible cases. The third case, whether subset preserves linear dependence, is covered by Example 1.17, which gives a linearly dependent set S with one subset that is linearly dependent and another that is independent. The fourth case, whether superset preserves linear independence, is covered by Example 1.15, which gives cases where a linearly independent set has both an independent and a dependent superset. This table summarizes.

	$\hat{S} \subset S$	$\hat{S} \supset S$
S *independent*	$\hat{S}$ must be independent	$\hat{S}$ may be either
S *dependent*	$\hat{S}$ may be either	$\hat{S}$ must be dependent

Example 1.15 has something else to say about the interaction between linear independence and superset. It names a linearly independent set that is maximal in that it has no supersets that are linearly independent. By Lemma 1.14 a linearly independent set is maximal if and only if it spans the entire space, because that is when all the vectors in the space are already in the span. This nicely complements Lemma 1.13, that a spanning set is minimal if and only if it is linearly independent.

Exercises

✓ 1.20 Decide whether each subset of $\mathbb{R}^3$ is linearly dependent or linearly independent.

(a) $\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\3\end{pmatrix}\}$ (b) $\{\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}$ (c) $\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}3\\0\\0\end{pmatrix}\}$
(d) $\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\5\end{pmatrix}\}$ (e) $\{\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}3\\0\\1\end{pmatrix},\begin{pmatrix}5\\1\\2\end{pmatrix},\begin{pmatrix}6\\0\\2\end{pmatrix}\}$

✓ 2.26 Parametrize each subspace's description. Then express each subspace as a span.

(a) The subset $\{(a \;\; b \;\; c) \mid a-c=0\}$ of the three-wide row vectors

(b) This subset of $\mathcal{M}_{2\times 2}$
$$\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid a+d=0\}$$

(c) This subset of $\mathcal{M}_{2\times 2}$
$$\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid 2a-c-d=0 \text{ and } a+3b=0\}$$

(d) The subset $\{a+bx+cx^3 \mid a-2b+c=0\}$ of $\mathcal{P}_3$

(e) The subset of $\mathcal{P}_2$ of quadratic polynomials p such that $p(7)=0$

✓ 2.27 Find a set to span the given subspace of the given space. (*Hint.* Parametrize each.)

(a) the xz-plane in $\mathbb{R}^3$

(b) $\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid 3x+2y+z=0\}$ in $\mathbb{R}^3$

(c) $\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \mid 2x+y+w=0 \text{ and } y+2z=0\}$ in $\mathbb{R}^4$

(d) $\{a_0+a_1x+a_2x^2+a_3x^3 \mid a_0+a_1=0 \text{ and } a_2-a_3=0\}$ in $\mathcal{P}_3$

(e) The set $\mathcal{P}_4$ in the space $\mathcal{P}_4$

(f) $\mathcal{M}_{2\times 2}$ in $\mathcal{M}_{2\times 2}$

2.28 Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

✓ 2.29 Decide if each is a subspace of the vector space of real-valued functions of one real variable.

(a) The *even* functions $\{f\colon \mathbb{R}\to\mathbb{R} \mid f(-x)=f(x) \text{ for all } x\}$. For example, two members of this set are $f_1(x)=x^2$ and $f_2(x)=\cos(x)$.

(b) The *odd* functions $\{f\colon \mathbb{R}\to\mathbb{R} \mid f(-x)=-f(x) \text{ for all } x\}$. Two members are $f_3(x)=x^3$ and $f_4(x)=\sin(x)$.

2.30 Example 2.16 says that for any vector $\vec{v}$ that is an element of a vector space V, the set $\{r\cdot\vec{v} \mid r\in\mathbb{R}\}$ is a subspace of V. (This is simply the span of the singleton set $\{\vec{v}\}$.) Must any such subspace be a proper subspace?

2.31 An example following the definition of a vector space shows that the solution set of a homogeneous linear system is a vector space. In the terminology of this subsection, it is a subspace of $\mathbb{R}^n$ where the system has n variables. What about a non-homogeneous linear system; do its solutions form a subspace (under the inherited operations)?

So far in this chapter we have seen that to study the properties of linear combinations, the right setting is a collection that is closed under these combinations. In the first subsection we introduced such collections, vector spaces, and we saw a great variety of examples. In this subsection we saw still more spaces, ones that are subspaces of others. In all of the variety there is a commonality. Example 2.19 above brings it out: vector spaces and subspaces are best understood as a span, and especially as a span of a small number of vectors. The next section studies spanning sets that are minimal.

Exercises

✓ 2.20 Which of these subsets of the vector space of 2×2 matrices are subspaces under the inherited operations? For each one that is a subspace, parametrize its description. For each that is not, give a condition that fails.

(a) $\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}\}$

(b) $\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a + b = 0\}$

(c) $\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a + b = 5\}$

(d) $\{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a + b = 0, c \in \mathbb{R}\}$

✓ 2.21 Is this a subspace of $\mathcal{P}_2$: $\{a_0 + a_1x + a_2x^2 \mid a_0 + 2a_1 + a_2 = 4\}$? If it is then parametrize its description.

2.22 Is the vector in the span of the set?

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$$

✓ 2.23 Decide if the vector lies in the span of the set, inside of the space.

(a) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$, in $\mathbb{R}^3$

(b) $x - x^3$, $\{x^2, 2x + x^2, x + x^3\}$, in $\mathcal{P}_3$

(c) $\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$, $\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}\}$, in $\mathcal{M}_{2\times 2}$

2.24 Which of these are members of the span $[\{\cos^2 x, \sin^2 x\}]$ in the vector space of real-valued functions of one real variable?

(a) $f(x) = 1$ (b) $f(x) = 3 + x^2$ (c) $f(x) = \sin x$ (d) $f(x) = \cos(2x)$

✓ 2.25 Which of these sets spans $\mathbb{R}^3$? That is, which of these sets has the property that any three-tall vector can be expressed as a suitable linear combination of the set's elements?

(e) Under the inherited operations,

$$\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2 \mid x+3y=4 \text{ and } 2x-y=3 \text{ and } 6x+4y=10\}$$

1.23 Define addition and scalar multiplication operations to make the complex numbers a vector space over $\mathbb{R}$.

1.24 Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

1.25 Show that the set of linear combinations of the variables x, y, z is a vector space under the natural addition and scalar multiplication operations.

1.26 Prove that this is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix} \qquad r\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}rx\\ry\end{pmatrix}$$

1.27 Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

(a) $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx\\ry\\rz\end{pmatrix}$

(b) $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

✓ 1.28 For each, decide if it is a vector space; the intended operations are the natural ones.

(a) The *diagonal* 2×2 matrices

$$\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \mid a,b\in\mathbb{R}\}$$

(b) This set of 2×2 matrices

$$\{\begin{pmatrix}x&x+y\\x+y&y\end{pmatrix} \mid x,y\in\mathbb{R}\}$$

(c) This set

$$\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\in\mathbb{R}^4 \mid x+y+w=1\}$$

(d) The set of functions $\{f\colon\mathbb{R}\to\mathbb{R} \mid df/dx+2f=0\}$

(e) The set of functions $\{f\colon\mathbb{R}\to\mathbb{R} \mid df/dx+2f=1\}$

✓ 1.29 Prove or disprove that this is a vector space: the real-valued functions f of one real variable such that $f(7)=0$.

✓ 1.30 Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x+y$' to mean the product of x and y (so that $2+3$ is 6), and we interpret '$r\cdot x$' as the r-th power of x.

1.31 Is $\{(x,y) \mid x,y\in\mathbb{R}\}$ a vector space under these operations?

(a) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$ and $r\cdot(x,y)=(rx,y)$

(b) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$ and $r\cdot(x,y)=(rx,0)$

(b) The space of 2×4 matrices.

(c) The space $\{f \colon [0..1] \to \mathbb{R} \mid f \text{ is continuous}\}$.

(d) The space of real-valued functions of one natural number variable.

✓ 1.18 Find the additive inverse, in the vector space, of the vector.

(a) In $\mathcal{P}_3$, the vector $-3 - 2x + x^2$.

(b) In the space 2×2,

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

(c) In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

✓ 1.19 For each, list three elements and then show it is a vector space.

(a) The set of linear polynomials $\mathcal{P}_1 = \{a_0 + a_1x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.

(b) The set of linear polynomials $\{a_0 + a_1x \mid a_0 - 2a_1 = 0\}$, under the usual polynomial addition and scalar multiplication operations.

Hint. Use Example 1.3 as a guide. Most of the ten conditions are just verifications.

1.20 For each, list three elements and then show it is a vector space.

(a) The set of 2×2 matrices with real entries under the usual matrix operations.

(b) The set of 2×2 matrices with real entries where the $2, 1$ entry is zero, under the usual matrix operations.

✓ 1.21 For each, list three elements and then show it is a vector space.

(a) The set of three-component row vectors with their usual operations.

(b) The set

$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x + y - z + w = 0\}$$

under the operations inherited from $\mathbb{R}^4$.

✓ 1.22 Show that each of these is not a vector space. (*Hint.* Check closure by listing two members of each set and trying some operations on them.)

(a) Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y + z = 1\}$$

(b) Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

(c) Under the usual matrix operations,

$$\{\begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

(d) Under the usual polynomial operations,

$$\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

Another answer is perhaps more satisfying. People in this area have worked to develop the right balance of power and generality. This definition is shaped so that it contains the conditions needed to prove all of the interesting and important properties of spaces of linear combinations. As we proceed, we shall derive all of the properties natural to collections of linear combinations from the conditions given in the definition.

The next result is an example. We do not need to include these properties in the definition of vector space because they follow from the properties already listed there.

1.16 Lemma In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof For (1) note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1+1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$. For (3), $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do. QED

The second item shows that we can write the additive inverse of $\vec{v}$ as '$-\vec{v}$' without worrying about any confusion with $(-1) \cdot \vec{v}$.

A recap: our study in Chapter One of Gaussian reduction led us to consider collections of linear combinations. So in this chapter we have defined a vector space to be a structure in which we can form such combinations, subject to simple conditions on the addition and scalar multiplication operations. In a phrase: vector spaces are the right context in which to study linearity.

From the fact that it forms a whole chapter, and especially because that chapter is the first one, a reader could suppose that our purpose in this book is the study of linear systems. The truth is that we will not so much use vector spaces in the study of linear systems as we instead have linear systems start us on the study of vector spaces. The wide variety of examples from this subsection shows that the study of vector spaces is interesting and important in its own right. Linear systems won't go away. But from now on our primary objects of study will be vector spaces.

Exercises

1.17 Name the zero vector for each of these vector spaces.
  (a) The space of degree three polynomials under the natural operations.

(a) $\begin{pmatrix}1&3\\2&4\end{pmatrix}$ (b) $\begin{pmatrix}1&5\\2&10\end{pmatrix}$ (c) $\begin{pmatrix}1&-1\\3&0\end{pmatrix}$ (d) $\begin{pmatrix}2&6\\4&10\end{pmatrix}$
(e) $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (f) $\begin{pmatrix}3&3\\2&2\end{pmatrix}$

2.13 Produce three other matrices row equivalent to the given one.

(a) $\begin{pmatrix}1&3\\4&-1\end{pmatrix}$ (b) $\begin{pmatrix}0&1&2\\1&1&1\\2&3&4\end{pmatrix}$

✓ 2.14 Perform Gauss's Method on this matrix. Express each row of the final matrix as a linear combination of the rows of the starting matrix.

$$\begin{pmatrix}1&2&1\\3&-1&0\\0&4&0\end{pmatrix}$$

2.15 Describe the matrices in each of the classes represented in Example 2.10.

2.16 Describe all matrices in the row equivalence class of these.

(a) $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ (b) $\begin{pmatrix}1&2\\2&4\end{pmatrix}$ (c) $\begin{pmatrix}1&1\\1&3\end{pmatrix}$

2.17 How many row equivalence classes are there?

2.18 Can row equivalence classes contain different-sized matrices?

2.19 How big are the row equivalence classes?
(a) Show that for any matrix of all zeros, the class is finite.
(b) Do any other classes contain only finitely many members?

✓ 2.20 Give two reduced echelon form matrices that have their leading entries in the same columns, but that are not row equivalent.

✓ 2.21 Show that any two $n \times n$ nonsingular matrices are row equivalent. Are any two singular matrices row equivalent?

✓ 2.22 Describe all of the row equivalence classes containing these.
(a) 2×2 matrices (b) 2×3 matrices (c) 3×2 matrices
(d) 3×3 matrices

2.23 (a) Show that a vector $\vec{\beta}_0$ is a linear combination of members of the set $\{\vec{\beta}_1, \ldots, \vec{\beta}_n\}$ if and only if there is a linear relationship $\vec{0} = c_0\vec{\beta}_0 + \cdots + c_n\vec{\beta}_n$ where c_0 is not zero. (*Hint.* Watch out for the $\vec{\beta}_0 = \vec{0}$ case.)
(b) Use that to simplify the proof of Lemma 2.5.

✓ 2.24 [Trono] Three truck drivers went into a roadside cafe. One truck driver purchased four sandwiches, a cup of coffee, and ten doughnuts for \$8.45. Another driver purchased three sandwiches, a cup of coffee, and seven doughnuts for \$6.30. What did the third truck driver pay for a sandwich, a cup of coffee, and a doughnut?

2.25 The Linear Combination Lemma says which equations can be gotten from Gaussian reduction of a given linear system.

(1) Produce an equation not implied by this system.

$$\begin{aligned}3x + 4y &= 8\\2x + y &= 3\end{aligned}$$

(2) Can any equation be derived from an inconsistent system?

because their reduced echelon forms are not equal.

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.9 Example Any nonsingular 3×3 matrix Gauss-Jordan reduces to this.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.10 Example We can describe all the classes by listing all possible reduced echelon form matrices. Any 2×2 matrix lies in one of these: the class of matrices row equivalent to this,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the infinitely many classes of matrices row equivalent to one of this type

$$\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$$

where $a \in \mathbb{R}$ (including $a = 0$), the class of matrices row equivalent to this,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the class of matrices row equivalent to this

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(this is the class of nonsingular 2×2 matrices).

Exercises

✓ 2.11 Decide if the matrices are row equivalent.

(a) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$

2.12 Which of these matrices are row equivalent to each other?

1.13 Give two distinct echelon form versions of this matrix.

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 6 & 4 & 1 & 2 \\ 1 & 5 & 1 & 5 \end{pmatrix}$$

✓ 1.14 List the reduced echelon forms possible for each size.
(a) 2×2 (b) 2×3 (c) 3×2 (d) 3×3

✓ 1.15 What results from applying Gauss-Jordan reduction to a nonsingular matrix?

1.16 Decide whether each relation is an equivalence on the set of 2×2 matrices.
(a) two matrices are related if they have the same entry in the first row and first column
(b) two matrices are related if their four entries sum to the same total
(c) two matrices are related if they have the same entry in the first row and first column, or the same entry in the second row and second column

1.17 [Cleary] Consider the following relationship on the set of 2×2 matrices: we say that A is *sum-what like* B if the sum of all of the entries in A is the same as the sum of all the entries in B. For instance, the zero matrix would be sum-what like the matrix whose first row had two sevens, and whose second row had two negative sevens. Prove or disprove that this is an equivalence relation on the set of 2×2 matrices.

1.18 The proof of Lemma 1.5 contains a reference to the $i\neq j$ condition on the row combination operation.
(a) Write down a 2×2 matrix with nonzero entries, and show that the $-1\cdot\rho_1+\rho_1$ operation is not reversed by $1\cdot\rho_1+\rho_1$.
(b) Expand the proof of that lemma to make explicit exactly where it uses the $i\neq j$ condition on combining.

1.19 [Cleary] Consider the set of students in a class. Which of the following relationships are equivalence relations? Explain each answer in at least a sentence.
(a) Two students x, y are related if x has taken at least as many math classes as y.
(b) Students x, y are related if they have names that start with the same letter.

1.20 Show that each of these is an equivalence on the set of 2×2 matrices. Describe the equivalence classes.
(a) Two matrices are related if they have the same product down the diagonal, that is, if the product of the entries in the upper left and lower right are equal.
(b) Two matrices are related if they both have at least one entry that is a 1, or if neither does.

1.21 Show that each is not an equivalence on the set of 2×2 matrices.
(a) Two matrices A, B are related if $a_{1,1}=-b_{1,1}$.
(b) Two matrices are related if the sum of their entries are within 5, that is, A is related to B if $|(a_{1,1}+\cdots+a_{2,2})-(b_{1,1}+\cdots+b_{2,2})|<5$.

One of the classes is the cluster of interrelated matrices from the start of this section sketched above (it includes all of the nonsingular 2×2 matrices).

The next subsection proves that the reduced echelon form of a matrix is unique. Rephrased in terms of the row-equivalence relationship, we shall prove that every matrix is row equivalent to one and only one reduced echelon form matrix. In terms of the partition what we shall prove is: every equivalence class contains one and only one reduced echelon form matrix. So each reduced echelon form matrix serves as a representative of its class.

Exercises

✓ 1.8 Use Gauss-Jordan reduction to solve each system.

(a) $\begin{aligned} x+y&=2\\ x-y&=0\end{aligned}$ (b) $\begin{aligned} x-z&=4\\ 2x+2y&=1\end{aligned}$ (c) $\begin{aligned} 3x-2y&=1\\ 6x+y&=1/2\end{aligned}$

(d) $\begin{aligned} 2x-y&=-1\\ x+3y-z&=5\\ y+2z&=5\end{aligned}$

1.9 Do Gauss-Jordan reduction.

(a) $\begin{aligned} x+y-z&=3\\ 2x-y-z&=1\\ 3x+y+2z&=0\end{aligned}$ (b) $\begin{aligned} x+y+2z&=0\\ 2x-y+z&=1\\ 4x+y+5z&=1\end{aligned}$

✓ 1.10 Find the reduced echelon form of each matrix.

(a) $\begin{pmatrix}2&1\\1&3\end{pmatrix}$ (b) $\begin{pmatrix}1&3&1\\2&0&4\\-1&-3&-3\end{pmatrix}$ (c) $\begin{pmatrix}1&0&3&1&2\\1&4&2&1&5\\3&4&8&1&2\end{pmatrix}$

(d) $\begin{pmatrix}0&1&3&2\\0&0&5&6\\1&5&1&5\end{pmatrix}$

1.11 Get the reduced echelon form of each.

(a) $\begin{pmatrix}0&2&1\\2&-1&1\\-2&-1&0\end{pmatrix}$ (b) $\begin{pmatrix}1&3&1\\2&6&2\\-1&0&0\end{pmatrix}$

✓ 1.12 Find each solution set by using Gauss-Jordan reduction and then reading off the parametrization.

(a) $\begin{aligned} 2x+y-z&=1\\ 4x-y&=3\end{aligned}$

(b) $\begin{aligned} x-z&=1\\ y+2z-w&=3\\ x+2y+3z-w&=7\end{aligned}$

(c) $\begin{aligned} x-y+z&=0\\ y+w&=0\\ 3x-2y+3z+w&=0\\ -y-w&=0\end{aligned}$

(d) $\begin{aligned} a+2b+3c+d-e&=1\\ 3a-b+c+d+e&=3\end{aligned}$

As always, you must back any assertion with either a proof or an example.

2.19 Verify the equality condition in Corollary 2.6, the Cauchy-Schwarz Inequality.

(a) Show that if $\vec{u}$ is a negative scalar multiple of $\vec{v}$ then $\vec{u} \cdot \vec{v}$ and $\vec{v} \cdot \vec{u}$ are less than or equal to zero.

(b) Show that $|\vec{u} \cdot \vec{v}| = |\vec{u}\,|\,|\vec{v}\,|$ if and only if one vector is a scalar multiple of the other.

2.20 Suppose that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ and $\vec{u} \neq \vec{0}$. Must $\vec{v} = \vec{w}$?

✓ 2.21 Does any vector have length zero except a zero vector? (If "yes", produce an example. If "no", prove it.)

✓ 2.22 Find the midpoint of the line segment connecting (x_1, y_1) with (x_2, y_2) in $\mathbb{R}^2$. Generalize to $\mathbb{R}^n$.

2.23 Show that if $\vec{v} \neq \vec{0}$ then $\vec{v}/|\vec{v}\,|$ has length one. What if $\vec{v} = \vec{0}$?

2.24 Show that if $r \geqslant 0$ then $r\vec{v}$ is r times as long as $\vec{v}$. What if $r < 0$?

✓ 2.25 A vector $\vec{v} \in \mathbb{R}^n$ of length one is a *unit* vector. Show that the dot product of two unit vectors has absolute value less than or equal to one. Can 'less than' happen? Can 'equal to'?

2.26 Prove that $|\vec{u} + \vec{v}\,|^2 + |\vec{u} - \vec{v}\,|^2 = 2|\vec{u}\,|^2 + 2|\vec{v}\,|^2$.

2.27 Show that if $\vec{x} \cdot \vec{y} = 0$ for every $\vec{y}$ then $\vec{x} = \vec{0}$.

2.28 Is $|\vec{u}_1 + \cdots + \vec{u}_n| \leqslant |\vec{u}_1| + \cdots + |\vec{u}_n|$? If it is true then it would generalize the Triangle Inequality.

2.29 What is the ratio between the sides in the Cauchy-Schwarz inequality?

2.30 Why is the zero vector defined to be perpendicular to every vector?

2.31 Describe the angle between two vectors in $\mathbb{R}^1$.

2.32 Give a simple necessary and sufficient condition to determine whether the angle between two vectors is acute, right, or obtuse.

2.33 Generalize to $\mathbb{R}^n$ the converse of the Pythagorean Theorem, that if $\vec{u}$ and $\vec{v}$ are perpendicular then $|\vec{u} + \vec{v}\,|^2 = |\vec{u}\,|^2 + |\vec{v}\,|^2$.

2.34 Show that $|\vec{u}\,| = |\vec{v}\,|$ if and only if $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are perpendicular. Give an example in $\mathbb{R}^2$.

2.35 Show that if a vector is perpendicular to each of two others then it is perpendicular to each vector in the plane they generate. (*Remark.* They could generate a degenerate plane — a line or a point — but the statement remains true.)

2.36 Prove that, where $\vec{u}, \vec{v} \in \mathbb{R}^n$ are nonzero vectors, the vector

$$\frac{\vec{u}}{|\vec{u}\,|} + \frac{\vec{v}}{|\vec{v}\,|}$$

bisects the angle between them. Illustrate in $\mathbb{R}^2$.

2.37 Verify that the definition of angle is dimensionally correct: (1) if $k > 0$ then the cosine of the angle between $k\vec{u}$ and $\vec{v}$ equals the cosine of the angle between $\vec{u}$ and $\vec{v}$, and (2) if $k < 0$ then the cosine of the angle between $k\vec{u}$ and $\vec{v}$ is the negative of the cosine of the angle between $\vec{u}$ and $\vec{v}$.

✓ 2.38 Show that the inner product operation is *linear*: for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $k, m \in \mathbb{R}$, $\vec{u} \cdot (k\vec{v} + m\vec{w}) = k(\vec{u} \cdot \vec{v}) + m(\vec{u} \cdot \vec{w})$.

the angle is

$$\arccos\left(\frac{(1)(0)+(1)(3)+(0)(2)}{\sqrt{1^2+1^2+0^2}\sqrt{0^2+3^2+2^2}}\right)=\arccos\left(\frac{3}{\sqrt{2}\sqrt{13}}\right)$$

approximately 0.94 radians. Notice that these vectors are not orthogonal. Although the yz-plane may appear to be perpendicular to the xy-plane, in fact the two planes are that way only in the weak sense that there are vectors in each orthogonal to all vectors in the other. Not every vector in each is orthogonal to all vectors in the other.

Exercises

✓ 2.11 Find the length of each vector.

(a) $\begin{pmatrix}3\\1\end{pmatrix}$ (b) $\begin{pmatrix}-1\\2\end{pmatrix}$ (c) $\begin{pmatrix}4\\1\\1\end{pmatrix}$ (d) $\begin{pmatrix}0\\0\\0\end{pmatrix}$ (e) $\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$

✓ 2.12 Find the angle between each two, if it is defined.

(a) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}$ (b) $\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\4\\1\end{pmatrix}$ (c) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\4\\-1\end{pmatrix}$

✓ 2.13 [Ohanian] During maneuvers preceding the Battle of Jutland, the British battle cruiser *Lion* moved as follows (in nautical miles): 1.2 miles north, 6.1 miles 38 degrees east of south, 4.0 miles at 89 degrees east of north, and 6.5 miles at 31 degrees east of north. Find the distance between starting and ending positions. (Ignore the earth's curvature.)

2.14 Find k so that these two vectors are perpendicular.

$$\begin{pmatrix}k\\1\end{pmatrix}\qquad\begin{pmatrix}4\\3\end{pmatrix}$$

2.15 Describe the set of vectors in $\mathbb{R}^3$ orthogonal to the one with entries 1, 3, and -1.

✓ 2.16 (a) Find the angle between the diagonal of the unit square in $\mathbb{R}^2$ and any one of the axes.

(b) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the axes.

(c) Find the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.

(d) What is the limit, as n goes to ∞, of the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and any one of the axes?

2.17 Is any vector perpendicular to itself?

2.18 Describe the algebraic properties of dot product.

(a) Is it right-distributive over addition: $(\vec{u}+\vec{v})\cdot\vec{w}=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$?

(b) Is it left-distributive (over addition)?

(c) Does it commute?

(d) Associate?

(e) How does it interact with scalar multiplication?

is a *degenerate* plane because it is actually a line, since the vectors are multiples of each other and we can omit one.

$$L = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} + r \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \mid r \in \mathbb{R} \}$$

We shall see in the Linear Independence section of Chapter Two what relationships among vectors causes the linear surface they generate to be degenerate.

We now can restate in geometric terms our conclusions from earlier. First, the solution set of a linear system with n unknowns is a linear surface in $\mathbb{R}^n$. Specifically, it is a k-dimensional linear surface, where k is the number of free variables in an echelon form version of the system. For instance, in the single equation case the solution set is an $n-1$-dimensional hyperplane in $\mathbb{R}^n$, where $n \geqslant 1$. Second, the solution set of a homogeneous linear system is a linear surface passing through the origin. Finally, we can view the general solution set of any linear system as being the solution set of its associated homogeneous system offset from the origin by a vector, namely by any particular solution.

Exercises

✓ **1.1** Find the canonical name for each vector.

(a) the vector from $(2, 1)$ to $(4, 2)$ in $\mathbb{R}^2$

(b) the vector from $(3, 3)$ to $(2, 5)$ in $\mathbb{R}^2$

(c) the vector from $(1, 0, 6)$ to $(5, 0, 3)$ in $\mathbb{R}^3$

(d) the vector from $(6, 8, 8)$ to $(6, 8, 8)$ in $\mathbb{R}^3$

✓ **1.2** Decide if the two vectors are equal.

(a) the vector from $(5, 3)$ to $(6, 2)$ and the vector from $(1, -2)$ to $(1, 1)$

(b) the vector from $(2, 1, 1)$ to $(3, 0, 4)$ and the vector from $(5, 1, 4)$ to $(6, 0, 7)$

✓ **1.3** Does $(1, 0, 2, 1)$ lie on the line through $(-2, 1, 1, 0)$ and $(5, 10, -1, 4)$?

✓ **1.4** (a) Describe the plane through $(1, 1, 5, -1)$, $(2, 2, 2, 0)$, and $(3, 1, 0, 4)$.

(b) Is the origin in that plane?

1.5 Give a vector description of each.

(a) the plane subset of $\mathbb{R}^3$ with equation $x - 2y + z = 4$

(b) the plane in $\mathbb{R}^3$ with equation $2x + y + 4z = -1$

(c) the hyperplane subset of $\mathbb{R}^4$ with equation $x + y + z + w = 10$

1.6 Describe the plane that contains this point and line.

$$\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \qquad \{ \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} t \mid t \in \mathbb{R} \}$$

✓ **1.7** Intersect these planes.

$$\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} s \mid t, s \in \mathbb{R} \} \qquad \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} k + \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} m \mid k, m \in \mathbb{R} \}$$

Exercises

3.14 Solve this system. Then solve the associated homogeneous system.

$$\begin{aligned} x + y - 2z &= 0 \\ x - y \phantom{{}-2z} &= -3 \\ 3x - y - 2z &= -6 \\ 2y - 2z &= 3 \end{aligned}$$

✓ 3.15 Solve each system. Express the solution set using vectors. Identify a particular solution and the solution set of the homogeneous system.

(a) $\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned}$ (b) $\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned}$ (c) $\begin{aligned} x_1 \phantom{{}-x_2} + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned}$

(d) $\begin{aligned} 2a + b - c &= 2 \\ 2a \phantom{{}+b} + c &= 3 \\ a - b \phantom{{}+c} &= 0 \end{aligned}$ (e) $\begin{aligned} x + 2y - z \phantom{{}+w} &= 3 \\ 2x + y \phantom{{}-z} + w &= 4 \\ x - y + z + w &= 1 \end{aligned}$ (f) $\begin{aligned} x \phantom{{}+y} + z + w &= 4 \\ 2x + y \phantom{{}+z} - w &= 2 \\ 3x + y + z \phantom{{}+w} &= 7 \end{aligned}$

3.16 Solve each system, giving the solution set in vector notation. Identify a particular solution and the solution of the homogeneous system.

(a) $\begin{aligned} 2x + y - z &= 1 \\ 4x - y \phantom{{}-z} &= 3 \end{aligned}$ (b) $\begin{aligned} x \phantom{{}+2y} - z \phantom{{}-w} &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$ (c) $\begin{aligned} x - y + z \phantom{{}+w} &= 0 \\ y \phantom{{}+z} + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \phantom{{}+z} - w &= 0 \end{aligned}$

(d) $\begin{aligned} a + 2b + 3c + d - e &= 1 \\ 3a - b + c + d + e &= 3 \end{aligned}$

✓ 3.17 For the system

$$\begin{aligned} 2x - y \phantom{{}-z} - w &= 3 \\ y + z + 2w &= 2 \\ x - 2y - z \phantom{{}+2w} &= -1 \end{aligned}$$

which of these can be used as the particular solution part of some general solution?

(a) $\begin{pmatrix} 0 \\ -3 \\ 5 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ -4 \\ 8 \\ -1 \end{pmatrix}$

✓ 3.18 Lemma 3.7 says that we can use any particular solution for $\vec{p}$. Find, if possible, a general solution to this system

$$\begin{aligned} x - y \phantom{{}-z} + w &= 4 \\ 2x + 3y - z \phantom{{}+w} &= 0 \\ y + z + w &= 4 \end{aligned}$$

that uses the given vector as its particular solution.

(a) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

3.19 One is nonsingular while the other is singular. Which is which?

(a) $\begin{pmatrix} 1 & 3 \\ 4 & -12 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

✓ 3.20 Singular or nonsingular?

revenue of \$100 per acre for corn, \$300 per acre for soybeans and \$80 per acre for oats. Which of your two solutions in the prior part would have resulted in a larger revenue?

2.24 Parametrize the solution set of this one-equation system.

$$x_1 + x_2 + \cdots + x_n = 0$$

✓ 2.25 (a) Apply Gauss's Method to the left-hand side to solve

$$\begin{aligned} x + 2y - w &= a \\ 2x + z &= b \\ x + y + 2w &= c \end{aligned}$$

for x, y, z, and w, in terms of the constants a, b, and c.

(b) Use your answer from the prior part to solve this.

$$\begin{aligned} x + 2y - w &= 3 \\ 2x + z &= 1 \\ x + y + 2w &= -2 \end{aligned}$$

2.26 Why is the comma needed in the notation '$a_{i,j}$' for matrix entries?

✓ 2.27 Give the 4×4 matrix whose i,j-th entry is

(a) $i + j$; (b) -1 to the $i + j$ power.

2.28 For any matrix A, the *transpose* of A, written A^{T}, is the matrix whose columns are the rows of A. Find the transpose of each of these.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

✓ 2.29 (a) Describe all functions $f(x) = ax^2 + bx + c$ such that $f(1) = 2$ and $f(-1) = 6$.

(b) Describe all functions $f(x) = ax^2 + bx + c$ such that $f(1) = 2$.

2.30 Show that any set of five points from the plane $\mathbb{R}^2$ lie on a common conic section, that is, they all satisfy some equation of the form $ax^2 + by^2 + cxy + dx + ey + f = 0$ where some of $a, \ldots, f$ are nonzero.

2.31 Make up a four equations/four unknowns system having

(a) a one-parameter solution set;

(b) a two-parameter solution set;

(c) a three-parameter solution set.

? 2.32 [Shepelev] This puzzle is from a Russian web-site `http://www.arbuz.uz/` and there are many solutions to it, but mine uses linear algebra and is very naive. There's a planet inhabited by arbuzoids (watermeloners, to translate from Russian). Those creatures are found in three colors: red, green and blue. There are 13 red arbuzoids, 15 blue ones, and 17 green. When two differently colored arbuzoids meet, they both change to the third color.

The question is, can it ever happen that all of them assume the same color?

? 2.33 [USSR Olympiad no. 174]

(a) Solve the system of equations.

$$\begin{aligned} ax + y &= a^2 \\ x + ay &= 1 \end{aligned}$$

For what values of a does the system fail to have solutions, and for what values of a are there infinitely many solutions?

(a) $\begin{aligned} 2x+y-z&=1\\ 4x-y\quad&=3 \end{aligned}$ (b) $\begin{aligned} x\quad-z\qquad&=1\\ y+2z-w&=3\\ x+2y+3z-w&=7 \end{aligned}$ (c) $\begin{aligned} x-y+z\qquad&=0\\ y\qquad+w&=0\\ 3x-2y+3z+w&=0\\ -y\qquad-w&=0 \end{aligned}$

(d) $\begin{aligned} a+2b+3c+d-e&=1\\ 3a-b+c+d+e&=3 \end{aligned}$

2.20 Solve each system using matrix notation. Express the solution set using vectors.

(a) $\begin{aligned} 3x+2y+z&=1\\ x-y+z&=2\\ 5x+5y+z&=0 \end{aligned}$ (b) $\begin{aligned} x+y-2z&=0\\ x-y\qquad&=-3\\ 3x-y-2z&=-6\\ 2y-2z&=3 \end{aligned}$ (c) $\begin{aligned} 2x-y-z+w&=4\\ x+y+z\qquad&=-1 \end{aligned}$

(d) $\begin{aligned} x+y-2z&=0\\ x-y\qquad&=-3\\ 3x-y-2z&=0 \end{aligned}$

✓ 2.21 The vector is in the set. What value of the parameters produces that vector?

(a) $\begin{pmatrix}5\\-5\end{pmatrix}$, $\{\begin{pmatrix}1\\-1\end{pmatrix}k \mid k\in\mathbb{R}\}$

(b) $\begin{pmatrix}-1\\2\\1\end{pmatrix}$, $\{\begin{pmatrix}-2\\1\\0\end{pmatrix}i+\begin{pmatrix}3\\0\\1\end{pmatrix}j \mid i,j\in\mathbb{R}\}$

(c) $\begin{pmatrix}0\\-4\\2\end{pmatrix}$, $\{\begin{pmatrix}1\\1\\0\end{pmatrix}m+\begin{pmatrix}2\\0\\1\end{pmatrix}n \mid m,n\in\mathbb{R}\}$

2.22 Decide if the vector is in the set.

(a) $\begin{pmatrix}3\\-1\end{pmatrix}$, $\{\begin{pmatrix}-6\\2\end{pmatrix}k \mid k\in\mathbb{R}\}$

(b) $\begin{pmatrix}5\\4\end{pmatrix}$, $\{\begin{pmatrix}5\\-4\end{pmatrix}j \mid j\in\mathbb{R}\}$

(c) $\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $\{\begin{pmatrix}0\\3\\-7\end{pmatrix}+\begin{pmatrix}1\\-1\\3\end{pmatrix}r \mid r\in\mathbb{R}\}$

(d) $\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\{\begin{pmatrix}2\\0\\1\end{pmatrix}j+\begin{pmatrix}-3\\-1\\1\end{pmatrix}k \mid j,k\in\mathbb{R}\}$

2.23 [Cleary] A farmer with 1200 acres is considering planting three different crops, corn, soybeans, and oats. The farmer wants to use all 1200 acres. Seed corn costs \$20 per acre, while soybean and oat seed cost \$50 and \$12 per acre respectively. The farmer has \$40 000 available to buy seed and intends to spend it all.

(a) Use the information above to formulate two linear equations with three unknowns and solve it.

(b) Solutions to the system are choices that the farmer can make. Write down two reasonable solutions.

(c) Suppose that in the fall when the crops mature, the farmer can bring in

1.33 Finish the proof of Theorem 1.5.

1.34 Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

✓ 1.35 Are any of the operations used in Gauss's Method redundant? That is, can we make any of the operations from a combination of the others?

1.36 Prove that each operation of Gauss's Method is reversible. That is, show that if two systems are related by a row operation $S_1 \to S_2$ then there is a row operation to go back $S_2 \to S_1$.

? 1.37 [Anton] A box holding pennies, nickels and dimes contains thirteen coins with a total value of 83 cents. How many coins of each type are in the box? (These are US coins; a penny is 1 cent, a nickel is 5 cents, and a dime is 10 cents.)

? 1.38 [Con. Prob. 1955] Four positive integers are given. Select any three of the integers, find their arithmetic average, and add this result to the fourth integer. Thus the numbers 29, 23, 21, and 17 are obtained. One of the original integers is:

(a) 19 (b) 21 (c) 23 (d) 29 (e) 17

? 1.39 [Am. Math. Mon., Jan. 1935] Laugh at this: $\mathrm{AHAHA} + \mathrm{TEHE} = \mathrm{TEHAW}$. It resulted from substituting a code letter for each digit of a simple example in addition, and it is required to identify the letters and prove the solution unique.

? 1.40 [Wohascum no. 2] The Wohascum County Board of Commissioners, which has 20 members, recently had to elect a President. There were three candidates (A, B, and C); on each ballot the three candidates were to be listed in order of preference, with no abstentions. It was found that 11 members, a majority, preferred A over B (thus the other 9 preferred B over A). Similarly, it was found that 12 members preferred C over A. Given these results, it was suggested that B should withdraw, to enable a runoff election between A and C. However, B protested, and it was then found that 14 members preferred B over C! The Board has not yet recovered from the resulting confusion. Given that every possible order of A, B, C appeared on at least one ballot, how many members voted for B as their first choice?

? 1.41 [Am. Math. Mon., Jan. 1963] "This system of n linear equations with n unknowns," said the Great Mathematician, "has a curious property."

"Good heavens!" said the Poor Nut, "What is it?"

"Note," said the Great Mathematician, "that the constants are in arithmetic progression."

"It's all so clear when you explain it!" said the Poor Nut. "Do you mean like $6x + 9y = 12$ and $15x + 18y = 21$?"

"Quite so," said the Great Mathematician, pulling out his bassoon. "Indeed, the system has a unique solution. Can you find it?"

"Good heavens!" cried the Poor Nut, "I am baffled."

Are you?

(a) $\begin{aligned} x - 3y &= b_1 \\ 3x + y &= b_2 \\ x + 7y &= b_3 \\ 2x + 4y &= b_4 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + 8x_3 &= b_3 \end{aligned}$

1.25 True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.26 Must any Chemistry problem like the one that starts this subsection — a balance the reaction problem — have infinitely many solutions?

✓ 1.27 Find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.28 After Theorem 1.5 we note that multiplying a row by 0 is not allowed because that could change a solution set. Give an example of a system with solution set S_0 where after multiplying a row by 0 the new system has a solution set S_1 and S_0 is a proper subset of S_1, that is, $S_0 \neq S_1$. Give an example where $S_0 = S_1$.

1.29 Gauss's Method works by combining the equations in a system to make new equations.

(a) Can we derive the equation $3x - 2y = 5$ by a sequence of Gaussian reduction steps from the equations in this system?

$$\begin{aligned} x + y &= 1 \\ 4x - y &= 6 \end{aligned}$$

(b) Can we derive the equation $5x - 3y = 2$ with a sequence of Gaussian reduction steps from the equations in this system?

$$\begin{aligned} 2x + 2y &= 5 \\ 3x + y &= 4 \end{aligned}$$

(c) Can we derive $6x - 9y + 5z = -2$ by a sequence of Gaussian reduction steps from the equations in the system?

$$\begin{aligned} 2x + y - z &= 4 \\ 6x - 3y + z &= 5 \end{aligned}$$

1.30 Prove that, where a, b, c, d, e are real numbers with $a \neq 0$, if this linear equation

$$ax + by = c$$

has the same solution set as this one

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

1.31 Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

✓ 1.32 In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities: there is a unique solution, there is no solution, and there are infinitely many solutions.

✓ 1.19 Use Gauss's Method to solve each system or conclude 'many solutions' or 'no solutions'.

(a) $\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$ (b) $\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$ (c) $\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$ (d) $\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$

(e) $\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x + z &= 5 \\ x + y - z &= 10 \end{aligned}$ (f) $\begin{aligned} 2x + z + w &= 5 \\ y - w &= -1 \\ 3x - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$

1.20 Solve each system or conclude 'many solutions' or 'no solutions'. Use Gauss's Method.

(a) $\begin{aligned} x + y + z &= 5 \\ x - y &= 0 \\ y + 2z &= 7 \end{aligned}$ (b) $\begin{aligned} 3x + z &= 7 \\ x - y + 3z &= 4 \\ x + 2y - 5z &= -1 \end{aligned}$ (c) $\begin{aligned} x + 3y + z &= 0 \\ -x - y &= 2 \\ -x + y + 2z &= 8 \end{aligned}$

✓ 1.21 We can solve linear systems by methods other than Gauss's. One often taught in high school is to solve one of the equations for a variable, then substitute the resulting expression into other equations. Then we repeat that step until there is an equation with only one variable. From that we get the first number in the solution and then we get the rest with back-substitution. This method takes longer than Gauss's Method, since it involves more arithmetic operations, and is also more likely to lead to errors. To illustrate how it can lead to wrong conclusions, we will use the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned}$$

from Example 1.13.

(a) Solve the first equation for x and substitute that expression into the second equation. Find the resulting y.

(b) Again solve the first equation for x, but this time substitute that expression into the third equation. Find this y.

What extra step must a user of this method take to avoid erroneously concluding a system has a solution?

✓ 1.22 For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

1.23 This system is not linear in that it says $\sin\alpha$ instead of α

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 10 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

and yet we can apply Gauss's Method. Do so. Does the system have a solution?

✓ 1.24 What conditions must the constants, the b's, satisfy so that each of these systems has a solution? *Hint.* Apply Gauss's Method and see what happens to the right side.

any solutions at all despite that in echelon form it has a $0 = 0$ row.

$$\begin{array}{rcrcrcr} 2x & & & - & 2z & = & 6 \\ & & y & + & z & = & 1 \\ 2x & + & y & - & z & = & 7 \\ & & 3y & + & 3z & = & 0 \end{array} \xrightarrow{-\rho_1+\rho_3} \begin{array}{rcrcrcr} 2x & & & - & 2z & = & 6 \\ & & y & + & z & = & 1 \\ & & y & + & z & = & 1 \\ & & 3y & + & 3z & = & 0 \end{array}$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{array}{rcrcrcr} 2x & & & - & 2z & = & 6 \\ & & y & + & z & = & 1 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & -3 \end{array}$$

In summary, Gauss's Method uses the row operations to set a system up for back substitution. If any step shows a contradictory equation then we can stop with the conclusion that the system has no solutions. If we reach echelon form without a contradictory equation, and each variable is a leading variable in its row, then the system has a unique solution and we find it by back substitution. Finally, if we reach echelon form without a contradictory equation, and there is not a unique solution — that is, at least one variable is not a leading variable — then the system has many solutions.

The next subsection explores the third case. We will see that such a system must have infinitely many solutions and we will describe the solution set.

Note. *In the exercises here, and in the rest of the book, you must justify all of your answers. For instance, if a question asks whether a system has a solution then you must justify a yes response by producing the solution and must justify a no response by showing that no solution exists.*

Exercises

✓ 1.17 Use Gauss's Method to find the unique solution for each system.

(a) $\begin{array}{rcrcr} 2x & + & 3y & = & 13 \\ x & - & y & = & -1 \end{array}$ (b) $\begin{array}{rcrcrcr} x & & & - & z & = & 0 \\ 3x & + & y & & & = & 1 \\ -x & + & y & + & z & = & 4 \end{array}$

1.18 Each system is in echelon form. For each, say whether the system has a unique solution, no solution, or infinitely many solutions.

(a) $\begin{array}{rcrcr} -3x & + & 2y & = & 0 \\ & & -2y & = & 0 \end{array}$ (b) $\begin{array}{rcrcrcr} x & + & y & & & = & 4 \\ & & y & - & z & = & 0 \end{array}$ (c) $\begin{array}{rcrcrcr} x & + & y & & & = & 4 \\ & & y & - & z & = & 0 \\ & & & & 0 & = & 0 \end{array}$ (d) $\begin{array}{rcrcr} x & + & y & = & 4 \\ & & 0 & = & 4 \end{array}$

(e) $\begin{array}{rcrcrcr} 3x & + & 6y & + & z & = & -0.5 \\ & & & & -z & = & 2.5 \end{array}$ (f) $\begin{array}{rcrcr} x & - & 3y & = & 2 \\ & & 0 & = & 0 \end{array}$ (g) $\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ & & y & = & 1 \\ & & 0 & = & 4 \end{array}$ (h) $2x + y = 0$

(i) $\begin{array}{rcrcr} x & - & y & = & -1 \\ & & 0 & = & 0 \\ & & 0 & = & 4 \end{array}$ (j) $\begin{array}{rcrcrcr} x & + & y & - & 3z & = & -1 \\ & & y & - & z & = & 2 \\ & & & & z & = & 0 \\ & & & & 0 & = & 0 \end{array}$