Design of IIR notch filter with maximally flat or equiripple Magnitude Characteristics

2 authors:

A. Fernandez-Vazquez
Instituto Politécnico Nacional

Gordana Jovanovic-Dolecek
Instituto Nacional de Astrofísica, Óptica y Electrónica (INAOE)

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DESIGN OF IIR NOTCH FILTERS WITH MAXIMALLY FLAT OR EQUIRIPPLE MAGNITUDE CHARACTERISTICS

Alfonso Fernandez-Vazquez and Gordana Jovanovic-Dolecek

Department of Electronics, INAOE
P.O. Box 51 and 216, 72000, Puebla, Mexico
Email: `afernan@inaoep.mx, gordana@inaoep.mx`

ABSTRACT

This paper presents the design of IIR (Infinite Impulse Response) notch filters with desired magnitude characteristic, which can be either maximally flat or equiripple. Butterworth polynomial, used for designing the allpass filter, will result in a maximally flat magnitude. Similarly, an equiripple characteristic is obtained by using Chebyshev I, Chebyshev II and Elliptic polynomials. The parameters of the design are notch frequency, rejection bandwidth and passbands ripple.

1. INTRODUCTION

Digital notch filters remove a single non-desired frequency component from signal as for example unmodulated carrier in communication systems or power line interference from a sampled signal [1].

Figure 1 shows a general characteristic of a notch filter. The frequency ω_0 is the notch frequency, while A_{1p} and A_{2p} are the passband droops for the first and the second passband, respectively. The frequencies ω_{1p} and ω_{2p} define the rejection bandwidth B.

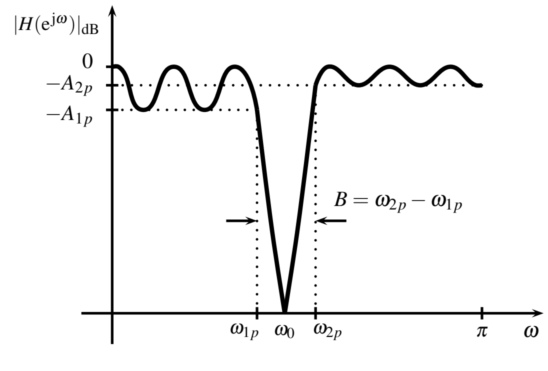


Figure 1: Gain characteristic of a notch filter.

A stable analog notch filter of the second order can be presented as, [2]

$$H(s) = \frac{1}{2}\left[1 + A(s)\right], \tag{1}$$

where $A(s)$ is an analog allpass filter. Its phase response is a decreasing function of Ω which goes from 0 to -2π for $0 \le \Omega < \infty$.

To design the corresponding digital IIR notch filter $H(z)$, we apply bilinear transformation [2].

It is possible to design real and complex IIR multiple notch filters by changing the order of the corresponding allpass filter [3, 4]. The filter coefficients are obtained by solving a set of linear equations. The problem arises if the set of linear equations is ill-conditioned.

To avoid this problem an alternative approach is proposed in [5]. In this case, a high order allpass filter is formed as a cascade of second order allpass sections.

In all mentioned cases, the resulting notch filters are not maximally flat.

In the proposed design, the notch filters can be either maximally flat or equiripple in one or both passbands.

Unlike notch filter (1), we express a stable notch filter $G_1(s)$ as,

$$G_1(s) = \frac{1}{2}\left[A_0^2(s) + A_1^2(s)\right], \tag{2}$$

where $A_0(s)$ and $A_2(s)$ are stable allpass filters.

Equation (2) can be rewritten as,

$$G_1(s) = \frac{1}{2}\left[1 + A_G^2(s)\right] A_0^2(s), \tag{3}$$

where $A_G(s) = A_1(s)/A_0(s)$.

We define an analog notch filter $G(s)$, which has the same magnitude response as $G_1(s)$,

$$G(s) = \frac{1}{2}\left[1 + A_G^2(s)\right]. \tag{4}$$

Therefore, the problem of a notch filter $G(s)$ design is reduced to the design of an analog allpass filter $A_G(s)$.

Using (4), the magnitude response $|G(\mathrm{j}\Omega)|$ is given by,

$$|G(\mathrm{j}\Omega)| = \left|\cos\left(2\phi_{A_G}(\Omega)\right)\right|, \tag{5}$$

where $\phi_{A_G}(\Omega)$ is the phase response of $A_G(s)$.

In order to obtain an ideal notch characteristic ($B = A_{1p} = A_{2p} = 0$), the phase response $\phi_{A_G}(\mathrm{j}\Omega)$ has to satisfy,

$$\phi_{A_G}(\mathrm{j}\Omega) = \begin{cases} 0 & 0 \le \Omega \le \Omega_{1p}, \\ \frac{\pi}{2} & \Omega = \Omega_0, \\ \pi & \Omega_{2p} \le \Omega < \infty. \end{cases} \tag{6}$$

To approximate the desired phase (6), we could chose the following allpass filter $A_G(s)$,

$$A_G(s) = \frac{1 + \mathrm{j}\widetilde{F}_N(s)}{1 - \mathrm{j}F_N(s)}, \tag{7}$$

where $\widetilde{F}_N(s)$ is the paraconjugate of $F_N(s)$ [6], and $F_N(s)$ is a polynomial of order N, which has the following frequency characteristic,

$$F_N(\mathrm{j}\Omega) = \begin{cases} 0 & \Omega \approx 0, \\ 1 & \Omega = \Omega_0, \\ \infty & \Omega \to \infty. \end{cases} \tag{8}$$

We chose Butterworth, Chebyshev I, Chebyshev II and Elliptic polynomials [7], to satisfy (8).

The resulting allpass filter $A_G(s)$ can be real or complex depending on the parity of the order N i.e. $A_G(s)$ is real if N is odd, otherwise is complex.

From (2) and (7), we arrive at,

$$G(s) = \frac{1 - F_N(s)}{1 - 2\mathrm{j}F_N(s) - F_N^2(s)}. \tag{9}$$

Therefore, the magnitude response of $G(s)$ is given by,

$$|G(\mathrm{j}\Omega)| = \left| \frac{1 - \widetilde{F}_N(\mathrm{j}\Omega)F_N(\mathrm{j}\Omega)}{1 + \widetilde{F}_N(\mathrm{j}\Omega)F_N(\mathrm{j}\Omega)} \right|. \tag{10}$$

The rest of the paper is organized as follows. In the next Section, we consider the design of real and complex allpass filters $A_G(s)$ using Butterworth, Chebyshev and Elliptic polynomials. Section III describes the proposed algorithm which is illustrated with one example.

2. DESIGN OF ALLPASS FILTERS USING DIFFERENT POLYNOMIALS

2.1 Butterworth polynomials

Using Butterworth polynomial [7], the allpass filter $A_G(s)$ can be expressed as,

$$A_G(s) = \frac{1 + \mathrm{j}\left(\frac{s}{\mathrm{j}\Omega_0}\right)^N}{1 - \mathrm{j}\left(\frac{s}{\mathrm{j}\Omega_0}\right)^N}, \tag{11}$$

where Ω_0 is the notch frequency (see Fig. 1).

The poles of the allpass filter are,

$$s_k = \mathrm{j}\Omega_0 \mathrm{e}^{-\mathrm{j}\frac{1-4k}{2N}\pi}, \qquad k = 0, \ldots, N-1. \tag{12}$$

Using (9), the zeros of $G(s)$ are,

$$s_l = \mathrm{j}\Omega_0 \mathrm{e}^{\mathrm{j}\frac{\pi l}{N}}, \qquad l = 0, \ldots, 2N-1. \tag{13}$$

From (10), the approximation of the allpass filter order is,

$$N = \left\lceil \frac{\log\left(\frac{10^{A_p/20}-1}{10^{A_p/20}+1}\right)}{\log\left(\frac{\Omega_1}{\Omega_2}\right)} \right\rceil, \tag{14}$$

where $\lceil \cdot \rceil$ is the ceiling function, A_p is the minimum value in dB, defined as $A_p = \min(A_{1p}, A_{2p})$, and

$$\Omega_1 = \Omega_0 - B_a/2, \tag{15}$$

$$\Omega_2 = \Omega_0 + B_a/2, \tag{16}$$

where B_a is the rejection bandwidth.

It is easily shown that the number of null derivatives of the square magnitude response $|G(\mathrm{j}\Omega)|^2$ at $\Omega = 0$ is $2N-1$ i.e. the filter has maximally flat characteristic.

2.2 Chebyshev I polynomials

Using the Chebyshev I polynomials and (7), we arrive at

$$A_G(s) = \frac{1 + \mathrm{j}\varepsilon_{1p} C_N\left(\frac{s}{\mathrm{j}\Omega_{1p}}\right)}{1 - \mathrm{j}\varepsilon_{1p} C_N\left(\frac{s}{\mathrm{j}\Omega_{1p}}\right)}, \tag{17}$$

where $C_N(\cdot)$ are the Chebyshev polynomials [7], and ε_{1p} controls the passband ripple in the first passband.

The corresponding poles of $A_G(s)$ are

$$s_k = \mathrm{j}\Omega_{1p}\left[a\cos\left(\frac{u_k}{N}\right) - \mathrm{j}b\sin\left(\frac{u_k}{N}\right)\right], \tag{18}$$

where

$$u_k = \frac{4k-1}{2}\pi, \tag{19}$$

$$v = \operatorname{arcsinh}\left(\frac{1}{\varepsilon_{1p}}\right), \tag{20}$$

$$a = \cosh\left(\frac{v}{N}\right), \tag{21}$$

$$b = \sinh\left(\frac{v}{N}\right). \tag{22}$$

From (9), it follows that the zeros of $G(s)$ are,

$$s_l = \mathrm{j}\Omega_{1p}\cos\left(\frac{\pm 1}{\varepsilon_{1p}} + \frac{2l\pi}{N}\right), \qquad l = 0, \ldots, N-1. \tag{23}$$

Using (10) and A_{1p} expressed in dB, the value of ε_{1p} is given by,

$$\varepsilon_{1p} = \sqrt{\frac{10^{A_{1p}/20}-1}{10^{A_{1p}/20}+1}}. \tag{24}$$

Substituting $l = 0$ into (23), the corresponding zero gives the notch frequency Ω_0. Therefore, we obtain the value Ω_{1p} as,

$$\Omega_{1p} = \frac{\Omega_0}{\cos(\arccos(1/\varepsilon_{1p})/N)}. \tag{25}$$

From (10), the order of the allpass filter can be estimated as,

$$N = \left\lceil \frac{\operatorname{arccosh}(\varepsilon_{1p}^2)}{\operatorname{arccosh}\left(\frac{\Omega_2}{\Omega_1}\right)} \right\rceil, \tag{26}$$

where Ω_1 and Ω_2 are given in (15) and (16), respectively.

2.3 Chebyshev II polynomials

In this case the allpass filter (7) can be rewritten as,

$$A_G(s) = \frac{1 + \mathrm{j}/\varepsilon_{2p} C_N\left(\frac{\mathrm{j}\Omega_{2p}}{s}\right)}{1 - \mathrm{j}/\varepsilon_{2p} C_N\left(\frac{\mathrm{j}\Omega_{2p}}{s}\right)}, \tag{27}$$

where ε_{2p} controls the passband ripple in the second passband.

The poles of $A_G(s)$ are given as,

$$s_k = \frac{\mathrm{j}\Omega_{2p}}{a\cos\left(\frac{u_k}{N}\right) - \mathrm{j}b\sin\left(\frac{u_k}{N}\right)}, \tag{28}$$

where

$$u_k = \frac{4k+1}{2}\pi, \tag{29}$$

$$w = \operatorname{arcsinh}\left(\frac{1}{\varepsilon_{2p}}\right), \tag{30}$$

$$a = \cosh\left(\frac{w}{N}\right), \tag{31}$$

$$b = \sinh\left(\frac{w}{N}\right). \tag{32}$$

The corresponding zeros of $G(s)$ are expressed as,

$$s_l = \frac{\mathrm{j}\Omega_{2p}}{\cos\left(\arccos\left(\frac{\pm 1}{\varepsilon_{2p}}\right) + \frac{2\pi k}{N}\right)}, \qquad l = 0,\dots,N-1. \tag{33}$$

Using (10) and A_{2p} in dB, we have,

$$\varepsilon_{2p} = \sqrt{\frac{10^{A_{2p}/20} - 1}{10^{A_{2p}/20} + 1}}. \tag{34}$$

Substituting $l = 0$ into (33), we obtain the value of Ω_{2p} from the notch frequency as,

$$\Omega_{2p} = \Omega_0 \cos(\arccos(1/\varepsilon_{2p})/N). \tag{35}$$

To estimate the order of the allpass filter we use (26).

2.4 Elliptic

Using Elliptic polynomial in (7), we get the following allpass filter,

$$A_G(s) = \frac{1 + \mathrm{j}\varepsilon_{1p}\,\mathrm{cd}\left(N\frac{K_\varepsilon}{K_\Omega}\,\mathrm{cd}^{-1}\left(\frac{s}{\mathrm{j}\Omega_{1p}}, \frac{\Omega_{1p}}{\Omega_{2p}}\right), \varepsilon_{1p}\varepsilon_{2p}\right)}{1 - \mathrm{j}\varepsilon_{1p}\,\mathrm{cd}\left(N\frac{K_\varepsilon}{K_\Omega}\,\mathrm{cd}^{-1}\left(\frac{s}{\mathrm{j}\Omega_{1p}}, \frac{\Omega_{1p}}{\Omega_{2p}}\right), \varepsilon_{1p}\varepsilon_{2p}\right)}, \tag{36}$$

where $\mathrm{cd}(\cdot)$ is the elliptic function, K_ε and K_Ω are the complete elliptic integrals with modulus $\varepsilon_{1p}\varepsilon_{2p}$ and Ω_{1p}/Ω_{2p}, respectively [8].

Unfortunately, for this case there are no closed form equations for the computation of the poles and zeros of $G(s)$.

We use the method proposed in [8], which is based on Landen transformation, to compute the poles of $A_G(s)$.

The order of the allpass filter is estimated as,

$$N = \left\lceil \frac{K_\Omega K'_\varepsilon}{K'_\Omega K_\varepsilon} \right\rceil, \tag{37}$$

where their corresponding complementary elliptic integrals of K_ε and K_Ω are denoted as K'_ε and K'_Ω, respectively.

3. PROPOSED ALGORITHM

In this Section we use the results from Section II to design IIR notch filters, which are causal, stable, and real.

The parameters of the filter are the notch frequency ω_0, the passband ripple in both bands A_{1p} and A_{2p}, and the rejection bandwidth B.

The proposed algorithm has the following steps:

1. Prewarp the notch frequency ω_0 and compute the frequencies Ω_1 and Ω_2 as, (see Fig. 1)

$$\Omega_0 = \tan\left(\frac{\omega_0}{2}\right), \tag{38a}$$

$$\Omega_1 = \tan\left(\frac{\omega_0 - B/2}{2}\right), \tag{38b}$$

$$\Omega_2 = \tan\left(\frac{\omega_0 + B/2}{2}\right). \tag{38c}$$

2. Select the type of the filter (Butterworth, Chebyshev I, Chebyshev II or Elliptic).
3. Compute the order N of the allpass filter using (14), (26), or (37) and (38a)–(38c).
4. Compute the corresponding poles of $A_G(s)$ and design the filters $A_0(s)$ and $A_1(s)$, i.e.

$$\begin{aligned} A_G(s) &= \frac{A_1(s)}{A_0(s)} \\ &= \frac{\alpha}{\alpha^*}\frac{\prod_{k=0}^{n_1-1}(s/p_{ok}^*+1)\prod_{k=0}^{n_2-1}(s/p_{ik}^*+1)}{\prod_{k=0}^{n_1-1}(s/p_{ok}-1)\prod_{k=0}^{n_2-1}(s/p_{ik}-1)}, \end{aligned} \tag{39}$$

where (p_{ik}, p_{ik}^*) and (p_{ok}, p_{ok}^*) are complex conjugate pair of poles in the left and right side, respectively, of the s plane, $n_1 + n_2 = N$, and

$$\alpha = \begin{cases} 1 - \mathrm{j}\varepsilon_{1p}\cos(\frac{N\pi}{2}), & \text{Chebyshev I or Elliptic} \\ 1 & \text{otherwise} \end{cases} \tag{40}$$

The complex conjugate of α is denoted as α^*.

$$A_0(s) = \frac{\alpha^*}{|\alpha|}\frac{\prod_{k=0}^{n_1-1}(s/p_{ok}-1)}{\prod_{k=0}^{n_1-1}(s/p_{ok}^*+1)}, \tag{41}$$

$$A_1(s) = \frac{\alpha}{|\alpha|}\frac{\prod_{k=0}^{n_2-1}(s/p_{ik}^*+1)}{\prod_{k=0}^{n_2-1}(s/p_{ik}-1)}. \tag{42}$$

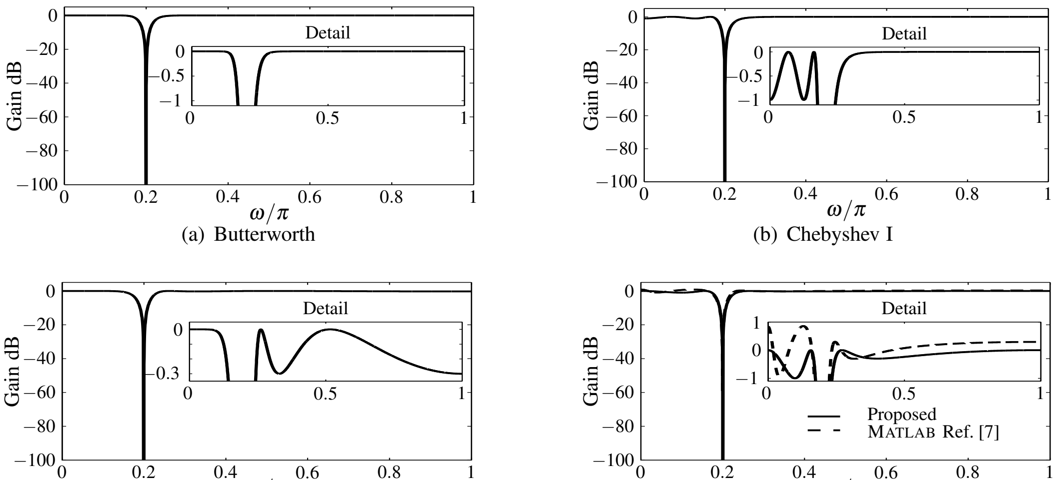
5. Apply bilinear transformation to get the poles in z-plane.
6. Finally, using the digital counterpart of (2), i.e.

$$G_1(z) = \frac{1}{2}\left[A_0^2(z) + A_1^2(z)\right], \tag{43}$$

design the IIR notch filter.

Example. This example illustrates the design of notch IIR filter using Butterworth, Chebyshev I, Chebyshev II and Elliptic polynomials. The parameters of the design are: the notch frequency $\omega_0 = 0.2\pi$, the first passband ripple $A_{1p} = 1$ dB, the second passband ripple $A_{2p} = 0.3$ dB and $B = 0.1\pi$.

1. From (38) it follows, $\Omega_0 = 0.324920$, $\Omega_1 = 0.240079$ and $\Omega_2 = 0.414213$.
2. For the Chebyshev and Elliptic filter, we calculate the values of ε_{1p} and ε_{2p}, which are shown in Table 1.
3. We estimate the order of the allpass filter for all types of filters and the frequencies Ω_{1p} and Ω_{2p} for Chebyshev and Elliptic filters given in Table 1.
4. We compute the poles of $A_G(s)$.
5. We transform the poles from s-plane to z-plane.
6. Finally, we get the desired notch filters using (43). Figure 2 shows the magnitude responses of the designed filters for different polynomials.



(a) Butterworth (b) Chebyshev I (c) Chebyshev II (d) Elliptic

Figure 2: Magnitude responses of the designed filters.

	Butt.	Cheb. I	Cheb. II	Elliptic
ε_{1p}		0.239794		0.240885
ε_{2p}			0.131407	0.130812
N	8	4	4	3
Ω_{1p}		0.284545		0.279670
Ω_{2p}			0.402877	0.407939

Table 1: Order of the allpass filter and values of ε_{1p}, ε_{2p}, Ω_{1p} and Ω_{2p}.

Using Butterworth polynomial, the notch IIR filter has maximally flat magnitude response at the frequencies $\omega = 0$ and $\omega = \pi$ (Fig. 2(a)).

The magnitude response of the notch filter designed with Chebyshev type I polynomials is equiripple in the first passband and is maximally flat in the second passband (Fig. 2(b)). Similarly, using Chebyshev II polynomial, the magnitude response of the notch filter is equiripple in the second passband (Fig. 2(c)).

Finally, for elliptic polynomials the magnitude response is equiripple in both passbands (see Fig. 2(d)).

We compare the proposed design with MATLAB design (file `iirlpnormc.m`) which uses least-pth norm constrained optimization [7].

Figure 2(d) shows the magnitude responses of the designed filter using elliptic polynomials and `iirlpnormc` function. We can notice that the proposed filter has smaller ripples than the filter designed using `iirlpnormc.m` for the same order and maximum pole radius 0.83.

4. CONCLUSIONS

This paper presents the design of IIR notch filter based on proposed allpass analog filter and bilinear transform. Unlike other methods, our approach can design either maximally flat or equiripple notch filters in one or both passbands. The Butterworth polynomial results in maximally flat filter while Elliptic polynomial results in equiripple characteristic in both passbands. Similarly, Chebyshev I and II polynomials are equiripple in one passband and maximally flat in another one.

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