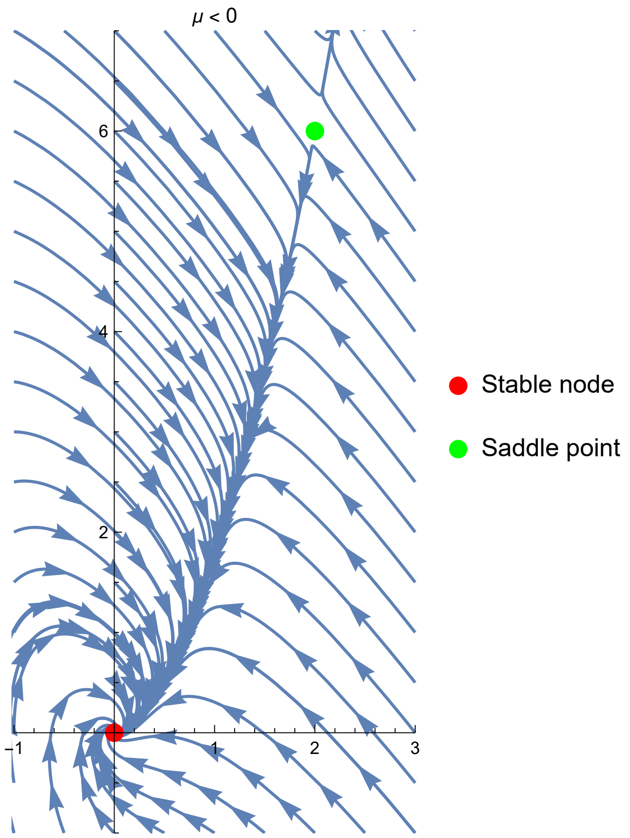
In[•]:=
```
Clear["Global`*"]
minx = -1; maxx = 3; miny = -1;  maxy = 7;  μ = -1;
sol = Solve[{0 == μ * x + y - x^2, 0 == -x + μ * y + 2 x^2}, {x, y}];
s[x0_, y0_] := NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2,
     y'[t] == -x[t] + μ * y[t] + 2 x[t]^2, x[0] == x0, y[0] == y0}, {x, y}, {t, 0, 10}];
initialCondition = Join[Table[{0, y}, {y, miny, maxy, 0.5}],
    Table[{minx, y}, {y, miny, maxy, 0.5}], Table[{maxx, y}, {y, miny, maxy, 0.5}],
    Table[{x, miny}, {x, minx, maxx, 0.5}], Table[{x, maxy}, {x, minx, maxx, 0.5}]];
Show[Table[ParametricPlot[Evaluate[{x[t], y[t]} /.
       s[initialCondition〚i, 1〛, initialCondition〚i, 2〛]], {t, 0, 10},
     PlotRange → {{minx, maxx}, {miny, maxy}}], {i, Length[initialCondition]}] /.
   Line[x_] :→ {Arrowheads[{0., 0.05, 0.05, 0.05, 0.05, 0.}], Arrow[x]},
  ListPlot[{{sol〚1, 1, 2〛, sol〚1, 2, 2〛}}, PlotMarkers → {Automatic, 10},
   PlotStyle → {Red}, PlotLegends → {"Stable node"}],
  ListPlot[{{sol〚2, 1, 2〛, sol〚2, 2, 2〛}}, PlotMarkers → {Automatic, 10},
   PlotStyle → {Green}, PlotLegends → {"Saddle point"}],
  PlotLabel → "μ < 0"]
```

Out[•]=



In[•]:=
```
minx = -1; maxx = 1; miny = -1;  maxy = 1; μ = 0;
sol = Solve[{0 == μ * x + y - x^2, 0 == -x + μ * y + 2 x^2}, {x, y}];
s[x0_, y0_] := NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2,
    y'[t] == -x[t] + μ * y[t] + 2 x[t]^2, x[0] == x0, y[0] == y0}, {x, y}, {t, 0, 1000}];
initialCondition = Join[Table[{0, y}, {y, miny, maxy, 0.1}],
   Table[{minx, y}, {y, miny, maxy, 0.1}], Table[{maxx, y}, {y, miny, maxy, 0.1}],
   Table[{x, miny}, {x, minx, maxx, 0.1}], Table[{x, maxy}, {x, minx, maxx, 0.1}]];
Show[Table[ParametricPlot[Evaluate[{x[t], y[t]} /.
       s[initialCondition[[i, 1]], initialCondition[[i, 2]]]], {t, 0, 10},
     PlotRange → {{minx, maxx}, {miny, maxy}}], {i, Length[initialCondition]}] /.
   Line[x_] :→ {Arrowheads[{0., 0.05, 0.05, 0.05, 0.05, 0.05, 0.05,
       0.05, 0.05, 0.05, 0.05, 0.05, 0.}], Arrow[x]},
  ParametricPlot[Evaluate[{x[t], y[t]} /. s[0.1, 0.1], {t, 0, 1000}],
   PlotRange → {{minx, maxx}, {miny, maxy}}, PlotStyle → {Orange},
   PlotLegends → {"A slow trajectory attracted to fixpoint (0,0)."}],
  ListPlot[{{sol[[1, 1, 2]], sol[[1, 2, 2]]}}, PlotMarkers → {Automatic, 10},
   PlotStyle → {Red}, PlotLegends → {"Stable spiral"}],
  ListPlot[{{sol[[2, 1, 2]], sol[[2, 2, 2]]}}, PlotMarkers → {Automatic, 10},
   PlotStyle → {Green}, PlotLegends → {"Saddle point"}],
  PlotLabel → "μ = 0, stable sprial with trajectories that look like closed orbits.
     Analytically the red fix point is a center but maybe numerical errors?"]
```

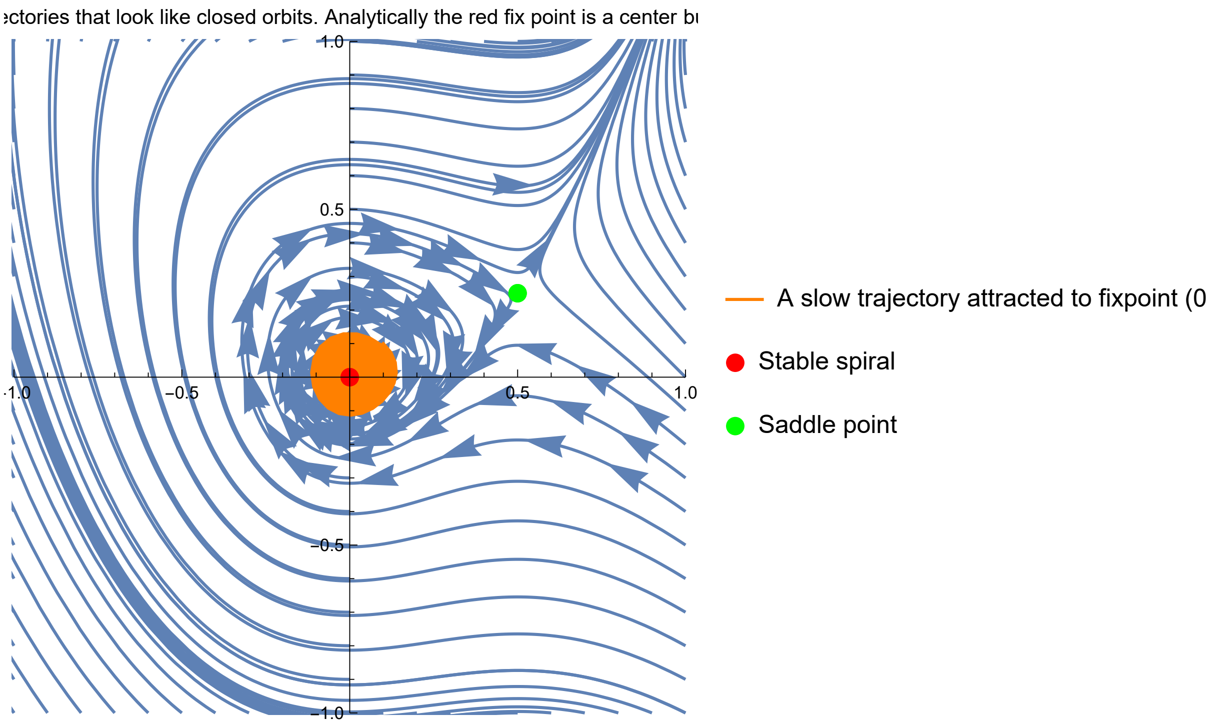
NDSolve: Error test failure at t == 27.306227206908037`; unable to continue.

NDSolve: Error test failure at t == 27.647589807288927`; unable to continue.

NDSolve: Error test failure at t == 27.941521806571348`; unable to continue.

General: Further output of NDSolve::nderr will be suppressed during this calculation.

Out[•]=



In[ ]:=
```
minx = -1; maxx = 1; miny = -1;  maxy = 1; μ = 0.03;
sol = Solve[{0 == μ * x + y - x^2, 0 == -x + μ * y + 2 x^2}, {x, y}];
s[x0_, y0_] := NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2,
    y'[t] == -x[t] + μ * y[t] + 2 x[t]^2, x[0] == x0, y[0] == y0}, {x, y}, {t, 0, 1000}];
initialCondition = Join[Table[{0, y}, {y, miny, maxy, 0.1}],
   Table[{minx, y}, {y, miny, maxy, 0.1}], Table[{maxx, y}, {y, miny, maxy, 0.1}],
   Table[{x, miny}, {x, minx, maxx, 0.1}], Table[{x, maxy}, {x, minx, maxx, 0.1}]];
Show[Table[ParametricPlot[Evaluate[{x[t], y[t]} /.
      s[initialCondition[[i, 1]], initialCondition[[i, 2]]]], {t, 0, 10},
    PlotRange → {{minx, maxx}, {miny, maxy}}], {i, Length[initialCondition]}] /.
  Line[x_] :→ {Arrowheads[{0., 0.05, 0.05, 0.05, 0.05, 0.05, 0.05,
      0.05, 0.05, 0.05, 0.05, 0.05, 0.}], Arrow[x]},
 ParametricPlot[Evaluate[{x[t], y[t]} /. s[0.05, 0.05], {t, 0, 1000}],
  PlotRange → {{minx, maxx}, {miny, maxy}}, PlotStyle → {Orange},
  PlotLegends → {"Trajectory approching a limit cycle"}],
 ListPlot[{{sol[[1, 1, 2]], sol[[1, 2, 2]]}}, PlotMarkers → {Automatic, 10},
  PlotStyle → {Red}, PlotLegends → {"Unstable spiral"}],
 ListPlot[{{sol[[2, 1, 2]], sol[[2, 2, 2]]}}, PlotMarkers → {Automatic, 10},
  PlotStyle → {Green}, PlotLegends → {"Saddle point"}],
 PlotLabel → "0 < μ < μ_c, limit cycles"]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

NDSolve: Error test failure at t == 26.786097258138987`; unable to continue.

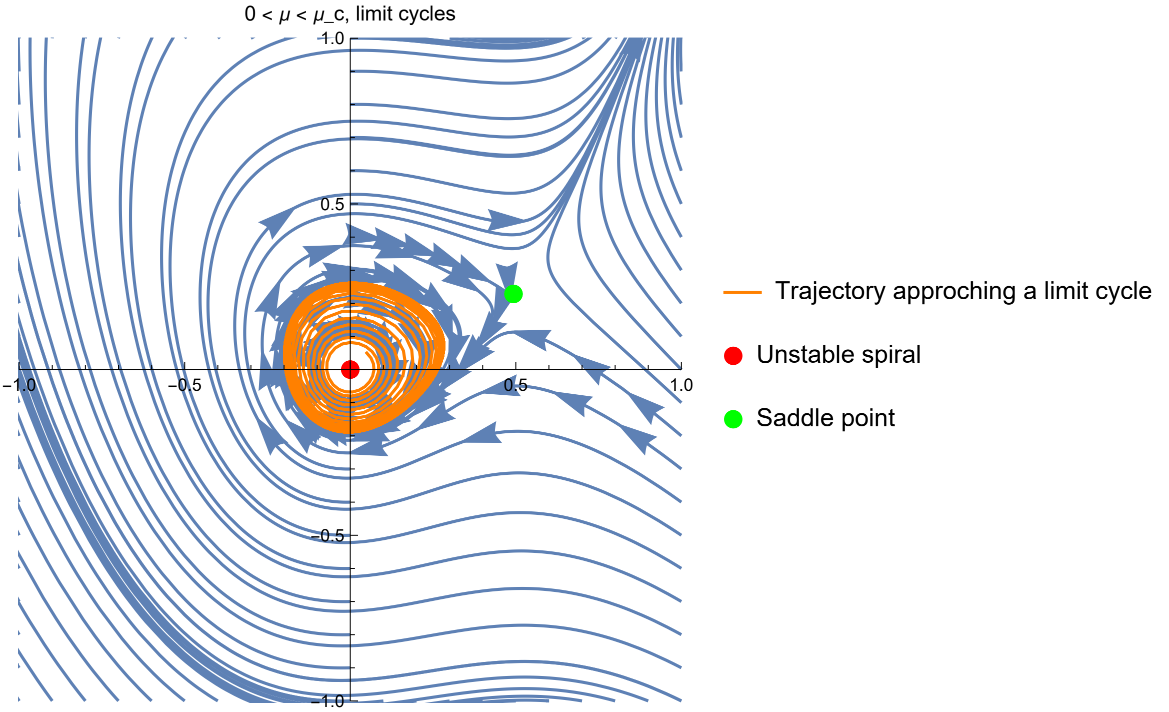
NDSolve: Error test failure at t == 27.14014608421554`; unable to continue.

NDSolve: Error test failure at t == 27.60082945891323`; unable to continue.

General: Further output of NDSolve::nderr will be suppressed during this calculation.

NDSolve: Repeated convergence test failure at t == 23.50827746067483`; unable to continue.

Out[ ]=



In[ ]:=
```
minx = -1; maxx = 1; miny = -1;  maxy = 1; μ = 0.066;
sol = Solve[{0 == μ * x + y - x^2, 0 == -x + μ * y + 2 x^2}, {x, y}];
s[x0_, y0_] := NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2,
    y'[t] == -x[t] + μ * y[t] + 2 x[t]^2, x[0] == x0, y[0] == y0}, {x, y}, {t, 0, 1000}];
initialCondition = Join[Table[{0, y}, {y, miny, maxy, 0.1}],
   Table[{minx, y}, {y, miny, maxy, 0.1}], Table[{maxx, y}, {y, miny, maxy, 0.1}],
   Table[{x, miny}, {x, minx, maxx, 0.1}], Table[{x, maxy}, {x, minx, maxx, 0.1}]];
Show[Table[ParametricPlot[Evaluate[{x[t], y[t]} /.
      s[initialCondition[[i, 1]], initialCondition[[i, 2]]]], {t, 0, 10},
    PlotRange → {{minx, maxx}, {miny, maxy}}], {i, Length[initialCondition]}] /.
  Line[x_] :→ {Arrowheads[{0., 0.05, 0.05, 0.05, 0.05, 0.05, 0.05,
      0.05, 0.05, 0.05, 0.05, 0.05, 0.}], Arrow[x]},
 ParametricPlot[Evaluate[{x[t], y[t]} /. s[0.1, 0.1], {t, 0, 1000}],
  PlotRange → {{minx, maxx}, {miny, maxy}}, PlotStyle → {Orange},
  PlotLegends → {"Trajectory reaching a homoclinic orbit"}],
 ListPlot[{{sol[[1, 1, 2]], sol[[1, 2, 2]]}}, PlotMarkers → {Automatic, 10},
  PlotStyle → {Red}, PlotLegends → {"Unstable spiral"}],
 ListPlot[{{sol[[2, 1, 2]], sol[[2, 2, 2]]}}, PlotMarkers → {Automatic, 10},
  PlotStyle → {Green}, PlotLegends → {"Saddle point"}],
 PlotLabel → "μ = μ_c, homoclinic orbit"]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

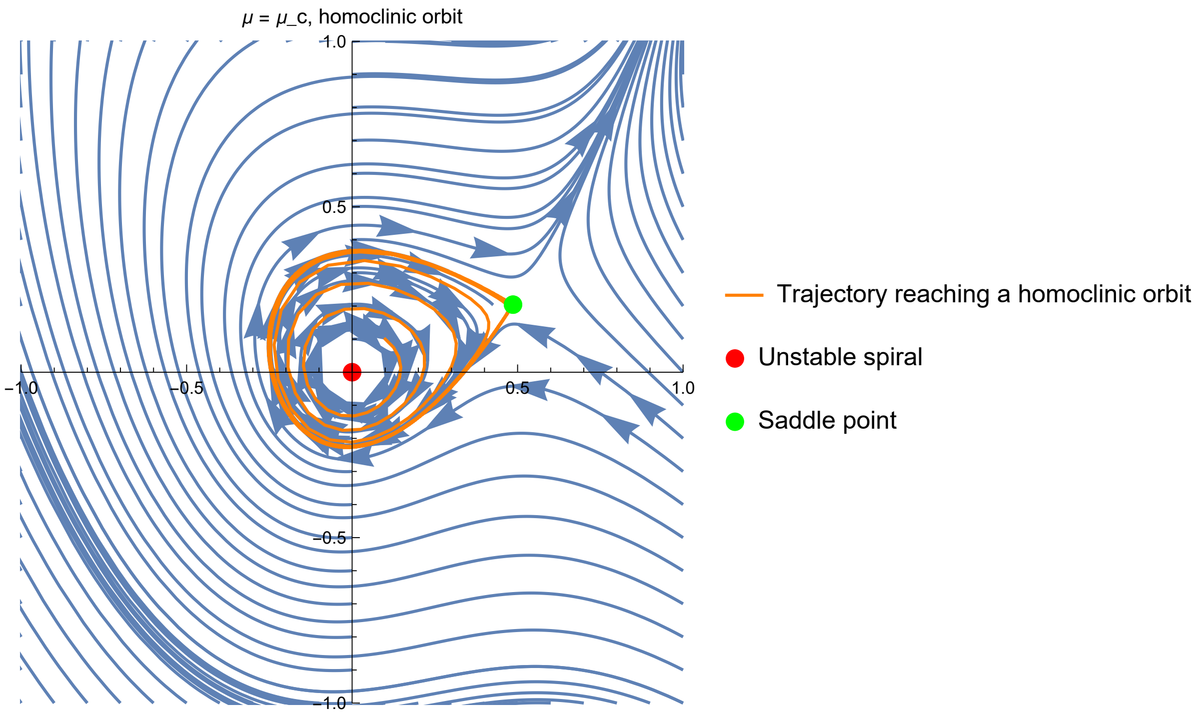
NDSolve: Error test failure at t == 26.522158985544042`; unable to continue.

NDSolve: Error test failure at t == 26.533396016902866`; unable to continue.

NDSolve: Error test failure at t == 26.685878252666257`; unable to continue.

General: Further output of NDSolve::nderr will be suppressed during this calculation.

Out[ ]=



In[∘]:=
```
minx = -1; maxx = 1; miny = -1;  maxy = 1; μ = 0.2;
sol = Solve[{0 == μ * x + y - x^2, 0 == -x + μ * y + 2 x^2}, {x, y}];
s[x0_, y0_] := NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2,
    y'[t] == -x[t] + μ * y[t] + 2 x[t]^2, x[0] == x0, y[0] == y0}, {x, y}, {t, 0, 100}];
initialCondition = Join[Table[{0, y}, {y, miny, maxy, 0.1}],
   Table[{minx, y}, {y, miny, maxy, 0.1}], Table[{maxx, y}, {y, miny, maxy, 0.1}],
   Table[{x, miny}, {x, minx, maxx, 0.1}], Table[{x, maxy}, {x, minx, maxx, 0.1}]];
Show[Table[ParametricPlot[Evaluate[{x[t], y[t]} /.
       s[initialCondition[[i, 1]], initialCondition[[i, 2]]]], {t, 0, 10},
    PlotRange → {{minx, maxx}, {miny, maxy}}], {i, Length[initialCondition]}] /.
  Line[x_] :→ {Arrowheads[{0., 0.05, 0.05, 0.05, 0.05, 0.05, 0.05,
       0.05, 0.05, 0.05, 0.05, 0.05, 0.}], Arrow[x]},
 ParametricPlot[Evaluate[{x[t], y[t]} /. s[0.01, 0.01], {t, 0, 100}],
  PlotRange → {{minx, maxx}, {miny, maxy}}, PlotStyle → {Orange},
  PlotLegends → {"Trajectory escaping to a unstable manifold of the saddle point"}],
 ListPlot[{{sol[[1, 1, 2]], sol[[1, 2, 2]]}}, PlotMarkers → {Automatic, 10},
  PlotStyle → {Red}, PlotLegends → {"Unstable spiral"}],
 ListPlot[{{sol[[2, 1, 2]], sol[[2, 2, 2]]}}, PlotMarkers → {Automatic, 10},
  PlotStyle → {Green}, PlotLegends → {"Saddle point"}],
 PlotLabel → "μ > μ_c, unstable spiral"]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

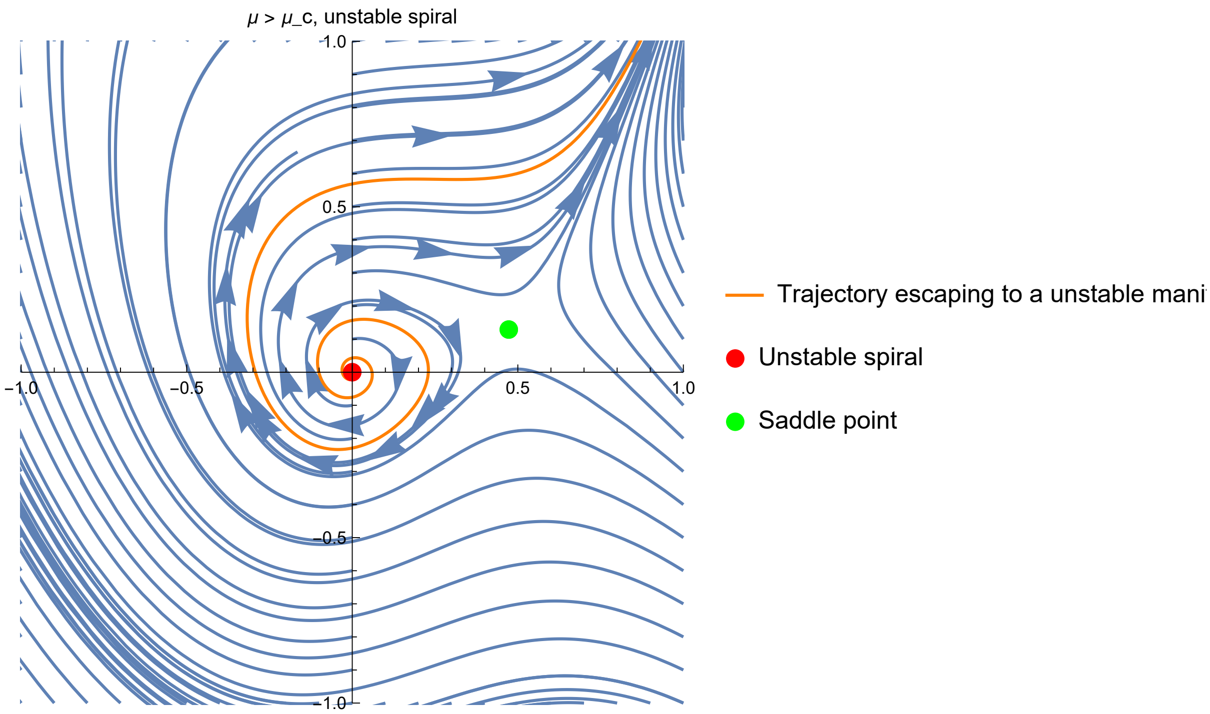
NDSolve: Error test failure at t == 24.141575999317233`; unable to continue.

NDSolve: Error test failure at t == 24.23493843398976`; unable to continue.

NDSolve: Error test failure at t == 24.60137975165016`; unable to continue.

General: Further output of NDSolve::nderr will be suppressed during this calculation.

Out[∘]=



In[ ]:=
```
s =.
sol = DSolve[{x'[t] == u * x[t], x[0] == γ}, x[t], t] // Flatten
```

Out[ ]= $\{x[t] \to e^{t\,u}\,\gamma\}$

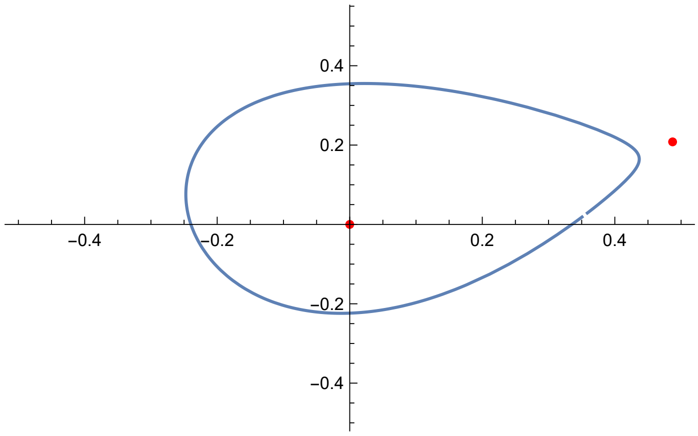
In[ ]:=
```
Clear["Global`*"]
f[x_, y_] := μ * x + y - x^2;
g[x_, y_] := -x + μ * y + 2 x^2;
sol = Solve[f[x, y] == 0 && g[x, y] == 0, {x, y}];
J[x_, y_] := {{D[f[x, y], x], D[f[x, y], y]}, {D[g[x, y], x], D[g[x, y], y]}};
eval = J[x, y] /. sol[[2]] // Eigenvalues
evec = J[x, y] /. sol[[2]] // Eigenvectors;
u = eval[[2]]
```

Out[ ]= $\left\{\frac{-1+2\mu-\sqrt{5+9\mu^2+4\mu^3+\mu^4}}{2+\mu}, \frac{-1+2\mu+\sqrt{5+9\mu^2+4\mu^3+\mu^4}}{2+\mu}\right\}$

Out[ ]= $\frac{-1+2\mu+\sqrt{5+9\mu^2+4\mu^3+\mu^4}}{2+\mu}$

In[305]:=
```
Clear["Global`*"]
criticalMu = 0.066;
μ = 0.060;
tMin = 170;
tMax = tMin - 0.1 + 9.8624007122786`;
f[x_, y_] := μ x + y - x^2;
g[x_, y_] := -x + μ y + 2 x^2;
fp = Solve[{f[x, y] == 0, g[x, y] == 0}, {x, y}];
data = {{fp[[1, 1, 2]], fp[[1, 2, 2]]}, {fp[[2, 1, 2]], fp[[2, 2, 2]]}};
line = Fit[data, {1, x}, x];
p1 = ListPlot[data, PlotStyle → Red];
s = NDSolve[{x'[t] == μ * x[t] + y[t] - x[t]^2,
    y'[t] == -x[t] + μ * y[t] + 2 x[t]^2, x[0] == y[0] == 0.1}, {x, y}, {t, tMin, tMax}];
p3 = ParametricPlot[Evaluate[{x[t], y[t]} /. s], {t, tMin, tMax},
   PlotRange → {{-1, 1}, {-1, 1}}] ;
Show[p1, p3, PlotRange → {{-0.5, 0.5}, {-0.5, 0.5}}]
tIntersect = Solve[{y[t] == fp[[2, 2, 2]]} /. s[[1, 2]], t][[1, 1, 2]];
xIf = s[[1, 1, 2]];
γ = xIf[tIntersect]
xVal = Log[Abs[μ - criticalMu]]
yVal = Log[γ]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[318]=


Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[321]= 0.413465

Out[322]= -5.116

Out[323]= -0.883183

In[81]:=
```
plotData = {{-5.115995809754081, -0.8831824175751058},
    {-5.115995809754081, -0.8831824175751058}, {-5.521460917862245,
     -0.8397348233958074}, {-5.809142990314027, -0.8169247714703272},
    {-6.2146080984221905, -1.6161665232963962}, {-6.2146080984221905,
     -1.6161665232963962}, {-6.2146080984221905, -1.6161665232963962}};
line2 = Fit[plotData, {1, x}, x]
gammaPlot = Show[ListPlot[plotData, PlotStyle → Red],
   Plot[line2, {x, -10, 0}, PlotLegends → {"log(γ) vs. log|μ-μ_c|, y = 2.8 + 0.7 x"}],
   PlotRange → {{-10, 0}, {-5, 5}}];
```

Out[82]= 2.78486 + 0.690574 x

In[59]:=
```
tConst = 0;
f = s[[1, 1, 2]];
g = s[[1, 2, 2]];
xStart = s[[1, 1, 2]][tMin];
yStart = s[[1, 2, 2]][tMin];
temp = FindRoot[f[t] + g[t] == xStart + yStart, {t, tMin + tConst}];

xStart
s[[1, 1, 2]][temp[[1, 2]]]
yStart
s[[1, 2, 2]][temp[[1, 2]]]
tPeriod = Abs[tMin - temp[[1, 2]]];
{Abs[μ - criticalMu], tPeriod}
```

Out[65]= 0.350527

Out[66]= 0.350527

Out[67]= 0.0186812

Out[68]= 0.0186812

Out[70]= {0.006, 0.}

In[84]:=
```
plotDataPeriodTime = {{0.006, 9.8624}, {0.005, 10.1301},
    {0.004, 10.4571}, {0.003, 10.8774}, {0.002, 11.4664}, {0.001, 12.4583}};
logPlotDataPeriodTime = Log[plotDataPeriodTime];
line3 = Fit[logPlotDataPeriodTime, {1, x}, x]
periodPlot = Show[ListPlot[logPlotDataPeriodTime, PlotStyle → Red],
   Plot[line3, {x, -10, 0}, PlotStyle → Orange, PlotLegends →
     {"log(Tμ) vs. log|μ-μ_c|, y = 1.6 - 0.1 x"}], PlotRange → {{-10, 0}, {-5, 5}}];
```

Out[86]= 1.62606 - 0.130313 x

In[76]:= `Show[gammaPlot, periodPlot, PlotRange → {{-10, 0}, {-5, 5}}]`

Out[76]=