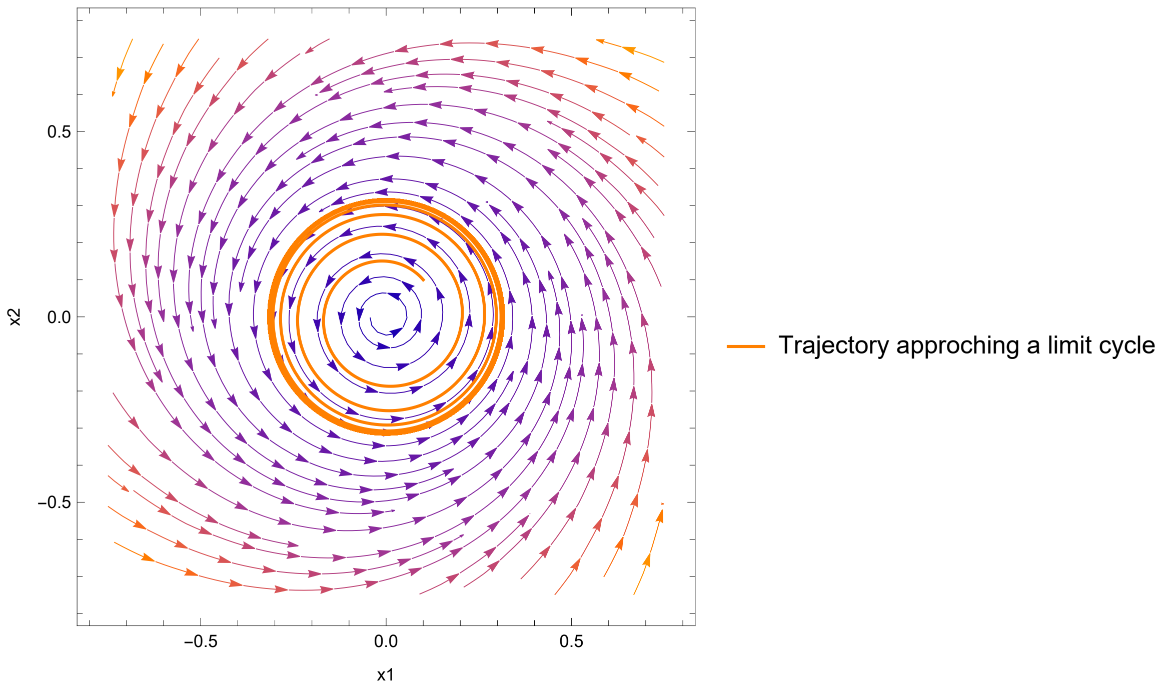
In[]:=
```
Clear["Global`*"]
r[r_] := μ*r - r^3;
θ[r_] := ω + ν*r^2;
s = Solve[r[r] == 0, {r}]
velocity = Sqrt[μ]*θ[Sqrt[μ]];
length = 2*π*Sqrt[μ];
periodTime = length / velocity
```

Out[]= $\{\{r \to 0\}, \{r \to -\sqrt{\mu}\}, \{r \to \sqrt{\mu}\}\}$

Out[]= $\frac{2\pi}{\mu \nu + \omega}$

In[]:=
```
Clear["Global`*"]
X1[x1_, x2_] := 1/10*x1 - x2^3 - x1*x2^2 - x1^2*x2 - x2 - x1^3;
X2[x1_, x2_] := x1 + 1/10*x2 + x1*x2^2 + x1^3 - x2^3 - x1^2*x2;
streamPlot = StreamPlot[{X1[x1, x2], X2[x1, x2]},
   {x1, -.75, .75}, {x2, -.75, .75}, FrameLabel → {"x1", "x2"}];
X1T := x1'[t] == 1/10*x1[t] - x2[t]^3 - x1[t]*x2[t]^2 -
    x1[t]^2*x2[t] - x2[t] - x1[t]^3;
X2T := x2'[t] == x1[t] + 1/10*x2[t] + x1[t]*x2[t]^2 +
    x1[t]^3 - x2[t]^3 - x1[t]^2*x2[t];
limitCycle = NDSolve[{X1T, X2T, x1[0] == x2[0] == 0.1}, {x1, x2}, {t, 0, 100}];
trajectory = ParametricPlot[
   Evaluate[{x1[t], x2[t]} /. {limitCycle}, {t, 0, 100}, PlotStyle → {Orange}],
   PlotLegends → {"Trajectory approching a limit cycle"}];
Show[streamPlot, trajectory]
```

Out[]=


```
In[•]:= Clear["Global`*"]
dr[r_] := μ * r - r^3;
dθ[r_] := ω + ν * r^2;
X1[x1_, x2_] := 1 / 10 * x1 - x2^3 - x1 * x2^2 - x1^2 * x2 - x2 - x1^3;
X2[x1_, x2_] := x1 + 1 / 10 * x2 + x1 * x2^2 + x1^3 - x2^3 - x1^2 * x2;
(*  r^2 = x^2 + y^2
    2rr' = 2xx' + 2yy'
     x' = (rr' - yy')/x. (1)
     θ = arctan(y/x)
      θ' = (xy' - yx')/r^2
       → x' = xy' - r^2θ'. (2)
          Put (1) equal to (2) and solve for x'.
        → x' = x/r*r' - yθ'.
           Similarly for y we get:
           y' = y/r*r' - xθ'. *)
dx1[x1_, x2_] :=
  x1 / Sqrt[x1^2 + x2^2] * dr[Sqrt[x1^2 + x2^2]] - x2 * dθ[Sqrt[x1^2 + x2^2]];
dx2[x1_, x2_] := x2 / Sqrt[x1^2 + x2^2] * dr[Sqrt[x1^2 + x2^2]] -
   x1 * dθ[Sqrt[x1^2 + x2^2]];
Simplify[dx1[x1, x2]]
X1[x1, x2]
(* From looking at the equations we get: μ = 1/10, ω = 1 and ν = 1 *)
```

Out[•]= $-x1^3 + x1\left(-x2^2 + \mu\right) - x1^2\, x2\, \nu - x2\left(x2^2\, \nu + \omega\right)$

Out[•]= $\frac{x1}{10} - x1^3 - x2 - x1^2\, x2 - x1\, x2^2 - x2^3$

In[•]:=
```
Clear["Global`*"]
μ = 1 / 10; ω = 1 ; ν = 1; T = 2π/(μ ν + ω);
F1[x1_, x2_] := 1 / 10 * x1 - x2^3 - x1 * x2^2 - x1^2 * x2 - x2 - x1^3;
F2[x1_, x2_] := x1 + 1 / 10 * x2 + x1 * x2^2 + x1^3 - x2^3 - x1^2 * x2;
J[x1_, x2_] :=
  {{D[F1[x1, x2], x1], D[F1[x1, x2], x2]}, {D[F2[x1, x2], x1], D[F2[x1, x2], x2]}};
M11[x1, x2] = J[x1, x2][[1]][[1]] M11 + J[x1, x2][[1]][[2]] M21
M12[x1, x2] = J[x1, x2][[1]][[1]] M12 + J[x1, x2][[1]][[2]] M22
M21[x1, x2] = J[x1, x2][[2]][[1]] M11 + J[x1, x2][[2]][[2]] M21
M22[x1, x2] = J[x1, x2][[2]][[1]] M12 + J[x1, x2][[2]][[2]] M22
s = NDSolve[{x1'[t] == 1 / 10 * x1[t] - x2[t]^3 -
      x1[t] * x2[t]^2 - x1[t]^2 * x2[t] - x2[t] - x1[t]^3,  x2'[t] ==
     x1[t] + 1 / 10 * x2[t] + x1[t] * x2[t]^2 + x1[t]^3 - x2[t]^3 - x1[t]^2 * x2[t],
    m11'[t] == m21[t] * (-1 - x1[t]^2 - 2 x1[t] × x2[t] - 3 x2[t]^2) +
      m11[t] * (1/10 - 3 x1[t]^2 - 2 x1[t] × x2[t] - x2[t]^2),
    m12'[t] == m22[t] * (-1 - x1[t]^2 - 2 x1[t] × x2[t] - 3 x2[t]^2) +
      m12[t] * (1/10 - 3 x1[t]^2 - 2 x1[t] × x2[t] - x2[t]^2),
    m21'[t] == m21[t] * (1 / 10 - x1[t]^2 + 2 x1[t] × x2[t] - 3 x2[t]^2) +
      m11[t] * (1 + 3 x1[t]^2 - 2 x1[t] × x2[t] + x2[t]^2),
    m22'[t] == m22[t] * (1 / 10 - x1[t]^2 + 2 x1[t] × x2[t] - 3 x2[t]^2) +
      m12[t] * (1 + 3 x1[t]^2 - 2 x1[t] × x2[t] + x2[t]^2),
    x1[0] == Sqrt[μ], x2[0] == 0, m11[0] == m22[0] == 1, m12[0] == m21[0] == 0},
   {x1, x2, m11, m12, m21, m22}, {t, 0, T}];
Plot[{Evaluate[x1[t] /. s], Evaluate[x2[t] /. s], Evaluate[m11[t] /. s],
  Evaluate[m12[t] /. s], Evaluate[m21[t] /. s], Evaluate[m22[t] /. s]},
 {t, 0, T}, PlotLegends → {"X1", "X2", "M11", "M12", "M21", "M22"}]
M = {m11[T], m12[T], m21[T], m22[T]} /. s
λ = 1 / T * Log[Eigenvalues[{{M[[1]][[1]], M[[1]][[2]]}, {M[[1]][[3]], M[[1]][[4]]}}]]
```

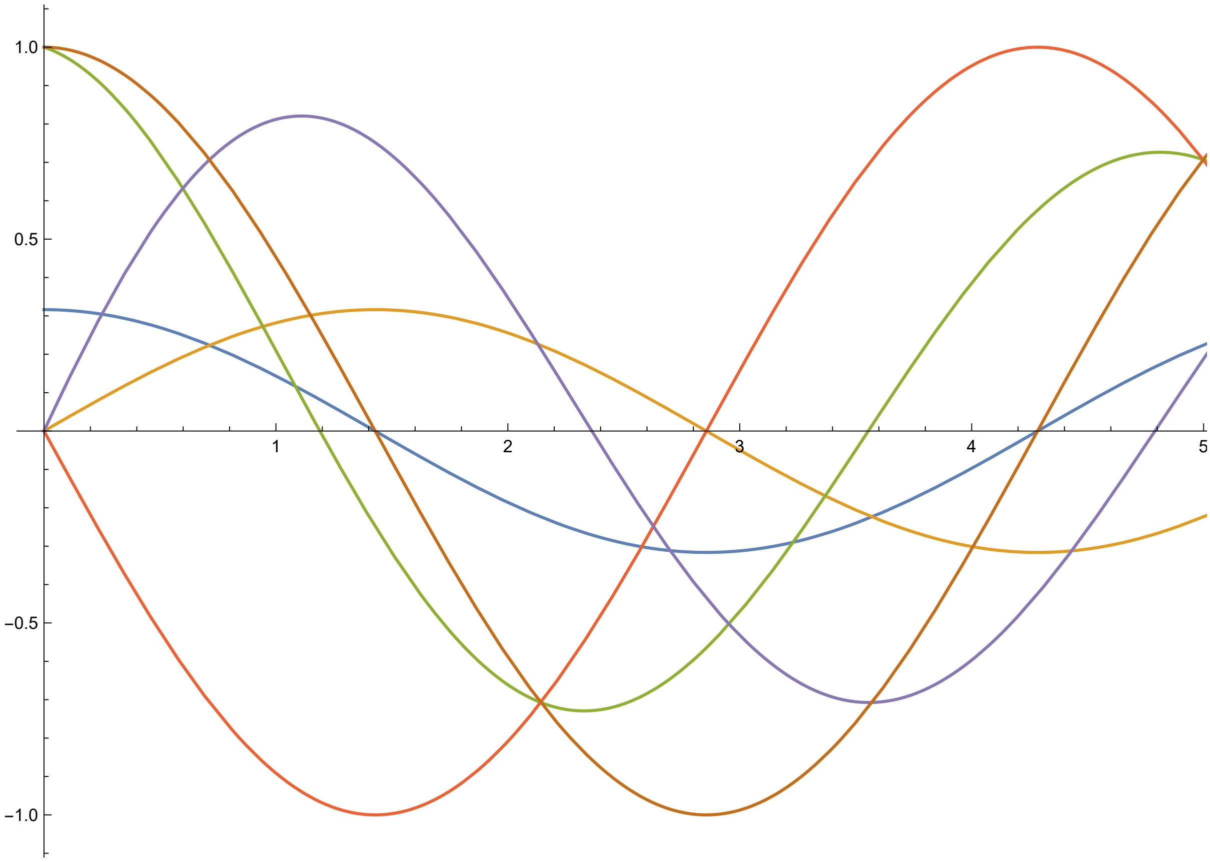
Out[•]= $M21\left(-1 - x1^2 - 2\,x1\,x2 - 3\,x2^2\right) + M11\left(\frac{1}{10} - 3\,x1^2 - 2\,x1\,x2 - x2^2\right)$

Out[•]= $M22\left(-1 - x1^2 - 2\,x1\,x2 - 3\,x2^2\right) + M12\left(\frac{1}{10} - 3\,x1^2 - 2\,x1\,x2 - x2^2\right)$

Out[•]= $M21\left(\frac{1}{10} - x1^2 + 2\,x1\,x2 - 3\,x2^2\right) + M11\left(1 + 3\,x1^2 - 2\,x1\,x2 + x2^2\right)$

Out[•]= $M22\left(\frac{1}{10} - x1^2 + 2\,x1\,x2 - 3\,x2^2\right) + M12\left(1 + 3\,x1^2 - 2\,x1\,x2 + x2^2\right)$

Out[ ]=



Out[ ]= $\{\{0.319053, 2.12317\times10^{-8}, 0.680947, 1.\}\}$

Out[ ]= $\{5.78753\times10^{-9}, -0.2\}$

In[1837]:=
```
Clear["Global`*"]
μ = 1 / 10; ω = 1 ; ν = 1; T = 2π/(μ ν + ω);
r[r_] := μ * r - r^3;
θ[r_] := ω + ν * r^2;
Jp[r_] := {{D[r[r], r], D[r[r], θ]}, {D[θ[r], r], D[θ[r], θ]}};
Jp = Jp[r];
Mpolar[r_] := IdentityMatrix[2].MatrixExp[Jp];
Mpolar = Mpolar[r]
JG[x1_, x2_] := {{D[Sqrt[x1^2 + x2^2], x1], D[Sqrt[x1^2 + x2^2], x2]},
   {D[ArcTan[x1, x2], x1], D[ArcTan[x1, x2], x2]}};
JG = JG[x1, x2] /. {x1 → r * Cos[θ], x2 → r * Sin[θ]}
JGinv[r_] :=
  {{D[r * Cos[θ], r], D[r * Cos[θ], θ]}, {D[r * Sin[θ], r], D[r * Sin[θ], θ]}};
JGinv = JGinv[r]
Mcart = JGinv.Mpolar.JG /. {r → Sqrt[μ], θ → 0};
Mcart = Simplify[Mcart]
λ = 1 / T * Log[Eigenvalues[Mpolar] /. r → Sqrt[μ]]
λ = 1 / T * Log[Eigenvalues[Mcart] /. r → Sqrt[μ]]
```

Out[1844]= $\left\{\left\{e^{\frac{1}{10}-3r^2}, 0\right\}, \left\{\frac{20\, e^{-3r^2}\left(-e^{1/10}+e^{3r^2}\right) r}{-1+30 r^2}, 1\right\}\right\}$

Out[1846]= $\left\{\left\{\frac{r\,\text{Cos}[\theta]}{\sqrt{r^2\,\text{Cos}[\theta]^2+r^2\,\text{Sin}[\theta]^2}}, \frac{r\,\text{Sin}[\theta]}{\sqrt{r^2\,\text{Cos}[\theta]^2+r^2\,\text{Sin}[\theta]^2}}\right\}, \left\{-\frac{r\,\text{Sin}[\theta]}{r^2\,\text{Cos}[\theta]^2+r^2\,\text{Sin}[\theta]^2}, \frac{r\,\text{Cos}[\theta]}{r^2\,\text{Cos}[\theta]^2+r^2\,\text{Sin}[\theta]^2}\right\}\right\}$

Out[1848]= $\{\{\text{Cos}[\theta], -r\,\text{Sin}[\theta]\}, \{\text{Sin}[\theta], r\,\text{Cos}[\theta]\}\}$

Out[1850]= $\left\{\left\{\frac{1}{e^{1/5}}, 0\right\}, \left\{1-\frac{1}{e^{1/5}}, 1\right\}\right\}$

Out[1851]= $\left\{-\frac{11}{100\pi}, 0\right\}$

Out[1852]= $\left\{0, -\frac{11}{100\pi}\right\}$