Optimization in Julia

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What is an optimization problem?

optimization problem: nonlinear form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m_1 \\ & h_i(x) = 0, \quad i = 1, \ldots, m_2 \\ & x \in \mathcal{C} \end{array}$$

- objective f_0
- inequality constraints f_i
- equality constraints h_i
- domain $\mathcal{C}$

advantages:

- easy to formulate

What is an optimization problem?

optimization problem: conic form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & b - Ax \in \mathcal{K}, \end{array}$$

where $\mathcal{K}$ is a **convex cone**:

$$x \in \mathcal{K} \iff rx \in \mathcal{K} \text{ for any } r > 0.$$

advantages:

- efficiently grok the structure of problem
- fast solvers

Structure determines solvers

How should we solve this problem?

- LP solver?
- conic solver?
- nonlinear derivative based solver?
- operator splitting?

Structure determines solvers

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What do we know about this problem's structure?

Structure

useful kinds of structure:

- is the problem convex?
  - is the objective convex?
  - is the domain convex?
  - are the inequality constraints convex?
  - are the equality constraints affine?
- is the problem representable in some standard form?
  - convex: LP, QP, SOCP, SDP, ...
  - nonconvex: MILP, MISOCP ...
- is the problem smooth?
- ...

Optimization in Julia

model specifies structure; solvers exploit structure

- model (*e.g.*, JuMP or Convex)
- glue (MathProgBase)
- solvers (*e.g.*, GLKP, Gurobi, Mosek, ECOS, ...)

JuliaOpt curates all these solvers: `http://www.juliaopt.org/`

Two major approaches

- JuMP: user specifies structure
- Convex: solver detects structure

JuMP vs Convex

JuMP

- lower level interface
- access to advanced solver features
- automatic differentiation
- support for conic and nonlinear programming

Convex

- automatic structure detection
- automatic convexity proof
- can only solve convex problems

JuMP: getting started

demo:

https://github.com/JuliaSystems/ACC-2017/JuMP-intro.ipynb

JuMP features

- automatic differentiation
- solver callbacks

JuMP: rocket control

demo:

`https://github.com/JuliaSystems/ACC-2017/JuMP-Rocket.ipynb`

JuMP extensions

- JuMPeR.jl: for robust optimization
- MultiJuMP.jl: for multi-objective optimization
- JuMPChance.jl: for probabilistic chance constraints
- StochDynamicProgramming.jl: for discrete-time stochastic optimal control problems
- PolyJuMP.jl: for polynomial optimization
- StructJuMP.jl: for block-structured optimization
- NLOptControl.jl: for formulating and solving nonlinear optimal control problems

`Convex.jl`: detecting and exploiting structure

`Convex.jl` is an extensible framework for detecting and exploiting structure.

three kinds of structure (so far):

- convexity
- conic form
- multiconvexity

Induction detects; recursion exploits. Let's see how.

`Convex.jl` in action

demo:

`https://github.com/JuliaSystems/ACC-2017/convex-intro.ipynb`

Basic types

Expressions are defined inductively:

- A **variable** is an expression.

```
x = Variable(4)
```

- A **constant** is an expression.

```
y = [1,2,3,4]
```

- A **composite expression** is formed by applying a **function** to other expressions.

```
f = norm(x - y)^2 + sum(abs(x))
```

Expressions: examples

(using prefix notation)

- $x + y \implies (+, (x, y))$
- $x[1] + x[2] \implies (+, ((\mathrm{index}, (x, 1)), (\mathrm{index}, (x, 2))))$
- $\log(x + 7y) \implies (\log, (+, (x, (*, (7, y)))))$

Every composite expression has a **head** (operation) and a (possibly empty) list of **children** (arguments).

Basic types

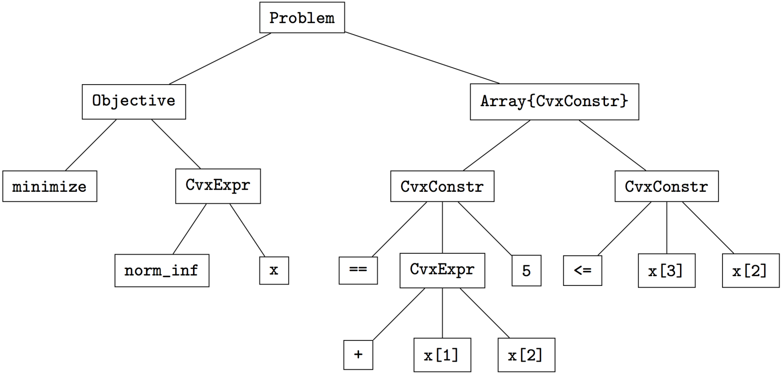
- A **constraint** is a comparison operator ($\leq$, $\geq$, $=$, $\preceq$, $\succeq$) applied to two expressions.

```
X <= 3 I
```

- A **problem** is a sense (minimize, maximize, or satisfy) applied to an objective (any expression), along with a list of constraints.

```
problem = minimize(norm(x - y)^2,
                  #= st =# x >= 1,
                           sum(x) <= 10)
```

Abstract expression tree for an optimization problem



Structure by induction

We use induction (and recursion) to move from properties of

- variables,
- constants, and
- functions

to properties of

- expressions,
- constraints, and
- problems.

Three case studies

- detect convexity
- transform to conic form
- detect multiconvexity

Disciplined convex programming

Disciplined convex programming (DCP) [Grant, Boyd & Ye, 2006] provides a set of simple inductive rules to verify (but not falsify) convexity:

- $f \circ g(x)$ is convex in x if
  - f is convex nondecreasing and g is convex
  - f is convex nonincreasing and g is concave
- $f \circ g(x)$ is concave in x if
  - f is concave nondecreasing and g is concave
  - f is concave nonincreasing and g is convex

cf., the chain rule:

$$(f \circ g)''(x) = f''(g(x))(g(x))^2 + f'(g(x))g''(x)$$

A function is **DCP** if its convexity can be inferred from these composition rules.

DCP: base case

A function `vexity` is defined on each data type (variable, constant, functions, constraints, problems) to return its **vexity**: constant, affine, convex, concave, or not DCP.

base case:

- *Constant.* Constants are **constant**.
- *Variable.* Variables are **affine**.

DCP: inductive rule

inductive rules:

- *Expressions.* Functions each have known **curvature** (convex, concave, or affine) and **monotonicity** (increasing, decreasing, or none) in each of their arguments. Expressions check their convexity by examining convexity of arguments and following composition rules.
- *Constraints.* Constraints check their convexity by determining their left and right hand sides define convex sets.
- *Problems.* Problems check their convexity by verifying the objective and constraints are all convex.

DCP: inductive rule

Composition rules are implemented as **arithmetic on vexities**:

$$\underbrace{\text{convex function}}_{\texttt{ConvexVexity}+}\ \underbrace{\text{nondecreasing}}_{\texttt{NonDecreasing}}\ \underbrace{\text{in}}_{*}\ \underbrace{\text{convex expression}}_{\texttt{ConvexVexity}}\ \underbrace{\text{is}}_{==}\ \underbrace{\text{convex}}_{\texttt{ConvexVexity}}$$

```
function vexity(x::AbstractExpr)
  monotonicities = monotonicity(x)
  vex = curvature(x)
  for i = 1:length(x.children)
    vex += monotonicities[i] * vexity(x.children[i])
  end
  return vex
end
```

DCP expressions might as well be convex

Observe:

- if f is convex and nonincreasing and g is concave,
- then define $\tilde{f}(x) = f(-x)$, $\tilde{g}(x) = -g(x)$
- so

$$f(g(x)) = f(-(-g(x))) = \tilde{f}(\tilde{g}(x)),$$

  $\tilde{f}$ is convex and nondecreasing and $\tilde{g}$ is convex.

So let's suppose all functions are **convex** and *nondecreasing* in their arguments.

(This will simplify our exposition of conic form.)

Conic form

A **conic form** optimization problem is written

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & b - Ax \in \mathcal{K}, \end{array}$$

where $\mathcal{K}$ is a **convex cone**:

$$x \in \mathcal{K} \iff rx \in \mathcal{K} \text{ for any } r > 0.$$

examples:

- zero cone $\mathcal{K} = \{0\}$
- positive orthant $\mathcal{K} = \{x : x_i >= 0, \; i = 1, \ldots, n\}$
- second order cone $\mathcal{K} = \{(x, t) : \|x\|_2 \leq t\}$
- positive semidefinite (PSD) cone
  $\mathcal{K} = \{X : X = X^T, \; v^T X v \geq 0, \; \forall v \in \mathbf{R}^n\}$
- products of cones

Conic form for expressions

epigraph conic form for expressions:

$$f(x) = \begin{array}{ll} \text{min} & C[x;t]+d \\ \text{with variable} & t \\ \text{subject to} & A[x;t]+b \in \mathcal{K} \end{array}$$

(note: “objective” can be vector valued)

- function can be represented by tuple

$$(C, d, A, b, \mathcal{K})$$

Conic form: base case

A function `conic_form` is defined on each data type (variable, constant, functions, constraints, problems) to return the tuple $(C, d, A, b, \mathcal{K})$.

base case:

- *Constant.*

$$3 = \begin{array}{ll} \text{min} & 3 \\ \text{with variable} & \emptyset \\ \text{subject to} & \emptyset \end{array}$$

- *Variable.*

$$x = \begin{array}{ll} \text{min} & x \\ \text{with variable} & \emptyset \\ \text{subject to} & \emptyset \end{array}$$

Conic form: inductive rule

inductive rule: if

$$f(y) = \begin{array}{ll} \text{min} & C^f[y;t^f] + d^f \\ \text{with variable} & t^f \\ \text{subject to} & A^f[y;t^f] + b^f \in \mathcal{K}^f, \end{array}$$

$$g(x) = \begin{array}{ll} \text{min} & C^g[x;t^g] + d^g \\ \text{with variable} & t^g \\ \text{subject to} & A^g[x;t^g] + b^g \in \mathcal{K}^g \end{array}$$

then

$$f(g(x)) = \begin{array}{ll} \text{min} & C^f[C^g I][x;t^g;t^f] + C^f d^g + d^f \\ \text{with variable} & t^g, t^f \\ \text{subject to} & A^f[C^g I][x;t^g;t^f] + A^f d^g + b^f \in \mathcal{K}^f \\ & A^g[x;t^g] + b^g \in \mathcal{K}^g \end{array}$$

proof: f is convex and increasing in its argument and g is convex, so partial minimizations over t^f and t^g commute.

Conic form: inductive rule

in math:

$$f(g(x)) = \begin{array}{ll} \text{min} & C^f[C^g I][x; t^g; t^f] + C^f d^g + d^f \\ \text{with variable} & t^g, t^f \\ \text{subject to} & A^f[C^g I][x; t^g; t^f] + A^f d^g + b^f \in \mathcal{K}^f \\ & A^g[x; t^g] + b^g \in \mathcal{K}^g \end{array}$$

in code:

```
conic_form(f::AbstractExpr)
    (Cg,dg,Ag,bg,Kg) = conic_form(f.children)
    (Cf,df,Af,bf,Kf) = conic_form(f.head)
    return(Cf*[Cg I], Cf*dg+df, [Af*[Cg I]; Ag],
           [Af*dg + bf; bg], [Kf; Kg])
```

Multiconvex functions

Definition (Restriction)

For $f : \mathbf{R}^n \to \mathbf{R}$ and $\omega \subseteq \{1, \ldots, n\}$, define the **restriction** $f_\omega(\cdot, \bar{x}) : \mathbf{R}^{|\omega|} \to \mathbf{R}$ of f to ω to be the function obtained by fixing the coefficients in ω^C to their values in $\bar{x} \in \mathbf{R}^n$:
$x \mapsto f_\omega(x; \bar{x})$.

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Definition (Multiconvex function)

A function $f : \mathbf{R}^n \to \mathbf{R}$ is k-**convex** if there exists a partition $\Omega = \{\omega_1, \ldots, \omega_k\}$ of $\{1, \ldots, n\}$ so that f_{ω_j} is convex for every $j = 1, \ldots, k$.

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Multiconvex functions generalize **biconvex** and **multilinear** functions.

- A 1-convex function is convex; a 2-convex function is biconvex; a 3-convex function is triconvex; etc.
- A multilinear function is multiconvex.

Multiconvex problems

Consider a (nonconvex) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x_{\beta_i}) \leq 0, \quad i = 1, \ldots, m \end{array} \qquad (\mathcal{P})$$

with variable $x \in \mathbf{R}^n$.

Definition (Multiconvex problem)

An optimization problem is k-**convex** if there exists a partition $\Omega = \{\omega_1, \ldots, \omega_k\}$ of $\{1, \ldots, n\}$ with the following properties:

- f_0 is k-convex with partition Ω;
- f_i is convex for every $i = 1, \ldots, m$;
- for every constraint $i = 1, \ldots, m$, there is an element j of the partition with $\beta_i \subseteq \omega_j$.

`MultiConvex.jl`

`MultiConvex.jl`extends `Convex.jl`to detect and (heuristically) solve multiconvex optimization problems using **disciplined** multiconvex programming:

- simple: less than 300 lines of code
- heuristic solution method: alternating minimization

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- simple: less than 300 lines of code
- heuristic solution method: alternating minimization

Definition (Disciplined multiconvex problem)

A multiconvex optimization problem is a **disciplined multiconvex problem** if

- f_0 is k-**convex** with partition $\Omega = \{\omega_1, \ldots, \omega_k\}$
- f_0 restricted to ω_j is a disciplined convex function for every $j = 1, \ldots, k$
- f_i is a disciplined convex function for $i = 1, \ldots, m$

`MultiConvex.jl` in action

```
using MultiConvex

# initialize nonconvex problem
n, k = 10, 1
A = rand(n, k) * rand(k, n)
x = Variable(n, k)
y = Variable(k, n)
problem = minimize(sum_squares(A - x*y), x>=0, y>=0)

# perform alternating minimization on the problem
altmin!(problem)
```

Conflict graphs

Definition

The **conflict graph** $G = (V, E)$ of a multiconvex expression e is a graph on the variables in the expression:

$$V = \textbf{variablesin}(e), \qquad E \subseteq V \times V$$

with the property that for any independent set of variables ω in the graph, the restriction f_ω of f to ω is convex.

Every multiconvex expression has a (unique) conflict graph.

Conflict graphs: recursion

- *Constant.* A constant c is multiconvex with conflict graph $(\emptyset, \emptyset)$
- *Variable.* A variable v is multiconvex with conflict graph $(v, \emptyset)$
- *Expressions.* The conflict graph of a composite expression is the union of the conflict graphs of its arguments, together with (possibly) a few more edges.
  - multiplication $(*, (x, y))$ adds complete bipartite graph on **variablesin**(x) and **variablesin**(y)
- *Constraints.* A constraint is multiconvex iff it is convex.
- *Problems.* Problems check their convexity by constructing a certifying partition Ω of the conflict graph of the objective that respects the constraints (if one exists).

Alternating minimization

Now that we've found a partition Ω, we can use alternating minimization:

```
for iter=1:AMiters
    for ω in Ω
        # free the variables in ω to optimize over just those variables
        for v in ω
            free!(v)
        end
        solve!(problem, warmstart=true)
        # now that we've found their values, fix them again
        for v in ω
            fix!(v)
        end
    end
end
```

(or ADMM, or . . .)

Convex: wrapping up

`Convex.jl` is

- a modelling language that detects structure
  - (disciplined) convexity
  - conic form
- a framework for recursive reasoning about optimization problems
  - (disciplined) multiconvexity
  - easy to extend to detect new structures ...

More information (and code!)

- `Convex.jl`: `http://www.github.com/JuliaOpt/Convex.jl`
- `MultiConvex.jl`: `http://www.github.com/madeleineudell/MultiConvex.jl`
- `Convex.jl`paper: `http://arxiv.org/abs/1410.4821`
- `MultiConvex.jl`paper: `https://arxiv.org/abs/1609.03285`