Conic Outer Approximation for Mixed-Integer Convex Optimization

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In preparation *A conic framework for solving MICPs via outer approximation*

Software **Pajarito** (open-source) `github.com/JuliaOpt/Pajarito.jl`

Mixed-integer convex optimization (MICP)

- a.k.a ‘convex mixed-integer nonlinear programming’ [BKL12]
- problems that are convex except for integrality constraints
- generalizes convex optimization and mixed-integer linear optimization

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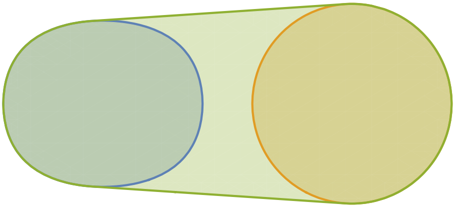
Many useful nonconvex sets are representable as feasible sets of MICPs, e.g. finite unions of compact convex sets [LZV16]



MICP general form and applications

$$\min_{\boldsymbol{x}\in\mathbb{R}^N} \quad \langle \boldsymbol{c}, \boldsymbol{x} \rangle : \qquad \text{(linear objective)}$$

$$\boldsymbol{x} \in \mathcal{S} \qquad \text{(convex set constraints)}$$

$$x_i \in \mathbb{Z} \qquad \forall i \in [I] \qquad \text{(integrality constraints)}$$

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\end{aligned}$$

- quadratic facility location, stochastic service system design, cutting stock and constrained layout problems [BKL12]
- optimal discrete experimental design (e.g. A-/D-/E-optimal designs)
- trajectory planning with spatial segmentation and sum-of-squares (SOS) control theory [DT15]
- portfolios with nonlinear risk measures and combinatorial constraints
- transistor gate-sizing for electrical circuit design [BKVH07]

A simple polyhedral outer approximation algorithm

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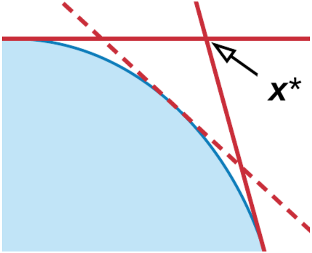
Build MILP OA model $\mathfrak{P}$ by replacing $\mathcal{S}$ with a polyhedral relaxation

solve $\mathfrak{P}$, let $\boldsymbol{x}^\star$ be optimal solution
if $\boldsymbol{x}^\star$ is 'close' to $\mathcal{S}$ **then**
    return $\boldsymbol{x}^\star$
else
    find separating hyperplane $(\boldsymbol{y}, z)$
    update $\mathfrak{P}$ with cut $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq z$
end if



Conic extended formulations

- an extended formulation (EF) for $\boldsymbol{x} \in \mathcal{S}$ is an equivalent representation as a projection of a set in a higher dimensional space
- EFs can greatly accelerate OA algorithms [TS05, VDHL16]

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[LYBV16] noted that

- disciplined convex programming (DCP) implementations can automate the construction of convex conic extended formulations
- all 333 MICPs in MINLPLIB2 can be encoded with about 4 convex nonpolyhedral cones
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In [CLV17] we detail a conic framework for solving MICPs via OA and extended formulations, and implement our algorithms in **Pajarito**

Mixed-integer conic form

$$\min_{\boldsymbol{x} \in \mathbb{R}^N} \quad \langle \boldsymbol{c}, \boldsymbol{x} \rangle : \tag{$\mathfrak{M}$}$$

$$\boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} \in \mathcal{C}_k \qquad \forall k \in [M]$$

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- $\mathcal{C}_{K+1}, \ldots, \mathcal{C}_M$ are polyhedral cones, e.g. $\mathbb{R}_+$, $\mathbb{R}_-$, $\{0\}$
- $\mathcal{C}_1, \ldots, \mathcal{C}_K$ are closed convex nonpolyhedral cones, e.g.

  $\mathcal{L}$ second-order cone (epi $\|\cdot\|_2$)

  $\mathcal{E}$ exponential cone (epi cl per exp)

  $\mathcal{P}$ positive semidefinite cone ($\mathbb{S}_+$ on $\mathbb{S}$)

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Assume that if $\mathfrak{M}$ is feasible then its optimal value is attained

Outer approximation with $\mathcal{K}^*$ cuts

The dual cone of a closed convex cone is also a closed convex cone

$$\mathcal{K}^* = \{\boldsymbol{z} \in \mathbb{R}^n : \langle \boldsymbol{y}, \boldsymbol{z} \rangle \geq 0, \forall \boldsymbol{y} \in \mathcal{K}\}$$

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For OA, we instead choose a finite subset $\mathcal{Z}_k$ of the dual cone points

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If added for each $k \in [K]$, these $\mathcal{K}^*$ cuts yield an MILP relaxation of $\mathfrak{M}$

Obtaining $\mathcal{K}^*$ cuts

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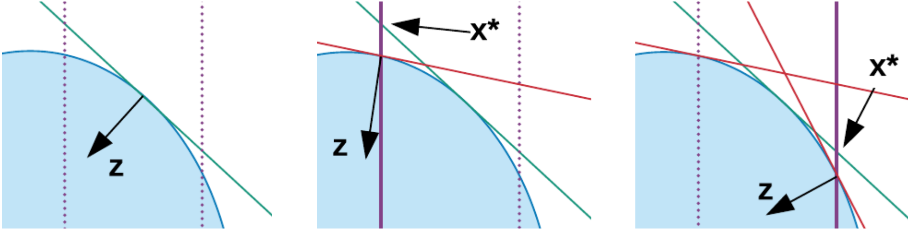
Define the following models

- $\mathfrak{M}$ the MICP problem
- $\mathfrak{P}$ the MILP OA model that we add $\mathcal{K}^*$ cuts to
- $\mathfrak{R}$ the continuous relaxation of $\mathfrak{M}$
- $\mathfrak{R}(\boldsymbol{x})$ a restriction of $\mathfrak{R}$ to the subspace in which the integer variables are fixed to $x_1, \ldots, x_I$

Geometric intuition

$\mathfrak{M}$: blue convex region intersected with purple dotted lines for integers
$\mathfrak{P}$: polyhedron under $\mathcal{K}^*$ cuts intersected with purple dotted lines



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1. solve $\mathfrak{P}$ for $\boldsymbol{x}^\star$
2. solve $\mathfrak{R}(\boldsymbol{x}^\star)$ for dual $\bar{\boldsymbol{z}}$
3. add $\bar{\boldsymbol{z}}$ cut to $\mathfrak{P}$
4. if $\mathfrak{R}(\boldsymbol{x}^\star)$ feasible check if $\bar{\boldsymbol{x}}$ is new incumbent

The continuous relaxation

The continuous conic relaxation of $\mathfrak{M}$ is $\mathfrak{R}$

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle : \\ \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} & \in \mathcal{C}_k \qquad \qquad \forall k \in [M] \\ \boldsymbol{x} & \in \mathbb{R}^N \end{aligned} \tag{$\mathfrak{R}$}$$

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Using standard conic duality [BV04], the conic dual is $\mathfrak{R}^*$

$$\max_{\boldsymbol{z}_1,\ldots,\boldsymbol{z}_K} \quad - \sum_{k \in [M]} \langle \boldsymbol{b}_k, \boldsymbol{z}_k \rangle : \qquad (\mathfrak{R}^*)$$
$$\boldsymbol{c} + \sum_{k \in [M]} \boldsymbol{A}_k^T \boldsymbol{z}_k \in \{0\}^N$$
$$\boldsymbol{z}_k \in \mathcal{C}_k^* \qquad \forall k \in [M]$$

Relaxation and subproblem $\mathcal{K}^*$ cuts

To obtain relaxation $\mathcal{K}^*$ cuts

- assume $\mathfrak{R}$ is bounded
- if $\mathfrak{R}$ is infeasible then $\mathfrak{M}$ is infeasible
- if $\mathfrak{R}$ is feasible, assume strong duality holds for $\mathfrak{R}, \mathfrak{R}^*$
    - exists primal-dual solutions with objective value C
    - from $\mathfrak{R}^*$ solution $(\bar{\boldsymbol{z}}_k)_{k \in [M]}$, we derive $\mathcal{K}^*$ cuts $\bar{\boldsymbol{z}}_k$ for $k \in [K]$
    - guarantee that $\mathfrak{P}$'s value is no worse than C

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  - guarantee that $\mathfrak{P}$'s value is no worse than C

To obtain subproblem $\mathcal{K}^*$ cuts given a feasible MILP solution $\boldsymbol{x}^\star$ for $\mathfrak{P}$

- note $\mathfrak{R}(\boldsymbol{x}^\star)$ is not unbounded
- if $\mathfrak{R}(\boldsymbol{x}^\star)$ is feasible, case is analogous to above for $\mathfrak{R}$
- if $\mathfrak{R}(\boldsymbol{x}^\star)$ is infeasible, we get a ray of $\mathfrak{R}^*(\boldsymbol{x}^\star)$ that defines $\mathcal{K}^*$ cuts excluding all $\boldsymbol{x}$ with the same integer assignment $x_1, \ldots, x_I$

Algorithmic extensions

See Appendix 6 for a detailed branch and bound algorithm with finite convergence guarantees (under reasonable assumptions)

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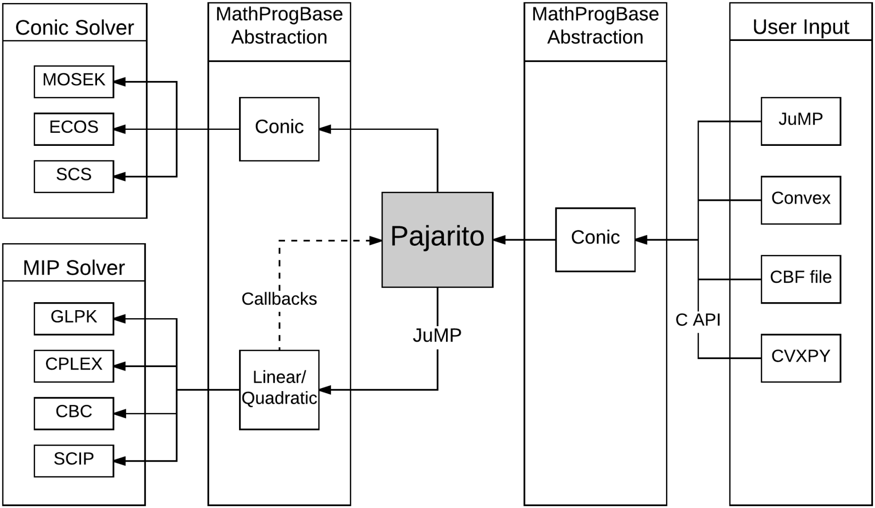
- extreme ray cuts, implemented for $\mathcal{P}$ based on eigenvector decompositions
- $\mathcal{L}$ cuts for $\mathcal{P}$, based on Schur complement [KKY03]

Pajarito solver

- public, open-source solver written in **Julia** and integrated with **MathProgBase**
- currently supports second order, semidefinite, and exponential cones
- *iterative* and *MIP-solver-driven* (single-tree) OA algorithms
- around 30 algorithmic options
- easily extensible

Integration with MathProgBase



Example code

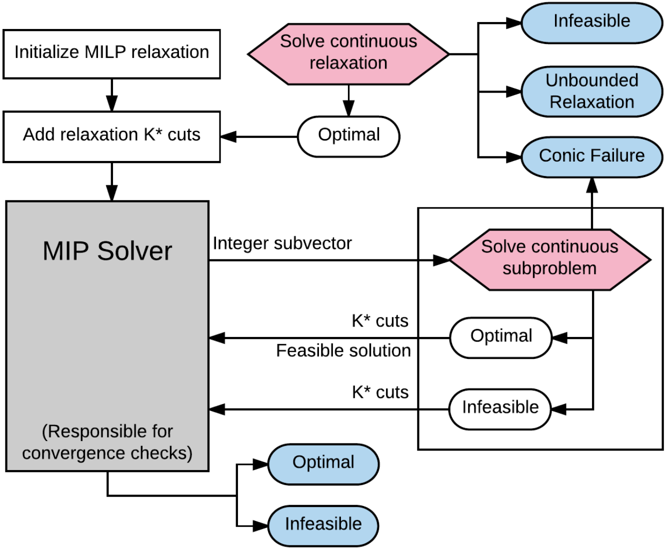
Model with **Convex.jl**, solve using **Pajarito** with **Cbc** and **ECOS**

```
x = Variable(1, :Int)        # Integer variable
y = Variable(1, Positive()) # Nonnegative variable

problem = minimize(
    -3x - y,                 # Objective (min)
    x >= 1, 3x + 2y <= 30,   # Linear constraints
    exp(y^2) + x <= 7)        # Convex constraint

solve!(problem,
    PajaritoSolver(rel_gap=1e-5,
    mip_solver=CbcSolver(),
    cont_solver=ECOSSolver())
```

MIP-solver-driven OA algorithm



Comparing subproblem and separation cuts

Termination statuses and shifted geomean of solve time and iteration count (for iterative algorithm only) on 120 MISOCPs, using **Pajarito** with **CPLEX** and **MOSEK**

options		termination status counts				conv only stats	
alg	cuts	conv	wrong	not conv	limit	time(s)	iterations
iter	sep	96	1	0	23	55.23	6.76
iter	subp	95	1	3	21	39.59	4.07
MSD	sep	95	1	0	24	20.86	–
MSD	subp	100	0	1	19	17.56	–

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Subproblem cuts should be used always, and separation cuts should be invoked when necessary for convergence

Comparisons with specialized MISOCP solvers

Termination statuses and shifted geometric mean of solve time on 120 MISOCPs, for **SCIP** and **CPLEX** MISOCP solvers, and default MSD and iterative **Pajarito** solvers using **CPLEX** and **MOSEK**

	termination status counts				
solver	conv	wrong	not conv	limit	time(s)
SCIP	78	1	0	41	43.36
CPLEX	96	3	5	16	14.30
Paj-iter	96	1	0	23	38.70
Paj-MSD	101	0	0	19	18.12

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Pajarito's MSD algorithm solves more instances in the time limit and has no incorrect solutions

Open-source solver comparisons for MISOCP

Termination statuses and shifted geomean of solve time on 120 MISOCPs for **BONMIN** [BBC+08] with **Cbc** and **IPOPT**, and **Pajarito** using **Cbc** or **GLPK** and **ECOS** (iterative algorithm with default options)

	termination status counts				
solver	conv	wrong	not conv	limit	time(s)
BONMIN-BB	37	27	10	46	82.95
BONMIN-OA	30	8	29	53	72.12
BONMIN-OA-D	35	8	29	48	64.25
Paj-CBC-ECOS	81	8	0	31	51.48
Paj-GLPK-ECOS	68	0	2	50	42.75

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On instances with 20 portfolios and up to 100 stocks per portfolio, running **Pajarito**'s MSD algorithm using default options and **CPLEX**

- with ℓ_2 norm, using **MOSEK**, several minutes
- with ℓ_2 norm and entropic ball, using **ECOS**, 5-10 minutes
- with ℓ_2 norm and robust norm, using **MOSEK**, 20-30 minutes
- with all three risk constraints, using **SCS**, hours

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Problems with $\mathcal{P}$ scale poorly - no disaggregated extended formulation

Thank you

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References IV

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Experimental design optimization

From [BV04]

- estimate a vector $\boldsymbol{x} \in \mathbb{R}^Q$

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$$\boldsymbol{E}(\boldsymbol{m}) \equiv \left(\sum_{p \in [P]} m_p \boldsymbol{u}_p \boldsymbol{u}_p' \right)^{-1}$$

Mixed-integer convex experimental design

If $f : \mathbb{S}_+^Q \to \mathbb{R}$ measures the 'size' of the error covariance matrix $\boldsymbol{E}(\boldsymbol{m})$

$$
\begin{aligned}
\min_{\boldsymbol{m} \in \mathbb{R}^P} \quad & f(\boldsymbol{E}(\boldsymbol{m})) : && \text{minimize error covariance} \\
& \mathbf{1}'\boldsymbol{m} \leq M && \text{budget of experiments} \\
& \boldsymbol{m} \in \mathbb{Z}_+^P && \text{integrality restriction}
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If $\mathcal{E}$ is a confidence ellipsoid for $\boldsymbol{x}$ given $\boldsymbol{E}$, there are many choices for $f(\boldsymbol{E})$

- E-opt minimizes the diameter of $\mathcal{E}$: $\min \ \|\boldsymbol{E}\|_2$
- A-opt minimizes mean squared error: $\min \ \operatorname{tr} \boldsymbol{E}$
- D-opt minimizes the volume of $\mathcal{E}$: $\min \ \log\det \boldsymbol{E}$, or by [BTN01], maximizes scaled geomean eigenvalues of $\sum_{p \in [P]} m_p \boldsymbol{u}_p \boldsymbol{u}_p'$

A branch and bound OA algorithm

- a conic analogue of [QG92] (convex MINLP)
- assume we have explicit bounds $\boldsymbol{l}^0, \boldsymbol{u}^0$ on the integer variables $(x_i)_{i \in [I]}$
- recursively partition the possible assignments of integer variables by lower and upper bound vectors $\boldsymbol{l}, \boldsymbol{u}$
- add subproblem $\mathcal{K}^*$ cuts when we get integer solutions for $x_1, \ldots, x_I$ - globally valid and, if added to the LP relaxation, contain enough information to properly process the node
- solve linear programming relaxations with reliable (dual) simplex
  - requires few pivots after adding cuts
  - achieve very tight feasibility and optimality tolerances
- finite convergence if there is a finite number of integer assignments
  - finite number of nodes, each examined a finite number of times
  - if we add subproblem cuts at every node, assuming strong duality
  - then the optimal objective value of the final polyhedral OA model will equal that of the MICP problem

Processing nodes

Suppose we are at a node $(\boldsymbol{l}, \boldsymbol{u}, L)$ of the branch and bound tree

- $\boldsymbol{l}, \boldsymbol{u}$ are the node's lower, upper variable bounds for $\hat{\boldsymbol{x}} = (x_1, \ldots, x_I)$
- L is a lower objective bound for $\mathfrak{M}$ restricted to $x_i \in [l_i, u_i], \forall i \in [I]$
- so if $L > U$ then we can discard the node
- otherwise we try to tighten L by solving a polyhedral OA relaxation

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- otherwise we try to tighten L by solving a polyhedral OA relaxation

Given current $\mathcal{K}^*$ cut sets $(\mathcal{Z}_k)_{k \in [K]}$, we solve the LP $\mathfrak{P}\left((\mathcal{Z}_k)_{k \in [K]}, \boldsymbol{l}, \boldsymbol{u}\right)$

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad \langle \boldsymbol{c}, \boldsymbol{x} \rangle : & & \left(\mathfrak{P}\left((\mathcal{Z}_k)_{k \in [K]}, \boldsymbol{l}, \boldsymbol{u}\right)\right) \\
\langle \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x}, \boldsymbol{z}_k \rangle \in \mathbb{R}_+ & & \forall \boldsymbol{z}_k \in \mathcal{Z}_k, k \in [K] \\
\boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} \in \mathcal{C}_k & & \forall k \in [M] \backslash [K] \\
x_i \in [l_i, u_i] & & \forall i \in [I]
\end{aligned}
$$

Branch and bound algorithm

initialize global upper bound U to ∞
solve $\mathfrak{R}$ for optimal value $C_{\mathfrak{R}}$ and dual solution $(\bar{\boldsymbol{z}}_k)_{k\in[M]}$
initialize $\mathcal{K}^*$ cut sets $(\mathcal{Z}_k)_{k\in[K]}$ with relaxation cuts $(\bar{\boldsymbol{z}}_k)_{k\in[K]}$
initialize node list $\mathcal{N}$ with most relaxed node $(\boldsymbol{l}^0, \boldsymbol{u}^0, C_{\mathfrak{R}})$
while $\mathcal{N}$ contains nodes **do**
    remove a node $(\boldsymbol{l}, \boldsymbol{u}, L)$ from $\mathcal{N}$
    if node's lower bound $L \leq U$ **then**
        solve LP $\mathfrak{P}\left((\mathcal{Z}_k)_{k\in[K]}, \boldsymbol{l}, \boldsymbol{u}\right)$ and update U, $(\mathcal{Z}_k)_{k\in[K]}$, $\mathcal{N}$
    end if
end while

LP procedure at a node

if $\mathfrak{P}\left((\mathcal{Z}_k)_{k\in[K]}, \boldsymbol{l}, \boldsymbol{u}\right)$ is feasible **&** optimal value $C_{\mathfrak{P}} < U$ **then**
  let $\bar{\boldsymbol{x}}^\star$ be the integer variable subvector of an optimal solution
  if integrality $\bar{\boldsymbol{x}}^\star \in \mathbb{Z}^I$ is satisfied **then**
    solve $\mathfrak{R}(\bar{\boldsymbol{x}}^\star, \bar{\boldsymbol{x}}^\star)$ for an optimal dual solution or ray $(\bar{\boldsymbol{z}}_k)_{k\in[M]}$
    add $\mathcal{K}^*$ cuts $(\bar{\boldsymbol{z}}_k)_{k\in[K]}$ to $(\mathcal{Z}_k)_{k\in[K]}$
    if $\mathfrak{R}(\bar{\boldsymbol{x}}^\star, \bar{\boldsymbol{x}}^\star)$ is feasible **&** optimal value $C_{\mathfrak{R}}(\bar{\boldsymbol{x}}^\star, \bar{\boldsymbol{x}}^\star) < U$ **then**
      update U to new best feasible value $C_{\mathfrak{R}}(\bar{\boldsymbol{x}}^\star, \bar{\boldsymbol{x}}^\star)$
    end if
    add node $(\boldsymbol{l}, \boldsymbol{u}, C_{\mathfrak{P}})$ to $\mathcal{N}$ for reprocessing
  else
    choose a fractional variable $i : x_i^\star \notin \mathbb{Z}$ to branch on
    add left branch node $\left(\boldsymbol{l}, \left(u_1, \ldots, \lfloor x_i^\star \rfloor, \ldots, u_I\right), C_{\mathfrak{P}}\right)$ to $\mathcal{N}$
    add right branch node $\left(\left(l_1, \ldots, \lceil x_i^\star \rceil, \ldots, l_I\right), \boldsymbol{u}, C_{\mathfrak{P}}\right)$ to $\mathcal{N}$
  end if
end if

A continuous subproblem

Consider restricting the (relaxed) integer variables of $\mathfrak{R}$ to a box $(\boldsymbol{l}, \boldsymbol{u})$

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle : & & (\mathfrak{R}(\boldsymbol{l}, \boldsymbol{u})) \\
\boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} & \in \mathcal{C}_k & \forall k \in [M] & \\
x_i & \in [l_i, u_i] & \forall i \in [I] & \\
\boldsymbol{x} & \in \mathbb{R}^N & &
\end{aligned}$$

A continuous subproblem

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$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle : && (\mathfrak{R}(\boldsymbol{l}, \boldsymbol{u})) \\ \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} &\in \mathcal{C}_k && \forall k \in [M] \\ x_i &\in [l_i, u_i] && \forall i \in [I] \\ \boldsymbol{x} &\in \mathbb{R}^N \end{aligned}$$

After encoding the box constraints conically, the conic dual is

$$\begin{aligned} \max_{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K, \boldsymbol{\alpha}, \boldsymbol{\beta}} \quad & \sum_{i \in [I]} (l_i \alpha_i - u_i \beta_i) - \sum_{k \in [M]} \langle \boldsymbol{b}_k, \boldsymbol{z}_k \rangle : && (\mathfrak{R}^*(\boldsymbol{l}, \boldsymbol{u})) \\ \boldsymbol{c} + \sum_{i \in [I]} (\beta_i - \alpha_i) \boldsymbol{e}(i) + \sum_{k \in [M]} \boldsymbol{A}_k^T \boldsymbol{z}_k &\in \{0\}^N \\ \boldsymbol{z}_k &\in \mathcal{C}_k^* && \forall k \in [M] \\ \boldsymbol{\alpha}, \boldsymbol{\beta} &\in \mathbb{R}_+^I \end{aligned}$$

Subproblem $\mathcal{K}^*$ cuts: feasible primal case

Assume $\mathfrak{R}(\boldsymbol{l}, \boldsymbol{u})$ is feasible and bounded, and strong duality holds, thus we have an optimal primal-dual solution $(\boldsymbol{x}^\star, \boldsymbol{z}_1^\star, \ldots, \boldsymbol{z}_K^\star, \boldsymbol{\alpha}^\star, \boldsymbol{\beta}^\star)$

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From the dual solution subvector $(\bar{\boldsymbol{z}}_k)_{k \in [M]}$, we derive $\mathcal{K}^*$ cuts

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From the dual solution subvector $(\bar{\boldsymbol{z}}_k)_{k \in [M]}$, we derive $\mathcal{K}^*$ cuts

$$
\begin{aligned}
\langle \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x}, \bar{\boldsymbol{z}}_k \rangle &\geq 0 && \forall k \in [K] \\
\boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} &\in \mathcal{C}_k && \forall k \in [M] \backslash [K] \\
x_i &\in [l_i, u_i] && \forall i \in [I]
\end{aligned}
$$

Any $\boldsymbol{x}$ satisfying these linear constraints satisfies an objective bound

$$\langle \boldsymbol{c}, \boldsymbol{x} \rangle \geq \langle \boldsymbol{c}, \boldsymbol{x}^\star \rangle$$

Subproblem $\mathcal{K}^*$ cuts: infeasible primal case

Assume now $\mathfrak{R}(\boldsymbol{l}, \boldsymbol{u})$ is infeasible, so we have a certificate of infeasibility i.e. a ray $\left((\boldsymbol{z}_k)_{k \in [M]}, \boldsymbol{\alpha}, \boldsymbol{\beta}\right)$ of $\mathfrak{R}^*(\boldsymbol{l}, \boldsymbol{u})$ satisfying

$$\sum_{i \in [I]} (\beta_i - \alpha_i) \boldsymbol{e}(i) + \sum_{k \in [K]} \boldsymbol{A}_k^T \boldsymbol{z}_k \in \{0\}^N$$

$$\sum_{i \in [I]} (u_i \beta_i - l_i \alpha_i) + \sum_{k \in [M]} \langle \boldsymbol{b}_k, \boldsymbol{z}_k \rangle < 0$$

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$$\sum_{i\in[I]}(u_i\beta_i - l_i\alpha_i) + \sum_{k\in[M]} \langle \boldsymbol{b}_k, \boldsymbol{z}_k \rangle < 0$$

From the ray subvector $(\bar{\boldsymbol{z}}_k)_{k\in[M]}$, we derive $\mathcal{K}^*$ cuts that exclude all solutions for the bounds $(\boldsymbol{l}, \boldsymbol{u})$

Subproblem $\mathcal{K}^*$ cuts: infeasible primal case

For all $\boldsymbol{x}$ satisfying

$$\begin{aligned} \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} &\in \mathcal{C}_k & \forall k \in [M] \backslash [K] \\ x_i &\in [l_i, u_i] & \forall i \in [I] \end{aligned}$$

there exists a $k \in [K]$ such that $\langle \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x}, \boldsymbol{z}_k \rangle < 0$

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there exists a $k \in [K]$ such that $\langle \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x}, \boldsymbol{z}_k \rangle < 0$

$$\begin{aligned} &\sum_{k \in [K]} \langle \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x}, \boldsymbol{z}_k \rangle \\ &\leq \sum_{k \in [M]} \langle \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x}, \boldsymbol{z}_k \rangle + \sum_{i \in [I]} (-l_i + x_i) \alpha_i + \sum_{i \in [I]} (u_i - x_i) \beta_i \\ &= \left\langle \boldsymbol{x}, \sum_{i \in [I]} (\alpha_i - \beta_i) \boldsymbol{e}(i) - \sum_{k \in [M]} \boldsymbol{A}_k^T \boldsymbol{z}_k \right\rangle + \sum_{k \in [M]} \langle \boldsymbol{b}_k, \boldsymbol{z}_k \rangle + \sum_{i \in [I]} (u_i \beta_i - l_i \alpha_i) \\ &= \sum_{k \in [M]} \langle \boldsymbol{b}_k, \boldsymbol{z}_k \rangle + \sum_{i \in [I]} (u_i \beta_i - l_i \alpha_i) < 0 \end{aligned}$$