External Lecture yuecao@purdue.edu

Day 8-3 P1

Recap Convert to Discrete Time using ZOH.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Continuous Time

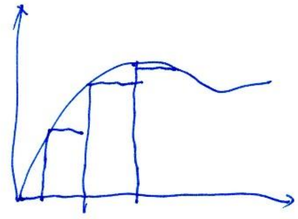
$\longrightarrow$

$$x[k+1] = A_D x[k] + B_D u[k]$$
$$y[k] = C_D x[k] + D_D u[k]$$

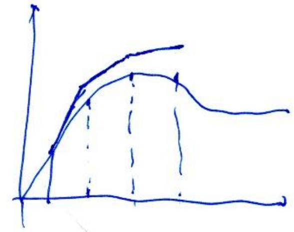
$$A_D = e^{AT},$$
$$B_D = \int_0^T e^{A\lambda} d\lambda B$$
$$C_D = C, \quad D_D = D.$$

Discrete Time

Topic 1. Convert to Discrete Time using Forward Euler.



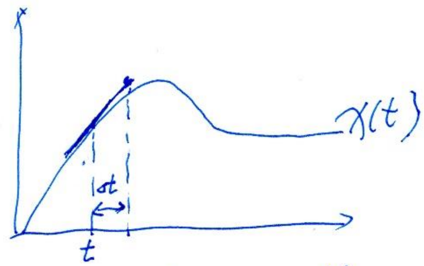
ZOH



Forward Euler

Day 8-3

P2



Calculus: How to define Derivative?

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

Let's assume: Δt is our sampling time.

and Δt is Very Small.

$$\dot{x}(t) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

$$\Delta t\, \dot{x}(t) \approx x(t+\Delta t) - x(t)$$

$$x(t+\Delta t) \approx \Delta t\, \dot{x}(t) + x(t)$$

Plug in the Continuous Time System. $\dot{x}(t) = Ax(t) + Bu(t)$

$$x(t+\Delta t) \approx \Delta t\left(Ax(t) + Bu(t)\right) + x(t)$$

Day 8-3
p 3

$x(t+\Delta t) \approx \Delta t \cdot A \cdot x(t) + \Delta t B u(t) + x(t)$

$\approx (I + A \cdot \Delta t) x(t) + B \cdot \Delta t \cdot u(t)$

Continuos → Discrete

$t \rightarrow k \cdot \text{step}$

$t+\Delta t \rightarrow k+1 \text{ step}$

$$x[k+1] = \underbrace{(I + A \cdot \Delta t)}_{A_D} x[k] + \underbrace{B \cdot \Delta t}_{B_D} \cdot u[k]$$

$y[k]$ is the same as ZOH method

$y[k] = C x[k] + D u[k]$

Day 8-3

P4

Example. Continuous Time.

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -5 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 4 \\ 10 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x(t)$$

Sampling time $= 0.01$ sec.

Solution: $$A_D = I + A\cdot \Delta t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -5 & -3 \end{bmatrix} \times 0.01$$

$$= \begin{bmatrix} 1 & 0.02 \\ -0.05 & 0.97 \end{bmatrix}$$

$$B_D = B\cdot \Delta t = \begin{bmatrix} 4 \\ 10 \end{bmatrix} \times 0.01 = \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix}$$

C. D matrix unchanged

$$x[k+1] = \begin{bmatrix} 1 & 0.02 \\ -0.05 & 0.97 \end{bmatrix} x[k] + \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 2 & 1 \end{bmatrix} x[k].$$

Check Python visulization ...

Topic 2.

Day 8-3
~~P4~~ P5

Transfer function for Discrete Time System

	Continuous		Discrete
Time	$\dot{x}(t) = Ax(t) + Bu(t)$	Time	$x[k+1] = A_D x[k] + B_D u[k]$
$\downarrow$	$y(t) = Cx(t) + Du(t)$	$\downarrow$	$y[k] = C_D x[k] + D_D u[k]$
Laplace	$H(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$	Z.	$H(z) = \frac{Y(z)}{U(z)} = ?$

Look up Z Transform table.

$x[k] \xrightarrow{Z} X(z)$; $a x[k] \xrightarrow{Z} aX(z)$; $x[k+1] \xrightarrow{Z} zX(z) - zx[0]$

Back to our system

$$x[k+1] = A_D x[k] + B_D u[k]$$

$\downarrow$ Z Transform.

$$zX(z) - z x[0] = A_D X(z) + B_D U(z)$$

$$(zI - A_D)X(z) = B_D U(z) + z\,x[0]$$

Day 8-3
p 6

$$X(z) = (zI - A_D)^{-1} B_D U(z) + (zI - A_D)^{-1} z \cdot x[0]$$

If zero initial condition

$$X(z) = (zI - A_D)^{-1} B_D U(z) \quad ①$$

Also we have $Y(z) = C_D X(z) + D_D U(z)$ ②

Plug ① into ②:

$$Y(z) = C_D \cdot (zI - A_D)^{-1} B_D U(z) + D_D U(z)$$

Transfer function

$$H(z) = \frac{Y(z)}{U(z)} = C_D (zI - A_D)^{-1} B_D + D_D$$

Day 8-3

P7

Example.

$$x[k+1] = \begin{bmatrix} 1 & 0.02 \\ -0.05 & 0.97 \end{bmatrix} x[k] + \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 2 & 1 \end{bmatrix} x[k]$$

Solution. $H(z) = C_D(zI - A_D)^{-1}B_D + D_D$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z-1 & -0.02 \\ 0.05 & z-0.97 \end{bmatrix}^{-1} \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix}$$

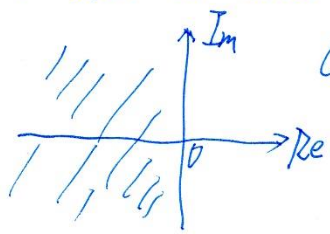
$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \frac{1}{(z-1)(z-0.97)+0.02\times0.05} \begin{bmatrix} z-0.97 & 0.02 \\ -0.05 & z-1 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix}$$

$$= \frac{1}{z^2-1.97z+0.971} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z-0.97 & 0.02 \\ -0.05 & z-1 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix}$$

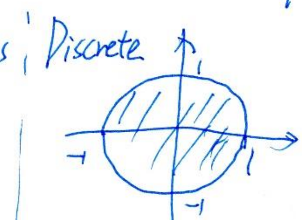
$$= \frac{\cancel{+}}{z^2-1.97+0.971}$$

Stable? check all roots of characteristic equation

Open left half plane.



Continuous | Discrete



Inside Unit Circle.