Plan

Monday (1/22): Set Theory $\begin{cases} \text{1.5–1.6: Set Operators (Con't)} \\ \text{1.7: Venn Diagrams} \\ \text{1.8: Indexed Sets} \end{cases}$

Wed (1/24): Mathematical Logic $\begin{cases} \text{Truth Table} \\ \text{Quantifiers} \\ \text{Open Statements} \end{cases}$

In this week's section, we will focus more on examples. Examples (marked with (*)) are the problems you need to submit for this week's participation points.

Monday (1/22):

① Ch 1.5 − 1.6: Set Operators

Recall that from last week's lec/sec,

(1) $\cup$: union

(2) $\cap$: intersection

(3) $\times$: Cartesian Product

(4) $-$: Diff

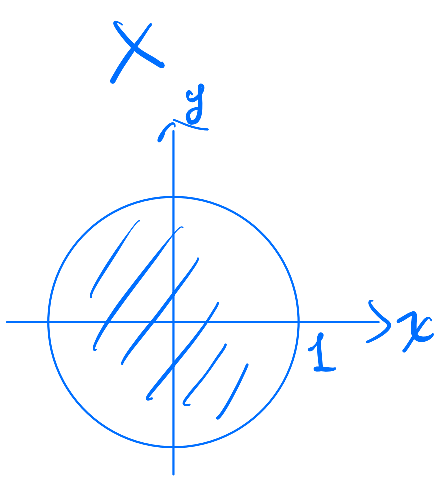
(5) $\mathcal{U}$: universal set, $X \subseteq \mathcal{U}$, $\overline{X} = \mathcal{U} - X$: complement

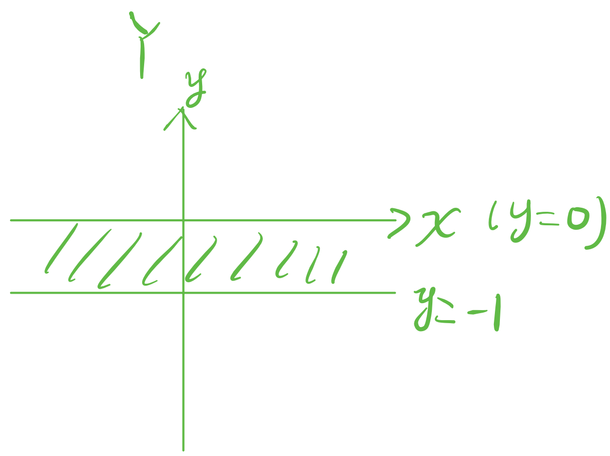
e.g. Sketch $X = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

$Y = \{(x,y) \in \mathbb{R}^2 : -1 \leq y \leq 0\}$ on $\mathbb{R}^2$

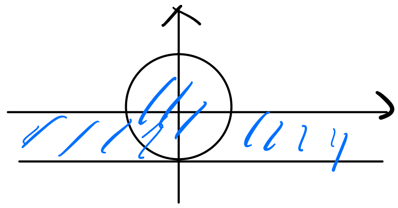
Recall that $(x-a)^2 + (y-b)^2 \leq r^2$ refer to a circle

with center (a,b) and radius r^2

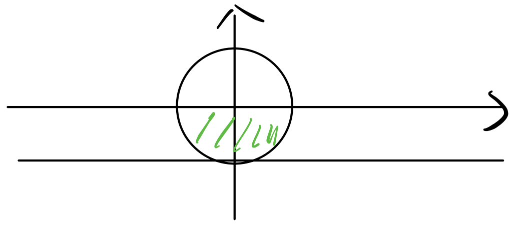




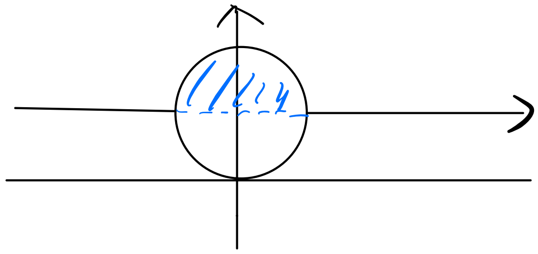
$X \cup Y \Rightarrow$



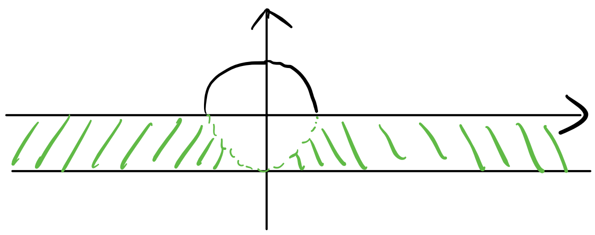
$X \cap Y \Rightarrow$



$X - Y \Rightarrow$



$Y - X \Rightarrow$



e.g.(*) $X = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2+y^2 \leq 4\}$ on $\mathbb{R}^2$

$\overline{X} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \text{ or } x^2+y^2 > 4\}$

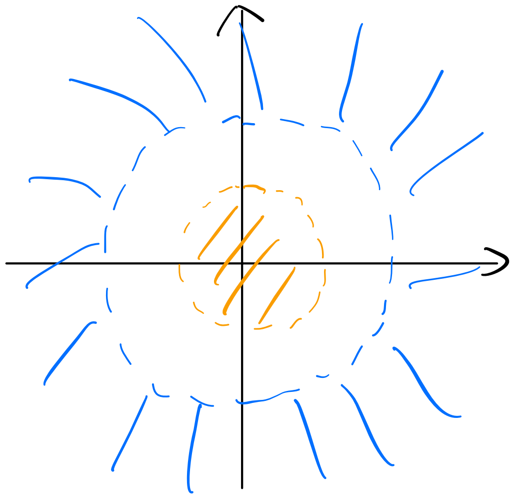
(X is a set of points in $\mathbb{R}^2 \Rightarrow$ Default universal: $\mathbb{R}^2$)

Small circle:

$(x,y): x^2+y^2=1$

inside the orange circle

$\Rightarrow x^2+y^2<1$



Big circle:

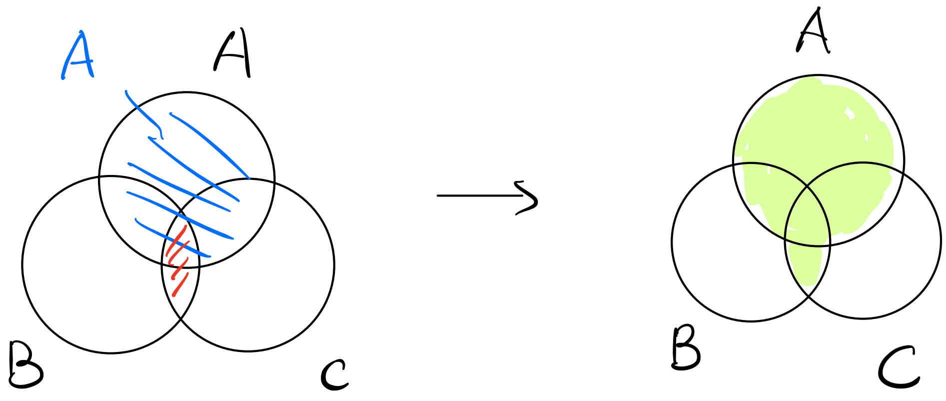
$(x,y): x^2+y^2=4.$

outside the blue circle

$\Rightarrow x^2+y^2>4.$

② Ch 1.7: Venn Diagrams

e.g. $A \cup (B \cap C)$

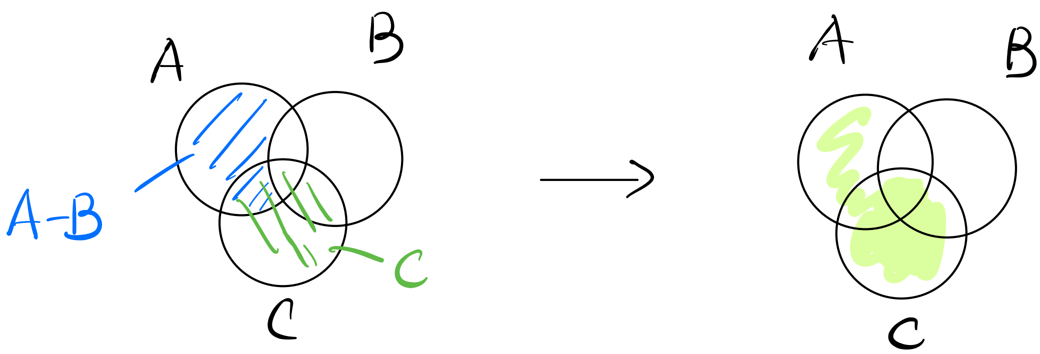


extra exercises: Use Venn Diagrams to check that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$$

Distributive law.

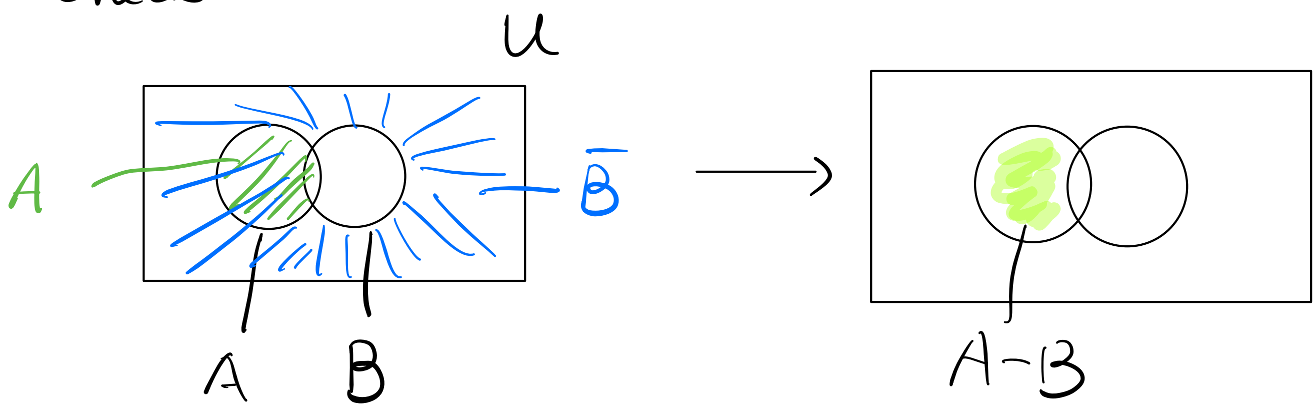
e.g. $(A - B) \cup C$



Remark: For set A, B, given a universal set U,

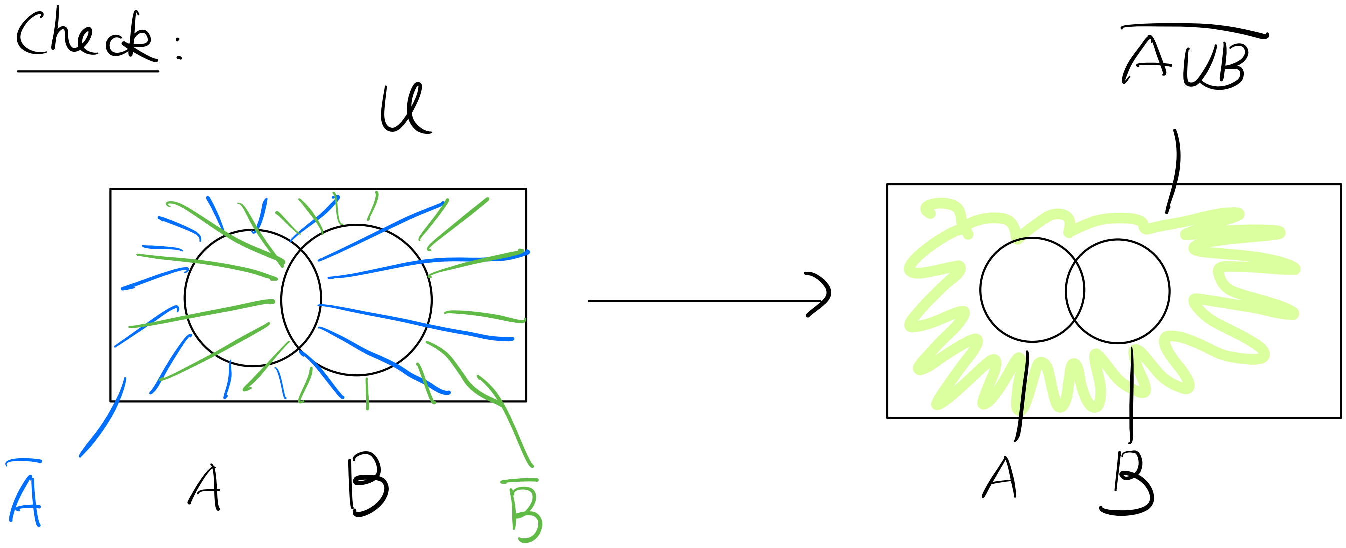
$$A - B = A \cap \overline{B}, \quad B - A = B \cap \overline{A}$$

Check:



e.g. De Morgan's Law
$$\begin{cases} \overline{A \cup B} = \bar{A} \cap \bar{B} \\ \overline{A \cap B} = \bar{A} \cup \bar{B} \end{cases}$$

Check:



③ Indexed sets

e.g.(*) $A_n = \{0, 1, 2, \cdots, n\}$ for $n \in \mathbb{N}$

(1) $\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i$ for at least one $A_i\}$

Take $n \in \mathbb{N}$, $n \in A_n = \{0, 1, \cdots, n\}$

$\Rightarrow$ any natural number belongs to some A_i

Also, $0 \in A_1 = \{0, 1\}$

Thus: $\bigcup_{i=1}^{\infty} A_i = \mathbb{N} \cup \{0\}$

(2) $\bigcap_{i=1}^{\infty} A_i = \{x : x$ belongs to every $A_i\}$

$0, 1 \in A_n = \{0, 1, \cdots, n\}$ for every n

but $n \geq 2$, $n \notin A_1$,

only 0 and 1 has this property.

(Wednesday)

Indexed Sets (Continued)

e.g. For $\mathbb{N}$, given $\mathcal{P}(\mathbb{N})$:

- $\bigcup\limits_{X \in \mathcal{P}(\mathbb{N})} = \mathbb{N}$

  check: for every natural number n, $n \in \{n\}$

  also, $\{n\} \subseteq \mathbb{N}$ so $\{n\} \in \mathcal{P}(\mathbb{N})$, in other words,

  n belongs to at least one X in $\mathcal{P}(\mathbb{N})$, so $n \in \bigcup\limits_{X \in \mathcal{P}(\mathbb{N})} X$.

- $\bigcap\limits_{X \in \mathcal{P}(\mathbb{N})} X = \phi$

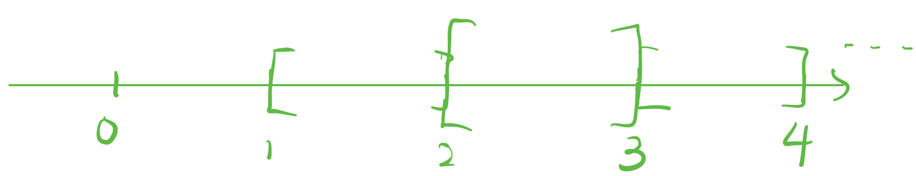
  For every natural number n, $n \notin \mathbb{N} - \{n\}$, but

  $\mathbb{N} - \{n\} \in \mathcal{P}(\mathbb{N})$, so $n \notin \bigcap\limits_{X \in \mathcal{P}(\mathbb{N})} X$

  in other words, $\bigcap\limits_{X \in \mathcal{P}(\mathbb{N})} X$ contains no elements.

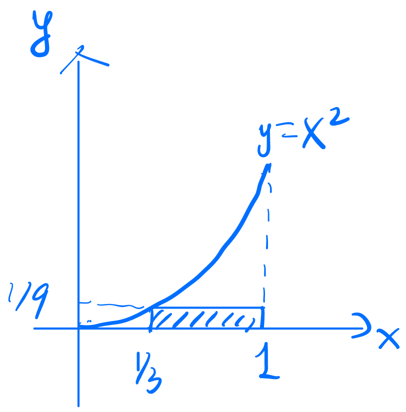
Fact: Suppose $A \subseteq B$, then $A \cup B = B$, $A \cap B = A$

e.g. (a) $\bigcup\limits_{i \in \mathbb{N}} [i, i+1] = [1, \infty)$

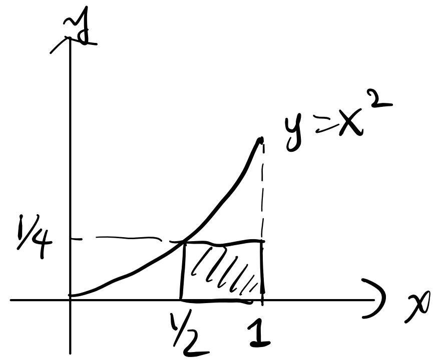


(b) $\bigcap\limits_{i \in \mathbb{N}} [i, i+1] = \phi$

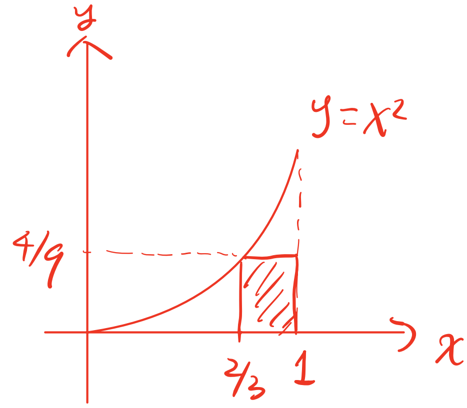
e.g. Sketch $[x,1]\times[0,x^2]$ for $x=\frac{1}{3}$, $x=\frac{1}{2}$, $x=\frac{2}{3}$



$x=1/3$



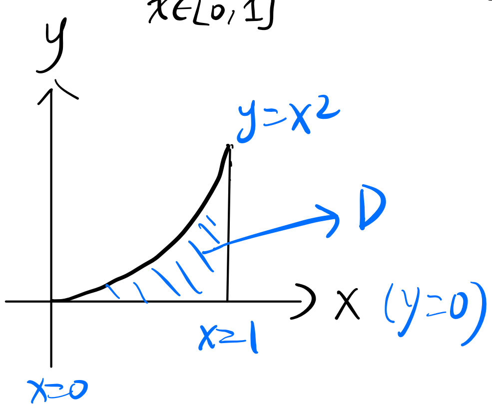
$x=\frac{1}{2}$



$x=2/3$

Consider index sets $A_x=[x,1]\times[0,x^2]$ for $0\le x\le 1$

(a) $\bigcup_{x\in[0,1]} A_x=\{(x,y)\in\mathbb{R}^2:\ 0\le y\le x^2,\ 0\le x\le 1\}=D$



Check: For all $(x,y)\in D$,

we have $x\in[x,1]$ and $y\in[0,x^2]$

$\Rightarrow (x,y)\in A_x=[x,1]\times[0,x^2]$

$\Rightarrow (x,y)$ belongs to at least one $A_x \Rightarrow (x,y)\in\bigcup_{x\in[0,1]} A_x$

(b) $\bigcap_{x\in[0,1]} A_x=\{(1,0)\}$

Check: Note that $(1,0)\in[x,1]\times[0,x^2]$ for all $0\le x\le 1$

But for any other points $(x,y)\neq(1,0)$:

Case 1: $x<1$, then $x<\frac{x+1}{2}<1$ so $(x,y)\notin A_{\frac{x+1}{2}}$

Case 2: $y>0$, then choose x_0 s.t. $y=\frac{x_0^2}{2}$, $(x,y)\notin A_{x_0}$

Mathematical Logic

- **Statement**: A sentence which is either True or False.

  e.g. If a circle has radius r, then its area is πr^2 square units. (T)

  e.g. $\sqrt{2} \in \mathbb{Q}$ (F)

- **Open Statement**: A statement whose truth depends on the value of one or more variables.

  e.g. $R(f, g)$: The function f is the derivative of the function g.

- And : $\wedge$

  Or : $\vee$

  Not : $\sim$

- Conditional Statements ($\Rightarrow$)

- Biconditional Statement ($\Leftrightarrow$)

  $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

e.g. A geometric series with ratio r converges if $|r|<1$.

Translation: If $|r|<1$, then a geometric series with ratio r converges.

or $\underbrace{|r|<1}_{P(r)} \Rightarrow \underbrace{\sum_{n=0}^{m} r^n \text{ converges as } m \to \infty}_{Q(r)}$

e.g. For matrix A to be invertible, it is necessary and sufficient that $\det(A) \neq 0$.

Translation: Matrix A is invertible iff $\det(A) \neq 0$

OR $\underbrace{A^{-1} \text{ exists}}_{P(A)} \Leftrightarrow \underbrace{\det(A) \neq 0}_{Q(A)}$

<u>Remark</u>: $P \Rightarrow Q$ $\begin{cases} Q \text{ is a necessary condition for } P. \\ P \text{ is a sufficient condition for } Q. \end{cases}$

P/Q is the necessary and sufficient condition for Q/P, then $P \Leftrightarrow Q$.

- Truth table

e.g. (*) $(P \wedge \sim P) \Rightarrow Q$

P	Q	$\sim P$	$P \wedge \sim P$	$(P \wedge \sim P) \Rightarrow Q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T