Solutions for Suggested Problems (Introduction)

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Basic Statistics

1.2

Set Theory and Axioms of Probability

$1.13, 1.14, B.1, B.5(a)$

Counting Methods

$1.7, 1.8, 1.26$

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu

(1.2) Bob has three options: Cereal, Eggs, or fruit. He has to choose exactly two out of three.

(a) Sample space Ω consists of the 2-element subsets of the set {cereal, eggs, fruits}.

i.e.

$$\Omega = \{\{\text{cereal}, \text{eggs}\}, \{\text{cereal}, \text{fruits}\}, \{\text{eggs}, \text{fruits}\}\}.$$

(b) $A = \{\text{Bob's breakfast includes cereal}\}$

$$= \{\{\text{cereal}, \text{eggs}\}, \{\text{cereal}, \text{fruits}\}\}.$$

(1.4)

(a) $\Omega = \{(x_1, x_2, x_3) : x_i \in \{\text{States in the U.S.}\}, i=1,2,3\}$

x_1: the state flag on Mon

x_2: 〃 〃 〃 〃 Tue

x_3: 〃 〃 〃 〃 Wed

$\# \Omega = 50^3$

For any event A in which $A \subseteq \Omega$,

we model the probability measure on A as

$$\mathbb{P}(A) = \frac{\# A}{\# \Omega}$$

(b) Let event A be "

Mon	Tue	Wed
Wis	Michigan	Cal

"

Obviously $\# A = 1$ (one outcome in event A)

so $\mathbb{P}(A) = \frac{1}{50^3}$

(c) $A = \{\text{At least two of three days we hang Wis}\}$

Let $A_1 = \{\text{exactly two days we hang Wis}\}$

$A_2 = \{\text{exactly three days we hang Wis}\}$

then $\quad A = A_1 \cup A_2, \quad A_1 \cap A_2 = \phi$

$\Rightarrow \#A = \#A_1 + \#A_2$

$\#A_1 = \underbrace{1 \cdot 1 \cdot 49} + 1 \cdot 49 \cdot 1 + 49 \cdot 1 \cdot 1$

Mon, Tue : Wisconsin

Wed : non-Wisconsin (choose any of 49 remaining state flags)

$\#A_2 = 1 \cdot 1 \cdot 1$

So $\mathbb{P}(A) = \dfrac{3 \cdot 49 + 1}{50^3} = \dfrac{37}{31250}$

(1.7):

3 Gs
4 Ys

urn

draw 3 balls without replacement

(a) Label the balls 1 through 7, with

green balls: 1, 2, 3

yellow balls: 4, 5, 6, 7.

$$\Omega = \{(i,j,k) : i,j,k \in \{1,\cdots,7\},\ i\neq j,\ i\neq k,\ j\neq k\}$$

Note that order matters for this problem.

$$\#\Omega = 7\cdot 6\cdot 5$$

Let $A = \{$green ball first, then a yellow ball, then a green ball$\}$

$$\#A = \underbrace{3}_{\text{3 greens in total}}\cdot\underbrace{4}_{\text{4 yellow in total}}\cdot 2 \rightarrow \text{2 green remaining after the first ball}$$

so $$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{3\cdot 4\cdot 2}{7\cdot 6\cdot 5} = \frac{4}{35}$$

(b) MTD 1: Continuing using Ω as defined in part (a) (in which the order matters)

$\mathbb{P}$(2 greens and One yellow)

$= \mathbb{P}(\text{Green, Green, Yellow}) + \mathbb{P}(\text{Green, Yellow, Green})$

$+ \mathbb{P}(\text{Yellow, Green, Green})$

$$= \frac{3\cdot 2\cdot 4 + 3\cdot 4\cdot 2 + 4\cdot 3\cdot 2}{7\cdot 6\cdot 5} = \frac{72}{210} = \frac{12}{35}$$

MTD 2: This question does NOT require the ordering, so we can take

$$\Omega = \{\{i,j,k\} : i,j,k \in \{1,\cdots,7\},\ i\neq j,\ i\neq k,\ j\neq k\}$$

$\#\Omega = \binom{7}{3}$

- Choose two green balls out of 3: $\binom{3}{2}$
- Choose one yellow balls out of 4: $\binom{4}{1}$

then $\mathbb{P}(\text{2 Greens, one yellow}) = \dfrac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = \dfrac{4}{35}$

$\boxed{3}$

(1-8) Label the letters from 1 and 14

5 Es: be labeled as 1, 2, 3, 4, 5

4 As: be labeled as 6, 7, 8, 9

3 Ns: be labeled as 10, 11, 12

2 Bs: be labeled as 13, 14

Draw four letters from a bag, without replacement.

$C = \{$ I got 2 Es, 1 A and 1 N $\}$

(a) If we consider ordering, then

$$\Omega = \{ (a_1, a_2, a_3, a_4) : a_i \neq a_j ,\ a_i = \{1, 2, \cdots, 14\} \}$$

$$\# \Omega = 14 \times 13 \times 12 \times 11$$

$C = \{ (a_1, a_2, a_3, a_4) : a_i \neq a_j,$

two of $(a_1, a_2, a_3, a_4) \in \{1, 2, \cdots, 5\}$

one of $(a_1, a_2, a_3, a_4) \in \{6, \cdots, 9\}$

one of $(a_1, a_2, a_3, a_4) \in \{10, 11, 12\} \}$

To calculate $\#C$, we do in a step-by-step way:

Step 1: Choose the positions for two Es: $\binom{4}{2}$

Step 2: Choose a first E out of 5 choices and place it into the first chosen position; Then choose a second E out of the remaining 4 choices and place it into the second chosen position.

Step 3: Choose A out of 4 choices, and one of the remaining 2 positions

Step 4: Choose N out of 3 choices, and put it into the remaining one position.

$$\#C = \binom{4}{2} \cdot 5 \cdot 4 \cdot 4 \cdot 2 \cdot 3 \cdot 1 = 2880$$

$$\mathbb{P}(C) = \frac{\#C}{\#\Omega} = \frac{120}{1001}$$

(b) Another way to see this position is not to consider ordering, i.e.

$$\Omega = \{\{a_1, a_2, a_3, a_4\} : a_i \neq a_j,\ a_i \in \{1, 2, \cdots, 14\}\}$$

$$\#\Omega = \binom{14}{4} = 1001$$

$C = \{\{a_1, a_2, a_3, a_4\} : a_i \neq a_j$, two numbers $\in \{1, \cdots, 5\}$, one number $\in \{6, \cdots, 9\}$, one number $\in \{10, 11, 12\}\}$.

$$\#C = \underbrace{\binom{5}{2}}_{\text{choose two of } \{1,\cdots,5\}} \underbrace{\binom{4}{1}}_{\text{choose one of } \{6,\cdots,9\}} \binom{3}{1} \rightarrow \text{choose one of } \{10, 11, 12\}$$

$$\mathbb{P}(C) = \frac{\#C}{\#\Omega} = \frac{120}{1001}$$

(1.13) $W = \{\text{wear a watch}\}$ $\mathbb{P}(W) = .25$

$B = \{\text{wear a bracelet}\}$ $\mathbb{P}(B) = .3$

$\mathbb{P}(B^c \cap W^c) = .6$

(a) $\mathbb{P}(\text{wear a watch or bracelet})$

$= \mathbb{P}(B \cup W)$ (De Morgan's: $(B \cup W)^c = B^c \cap W^c$)

$= 1 - \mathbb{P}(B^c \cap W^c)$

$= 1 - .6 = .4$

(b) $\mathbb{P}(\text{wear both a watch and a bracelet})$

$= \mathbb{P}(B \cap W)$ ($\mathbb{P}(B \cup W) = \mathbb{P}(B) + \mathbb{P}(W) - \mathbb{P}(B \cap W)$)

$= \mathbb{P}(B) + \mathbb{P}(W) - \mathbb{P}(B \cup W)$

$= (.25) + (.3) - (.4)$

$= .15$

(1.14) $\mathbb{P}(A) = .4 \quad \mathbb{P}(B) = .7$

$$\mathbb{P}(AB) \le \min\{\mathbb{P}(A), \mathbb{P}(B)\} = .4$$

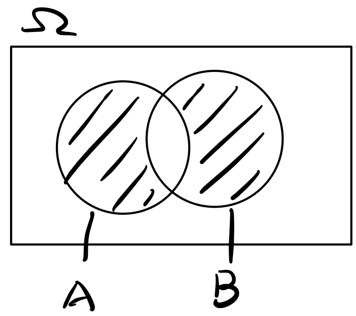
$$\mathbb{P}(AB) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$$

$$\ge \mathbb{P}(A) + \mathbb{P}(B) - 1 \quad (\mathbb{P}(A \cup B) \le 1)$$

$$= (.4) + (.7) - 1 = .1$$

(B5) (a) $A \Delta B = (A \cup B) \setminus (A \cap B)$

or $A \Delta B = (AB^c) \cup (A^cB)$



(B1) $A, B, C \subseteq \Omega$.

(a) $D = \{$Exactly Two of A, B, C happen$\}$

$$= \underbrace{(ABC^c)}_{\text{A, B happen but not C}} \cup \underbrace{(AB^cC)}_{\text{A,C happen but not B}} \cup \underbrace{(A^cBC)}_{\text{B,C happen but not A}}$$

(b) $E = \{$At least two of A, B, C happen$\}$

$$= \underbrace{(ABC^c) \cup (AB^cC) \cup (A^cBC)}_{\text{Exactly two of them happen}} \cup \underbrace{(ABC)}_{\text{All three happen.}}$$

(1.26)

15 people $\begin{cases} 10 \text{ men} \\ 5 \text{ women} \end{cases}$ Choose 4 people at random to form a committee

(a) Let $A = \{$ Two men and two women are chosen $\}$

<u>MTD 1</u>: Without order

$$\mathbb{P}(A) = \frac{\binom{10}{2}\binom{5}{2}}{\binom{15}{4}} = \frac{30}{91}$$

<u>MTD 2</u>: With order: $\#\Omega = 15 \cdot 14 \cdot 13 \cdot 12$ possible outcomes

To calculate $\#A$, need to do step-by-step:

<u>Step 1</u>: Choose two position for men, so $\binom{4}{2}$ ways

<u>Step 2</u>: For the first position, 10 ways to choose a man;

For the second position, 9 way to choose another one.

<u>Step 3</u>: Among the remaining two positions:

For the first position, 5 ways to choose a man;

For the second position, 4 ways to choose another one.

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{\binom{4}{2} \cdot 10 \cdot 9 \cdot 5 \cdot 4}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{30}{91}$$

(b) $B = \{$ A couple, Bob and Jane, are chosen $\}$

MTD 1 : Without order

$$\mathbb{P}(B) = \frac{\binom{2}{2}\boxed{\binom{13}{2}}}{\binom{15}{4}}$$ → choose two people among the remaining 13 people.

$$= 2/35$$

MTD 2 : With order, to calculate $\#B$, we need to do step-by-step.

Step 1 : Bob's position (4 choices, since there are four positions)

Step 2 : Jane's position (3 choices b/c three positions remaining)

Step 3 : For the remaining two positions

13 choices for the first position (13 members remaining)

12 choices for the second position (12 members remaining)

$$\mathbb{P}(B) = \frac{4 \cdot 3 \cdot 13 \cdot 12}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{2}{35}$$

(c) $C = \{$ Bob is in the committee, but Jane is NOT $\}$

MTD 1 : Without order:

$$\mathbb{P}(C) = \frac{\binom{13}{3}}{\binom{15}{4}}$$

choose 3 additional members beside Bob, out of the 13 possibilities.

$$= 22/105$$

MTD 2 : With order :

Step 1 : Choose Bob's position (4 choices)

Step 2 : For each of the remaining three positions, choose 3 addition members, respectively ($13 \cdot 12 \cdot 11$ choices)

$$\mathbb{P}(C) = \frac{4 \cdot 13 \cdot 12 \cdot 11}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{22}{105}$$

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