Continuous Random Variables

X: Continuous R.V.

1. Support of X:

S_X should be an interval

or $\mathbb{R}$ $\longrightarrow x \in \mathbb{R}$.

2. Probability Density Function (P.D.F.):

f is P.D.F. of X if

$$\mathbb{P}(X \in B) = \int_B f(x)\,dx \qquad B \subseteq \mathbb{R}: \text{ any subset in } \mathbb{R}.$$

3. Cumulative Density Function (C.D.F.):

Let F be the C.D.F. of X,

$$F(x) \overset{\text{def}}{=} \mathbb{P}(X \le x) = \int_{-\infty}^{x} f(t)\,dt \qquad (f: \text{P.D.F. of } X)$$

Properties:

1. P.D.F: $f(x) \ge 0$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, $f(x) = F'(x)$

2. C.D.F: $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

for $a < b$: $F(b) - F(a) = \mathbb{P}(X \le b) - \mathbb{P}(X \le a)$ $\Rightarrow F$ is non-decreasing.

$$= \mathbb{P}(a < X \le b) \ge 0$$

HW4, P1:

80 patients: test a new drug w.p. p to be effective.

with probability

X: # of patients for whom the drug is effective.

Event $S = \{$Trial is successful for two friends$\}$

Note $X \sim Bin(80, p)$ $\mathbb{P}(S) = p^2$

then $S \cap \{X=55\}$

$= \{$Trial is successful for two of your friends and the other 53 patients out of 78$\}$

Given the trial is successful for 55 patients,

$$\mathbb{P}(S \mid X=55) = \frac{\mathbb{P}(S \cap \{X=55\})}{\mathbb{P}(X=55)}$$

$$= \frac{\cancel{p^2} \binom{78}{53} \cancel{p^{53}} \cancel{(1-p)^{25}}}{\binom{80}{55} \cancel{p^{55}} \cancel{(1-p)^{25}}}$$

Please simplify the fraction.

$$= \frac{78!}{53! \cdot 25!} \cdot \frac{55! \cdot 25!}{80!} = \frac{55 \cdot 54}{80 \cdot 79}$$

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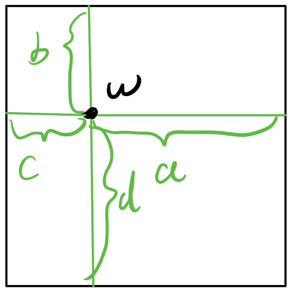
HW4, P4: A point w is randomly chosen in a unit square Ω.

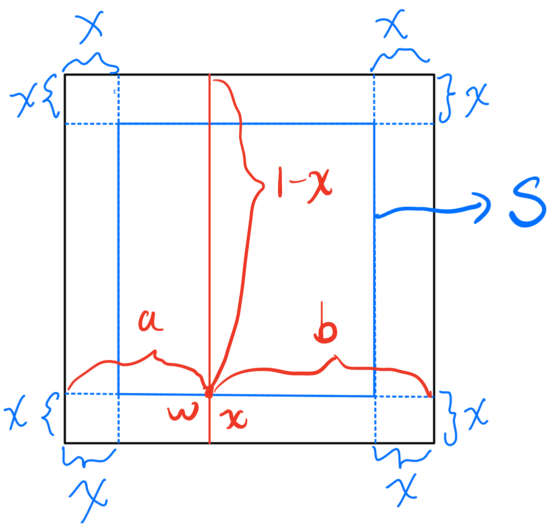


unit square Ω

Then the random variable

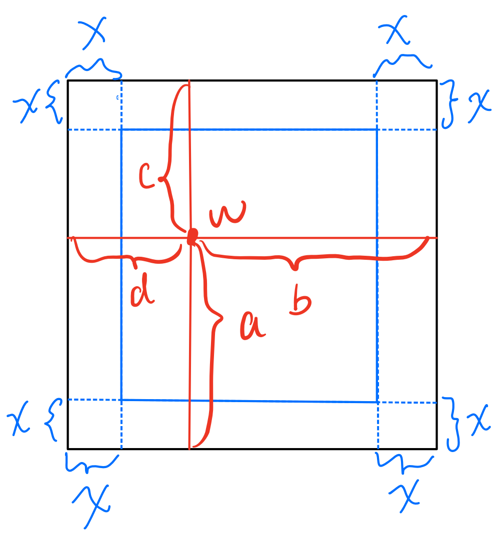
$$X(w) \stackrel{def}{=} \min\{a, b, c, d\}$$

Remark: For one number $x \in [0, \frac{1}{2}]$, cut down the unit square Ω as follows, and denote the edge of small square as S.



(1) $S = \{w \in \Omega : X(w) = x\}$ since (see the graph in the left), $w \in S \Rightarrow a > x,\ b > x$, and $1-x \geq \frac{1}{2} \geq x$, so

$X(w) = \min\{a, b, x, 1-x\} = x$

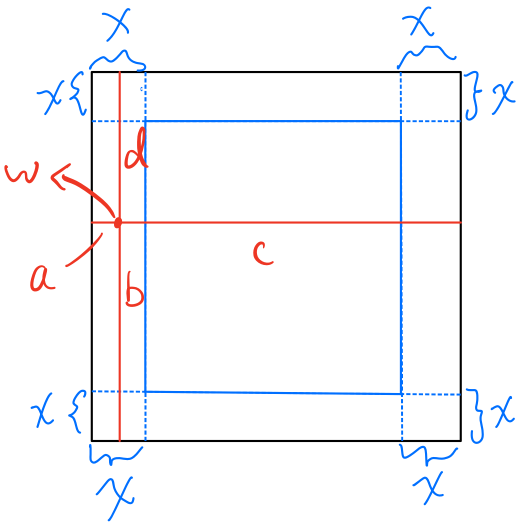


(2) If w is inside S, then

$a > x,\ b > x,\ c > x,\ d > x$

$\Rightarrow X(w) = \min\{a, b, c, d\}$

$> x$



(3) If w is outside S, then

$a < x$ implies

$$X(w) = \min\{a, b, c, d\}$$
$$\le a < x$$

Based on the argument (1), (2), (3) above, we have $\{X(w) \le x\} \Leftrightarrow \{w \text{ is outside } S\}$

$$\text{Area}(S) = (1-2x)^2$$

$$\mathbb{P}(X \le x) = \frac{\text{Area}(\Omega \setminus S)}{\text{Area}(\Omega)}$$

$$= \frac{1-(1-2x)^2}{1 \times 1} \quad \text{for} \quad 0 \le x \le \frac{1}{2}$$

So C.D.F $F_X(x) = 1-(1-2x)^2 \quad 0 \le x \le \frac{1}{2}$

P.D.F $f_X(x) = 4 - 8x \quad 0 \le x \le \frac{1}{2}$

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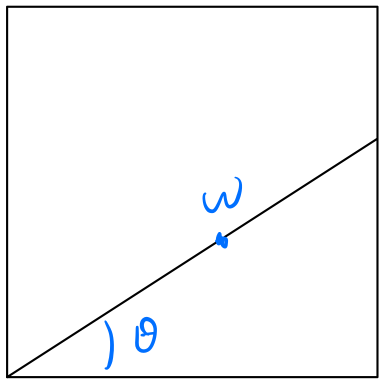
HW 4, P8: A point w is randomly chose in a unit square Ω.



Unit Square Ω

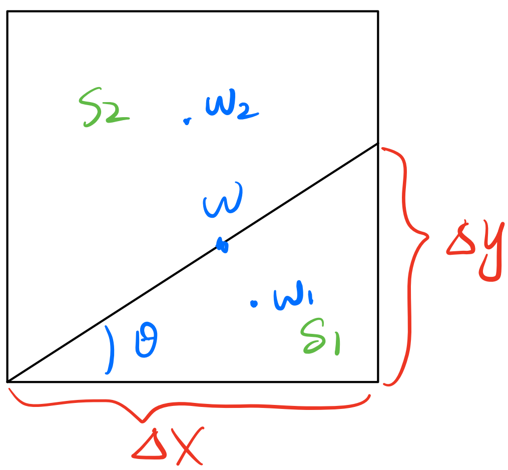
$X(w) =$ slop of line

$= \tan\theta$

Support: $S_X = [0, \infty]$

Two cases:

(1) Slope $X(w) = x \le 1$



$w_1 \in S_1$: Slope $X(w_1) \le X(w) = x$

$w_2 \in S_2$: slope $X(w_2) \ge X(w) = x$

So $X(w) \le x \Leftrightarrow w \in S_1$

slope $= x = \frac{\Delta y}{\Delta x}$ $\quad \Delta x = 1, \ \Delta y = x$

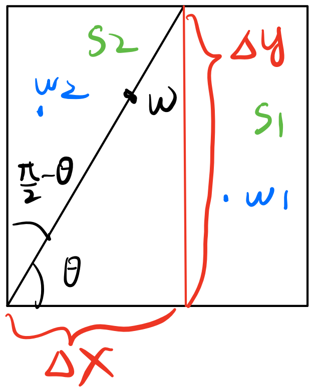
$$\mathbb{P}(X \le x) = \frac{\text{Area}(S_1)}{\text{Area}(\Omega)}$$

$\Rightarrow \text{Area}(S_1) = \frac{(\Delta x)(\Delta y)}{2} = \frac{x}{2}$

$$= \frac{1 \cdot x \cdot (1/2)}{1 \times 1} = x/2$$

C.D.F. $F_X(x) = x/2 \Rightarrow$ P.D.F $f_X(x) = \frac{1}{2} \quad 0 \le x \le 1.$

(2) Slope $X(\omega) = x > 1$



$\omega_1 \in S_1$: Slope $X(\omega_1) \le X(\omega) = x$

$\omega_2 \in S_2$: slope $X(\omega_2) \ge X(\omega) = x$

So $X(\omega) \le x \iff \omega \in S_1$ and

slope $= x = \dfrac{\Delta y}{\Delta x}$, $\Delta y = 1 \Rightarrow \Delta x = \dfrac{1}{x}$

$\Rightarrow \text{Area}(S_2) = \dfrac{(\Delta x)(\Delta y)}{2} = \dfrac{1}{2x}$

$$\mathbb{P}(X \le x) = \frac{\text{Area}(S_1)}{\text{Area}(\Omega)}$$

$$= \frac{\text{Area}(\Omega) - \text{Area}(S_2)}{\text{Area}(\Omega)}$$

$$= \frac{1 - 1 \cdot \frac{1}{x} \cdot \frac{1}{2}}{1 \cdot 1} = 1 - \frac{1}{2x}$$

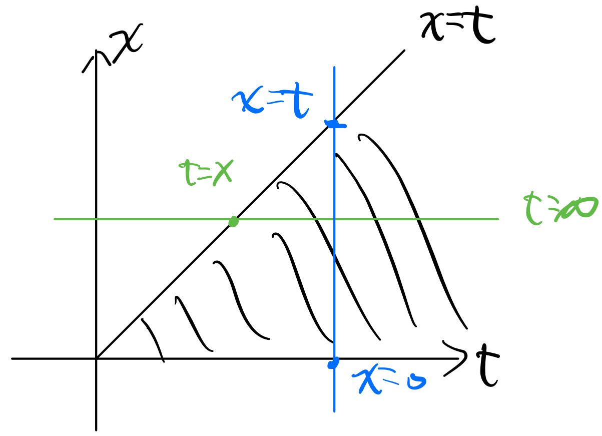
C.D.F. $F_X(x) = 1 - \dfrac{1}{2x}$ $\quad x > 1$

$\Rightarrow$ P.D.F. $f_X(x) = \dfrac{1}{2x^2}$ $\quad x > 1$

so
$$f_X(x) = \begin{cases} 1/2 & 0 \le x \le 1 \\ \dfrac{1}{2x^2} & x > 1 \\ 0 & \text{o.w.} \end{cases}$$

HW4 P7 :

$\mathbb{P}(X \geq 0) = 1 \Rightarrow x > 0, \quad F_X(x) = \int_0^x f_X(t)\,dt$



$$\int_0^\infty [1 - F_X(x)]\,dx$$

$$= \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx \quad \text{by } (*)$$

$$= \int_0^\infty \int_0^t f_X(t)\,dx\,dt$$

$$= \int_0^\infty t f_X(t)\,dt$$

$(*)$ $1 - F_X(x) = \int_{-\infty}^{\infty} f(t)\,dt - \int_{-\infty}^{x} f(t)\,dt$

$$= \int_x^\infty f(t)\,dt$$