Problem 4bc Homework 5

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1 Definition of Gamma Function and Gamma Distribution

Define the Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

for $\alpha > 0$. Here are the claims for Gamma function:

a. $\Gamma(1) = 1$.

b. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$.

c. $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{N}$.

d. For any $\alpha, \beta > 0$, we have

$$\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

Proof. By definition, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ for $\alpha > 0$,

a. For $\alpha = 1$, we have

$$\int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$$

b. For $\alpha > 1$, we can use the integration by parts, we have

$$\begin{aligned}
\int_0^\infty x^{\alpha-1} e^{-x} dx &= -x^{\alpha-1} e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}(\alpha-1)x^{\alpha-2} dx \\
&= -(0-0) + (\alpha-1)\int_0^\infty x^{\alpha-2} e^{-x} dx \\
&= (\alpha-1)\int_0^\infty x^{(\alpha-1)-1} e^{-x} dx \\
&= (\alpha-1)\Gamma(\alpha-1)
\end{aligned}$$

c. To show that $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{N}$, we need to use mathematical induction, which defines $\mathcal{P}(n)$ as $\Gamma(n) = (n-1)!$:

(1) Base Case: when $n = 1$, we have left-hand side is 1 (shown above), and the right-hand side is $(1-1)! = 0! = 1$.

(2) Induction Hypothesis: Assume that $\Gamma(n) = (n-1)!$ for some $n \geq 1$.

(3) Induction Step: We wish to show that $\Gamma(n+1) = n!$.

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \quad \text{By properties of Gamma function} \\
&= n(n-1)! \quad \text{Induction Hypothesis} \\
&= n!
\end{aligned}$$

(4) Conclusion: For any $n \in \mathbb{N}$, we have $\Gamma(n) = (n-1)!$

d. Using the change of variables, let $t = \beta x$, we have

$$\begin{aligned}\int_0^\infty x^{\alpha-1} e^{-\beta x} dx &= \int_0^\infty \left(\frac{t}{\beta}\right)^{\alpha-1} e^{-t} d(t/\beta) \\ &= \left(\frac{1}{\beta}\right)^{\alpha-1} \frac{1}{\beta} \int_0^\infty t^{\alpha-1} e^{-t} dt \\ &= \frac{1}{\beta^\alpha} \Gamma(\alpha)\end{aligned}$$

which completes the proof of these four claims. $\square$

2 Solution to Problem 4bc Homework 5

Given $X \sim \exp(1)$, the C.D.F of random variable X follows the density function $f(x) = e^{-x}$ for $x > 0$. Let $Y = X^\beta$ and $\beta = 3$, we want to calculate the n-th moment of Y, which is

$$\begin{aligned}\mathbb{E}[Y^n] &= \mathbb{E}[X^{3n}] \\ &= \int_0^\infty x^{3n} e^{-x} dx \\ &= \int_0^\infty x^{(3n+1)-1} e^{-x} dx \\ &= \Gamma(3n+1) = (3n)!\end{aligned}$$

where the last two equalities follows from the definition of Gamma function and the fact that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, respectively.

Remark 1. Suppose a random variable $X \sim \Gamma(\alpha, \lambda)$, then it has a density

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & o.w. \end{cases}$$

Note that exponential distribution is a special case for Gamma distribution, in which $\alpha = 1$ and the density becomes

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & o.w. \end{cases}$$

Remark 2. Going back to the homework problem, extend the case for $\beta = 3$ to a general case in which $\beta > 0$, then we have

$$\begin{aligned}\mathbb{E}[Y^n] &= \mathbb{E}[X^{\beta n}] \\ &= \int_0^\infty x^{\beta n} e^{-x} dx \\ &= \int_0^\infty x^{(\beta n+1)-1} e^{-x} dx \\ &= \Gamma(\beta n+1)\end{aligned}$$

where the last equality follows from the definition of Gamma function.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu