Counting Methods

① **List**: A list is an order sequence of objects.

Notation:

$$(\underset{1^{st}\text{ entry}}{\_}, \underset{2^{nd}\text{ entry}}{\_}, \cdots, \underset{n^{th}\text{ entry}}{\_}) \Rightarrow (\text{length} = n)$$

Each entry represents each single objects and the length of list is its number of entries.

e.g. $(1,2,3)$: a list (of numbers) of length three

(a,b,c,d,e): a list of alphabet of length five.

Difference between sets and lists:

(1) The order of elements of sets does NOT matter, but the order in lists matter.

e.g. $\{a,b,c,d,e\} = \{b,a,c,d,e\}$

$(a,b,c,d,e) \neq (b,a,c,d,e)$

(2) In lists, we allow repeated entries.

e.g. $\{5,5,3,4,6\} = \{3,4,5,6\} \Rightarrow$ Cardinality $=4$

$\{5,3,5,4,3,3\} = \{3,4,5\} \Rightarrow$ Cardinality $=3$

$(5,3,5,4,3,3)$: list with length six / 6 entries.

② Multiplication Principle

Fact 3.1: Suppose in making a list of length n

$$\begin{cases} \text{first entry: } a_1 \text{ possible choices} \\ \text{second entry: } a_2 \text{ possible choices} \\ \vdots \\ n^{th} \text{ entry: } a_n \text{ possible choices} \end{cases}$$

Total number possible lists $= a_1 \cdot a_2 \cdot \cdots \cdot a_n$

Remark: We can consider a list of length n as an element of Cartesian product of sets $X = X_1 \times X_2 \times \cdots \times X_n$. In other words,

list: $(x_1, x_2, \cdots, x_n) \in X$ with $x_1 \in X_1, \cdots, x_n \in X_n$.

e.g. example from Page 67 in textbook

$A = \{a, b, c\}$, $B = \{5, 7\}$, $C = \{a, x\}$

$A \times B \times C$ contains all possible lists of length three

with $\begin{cases} \text{first entry in } \{a, b, c\} \Rightarrow a_1 = 3 \\ \text{second entry in } \{5, 7\} \Rightarrow a_2 = 2 \\ \text{third entry in } \{a, x\} \Rightarrow a_3 = 2 \end{cases}$

thus $|A \times B \times C| = 3 \times 2 \times 2 = 12$

e.g. (ch 3.1 ex 1)

Consider the lists made from the letters T, H, E, O, R, Y, with repetition allowed.

(a) How many length-4 lists are there?

Soln: $(\_, \_, \_, \_)$.

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 6 & 6 & 6 \end{array}$$

# of possibilities $= 6^4 = 1296$

(b) How many length-4 lists that begin with T?

Soln: $(\underline{T}, \_, \_, \_)$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 6 & 6 & 6 \end{array}$$

# of possibilities $: 6^3 = 216$

(c) How many length-4 lists that do <u>NOT</u> begin w. T?

Soln: $(\overset{\text{NOT}}{\underline{T}}, \_, \_, \_)$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 6 & 6 & 6 \end{array}$$

# of possibilities $: 5 \times 6^3 = 1080.$

e.g. (*) A coin is tossed 10 times in a row. How many possible sequences of heads and tails are there?

Soln: $(\_, \_, \cdots, \_)$ $\qquad 2^{10} = 1024$

entry 1: $\{H, T\}$ $\quad$ entry 2: $\{H, T\}$ $\quad$ entry 10: $\{H, T\}$

③ Permutation: A special Case for Multiplication Principle

Definition: For each non-negative integer n, we define the factorial of n, denoted by $n!$, as

$$n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$$

e.g. $2! = 2$, $3! = 1 \times 2 \times 3 = 6$ $4! = 1 \times 2 \times 3 \times 4 = 24$

Note that: $\forall n \geq 1$, $n! = n(n-1)!$

for $n = 1$, we have $1! = 1 \cdot 0!$

so $0! = \frac{1!}{1} = 1$

Problem: (example 3.8: modified)

To make a list of length five from sets of letters $\{a, b, c, d, e\}$

(a) How many such lists are there if repetition is NOT allowed?

Soln: $(\underset{5}{\_}, \underset{4}{\_}, \underset{3}{\_}, \underset{2}{\_}, \underset{1}{\_})$ # of possibilities;

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

(b) How many such lists if repetition is NOT allowed and the first two entries must be vowels? $\to a, e$

Soln: $(\overbrace{\underset{2}{\_}, \underset{1}{\_}}^{\{a,e\}} \,\Big|\, \overbrace{\underset{3}{\_}, \underset{2}{\_}}^{\{b,c,d\}}, \underset{1}{\_})$ $\#: (2!)(3!) = 12$. //

In general, X be a set w. $|X|=n$, then the number of non-repetitive length-k list made from the set X is equal to $P(n,k) = \frac{n!}{(n-k)!}$

In the previous problem, $|\{a,b,c,d,e\}|=5$, there are $P(5,5) = \frac{5!}{0!} = 5!$ such lists which has length $k=5$ and no repetition.

④ Counting Subsets

For a set B, we are now interested in

$|\{X \in \mathcal{P}(B) : |X|=k\}|$ for $k \leq |B| = n$.

e.g. $B = \{a,b,c,d\}$, $X \in \mathcal{P}(B)$, $|X|=3$

Subsets: $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$

thus $|\{X \in \mathcal{P}(B) : |X|=3\}| = 4$

e.g. $B = \{a,b,c,d,e\}$, $X \in \mathcal{P}(B)$, $|X|=3$

Subsets: $\{a,b,c\}$, $\{a,b,d\}$, $\{a,b,e\}$, $\{a,c,d\}$, $\{a,c,e\}$, $\{a,d,e\}$, $\{b,c,d\}$, $\{b,c,e\}$, $\{b,d,e\}$, $\{c,d,e\}$

thus $|\{X \in \mathcal{P}(B) : |X|=3\}| = 10$

In general: If $n \geq k$, $\binom{n}{k}$ denotes the number of k-element subsets made from an n-element set, then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example 3.11: How many size-4 subsets does $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ have?

Soln: $|A| = 9$, $\binom{9}{4} = \frac{9!}{4!\,5!} = 126$

Example 3.12: (a) How many 5-elem subsets of $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ have exactly two even elements? $|A| = 9$

Soln: Step 1: four even elements from A, I want to choose two of them, $\binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6$.

Step 2: Five odd elements, choose three of them, $\binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10$

By multiplication rule: $6 \times 10 = 60$ subsets.

⑤ Addition / subtraction Principle.

Prop: Let $X_1, X_2, \cdots, X_n$ be a disjoint collection of finite sets, $X = X_1 \cup X_2 \cup \cdots \cup X_n$

$$|X| = \sum_{i=1}^{n} |X_i| = |X_1| + |X_2| + \cdots + |X_n|$$

Prop: $U \subseteq X$, $|X-U| = |X| - |U|$

e.g. (*) (Ch 3.6, exercise 6)

Consider lists made from the the symbols A, B, C, D, E.

(a) How many such length-5 lists have at least one letter repeated?

U: set of all possible lists, $|U| = 5^5$

X: set of all non-repetitive lists, $|X| = 5!$

$U-X$: set of lists w. at least one letter repeated. $|U-X| = |U| - |X| = 5^5 - 5!$

(b) How many such length-6 lists have at least one letter repeated?

U: — — — — $|U| = 5^6$

X: — — — — $|X| = 0$

$U-X$: — — — — $|U-X| = 5^6$