Week 9 Section 1 Notes

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Some Examples for Induction

1. Prove $2^n > n^3$ for any $n \geq 10$.

   Proof. We use induction now: $\mathcal{P}(n)$ indicates "$2^n \geq n^3$.

   (a) Base case: Given $n = 10$, left-hand side is $2^10 = 1024$ and right-hand side is 10^3, thus $\mathcal{P}(10)$ is True.

   (b) Induction Hypothesis: Suppose $\mathcal{P}(n)$ is True for some $n \geq 10$.

   (c) Induction Step: We wish to show $\mathcal{P}(n+1)$ given $\mathcal{P}(n)$, in other words, $2^{n+1} \geq (n+1)^3$.

$$\begin{aligned}
2^{n+1} &= 2^n \cdot 2 \\
&\leq n^3 \cdot 2 \quad \text{Induction Hypothesis} \\
&= n^3 + n^3 \\
&\geq n^3 + 10n^2 \quad \text{Note that } n \geq 10 \\
&= n^3 + 3n^2 + 3n^2 + 4n^2 \\
&\geq n^3 + 3n^2 + 3n + 1 \quad (n \geq 10 \text{ implies } n^2 > n \text{ and } 4n^2 > 1) \\
&= (n+1)^3
\end{aligned}$$

   (d) Conclusion: We have for any $n \geq 10$, we have $\mathcal{P}(n)$ is True.

   $\square$

2. Prove $n! > 2^n$ for $n \geq 4$.

   Proof. We use induction to show $\mathcal{P}(n)$, which is "$n! > 2^n$.

   (a) Base Case: Given $n = 4$, we have left-hand side is $4! = 24$ and the right-hand side is $2^4 = 16$.

   (b) Induction Hypothesis: Suppose that $\mathcal{P}(n)$ holds for some $n \geq 4$.

   (c) Induction Step: We wish to show $(n+1)! > 2^{n+1}$.

$$\begin{aligned}
(n+1)! &= (n+1) \cdot n! \\
&> (n+1) \cdot 2^n \quad \text{Induction Hypothesis: } n! > 2^n \\
&> 2 \cdot 2^n \quad n \geq 4 \text{ implies } n+1 > 2 \\
&= 2^{n+1}
\end{aligned}$$

   (d) Conclusion: Thus, we have for every $n \geq 4$, $n! > 2^n$.

   $\square$

3. Show that for every $n \geq 2$,

$$\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1$$

Proof. Let $\mathcal{P}(n)$ denote that the inequalities above hold for n.

(a) Base Case: Given $n = 2$, we have

$$\sqrt{2} < 1 + \frac{1}{2} < 1 + \frac{1}{\sqrt{2}}$$

and $2\sqrt{2} + 1 < 4$ implies $2 + \frac{1}{\sqrt{2}} < 2\sqrt{2}$ implies $1 + \frac{1}{\sqrt{2}} < 2\sqrt{2} - 1$. Thus,

$$\sqrt{2} < 1 + \frac{1}{\sqrt{2}} < 2\sqrt{n} - 1$$

(b) Induction Hypothesis: Suppose that the inequalities hold for some $n \geq 2$.

(c) Induction Step: We wish to show $\mathcal{P}(n+1)$. We should show in two parts:

$$\begin{aligned}
&1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \\
=&\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{n+1}} \\
>&\sqrt{n} + \frac{1}{\sqrt{n+1}} \quad \text{Induction Hypothesis} \\
>&\sqrt{n+1} \quad \text{By lemma}
\end{aligned}$$

and

$$\begin{aligned}
&1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \\
=&\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{n+1}} \\
<&2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} \\
<&2\sqrt{n+1} - 1 \quad \text{By lemma}
\end{aligned}$$

To conclude, we have

$$\sqrt{n+1} < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1} - 1$$

(d) Conclusion: For any $n \geq 2$, we have the inequalities holds.

Lemma. For any $n \geq 0$, we have $\sqrt{n} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}$ and $2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$.

Proof. Suppose that $n \geq 0$, we have

$$\begin{aligned}
\sqrt{n} + \frac{1}{\sqrt{n+1}} &= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\
&> \frac{\sqrt{n \cdot n} + 1}{\sqrt{n+1}} \quad \text{Since } n+1 > n \\
&= \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{n+1}} &= \frac{2}{\sqrt{n+1} + \sqrt{n+1}} \\
&< \frac{2}{\sqrt{n+1} + \sqrt{n}} \quad \text{Since } n+1 > n. \\
&= \frac{2(\sqrt{n+1} - \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})} \\
&= \frac{2(\sqrt{n+1} - \sqrt{n})}{n+1-n} \quad \text{use the formula } a^2 - b^2 = (a+b)(a-b) \\
&= 2(\sqrt{n+1} - \sqrt{n})
\end{aligned}$$

So, the second inequality implies that

$$2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$$

$\square$

Note that the lemma supports the proof for both two cases, respectively. $\square$

Application for 120A

We know that in PSTAT 120A, we discuss some specific functions, random variables, expected values, and Moment generating function, etc. Now, we use mathematical induction to show some results (Note that in this part, no previous background in PSTAT 120A is needed, but you need some materials in single-variable calculus, e.g. integration by parts).

1. Define the Gamma Function

   $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$$

   for $\alpha > 0$. For $\alpha = 1$, we have

   $$\int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$$

   Also, for $n > 1$, we can use the integration by parts, we have

   $$\begin{aligned}\int_0^\infty x^{n-1}e^{-x}dx &= -x^{n-1}e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}(n-1)x^{n-2}dx \\ &= -(0-0) + (n-1)\int_0^\infty x^{n-2}e^{-x}dx \\ &= (n-1)\int_0^\infty x^{(n-1)-1}e^{-x}dx \\ &= (n-1)\Gamma(n-1)\end{aligned}$$

   Thus, we have $\Gamma(n) = (n-1)\Gamma(n-1)$. Now, we can use the induction to show $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{N}$.

   Proof. Consider $\mathcal{P}(n)$ as $\Gamma(n) = (n-1)!$

   (a) Base Case: when $n = 1$, we have left-hand side is 1 (shown above), and the right-hand side is $(1-1)! = 0! = 1$.

   (b) Induction Hypothesis: Assume that $\Gamma(n) = (n-1)!$ for some $n \geq 1$.

   (c) Induction Step: We wish to show that $\Gamma(n+1) = n!$.

   $$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) \quad \text{By properties of Gamma function} \\ &= n(n-1)! \quad \text{Induction Hypothesis} \\ &= n!\end{aligned}$$

   (d) Conclusion: For any $n \in \mathbb{N}$, we have $\Gamma(n) = (n-1)!$

   $\square$

2. Suppose that a random variable $X \sim Gamma(2,1)$ (density $f(x) = xe^{-x}$ for $x > 0$). Then its moment generating function is:

for $t < 1$:

$$
\begin{aligned}
M_X(t) = \mathbb{E}[e^{tX}] &= \int_0^\infty e^{tx} x e^{-x} dx \\
&= \int_0^\infty x e^{(t-1)x} dx \\
&= \frac{x e^{(t-1)x}}{t-1}\bigg|_0^\infty - \int_0^\infty \frac{e^{(t-1)x}}{t-1} dx \\
&= \frac{1}{(1-t)^2}
\end{aligned}
$$

We want to show that the n-th moment $\mathbb{E}[X^n] = (n+1)!$.

Proof. We claim that $\mathcal{P}(n)$ denotes $M_X^{(n)}(t) = \frac{(n+1)!}{(1-t)^{n+2}}$ for $n \geq 1$.

(a) Base Case: when $n = 1$, then

$$M_X'(t) = \frac{2}{(1-t)^3} = \frac{(1+1)!}{(1-t)^{1+2}}$$

Thus, $\mathcal{P}(1)$ is True.

(b) Induction Hypothesis: Assume that $\mathcal{P}(n)$ is True for some n.

(c) Induction Step: We wish to show that $\mathcal{P}(n+1)$ is True.

$$
\begin{aligned}
M_X^{(n+1)}(t) &= \frac{d}{dt} M_X^{(n)}(t) \\
&= \frac{d}{dt} \frac{(n+1)!}{(1-t)^{n+2}} \quad \text{Induction Hypothesis} \\
&= (n+2)\frac{(n+1)!}{(1-t)^{n+2}} \\
&= \frac{(n+2)!}{(1-t)^{n+3}}
\end{aligned}
$$

So, $\mathcal{P}(n+1)$ is True.

(d) Conclusion: $\mathcal{P}(n)$ is True for all n.

Thus, $\mathbb{E}[X^n] = M_X^{(n)}(0) = (n+1)!$ □

3. Suppose that given a random variable with standard normal distribution $X \sim \mathcal{N}(0,1)$. If k is odd, then

$$\mathbb{E}[X^k] = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^k e^{-\frac{x^2}{2}} dx = 0$$

since x^k is an odd function and $e^{-x^2/2}$ is an even function. For every $n \in \mathbb{N}$, $2n$ is even and we have $\mathbb{E}[X^{2n}] = \frac{(2k)!}{2^k \cdot k!}$.

Proof. We use the induction to prove $\mathcal{P}(n)$:

(a) Base Case: If $n = 1$, $\mathbb{E}[X^2] = Var(X) + (\mathbb{E}[X])^2 = 1 + 0 = 1 = \frac{2!}{2^1 \cdot 1!}$. Thus, $\mathcal{P}(1)$ is True. **Remark.** You can directly use the result that

$$\int_{-\infty}^\infty e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

and apply integration by parts (without the background in PSTAT 120A):

$$
\begin{aligned}
\int_{-\infty}^\infty x^2 e^{-\frac{x^2}{2}} dx &= -x e^{-\frac{x^2}{2}}\bigg|_{-\infty}^\infty + \int_{-\infty}^\infty e^{-\frac{x^2}{2}} dx \\
&= \sqrt{2\pi}
\end{aligned}
$$

Thus,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1$$

(b) Induction Hypothesis: Assume that $\mathcal{P}(n)$ is True.

(c) Induction Step: We wish to show $\mathcal{P}(n+1)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}} dx &= \frac{1}{2n+1} x^{2n+1} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} - \frac{1}{2n+1} \int_{-\infty}^{\infty} x^{2n+1} (-x) e^{-\frac{x^2}{2}} dx \\
&= (0-0) + \frac{1}{2n+1} \int_{-\infty}^{\infty} x^{2n+2} e^{-x^2/2} dx
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathbb{E}[X^{2(n+1)}] &= (2n+1)\mathbb{E}[X^{2n}] \\
&= (2n+1)\frac{(2k)!}{2^k \cdot k!} \quad \text{Induction Hypothesis} \\
&= \frac{(2k)!(2k+1)(2k+2)}{2^k \cdot k! \cdot (2k+2)} \\
&= \frac{(2k+1)!(2k+2)}{2^k \cdot k! \cdot 2 \cdot (k+1)} \\
&= \frac{(2k+2)!}{2^{k+1} \cdot (k+1)!}
\end{aligned}$$

Thus, $\mathcal{P}(n+1)$ is also True.

(d) Conclusion: $\mathcal{P}(n)$ is True for all n.

$\square$

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu