Topic Lists (Tentative):

Week 7 : Expectation

Week 8 : Moment Generating Functions

Week 9 : Normal Approximation and Transformation of R.V.s

Week 10 : Joint Distribution

Expectation

For a random variable X,

$$\mathbb{E}[X] \overset{\text{def}}{=} \begin{cases} \sum_{k \in S_X} k \, \mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{P.D.F}} dx & X: \text{continuous} \end{cases}$$

$E[X]$: the expected value X.

More generally, let $g: \mathbb{R} \to \mathbb{R}$, then $g(X)$ is also a R.V.

with

$$\mathbb{E}[g(X)] \overset{\text{def}}{=} \begin{cases} \sum_{k \in S_X} g(k) \, \mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f(x) dx & X: \text{continuous} \end{cases}$$

Moreover, $\text{Var}(X) \overset{\text{def}}{=} \mathbb{E}[(X-\mathbb{E}[X])^2]$

$$= \underbrace{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}_{\text{easier way to calculate}}$$

e.g. $f(t) = \begin{cases} 2t & t \in [0,1] \\ 0 & \text{o.w.} \end{cases}$

Claim 1: f is a valid density function.

Proof: ① $f(t) \geq 0 \quad \forall\, t \in \mathbb{R}$

② $\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^1 2t\,dt + \int_1^{\infty} 0\,dt$

$= \int_0^1 2t\,dt = t^2\Big|_0^1 = 1.$

Claim 2: Let X be a R.V. with density f, calculate $\mathbb{E}[X]$ and $Var(X)$.

Soln: $\mathbb{E}[X] = \int_{-\infty}^{\infty} t f(t)\,dt$

$= \int_0^1 t \cdot 2t\,dt = 2\int_0^1 t^2\,dt = \frac{2}{3}$

$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} t^2 f(t)\,dt = \int_0^1 t^2 \cdot 2t\,dt$

$= 2\int_0^1 t^3\,dt = \frac{1}{2}$

$Var(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = 5/18$ //

Normal Distribution

$X \sim N(\mu, \sigma^2)$

density: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \qquad x \in \mathbb{R}$

$\mathbb{E}[X] = \mu$, $\quad Var(X) = \sigma^2$

Facts:

① If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$ — standard Normal Dist

② For $Z \sim N(0,1)$, its CDF:

$\Phi(z) = \mathbb{P}(Z \le z)$ obtained from Normal Table.

e.g. $X \sim N(68, 16) \Rightarrow Z = \frac{X-68}{\sqrt{16}} = \frac{X-68}{4} \sim N(0,1)$

Type I Problem: Calculate the probability.

$$\mathbb{P}(X \ge 72) = \mathbb{P}\left(\frac{X-68}{4} \ge \frac{72-68}{4}\right)$$

$$= \mathbb{P}(Z \ge 1) \qquad Z \sim N(0,1)$$

$$= 1 - \underbrace{\Phi(1)}_{\text{Normal Table}} = 1 - .8413 = .1587$$

Type II Problem: Given the probability, obtain the threshold

Example: given $\mathbb{P}(X \ge x) = 28.1\%$, calculate x?

Soln: $\mathbb{P}(X \ge x) = \mathbb{P}\left(\frac{X-68}{4} \ge \frac{x-68}{4}\right)$

$$= \mathbb{P}\left(Z \ge \frac{x-68}{4}\right) = 28.1\%$$

$\Phi\left(\frac{x-68}{4}\right) = 1 - 28.1\% = 71.9\%$

Figure out $\frac{x-68}{4}$ by Normal Table. //.

Gamma Distribution

Gamma Function $\Gamma(\alpha) \overset{\text{def}}{=} \int_0^\infty x^{\alpha-1} e^{-x}\, dx \quad \alpha > 0$

Facts: ① $\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$

② $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \qquad \alpha > 1$

(Integration by parts)

③ $\forall n \in \mathbb{N}, \ \Gamma(n) = (n-1)!$ (Mathematical Induction)

④ $\alpha, \beta > 0: \ \int_0^\infty x^{\alpha-1} e^{-\beta x}\, dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$

$X \sim \text{Gamma}(r, \lambda)$ if its density

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & x > 0 \\ 0 & \text{o.w} \end{cases}$$

e.g. (From the Past Exam) X has a density

$$f(x) = \begin{cases} c x e^{-2x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

(a) Figure out c that makes f a valid p.d.f.

Soln: f is a form of Gamma distribution with

$r = \lambda = 2$, so $c = \frac{\lambda^r}{\Gamma(r)} = \frac{2^2}{\Gamma(2)} = 4$

$\Gamma(2) = (2-1)! = 1$. //

(b) Calculate $\mathbb{E}[X]$ and $Var(X)$.

Soln: By Distribution sheet, $X \sim Gamma(2,2)$

implies $\mathbb{E}[X] = \frac{r}{\lambda} = 1$, $Var(X) = \frac{r}{\lambda^2} = \frac{1}{2}$. //

(c) More generally, calculate the n-th moment of X, which is $\mathbb{E}[X^n]$, $n \in \mathbb{N}$.

Soln: $\mathbb{E}[X^n] = \int_0^\infty x^n \cdot 4xe^{-2x}\,dx$

$= 4\int_0^\infty x^{n+1}e^{-2x}\,dx$

$= 4 \cdot \frac{\Gamma(n+2)}{2^{n+2}}$ (use Fact ④, with $\begin{cases} \alpha = n+2 \\ \beta = 2 \end{cases}$)

$= \frac{4\,(n+1)!}{2^{n+2}} = \frac{(n+1)!}{2^n}$ //,