1. $X_1 \sim \exp(\beta_1)$, $X_2 \sim \exp(\beta_2)$, $X_1 \perp X_2$, $k>0$

$\hookrightarrow$ independent

(a) distribution of $\min\{X_1, kX_2\}$? For $x>0$:

$$\mathbb{P}(\min\{X_1, kX_2\} > x)$$

$$= \mathbb{P}(X_1 > x, kX_2 > x)$$

$$= \mathbb{P}(X_1 > x) \cdot \mathbb{P}(X_2 > x/k)$$

$$= e^{-\beta_1 x} \cdot e^{-\beta_2 x/k}$$

$$= e^{-(\beta_1 + \beta_2/k)x}$$

CDF: $\mathbb{P}(X_1 \le x) = 1 - e^{-\beta_1 x}$

$\mathbb{P}(X_2 \le x) = 1 - e^{-\beta_2 x}$

$x>0$

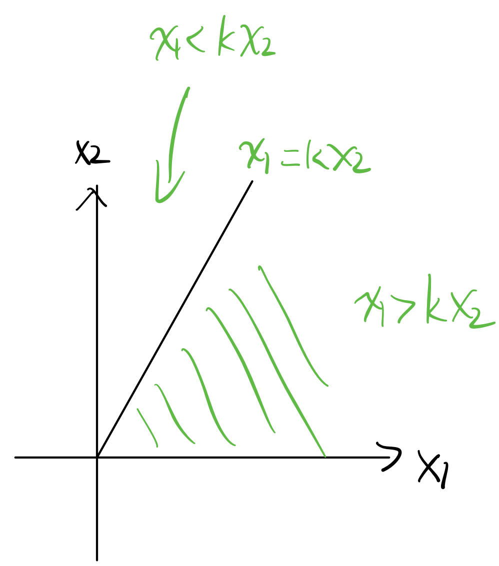
So CDF of $\min\{X_1, kX_2\}$ is

$$\mathbb{P}(\min\{X_1, kX_2\} \le x) = 1 - e^{-(\beta_1 + \beta_2/k)x}$$

Hence, $\min\{X_1, kX_2\} \sim \exp\left(\beta_1 + \frac{\beta_2}{k}\right)$

(b) Joint density of (X_1, X_2) is

$$f(x_1, x_2) = \begin{cases} \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} & x_1, x_2 > 0 \\ 0 & \text{o.w} \end{cases}$$



$$\mathbb{P}(X_1 > kX_2)$$

$$= \iint_{x_1 > kx_2} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^\infty \int_{kx_2}^\infty \beta_1 e^{-\beta_1 x_1} dx_1 \ \beta_2 e^{-\beta_2 x_2} dx_2$$

$$= \int_0^\infty e^{-\beta_1 (kx_2)} \beta_2 e^{-\beta_2 x_2} dx_2$$

$$= \beta_2 \int_0^\infty e^{-(k\beta_1 + \beta_2) x_2} dx_2$$

$$= \frac{\beta_2}{k\beta_1 + \beta_2}$$

2. X = number of toss needed to obtain five heads

$X \sim Negbin(r=5, p=1/30)$

(a) $\mathbb{E}[X] = \frac{r}{p} = \frac{5}{1/30} = 150$

(b) $Var(X) = \frac{r(1-p)}{p^2} = \frac{5 \times 29/30}{(1/30)^2} = 4350$

3. misprints in one page $\sim Poi(\lambda)$

$p = \mathbb{P}($ Exactly k misprints$) = e^{-\lambda} \frac{\lambda^k}{k!}$

$\mathbb{P}($ At least one page contains exactly k misprints$)$

$= 1 - \mathbb{P}($ no pages contains exactly k misprints$)$

$$= 1 - (1-p)^n = 1 - \left(1 - e^{-\lambda} \frac{\lambda^k}{k!}\right)^n$$

4. $f(x) = \begin{cases} c\,x^2 e^{-4x} & x>0 \\ 0 & \text{o.w.} \end{cases}$

(a) $X \sim \text{Gamma}(3, 4)$

$$c = \frac{4^3}{\Gamma(3)} = \frac{64}{2!} = 32$$

(b) $$\mathbb{E}\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} \cdot 32x^2 e^{-4x}\,dx$$

$$= 32\int_0^\infty x e^{-4x}\,dx$$

$$= 32\,\frac{\Gamma(2)}{4^2} \qquad \left(\int_0^\infty x^{\alpha-1} e^{-\beta x}\,dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \right)$$

$$= 2$$

(c) $$\mathbb{E}\left[\left(\frac{1}{X}\right)^2\right] = \int_0^\infty \frac{1}{x^2}\,32x^2 e^{-4x}\,dx$$

$$= 32\int_0^\infty e^{-4x}\,dx = 8$$

$$\text{Var}\left(\frac{1}{X}\right) = \mathbb{E}\left[\left(\frac{1}{X}\right)^2\right] - \left(\mathbb{E}\left[\frac{1}{X}\right]\right)^2$$

$$= 8 - 2^2 = 4$$

5. Let $Z \sim Bin(3, 2/3)$ and Z is independent of Y.

$$M_Z(t) = \left(\frac{2}{3}e^t + \frac{1}{3}\right)^3 = \frac{(2e^t+1)^3}{3^3} = \frac{1}{27}(2e^t+1)^3$$

Since $M_X(t) = M_Z(t) \cdot M_Y(t) = M_{Z+Y}(t)$, and

b/c Z and Y are independent

$\mathbb{E}[Y] = 10$, $Var(Y) = 2$, we have

$$\mathbb{E}[X] = \mathbb{E}[Z] + \mathbb{E}[Y]$$

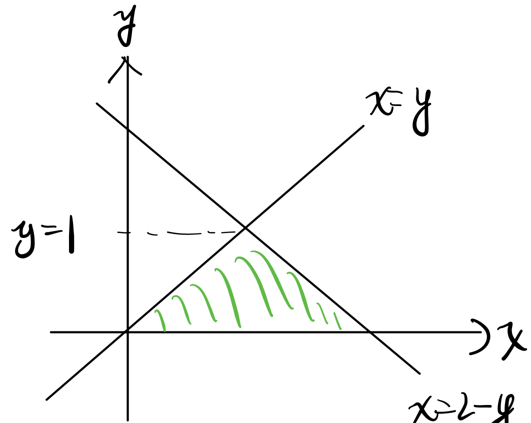
$$= 3 \cdot \frac{2}{3} + 10 = 12$$

$$Var(X) = Var(Z) + Var(Y)$$

again, Z and Y are independent

$$= 3 \cdot \frac{2}{3} \cdot \frac{1}{3} + 2 = \frac{2}{3} + 2$$

6. $$f(x,y) = \begin{cases} 3(2-x)y & 0<y<1,\ y<x<2-y \\ 0 & \text{o.w.} \end{cases}$$



(a) $\iint f(x,y)\,dx\,dy$

$$= \int_0^1 \int_y^{2-y} (2-x)\,dx\, 3y\, dy$$

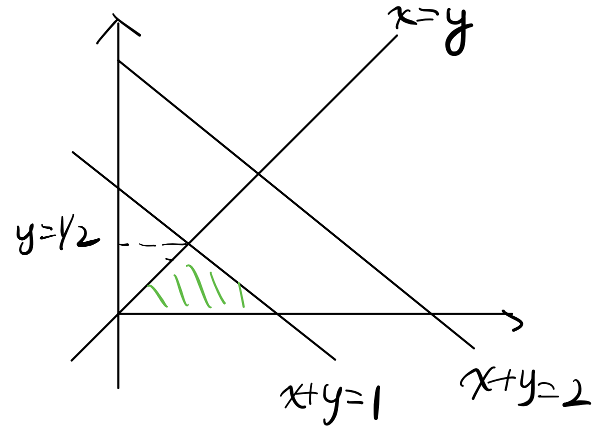
$$= \int_0^1 \left\{2x - \frac{1}{2}x^2 \Big|_y^{2-y}\right\} 3y\, dy$$

$$= \int_0^1 \left\{ 2(2-y) - \frac{1}{2}(2-y)^2 - 2y + \frac{1}{2}y^2 \right\} 3y \, dy$$

$$= \int_0^1 (2-2y) 3y \, dy$$

$$= \int_0^1 6y - 6y^2 \, dy = 3y^2 - 2y^3 \Big|_0^1 = 1$$

(b) $\displaystyle\iint_{x+y\le 1} f(x,y)\, dx\, dy$



$$= \int_0^{1/2} \int_y^{1-y} (2-x)\, dx\, (3y)\, dy$$

$$= 3 \int_0^{1/2} \left\{ 2x - \frac{1}{2}x^2 \Big|_{x=y}^{x=1-y} \right\} y \, dy$$

$$= 3 \int_0^{1/2} \left(2(1-y) - \frac{1}{2}(1-y)^2 - 2y + \frac{1}{2}y^2 \right) y \, dy$$

$$= 3 \int_0^{1/2} \left(\frac{3}{2} - 3y \right) y \, dy$$

$$= 3 \left(\frac{3}{4}y^2 - y^3 \right) \Big|_{y=0}^{y=1/2}$$

$$= \frac{3}{16}$$

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