<u>Example</u>: Now, suppose there are 20 distinct items

20 items
Group A ↙ ↘ Group B
$N_A = 4$ items $N - N_A = 16$ items

Select the items at random.

(a) If I select the items for 9 times, one item each time, with replacement, define

X = # of time I select the items from Group A.

calculate $\mathbb{P}(X \geq 1)$?

<u>Soln</u>: $X \sim Bin(9, p)$ $p = \frac{4}{20} = \frac{1}{5}$

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0) = 1 - \binom{9}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^9 = 1 - \left(\frac{4}{5}\right)^9 \quad //$$

(b) If I select 9 distinct items at random, calculate $\mathbb{P}(X \geq 1)$?

<u>Soln</u>: $\{X=0\} = \{$All 9 items come from Group B$\}$

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0) = 1 - \frac{\binom{4}{0}\binom{16}{9}}{\binom{20}{9}} = \text{simplify it} \quad //$$

Transformation of Random Variables

X : a R.V. $\quad g: \mathbb{R} \to \mathbb{R}$

Q: Given the distribution of X, how can I identify the distribution of $Y = g(X)$?

A: – If X is discrete, use P.M.F. Method.

– If X is continuous, use C.D.F. Method.

Example for Discrete R.V.: X has a p.m.f

$$\begin{cases} \mathbb{P}(X=-1) = 1/7 \\ \mathbb{P}(X=0) = 1/14 \\ \mathbb{P}(X=2) = 3/14 \\ \mathbb{P}(X=4) = 4/7 \end{cases}$$

Find the p.m.f of $Y = (X-1)^2$

Soln: Draw a Table

X	-1	0	2	4
$Y=(X-1)^2$	4	1	1	9
$\mathbb{P}$	$1/7$	$1/14$	$3/14$	$4/7$

we have $\mathbb{P}(Y=1) = \mathbb{P}(X=0) + \mathbb{P}(X=2) = \frac{1}{14} + \frac{3}{14} = \frac{2}{7}$

$\mathbb{P}(Y=4) = \mathbb{P}(X=-1) = \frac{1}{7}$

$\mathbb{P}(Y=9) = \mathbb{P}(X=4) = \frac{4}{7}$ //

Example for Continuous R.V.: $X \sim N(0,1)$ w.

p.d.f: $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, c.d.f $\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt$

Identify the density of $Y = X^2$. (Hint: use $\Gamma(1/2) = \sqrt{\pi}$)

Soln: Use C.D.F MTD: for $y>0$, C.D.F of Y

$$F_Y(y) = \mathbb{P}(Y \le y)$$
$$= \mathbb{P}(X^2 \le y)$$
$$= \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

Then, we have p.d.f:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$
$$= \frac{d}{dy}\{\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\}$$

$\phi(\sqrt{y}) = \phi(-\sqrt{y})$

$$= \frac{1}{2\sqrt{y}}\phi(\sqrt{y}) + \frac{1}{2\sqrt{y}}\phi(-\sqrt{y}) = \frac{1}{\sqrt{y}}\phi(\sqrt{y})$$
$$= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} = \frac{(1/2)^{1/2}}{\Gamma(1/2)} y^{\frac{1}{2}-1} e^{-y/2} \quad y>0$$

$$\Rightarrow Y \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$
$$= \chi_1^2 \text{ (Chi-Square distribution, 120B)}$$

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More Examples:

① $X \sim \text{Unif}[-2,3]$, $Y = |X-1|$, find density of Y.

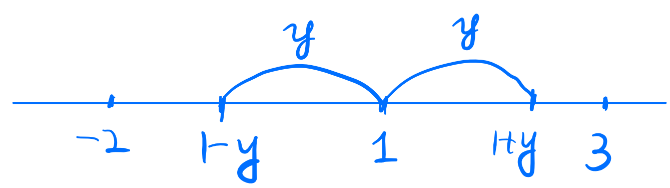
Soln: Use C.D.F again, for $y > 0$

$$F_Y(y) = \mathbb{P}(Y \le y)$$
$$= \mathbb{P}(|X-1| \le y)$$
$$= \mathbb{P}(-y \le X-1 \le y)$$
$$= \mathbb{P}(1-y \le X \le 1+y)$$
$$= F_X(1+y) - F_X(1-y)$$

p.d.f $f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(1+y) + f_X(1-y)$

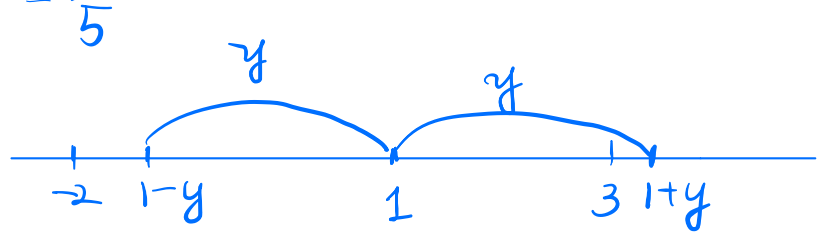
Note that $f_X(x) = \frac{1}{5}$ for $-2 < x < 3$, then

(1) $0 < y < 2$:



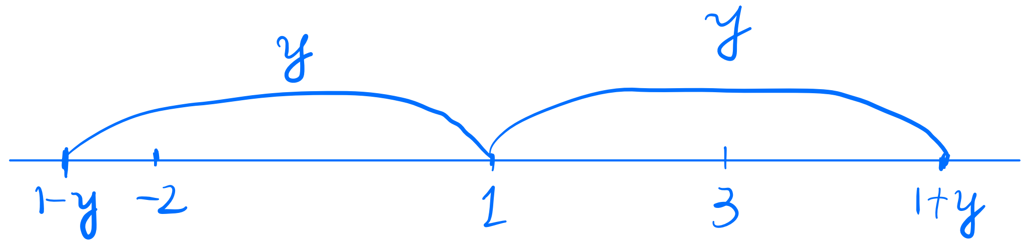
$$f_X(1+y) = f_X(1-y) = \frac{1}{5}$$

(2) $2 \le y < 3$:



$$f_X(1+y) = 0, \quad f_X(1-y) = \frac{1}{5}$$

(3) $y \geq 3$:



$$f_X(1+y) = f_X(1-y) = 0$$

Thus,

$$f_Y(y) = \begin{cases} 2/5 & 0 < y < 2 \\ 1/5 & 2 \leq y < 3 \\ 0 & \text{o.w.} \end{cases}$$

//

② $X \sim N(0,1)$, $Y = e^X$

(a) Calculate p.d.f of X.

<u>Soln</u>: $F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(e^X \leq t)$

$$= \mathbb{P}(X \leq \log t) = F_X(\log t) \qquad t > 0$$

so $f_Y(t) = \frac{d}{dt} F_Y(t) = \frac{1}{t} f_X(\log t)$

where $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$ $(x \in \mathbb{R})$ is p.d.f of X.

Thus, $f_Y(t) = \begin{cases} \frac{1}{t\sqrt{2\pi}} \exp\left\{-\frac{(\log t)^2}{2}\right\} & t > 0 \\ 0 & t \leq 0 \end{cases}$ //

(b) Find the n-th moment $\mathbb{E}[Y^n]$ of Y.

Soln: $\mathbb{E}[Y^n] = \mathbb{E}[(e^X)^n] = \mathbb{E}[e^{nX}] = M_X(n)$

where $M_X(t) = e^{\frac{1}{2}t^2}$ is the MGF of X.

Thus $\mathbb{E}[Y^n] = e^{n^2/2}$. //

Belows are extra exercises :