# Solutions for Practice Final

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## Final Exams

See exercise: 3.37, 3.67, 5.10, 5.22, 6.6, 6.10, 6.32, 8.6, 8.13, 10.3

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**Note**: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu

(3.37) Suppose $X$ has a C.D.F.

$$F(x) = \begin{cases} \dfrac{x}{1+x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) P.D.F $f(x) = F'(x)$

$x<0$: $f(x) = 0$

$x>0$: $f(x) = \left(1 - \frac{1}{1+x}\right)' = \frac{1}{(1+x)^2}$

so $f(x) = \begin{cases} 1/(1+x)^2 & x>0 \\ 0 & x<0 \end{cases}$

(b) $\mathbb{P}(2 < X < 3)$

$= F(3) - F(2) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

(c) $\mathbb{E}[(1+X)^2 e^{-2X}]$

$= \int_{-\infty}^{\infty} f(x)(1+x)^2 e^{-2x}\,dx$

$= \int_0^{\infty} e^{-2x}\,dx = 1/2$

(3.67) $Z \sim N(0,1)$, $X \sim N(\mu, \sigma^2)$

(a) $\mathbb{E}[Z^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{x^3}_{\text{odd}} \underbrace{\exp\{-\frac{x^2}{2}\}}_{\text{even}} \, dx = 0$

(b) $X = \sigma Z + \mu$

$$\mathbb{E}[X^3] = \mathbb{E}[(\sigma Z + \mu)^3]$$

$$= \mathbb{E}[\sigma^3 Z^3 + 3\sigma^2 Z^2 \mu + 3\sigma Z \mu^2 + \mu^3]$$

$$= \sigma^3 \cancel{\mathbb{E} Z^3}^{\,0} + 3\sigma^2 \mu \underbrace{\mathbb{E} Z^2}^{1} + 3\sigma\mu^2 \cancel{\mathbb{E}[Z]}^{\,0} + \mu^3$$

$$= 3\sigma^2 \mu + \mu^3$$

(5.10) $M(t) = \left(\frac{1}{5} + \frac{4}{5} e^t\right)^{30}$

by checking distribution Table, $X \sim Bin(30, 4/5)$

(5.22) $M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(3X-2)}]$

$$= e^{-2t} \mathbb{E}[e^{(3t)X}] = e^{-2t} \varphi_X(3t)$$

where $\varphi_X(t) = \mathbb{E}[e^{tX}]$

$= \frac{\lambda}{\lambda - t}$ (by checking Distribution Table)

So $M_Y(t) = \begin{cases} e^{-2t} \frac{\lambda}{\lambda - 3t} & t < \lambda/3 \\ \infty & o/w. \end{cases}$

be careful about the domain of X

(6.6) $f(x,y) = \begin{cases} xe^{-x(1+y)} & x>0,\ y>0 \\ 0 & o/w \end{cases}$

(a) $f_X(x) = \int_0^\infty xe^{-x(1+y)}\,dy$

$$= xe^{-x}\int_0^\infty e^{-xy}\,dy$$

$$= xe^{-x}\left(-\frac{1}{x}\right)e^{-xy}\Big|_{y=0}^{y=\infty} = e^{-x} \qquad x>0$$

$f_Y(y) = \int_0^\infty xe^{-x(1+y)}\,dx$

$$= \frac{1}{(1+y)^2} \qquad y>0$$

Remark: You can consider $(1+y)\,e^{-(1+y)x}$ as the density of $\exp(\lambda = 1+y)$, then use the fact that

$\int_0^\infty x(1+y)e^{-(1+y)x}\,dx = \frac{1}{1+y}$ (Expected value is $\frac{1}{1+y}$)

so $\int_0^\infty xe^{-(1+y)}\,dx = \frac{1}{(1+y)^2}$.

(b) $\mathbb{E}[XY] = \int_0^\infty\int_0^\infty xy\,f(x,y)\,dx\,dy$

$$= \int_0^\infty\int_0^\infty x^2 y e^{-x(1+y)}\,dx\,dy$$

$$= \int_0^\infty x^2e^{-x}\Big[\underbrace{\int_0^\infty ye^{-xy}\,dy}_{\frac{1}{x^2}}\Big]dx = \int_0^\infty e^{-x}\,dx = 1$$

(using same method as part a)

(c) $\mathbb{E}\left[\frac{X}{1+Y}\right] = \int_0^\infty \int_0^\infty \frac{x}{1+y} x e^{-x(1+y)} \, dx\, dy$

$$= \int_0^\infty \frac{1}{1+y} \underbrace{\left\{ \int_0^\infty x^2 e^{-x(1+y)} \, dx \right\}}_{} dy$$

$\frac{(1+y)^3}{\Gamma(3)} x^2 e^{-x(1+y)}$ : density of Gamma$(3, 1+y)$

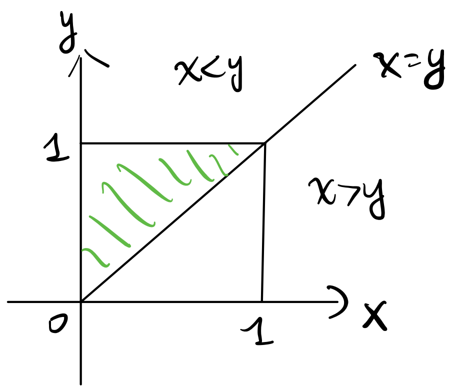
so $\int_0^\infty x^2 e^{-x(1+y)} \, dx = \frac{2}{(1+y)^3}$

$$= 2\int_0^\infty \frac{1}{(1+y)^4} \, dy$$

$$= -\frac{2}{3} \cdot \frac{1}{(1+y)^3} \Big|_0^\infty = \frac{2}{3}$$

(6.10) $X, Y \sim \text{Unif}(0,1)$ $X \perp Y$

$$f(x,y) = \begin{cases} 1 & 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$




$\mathbb{P}(X<Y)$

$$= \int_0^1 \int_0^y dx\, dy$$

$$= \int_0^1 y \, dy = \frac{1}{2}$$

(6.32) Support of $N = \mathbb{N}$

Support of $Y = \{1,2\}$

$\mathbb{P}(N=k, Y=1)$

$= \mathbb{P}(\text{White in the first } (k-1)\text{-th time, Green at the } k\text{-th time})$

$= \left(\frac{2}{9}\right)^{k-1} \cdot \frac{4}{9}$

Similarly, $\mathbb{P}(N=k, Y=2) = \left(\frac{2}{9}\right)^{k-1} \cdot \frac{1}{3}$

Note: $\mathbb{P}(N=k) = \left(\frac{2}{9}\right)^{k-1} \cdot \frac{7}{9}$ b/c $N \sim \text{Geo}(7/9)$

$\mathbb{P}(Y=1) = \sum_{k=1}^{\infty} \mathbb{P}(N=k, Y=1)$

$= \frac{4}{9} \sum_{k=1}^{\infty} \left(\frac{2}{9}\right)^{k-1} = \frac{4}{9} \cdot \frac{1}{1-2/9} = \frac{4}{7}$

$\mathbb{P}(Y=2) = \sum_{k=1}^{\infty} \mathbb{P}(N=k, Y=1)$

$= \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{2}{9}\right)^{k-1} = \frac{1}{3} \cdot \frac{1}{1-2/9} = \frac{3}{7}$

Thus for any $k \in \mathbb{N}$,

$\mathbb{P}(N=k, Y=1) = \mathbb{P}(N=k)\,\mathbb{P}(Y=1)$

$\mathbb{P}(N=k, Y=2) = \mathbb{P}(N=k)\,\mathbb{P}(Y=2)$

Hence, $N \perp Y$

$\blacksquare$

(8.6) $X \sim Geo(p)$, $Y \sim Bin(n, r)$

$\mathbb{E}[X] = \frac{1}{p}$ $Var(X) = \frac{1-p}{p^2}$ so $\mathbb{E}X^2 = (\mathbb{E}X)^2 + Var(X) = \frac{2-p}{p^2}$

$\mathbb{E}[Y] = nr$ $Var(Y) = nr(1-r)$ so $\mathbb{E}Y^2 = (\mathbb{E}Y)^2 + Var(Y)$

using the distribution sheet

$= n(n-1)r^2 + nr$

(a) $\mathbb{E}[X+Y] = \frac{1}{p} + nr$

(b) $\mathbb{E}[XY] = \mathbb{E}X \cdot \mathbb{E}Y = \frac{nr}{p}$

(c) $\mathbb{E}[X^2+Y^2] = \frac{2-p}{p^2} + n(n-1)r^2 + nr$

(d) $\mathbb{E}[(X+Y)^2]$

$$= \mathbb{E}[X^2+Y^2+2XY] = \frac{2-p}{p^2} + n(n-1)r^2 + nr + \frac{2nr}{p}$$

(8.13) Let $Z = X_1 + X_2 + \cdots + X_{36}$

with $X_1, \cdots, X_{36}$ are i.i.d. Random Variables

independent and identically distributed

and $M_{X_i}(t) = \frac{1}{2}e^{-t} + \frac{2}{5} + \frac{1}{10}e^{t/2}$

then P.M.F of $X_i$ is

| $X_i$ | $-1$ | $0$ | $1/2$ |
|---|---|---|---|
| $\mathbb{P}$ | $1/2$ | $2/5$ | $1/10$ |

for $i = 1, 2, \cdots, 36$

(10.3) $\mathbb{P}(Y=k)=\frac{1}{6}$ $\quad k=1,2,\cdots,6$

$$X \mid \underbrace{Y=k}_{} \sim \text{Bin}(k,\tfrac{1}{2})$$

flip the coins for k times

for $n=0,1,2,\cdots,6$

$$\mathbb{P}(X=n) = \sum_{k=n}^{6} \mathbb{P}(X=n \mid Y=k)\,\mathbb{P}(Y=k)$$ ← Law of Total Probability

$$= \sum_{k=n}^{6} \binom{k}{n}\left(\frac{1}{2}\right)^k \frac{1}{6}$$

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X \mid Y]\right]$$

$$= \sum_{k=1}^{6} \mathbb{E}[X \mid Y=k]\,\mathbb{P}(Y=k)$$

$$= \sum_{k=1}^{6} \frac{k}{2}\cdot\frac{1}{6} = \frac{7}{4}$$