Solution for Suggested Problems (Conditional Probability)

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Conditional probability

1. Exercise 2.2:

   Event A refers that the second flip is tail, and event B refers that at most one tails, then

   - $A = \{HTH, HTT, TTH, TTT\}$
   - $B = \{THH, HTH, HHT, HHH\}$
   - $A \cap B = \{HTH\}$

   By the definition of conditional probability:

   $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|AB|}{|B|} = \frac{1}{4}$$

2. Exercise 2.7: We need to use the fact that $P(A \cup B) + P(A^c \cap B) = P(B)$

   - Part a:
   $$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - P(A|B)$$
   - Part b: Use the result of part a we obtained
   $$P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - .6)(.5) = .2$$

Law of Total Probability

Exercise 2.6: Let A be the event that Alice watches TV and B represent the event that Betty watched TV.

1. Part a: $P(A \cap B) = P(A)P(B|A) = (.6)(.8) = .48$
2. Part b: Since Betty cannot watch TV given Alice does not watch, then $P(B|A^c) = 0$, so
   $$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = .48$$
3. Part c:
   - Method 1: using the result in exercise 2.7 that $P(B^c|A) = 1 - P(B|A)$, we have
   $$P(A \cap B^c) = P(A)P(B^c|A) = (.6)(1 - .2) = .12$$
   - Method 2: $P(A \cap B^c) = P(A) - P(A \cap B) = .12$

Bayes Formula

Exercise 2.33, 2.37, 2.40

1. Exercise 2.33:

   (a) Consider A be the event that red ball is drawn, B_k be the event that urn k is chosen for $k = 1, 2, 3, 4, 5$. Then $P(A|B_k) = k/10$ and $P(B_k) = 1/5$, using Law of Total Probability, we have

   $$P(A) = \sum_{k=1}^{5} P(A|B_k)P(B_k) = \frac{1}{5}\sum_{k=1}^{5}\frac{k}{10} = \frac{1}{50}\sum_{k=1}^{5} k = \frac{3}{10}$$

   (b) By Bayes Formula:

   $$\begin{aligned} P(B_k|A) &= \frac{P(A|B_k)P(B_k)}{P(A)} \\ &= \frac{(k/10)(1/5)}{3/10} \\ &= \frac{k}{15} \end{aligned}$$

2. Exercise 2.37:

   (a) Let A be the event that we get a five, B_k be the event that $k - sideddiceischosen$, for $k = 6, 8, 10, 20$, then we have

   - $P(A|B_k) = 1/k$
   - $P(B_6) = .1$, $P(B_8) = .2$, $P(B_{10}) = .3$, $P(B_{20}) = .4$

   By Law of total probability, we have

   $$\begin{aligned} P(A) &= P(A|B_6)P(B_6) + P(A|B_8)P(B_8) + P(A|B_{10})P(B_{10}) + P(A|B_{20})P(B_{20}) \\ &= \frac{1}{6}\frac{1}{10} + \frac{1}{8}\frac{2}{10} + \frac{1}{10}\frac{3}{10} + \frac{1}{20}\frac{4}{10} \\ &= \frac{11}{120} \end{aligned}$$

   (b) Let C be the event that seven is rolled, then Bayes formula, we have

   $$\begin{aligned} P(B_20|C) &= \frac{P(C|B_20)P(B_20)}{P(C)} \\ &= \frac{(1/20)(4/10)}{(1/8)(2/10) + (1/10)(3/10) + (1/20)(4/10)} \\ &= \frac{4/200}{15/200} = \frac{4}{15} \end{aligned}$$

   Remark. Be careful that given B_6, we have only six possible values, so $P(C|B_6) = 0$ and we only condition on B_8, B_{10}, and B_{20} to derive $P(C)$ using the law of total probability.

3. Exercise 2.40:

   (a) Let A be the event that the student choose mathematics major, and B_k be the event that the student's grade is k for $k = 1, 2, 3, 4$, then we have

   - $P(A|B_k) = \frac{k-1}{k+3}$
   - $P(B_1) = .1$, $P(B_2) = .2$, $P(B_3) = .6$, $P(B_4) = .1$

   By law of total probability, we have

   $$\begin{aligned} P(A) &= \sum_{k=1}^{4} P(A|B_k)P(B_k) \\ &= \frac{1-1}{1+3}(.1) + \frac{2-1}{2+3}(.2) + \frac{3-1}{3+3}(.6) + \frac{4-1}{4+3}(.1) \\ &= \frac{1}{5}(.2) + \frac{2}{6}(.6) + \frac{3}{7}(.1) \approx .2829 \end{aligned}$$

(b) By law of total probability:

$$\begin{aligned}P(B_4|A) &= \frac{P(A|B_4)P(B_4)}{P(A)} \\ &= \frac{(3/7)(.1)}{(1/5)(.2)+(2/6)(.6)+(3/7)(.1)} \approx .1515\end{aligned}$$

Independence

1. Exercise 2.13:

$$P(AB) = P(A) - P(AB^c) = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$$
$$P(A)P(B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Since $P(AB) = P(A)P(B)$, both events A and B are independent.

2. Exercise 2.17: Use the formula $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ for any events A_1, A_2:

$$\begin{aligned}P(AB \cup C) &= P(AB) + P(C) - P(ABC) \\ &= P(A)P(B) + P(C) - P(A)P(B)P(C) \\ &= \frac{1}{2 \times 3} + \frac{1}{4} - \frac{1}{2 \times 3 \times 4} = \frac{3}{8}\end{aligned}$$

The second equality holds because A, B, and C are mutually independent.

Conditional independence

Exercise 2.27: Let G be the event that green ball is drawn, R be the event that red ball is drawn, Y be the event that yellow ball is drawn, and B_i be the event that urn i is drawn for $i = 1, 2$.

1. Part a): Use the law of total probability:

$$P(G) = P(G|B_1)P(B_1) + P(G|B_2)P(B_2) = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{1}{2}$$

2. Part b): Since two experiments are independent, and in each experiment, the probability that the green balls is drawn is $1/2$ (from part a), then we have

$$P(G_1G_2) = P(G_1)P(G_2) = \frac{1}{4}$$

where G_1 is the event that we pick up the green ball in the first experiment, and G_2 represents the same event in the second round.

3. Part c): Let GG' be the event that green ball are chosen in both experiments (G represents the first round, and G' represents the second experiment), then using the law of total probability:

$$P(GG') = P(GG'|B_1)P(B_1) + P(GG'|B_2)P(B_2) = \left(\frac{1}{3}\right)^2 \frac{1}{2} + \left(\frac{2}{3}\right)^2 \frac{1}{2} = \frac{5}{18}$$

Remarks.

- In part b), G_1 and G_2 are independent, but in part c), G and G' are conditionally independent given B_1 or B_2. Independence and conditional independence are quite the different terms, so we need to be careful about which one we are choosing.
- In part c), in fact G and G' are not independent, to see this:

$$P(G)P(G') = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq \frac{5}{18} = P(GG')$$

- Extra exercise for part c): Given we draw the green ball in the first and second experiments, what is the probability that urn 2 is chosen?

  Solution: By Bayes formula:

$$P(B_2|GG') = \frac{P(GG'|B_2)P(B_2)}{P(GG')} = \frac{(2/3)^2(1/2)}{5/18} = \frac{4}{5}$$

4. Part d): $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu