Moment Generating Function

__Def__ : For a random variable X, the Moment Generating Function (M.G.F) is defined by

$$M_X(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_{k \in S_X} e^{tk} \, \mathbb{P}(X=k) & X : \text{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & X : \text{continuous} \end{cases}$$

Moreover, given the M.G.F of X, the n-th moment of X can be obtained by

$$\mathbb{E}[X^n] = M_X^{(n)}(t) \Big|_{t=0}$$

__Example 1__ : Suppose a R.V. X has density

$$f(x) = \frac{1}{2} e^{-|x|} = \begin{cases} \frac{1}{2} e^{-x} & x > 0 \\ \frac{1}{2} e^{x} & x < 0 \end{cases}$$

m.g.f : $M_X(t) = \mathbb{E}[e^{tX}]$

$$= \int_0^{\infty} \frac{1}{2} e^{(t-1)x} dx + \int_{-\infty}^{0} \frac{1}{2} e^{(t+1)x} dx$$

$$= \frac{1}{2} \Big\{ \underbrace{\int_0^{\infty} e^{(t-1)x} dx}_{①} + \underbrace{\int_{-\infty}^{0} e^{(t+1)x} dx}_{②} \Big\}$$

For ①:

- when $t-1>0$, $e^{(t-1)x} \nearrow \infty$ as $x \nearrow \infty$, ① $=\infty$
- when $t-1=0$, ① $= \int_0^\infty dx = \infty$
- when $t-1<0$, then

$$① = \int_0^\infty e^{(t-1)x} dx = \frac{1}{1-t}$$

For ②:

- When $t+1<0$, $e^{(t+1)x} \nearrow \infty$ as $x \searrow -\infty$, ② $=\infty$
- When $t+1=0$, ② $= \int_{-\infty}^0 dx = \infty$
- when $t+1>0$, then

$$② = \int_{-\infty}^0 e^{(t+1)x} dx = \frac{1}{1+t}$$

Combining ① and ②: when $-1<t<1$,

$$M_X(t) = \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) = \frac{1}{1-t^2}$$

$$E[X] = M_X'(0) = \left.\frac{2t}{(1-t^2)^2}\right|_{t=0} = 0$$

Remark: $E[X]$ can also be obtained by definition, which is

$$E[X] = \int_{-\infty}^\infty \underbrace{x \frac{1}{2} e^{-|x|}}_{\text{odd}} dx = 0$$

x: odd function; $\frac{1}{2}e^{-|x|}$: even function

Example 2: Suppose X has density

$$f(x) = \begin{cases} x e^{-x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

m.g.f $\quad M_X(t) = \mathbb{E}[e^{tX}]$

$$= \int_0^\infty x e^{-x} e^{tx} dx = \int_0^\infty x e^{-(1-t)x} dx$$

(1) when $1-t \le 0$, $\quad x e^{-(1-t)x} \uparrow \infty$ as $x \uparrow \infty$

(2) when $1-t > 0$, use the fact that

$$\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \quad \text{for } \alpha, \beta > 0$$

$$\Rightarrow \int_0^\infty x e^{-(1-t)x} dx = \frac{\Gamma(2)}{(1-t)^2} = \frac{1}{(1-t)^2} \quad (\Gamma(2) = 1! = 1)$$

$$M_X(t) = \left(\frac{1}{1-t}\right)^2 \quad t < 1$$

Given m.g.f, use $\mathbb{E}[X^n] = M_X^{(n)}(t) \big|_{t=0}$

$$M_X'(t) = \frac{2}{(1-t)^3}, \quad M_X''(t) = \frac{2\cdot 3}{(1-t)^4}$$

$$M_X'''(t) = \frac{2\cdot 3\cdot 4}{(1-t)^5}, \quad \cdots, \quad M_X^{(n)}(t) = \frac{(n+1)!}{(1-t)^{n+2}}$$

Thus, $\mathbb{E}[X^n] = (n+1)!$

Remark 1: Given $f(x) = xe^{-x}$ $x>0$, we know

$X \sim Gamma(r=2, \lambda=1)$, by distribution sheet:

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r = \frac{1}{(1-t)^2}$$

Remark 2: The n-th moment of X can also be obtained by

$$\mathbb{E}[X^n] = \int_0^\infty x^n \cdot xe^{-x} dx = \int_0^\infty x^{n+1} e^{-x} dx$$

$$= \Gamma(n+2) = (n+1)!$$

Q: Given the m.g.f of a R.V. X, how can we recover its probability distribution?

MTD 1: If the m.g.f. has the format

$$M_X(t) = p_1 e^{a_1 t} + \cdots + p_n e^{a_n t} = \sum_{k=1}^{n} p_k e^{a_k t}$$

with $p_1 + p_2 + \cdots + p_n = 1$, then X has p.m.f:

X	a_1	a_2	---	a_n
$\mathbb{P}$	P_1	P_2	---	P_n

$\Rightarrow$ X is now a discrete R.V.

e.g. $M_X(t) = \underbrace{\frac{1}{2}}_{\frac{1}{2}e^{0\cdot t}} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}$

then p.m.f of X: $\begin{cases} \mathbb{P}(X=-4) = 1/3 \\ \mathbb{P}(X=5) = 1/6 \\ \mathbb{P}(X=0) = 1/2 \end{cases}$

<u>MTD 2</u>: Use the Distribution Sheet:

Example 1: $M_X(t) = e^{\lambda(e^t-1)} \Rightarrow X \sim Poi(\lambda)$

Example 2: $M_Y(t) = \{(.2)+(.8)e^t\}^{10} \Rightarrow Y \sim Bin(10, .8)$

Example 3: $M_Z(t) = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\} \Rightarrow Z \sim N(\mu, \sigma^2)$

Extra exercise: Suppose that $M_X(t) = e^{2t+3(e^{5t}-1)}$, derive the p.m.f. of X.

<u>Soln</u>: let Y follows the m.g.f $M_Y(t) = e^{3(e^t-1)}$, $Y \sim Poi(3)$

Use $M_{aY+b} = \mathbb{E}[e^{(aY+b)t}] = e^{bt}\,\mathbb{E}[e^{atY}] = e^{bt}M_Y(at)$,

we have $X = aY+b$ with $a=5$, $b=2$

$\mathbb{P}(Y=k) = \mathbb{P}(5Y+2=k) = \mathbb{P}(Y=\frac{k-2}{5}) = \dfrac{e^{-3}\,3^{\frac{k-2}{5}}}{(\frac{k-2}{5})!}$

where $k = 5\ell+2$ with $\ell = 0, 1, 2, \cdots$