Introduction to Set

A set is a collection of things. The things are called elements of the set.

e.g. $A = \{2, 4, 6, 8\}$

A is a set, and 2, 4, 6, and 8 are elements of A.

Notations:

① $\in$: belongs to

e.g. $A = \{1, 3, 5\}$ is a set

$1 \in A$, $3 \in A$, $6 \notin A$.

② ϕ: Empty set

$\phi = \{\}$: a set with no elements.

③ Set-builder notation : used to describe sets

For a set X, set-builder notation for X is

$$X = \{\text{expression} : \text{rule}\}$$

expression: one notation to represent things / elements in X

Rule: the requirements that each element in set X should satisfy

If x satisfies the rule, $x \in X$; Otherwise, $x \notin X$

e.g. $X = \{n : n \text{ is a prime number}\} = \{2, 3, 5, 7, 11, \cdots\}$

number

Expression : n, represents the number

Rule : n is a prime number

$5 \in X$ b/c 5 is a prime number

$12 \notin X$ b/c 12 is NOT a prime number,

12 does NOT satisfy the rule.

e.g. $\{n^2 : n \in \mathbb{Z}\} = \{0, 1, 4, 9, \cdots\}$

integers

$\{x \in \mathbb{R} : x^2 - 2 = 0\} = \{\sqrt{2}, -\sqrt{2}\}$

real number

$\{x \in \mathbb{Z} : x^2 - 2 = 0\} = \phi$

④ $A \times B$: Cartesian Product

A, B : two sets

$A \times B = \{(a, b) : a \in A, b \in B\}$

(a, b) : ordered pair of a, b

e.g. $A = \{k, l, m\}$, $B = \{q, r\}$

$A \times B = \{(k, q), (k, r), (l, q), (l, r), (m, q), (m, r)\}$

⑤ A^n: Cartesian Power

e.g. $A^2 = A \times A$

$A^3 = A \times A \times A$

$A^k = \underbrace{A \times A \times \cdots \times A}_{k\ A\text{'s in total}}$

⑥ $\subseteq$: Subset

$A \subseteq B$: Every element of A is also an element of B.

$A \nsubseteq B$: At least one element of A is not in B.

e.g. $A = \{2, 3, 7\}$, $B = \{2, 4, 5, 6, 7\}$

$C = \{2, 3, 4, 5, 6, 7\}$

$A \subseteq C$ b/c $2, 3, 7 \in C$

$B \subseteq C$

$A \nsubseteq B$ b/c $3 \in A$ but $3 \notin B$

Important: $A = B$: the element of A and B are the same.

$A = B$ is equivalent to $A \subseteq B$, $B \subseteq A$

⑦ $\cap$: intersection

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

e.g. $A = \{1, 2\}$ $B = \{2, 3, 4\}$ $A \cap B = \{2\}$

⑧ $\cup$: union

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Rule: $x \in A$ or $x \in B$ $\begin{cases} x \in A, x \notin B \\ x \notin A, x \in B \\ x \in A, x \in B \end{cases}$

e.g. $A \cup B = \{1, 2, 3, 4\}$

<u>Exercise</u>: $A = \{a, b, c, d, e\}$, $B = \{d, e, f\}$, $C = \{1, 2, 3\}$

(a) $(A \cap B) \times B$

(b) $(A \times C) \cap (B \times C)$

<u>Soln</u>: (a) $A \cap B = \{d, e\}$

So $(A \cap B) \times B = \{(d,d), (d,e), (d,f), (e,d), (e,e), (e,f)\}$

(b) $A \times C = \{(a, c) : a \in A, c \in C\}$

$B \times C = \{(a, c) : a \in B, c \in C\}$

$(A \times B) \cap (B \times C)$

$= \{(a, c) : a \in A \cap B, c \in C\}$

$= \{(d,1), (d,2), (d,3), (e,1), (e,2), (e,3)\}$

⑨ Power Set

A : set

Power set of A, denoted by $\mathcal{P}(A)$, is the coll of all subset of A.

e.g.

(1) $A = \{1, 2, 3, 4\}$

$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\},$
$\{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\},$
$\{2,3,4\}, \{1,2,3,4\}\}$

(2) $A = \{\{a,b\}, \{c\}\}$

subset of A : ϕ, $\{\{a,b\}\}$, $\{\{c\}\}$, and
$\{\{a,b\}, \{c\}\}$

$\Rightarrow \mathcal{P}(A) = \{\phi, \{\{a,b\}\}, \{\{c\}\}, \{\{a,b\}, \{c\}\}\}$

<u>Remark</u>: For the set $\{a,b\}$.

$\{a,b\} \subseteq A$: wrong, $a \notin A$, $b \notin A$.

but $\{a,b\} \in A$.

⑪ 1.1: Cardinality of sets, the number of elements inside the set.

e.g. (1) $\{x \in \mathbb{Z} : |x| < 10\}$

$= \{0, 1, -1, 2, -2, \cdots, 9, -9\}$

$|\{x \in \mathbb{Z} : |x| < 10\}| = 19$

(2) $\{x \in \mathbb{N} : x^2 < 10\} = \{1, 2, 3\}$

$|\{x \in \mathbb{N} : x^2 < 10\}| = 3$

(3) $A = \{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\}$

elements of A: $\{1, 4\}$, a, b, $\{\{3, 4\}\}$, and $\{\phi\}$

$|A| = 5$

(4) $B = \{\{\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\}\}\}$

elements of B: $\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\}$

$|B| = 1$.

Note:

(1) If $|A|=m$, $|B|=n$ then $|A\times B|=m\cdot n$

(2) If $|A|=m$, $|\mathcal{P}(A)|=2^m$

e.g. Suppose $|A|=m$, $|B|=n$

$$|\mathcal{P}(A)|=2^m$$

$$|\mathcal{P}(\mathcal{P}(A))|=2^{2^m}$$

$$|\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))|=2^{2^{2^m}}$$

$$|\mathcal{P}(A\times B)|=2^{m\cdot n}$$

$$|A\times\mathcal{P}(B)|=|A|\,|\mathcal{P}(B)|=m\cdot 2^n$$

$$|\mathcal{P}(A\times\mathcal{P}(B))|=2^{m\cdot 2^n}$$

(11) $-$: Difference

$$A-B=\{x: x\in A,\ x\notin B\}$$

e.g. $A=\{0,2,4,6,8\}$, $C=\{2,8,4\}$

$A-C=\{0,6\}$ $\quad C-A=\phi$

(12) Complement

U: a universal set, $X \subseteq U$

$\overline{X} = U - X$

Next time: Mathematical logic.