Conditional Probability

Def: $\mathbb{P}(A|B)$ — Conditional probability of event A given event B has occurred,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} \quad \text{for } \mathbb{P}(B) > 0.$$

1. Law of Total Probability

Review of the set theory:

Sets A, $B_1, \cdots, B_n$, then by distributive Law,

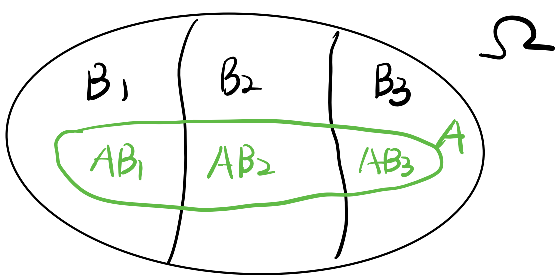
$$A\cap(B_1 \cup B_2 \cup \cdots \cup B_n) = (AB_1)\cup(AB_2)\cup\cdots\cup(AB_n)$$

Def: $\{B_1, B_2, \cdots, B_n\}$ is a partition of Ω if

$B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ $\Rightarrow$ make up Ω

$B_i \cap B_j = \phi$ for $i \neq j$ $\Rightarrow$ Pairwise disjoint

e.g. when $n=3$,



$\{B_1, B_2, B_3\}$ forms a partition of Ω.

$\{AB_1, AB_2, AB_3\}$ forms a partition of A.

— Now, Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of Ω, then

$$A = A\cap\Omega = A\cap(B_1\cup B_2\cup\cdots\cup B_n)$$

$$= (AB_1)\cup(AB_2)\cup\cdots\cup(AB_n)$$

Also, $AB_1, AB_2, \cdots, AB_n$ are pairwise disjoint, then

$$\mathbb{P}(A) = \mathbb{P}(AB_1) + \mathbb{P}(AB_2) + \cdots + \mathbb{P}(AB_n)$$

$$= \sum_{i=1}^{n} \mathbb{P}(AB_i) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$$

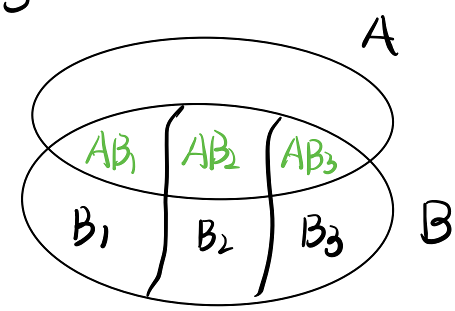
— Law of total Probability.

— Now, let $\{B_1, B_2, \cdots, B_n\}$ be a partition of $B \subseteq \Omega$

$$A \cap B = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n)$$

$$= (AB_1) \cup (AB_2) \cup \cdots \cup (AB_n)$$

e.g. $n = 3$



$\{B_1, B_2, B_3\}$: partition of B.

$\{AB_1, AB_2, AB_3\}$: partition of $A \cap B$.

$$\mathbb{P}(A \cap B) = \mathbb{P}(AB_1) + \mathbb{P}(AB_2) + \cdots + \mathbb{P}(AB_n)$$

$$= \sum_{i=1}^{n} \mathbb{P}(AB_i) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$\mathbb{P}(B_i|B)$

$$= \sum_{i=1}^{n} \mathbb{P}(A|B_i) \overbrace{\frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}} = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i|B).$$

Note: $B_i \subseteq B$, then $B_i = B_i \cap B$, and

$$\mathbb{P}(B_i|B) = \frac{\mathbb{P}(B_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}$$

2. Baye's Theorem

$\{B_1, B_2, \cdots, B_n\}$: partition of Ω.

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A \cap B_k)}{\mathbb{P}(A)}$$

where $\mathbb{P}(A \cap B_k) = \mathbb{P}(A|B_k)\,\mathbb{P}(B_k)$ — Def of Conditional Prob

$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$ — Law of Total Prob.

<u>Example</u>: 3 fair dice $\begin{cases} 4 \text{ sides } (1, 2, 3, 4) \\ 6 \text{ sides } (1, 2, \cdots, 6) \\ 12 \text{ sides } (1, 2, \cdots, 12) \end{cases}$

pick one dice at random, roll it twice.

$$\begin{cases} B_4 = \{4\text{-sided dice is chosen}\} \\ B_6 = \{6\text{-sided dice is chosen}\} \\ B_{12} = \{12\text{-sided dice is chosen}\} \\ A_1 = \{\text{The first roll is } 3\},\ A_2 = \{\text{The second roll is } 4\} \\ A = A_1 \cap A_2 \end{cases}$$

$\mathbb{P}(B_4) = \mathbb{P}(B_6) = \mathbb{P}(B_{12}) = 1/3$

By Law of total probability:

$$\mathbb{P}(A) = \underbrace{\mathbb{P}(A|B_4)}_{(1/4)^2}\underbrace{\mathbb{P}(B_4)}_{1/3} + \underbrace{\mathbb{P}(A|B_6)}_{(1/6)^2}\underbrace{\mathbb{P}(B_6)}_{1/3} + \underbrace{\mathbb{P}(A|B_{12})}_{(1/12)^2}\underbrace{\mathbb{P}(B_{12})}_{1/3}$$

$$\mathbb{P}(B_6|A) = \frac{\mathbb{P}(A \cap B_6)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_6)\,\mathbb{P}(B_6)}{\mathbb{P}(A)} \quad \text{(by Bayes thm).}$$

Extra exercise: Check that A_1 and A_2 are NOT indep.

in other words, $\mathbb{P}(A) \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$

Pf: by Law of total prob:

$$\mathbb{P}(A) = \frac{1}{3}\left(\frac{1}{16} + \frac{1}{36} + \frac{1}{144}\right) = \frac{7}{216}$$

$$\mathbb{P}(A_1) = \mathbb{P}(A_1 \mid B_4)\mathbb{P}(B_4) + \mathbb{P}(A_1 \mid B_6)\mathbb{P}(B_6) + \mathbb{P}(A_1 \mid B_{12})\mathbb{P}(B_{12})$$

$$= \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{3} + \frac{1}{12}\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{12}\right) = \frac{1}{6}$$

Similarly, $\mathbb{P}(A_2) = \frac{1}{6}$, so $\mathbb{P}(A_1 \cap A_2) \neq \mathbb{P}(A_1)\,\mathbb{P}(A_2)$

Example (In the Past Quiz):

Choose four words at random:

APPLE PEAR AND BANADA

and randomly choose one letter from the chosen word.

- $A = \{$The chosen letter is "A"$\}$

$$\mathbb{P}(A) = \mathbb{P}(A \mid APPLE)\,\mathbb{P}(APPLE) + \mathbb{P}(A \mid PEAR)\,\mathbb{P}(PEAR)$$
$$+ \mathbb{P}(A \mid AND)\,\mathbb{P}(AND) + \mathbb{P}(A \mid BANANA)\,\mathbb{P}(BANADA)$$
$$= \frac{1}{4}\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{3}{6}\right) = \frac{77}{240}$$

- $$\mathbb{P}(APPLE \mid A) = \frac{\mathbb{P}(A \mid APPLE)\,\mathbb{P}(APPLE)}{\mathbb{P}(A)}$$

$$= \frac{(1/5)(1/4)}{77/240} = \frac{12}{77}$$

3. Independence

Def: • Two events $A, B \subset \Omega$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

• n events: $A_1, A_2, \cdots, A_n$ are conditionally independent given event B if

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n \mid B) = \prod_{i=1}^{n} \mathbb{P}(A_i \mid B)$$

Example: 3 Jurors & 1 Defendant

$$\text{Defendant} \begin{cases} G = \{\text{Guilty}\} & \text{w.p. } 70\% \\ G^c = \{\text{Innocent}\} & \text{w.p. } 30\% \end{cases}$$

• Given G (he is guilty):

Each juror declares guilty w.p. .7, independently.

• Given G^c (he is innocent):

Each juror declares guilty w.p. .2, independently.

$A_1 = \{\text{Juror 1 declares guilty}\}$

$A_2 = \{\text{Juror 2 declares guilty}\}$

$A_3 = \{\text{Juror 3 declares guilty}\}$

Q: What is the prob. that Juror 3 declares guilty given the other two declare?

wrong way:

$$\mathbb{P}(A_3 | A_1 A_2) \cancel{=} \mathbb{P}(A_3)$$

b/c A_1, A_2, A_3 are conditionally independent given G (or G^c), which does NOT necessarily mean they are indep.

Correct:

$$\mathbb{P}(A_3 | A_1 A_2) = \frac{\mathbb{P}(A_1 A_2 A_3)}{\mathbb{P}(A_1 A_2)}$$

where by Law of total Prob:

$$\mathbb{P}(A_1 A_2 A_3)$$

$$= \underbrace{\mathbb{P}(A_1 A_2 A_3 | G)}_{\mathbb{P}(A_1|G)\,\mathbb{P}(A_2|G)\,\mathbb{P}(A_3|G) \,=\, (.7)^3} \overbrace{\mathbb{P}(G)}^{.7} + \underbrace{\mathbb{P}(A_1 A_2 A_3 | G^c)}_{\mathbb{P}(A_1|G^c)\,\mathbb{P}(A_2|G^c)\,\mathbb{P}(A_3|G^c) \,=\, (.2)^3} \overbrace{\mathbb{P}(G^c)}^{.3}$$

similarly,

$$\mathbb{P}(A_1 A_2) = (.7)^2 (.7) + (.2)^2 (.3)$$

Extra exercise (Based on the example above):

(a) Check that A_1, A_2, A_3 are NOT independent.

(b) What is the prob. that exactly TWO of them voted guilty?

(c) Calculate the prob that the defendant is guilty given juror 1 declares guilty and juror 3 declares non-guilty?

Soln (a) $\mathbb{P}(A_1) = \mathbb{P}(A_1|G)\mathbb{P}(G) + \mathbb{P}(A_1|G^c)\mathbb{P}(G^c)$

$= (.7)(.7) + (.2)(.3) = .49 + .06 = .55$

Similarly, $\mathbb{P}(A_2) = \mathbb{P}(A_3) = .55$

but $\mathbb{P}(A_1 A_2 A_3) = .2425 \neq (.55)^3 = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.

(b) $\mathbb{P}(\text{Exactly TWO of them voted guilty})$

$= \mathbb{P}(A_1^c A_2 A_3) + \mathbb{P}(A_1 A_2^c A_3) + \mathbb{P}(A_1 A_2 A_3^c)$

$= 3 \times .1125 = .3375$

(c) $\mathbb{P}(G|A_1 A_3^c) = \dfrac{\mathbb{P}(A_1 A_3^c|G)\mathbb{P}(G)}{\mathbb{P}(A_1 A_3^c)}$

where $\mathbb{P}(A_1 A_3^c)$

$= \underbrace{\mathbb{P}(A_1 A_3^c|G)}_{(.7)(.3)} \underbrace{\mathbb{P}(G)}_{(.7)} + \underbrace{\mathbb{P}(A_1 A_3^c|G^c)}_{(.2)(.8)} \underbrace{\mathbb{P}(G^c)}_{(.3)}$