Ex 1. Suppose a R.V. X has density

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

Calculate cumulative distribution function for X, $\mathbb{P}(X<1/4)$, and $\mathbb{P}(X>1/2)$.

Soln: By definition of c.d.f / p.d.f, we have

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

(i) $x \le 0$: $\mathbb{P}(X \le x) = \int_{-\infty}^{x} 0\,dt = 0$

(ii) $x \in (0,1)$: $\mathbb{P}(X \le x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} 6t(1-t)\,dt = 3t^2 - 2t^3 \Big|_0^x$$

$$= 3x^2 - 2x^3$$

(iii) $x \ge 1$: $F(x) = 1$

so $$F(x) = \begin{cases} 0 & x \le 0 \\ 3x^2 - 2x^3 & 0<x<1 \\ 1 & x \ge 1 \end{cases}$$

$$\mathbb{P}(X < 1/4) = F(1/4) = \frac{3}{16} - \frac{2}{64} = \frac{5}{32}$$

$$\mathbb{P}(X > 1/2) = 1 - F(1/2) = 1 - \left(\frac{3}{4} - \frac{2}{8}\right) = \frac{1}{2}$$

Ex 2: Suppose a R.V. X has density

$$f(x)=\begin{cases}\frac{x}{2} & 0<x\leq 1\\ \frac{1}{2} & 1<x\leq 2\\ \frac{3-x}{2} & 2<x<3\\ 0 & \text{elsewhere}\end{cases}$$

Calculate the distribution function of X.

Soln: (i) $x\leq 0$, $F(x)=0$

(ii) $0<x\leq 1$: $F(x)=\int_{-\infty}^{0} 0\,dt+\int_0^x \frac{t}{2}\,dt=\frac{t^2}{4}\Big|_0^x=\frac{x^2}{4}$

(iii) $1<x\leq 2$: $F(x)=\int_{-\infty}^{0} 0\,dt+\int_0^1 \frac{t}{2}\,dt+\int_1^x \frac{1}{2}\,dt$

$$=\frac{1}{4}+\frac{1}{2}(x-1)$$

(iv) $2<x\leq 3$: $F(x)=\int_{-\infty}^{0} 0\,dt+\int_0^1 \frac{t}{2}\,dt+\int_1^2 \frac{1}{2}\,dt+\int_2^x \frac{3-t}{2}\,dt$

$$=\frac{3}{4}+\frac{3t-t^2/2}{2}\Big|_2^x$$

$$=\frac{3}{4}+\left(\frac{6x-x^2}{4}-\frac{6-2}{2}\right)=\frac{3x}{2}-\frac{x^2}{4}-\frac{5}{4}$$

(v) $x>3$: $F(x)=1$

Thus, $F(x)=\begin{cases}0 & x\leq 0\\ x^2/4 & 0<x\leq 1\\ x/2-1/4 & 1<x\leq 2\\ 3x/2-x^2/4-5/4 & 2<x\leq 3\\ 1 & x>3\end{cases}$

Ex 3. Let Z has distribution function (c.d.f.)

$$F(z) = \begin{cases} 0 & z < -2 \\ \frac{z+4}{8} & -2 \le z < 2 \\ 1 & z \ge 2 \end{cases}$$

Calculate $\mathbb{P}(Z=-2)$, $\mathbb{P}(Z=2)$, $\mathbb{P}(-2<Z\leq 1)$, and $\mathbb{P}(2<Z\leq 3)$.

left limit

Soln: $\mathbb{P}(Z=-2) = F(-2) - \lim_{z\uparrow 2^-} F(z) = \frac{-2+4}{8} - 0 = \frac{1}{4}$

$\mathbb{P}(Z=2) = F(2) - \lim_{z\uparrow 2^-} F(z) = 1 - \frac{2+4}{8} = \frac{1}{4}$

$\mathbb{P}(-2<Z\leq 1) = F(1) - F(-2) = \frac{1+4}{8} - \frac{-2+4}{8} = \frac{3}{8}$

$\mathbb{P}(2<Z\leq 3) = F(3) - F(2) = 1-1 = 0.$

Remarks: For a R.V. Z with C.D.F F,

$\mathbb{P}(Z\leq x) = F(x)$

$\mathbb{P}(Z<x) = \lim_{z\uparrow x^-} F(z)$: left limit of F at x

$$\mathbb{P}(Z=x) = \mathbb{P}(Z\leq x) - \mathbb{P}(Z<x) = F(x) - \lim_{z\uparrow x^-} F(z)$$

Ex 4. The c.d.f. of X is given by

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & x>0 \\ 0 & x\leq 0 \end{cases}$$

then its density $f(x) = \begin{cases} xe^{-x} & x>0 \\ 0 & x<0 \end{cases}$ $\quad X \sim \text{Gamma}(2,1)$