Study Guide for Week 7-10 (120A)

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Week 7-8: Expected values

1. Expected value: $3.10, 3.30$ Challenging problems: $3.51 - 3.53$
2. Variance: $3.31, 3.15, 3.28$
3. Moment generating functions: $5.13, 5.15, 5.17, 5.19$
4. Transformation of random variables: $5.7, 5.8$, example 5.20 and remark 5.21

Hint: For exercise 5.19, use geometric series

$$\sum_{n=0}^{\infty} p^n = \begin{cases} \frac{1}{1-p} & |p| < 1 \\ \infty & |p| \geq 1 \end{cases}$$

For part b, since the second derivative is tedious, you can directly skip the variance or calculate variance by the same method as part b in 3.53

Week 9: Binomial Approximation

See exercise: $4.35, 4.41$

Week 10: Joint Distributions

1. Discrete case: $6.1, 6.19$
2. Continuous case: $6.5, 6.35$
3. Independence: $6.12, 6.27$
4. Covariance: $8.14 - 8.16$

Week 10: Conditional Distributions

See exercise $10.1, 10.2, 10.5, 10.9$

Expected values

1. Exercise 3.10:

   (a) Given the information on X, we have

X	-1	0	1
$Y = \|X\|$	1	0	1
P	$1/2$	$1/3$	$1/6$

   So p.m.f of $Y = |X|$ is

Y	0	1
P	$1/3$	$2/3$

   and therefore $E[|X|] = E[Y] = 0 \times (1/3) + 1 \times (2/3) = 2/3$

   (b) $E[|X|] = |-1| \times (1/2) + |0|(1/3) + |1|(1/6) = 2/3$

2. Exercise 3.30: For X be the number of missing shots

   - $P(X = 0) = P(\text{success in the first shot}) = 1/2$
   - $P(X = 1) = P(\text{miss the first shot, but make the second shot}) = (1 - 1/2)(1/3) = 1/6$
   - $P(X = 2) = P(\text{miss the first two shots, but make the third shot}) = (1-1/2)(1-1/3)(1/4) = 1/12$
   - $P(X = 3) = P(\text{miss the first three shots, but make the four shot}) = (1 - 1/2)(1 - 1/3)(1 - 1/4)(1/5) = 1/20$
   - $P(X = 4) = P(\text{miss all the shot}) = (1 - 1/2)(1 - 1/3)(1 - 1/4)(1 - 1/5) = 1/5$

   Thus the p.m.f of X is

X	0	1	2	3	4
P	$1/2$	$1/6$	$1/12$	$1/20$	$1/5$

   and the expected value is

$$E[X] = 0 \times \frac{1}{2} + 1 \times \frac{1}{6} + 2 \times \frac{1}{12} + 3 \times \frac{1}{20} + 4 \times \frac{1}{5} = \frac{77}{60}$$

Challenging Problems

1. Exercise 3.51: given $X \sim Geo(p)$, we have

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kP(X = k) \\
&= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\
&= p\sum_{k=1}^{\infty}\sum_{j=1}^{k}(1-p)^{k-1} \\
&= p\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}(1-p)^{k-1} \\
&= \frac{p}{1-p}\sum_{j=1}^{\infty}\frac{(1-p)^j}{p} \\
&= \frac{1}{1-p}\left(\sum_{j=0}^{\infty}(1-p)^j - 1\right) = \frac{1}{1-p}(1/p - 1) = \frac{1}{p}
\end{aligned}$$

2. Exercise 3.52:

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kP(X=k) \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{k} P(X=k) \\
&= \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} P(X=k) \\
&= \sum_{j=1}^{\infty} P(X \geq j)
\end{aligned}
$$

3. Exercise 3.53: use the first and second derivative of geometric series, we have

$$\sum_{k=1}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2}$$

and

$$\sum_{k=2}^{\infty} k(k-1)p^{k-2} = \frac{1}{(1-p)^3}$$

Thus we have

$$E[X] = \frac{1}{2}\sum_{k=1}^{\infty} k(1/3)^k = \frac{1}{6}k(1/3)^{k-1} = \frac{1}{6}\cdot\frac{1}{(1-1/3)^2} = \frac{3}{8}$$

and

$$E[X(X-1)] = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)(1/3)^k = \frac{1}{18}\sum_{k=2}^{\infty} k(k-1)(1/3)^{k-2} = \frac{1}{18}\cdot\frac{1}{(1-1/3)^3} = \frac{3}{16}$$

then $Var(X) = E[X^2] - (E[X])^2 = E[X(X-1)] + E[X] - (E[X])^2 = \frac{3}{16} + \frac{3}{8} - \frac{9}{64} = \frac{27}{64}$

Variance

1. Exercise 3.31: Note that $\int_1^\infty x^{-4}dx = 1/3$

   (a) By properties of p.d.f, $\int_1^\infty cx^{-4}dx = 1$. It implies that $c = 3$.

   (b) $P(0.5 < X < 1) = 0$

   (c) $P(0.5 < X < 2) = \int_1^2 3x^{-4}dx = 7/8$

   (d) $P(2 < X < 4) = \int_2^4 3x^{-4}dx = 7/64$

   (e) For $x > 1$: $\int_1^x 3t^{-4}dx = 1 - x^{-3}$, thus the c.d.f of X should be

$$F(x) = \begin{cases} 0 & x \leq 1 \\ 1 - x^{-3} & x > 1 \end{cases}$$

   (f)

$$E[X] = \int_1^\infty 3x \times x^{-4}dx = \int_1^\infty 3x^{-3}dx = (-3/2)x^{-2}|_1^\infty = 3/2$$
$$E[X^2] = \int_1^\infty 3x^2 \times x^{-4}dx = \int_1^\infty 3x^{-2}dx = (-3)x^{-1}|_1^\infty = 3$$
$$Var(X) = E[X^2] - (E[X])^2 = 3 - (3/2)^2 = 3/4$$

   (g)

$$E[5X^2 + 3X] = 5E[X^2] + 3E[X] = 5 \times 3 + 3 \times (3/2) = 19.5$$

(h) Note that when $\alpha \geq -1$, we have

$$\int_1^\infty x^- 1 dx = \log x|_1^\infty = \infty$$

and

$$\int_1^\infty x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1}|_1^\infty = \infty$$

for $\alpha > -1$ (so $\alpha + 1 > 0$), therefore, we have

$$E[X^n] = \begin{cases} 3/2 & n = 1 \\ 3 & n = 2 \\ \infty & n \geq 3 \end{cases} = \begin{cases} \frac{3}{3-n} & n \leq 2 \\ \infty & n \geq 3 \end{cases}$$

2. Exercise 3.15: $E[X] = 3$, $Var(X) = 4$.

   (a) $E[3X+2] = 3E[X] + 2 = 3 * 3 + 2 = 11$

   (b) $E[X^2] = Var(X) + (E[X])^2 = 4 + 3^2 = 13$

   (c) $E[(2X+3)^2] = E[4X^2 + 12X + 9] = 4E[X^2] + 12E[X] + 9 = 4 * 13 + 12 * 3 + 9 = 97$

   (d) $Var(4X - 2) = 16Var(X) = 16 * 4 = 64$

3. Exercise 3.28: X: the number of boxes you open until you get the prize

   (a)
   - $P(X = 1) = P(\text{The first box has the prize}) = 3/5$
   - $P(X = 2) = P(\text{The first box has no prize, but the second box has}) = (1 - 3/5)(3/4) = 3/10$
   - $P(X = 3) = P(\text{no prize in the first two box}) = (1 - 3/5)(1 - 3/4)(1) = 1/10$

   Thus the p.m.f of X is

X	1	2	3
P	3/5	3/10	1/10

   (b) $E[X] = 1 * (3/5) + 2 * (3/10) + 3 * (1/10) = 3/2$

   (c)

$$E[X^2] = 1^2 * (3/5) + 2^2 * (3/10) + 3^2 * (1/10) = 27/10$$
$$Var(X) = E[X^2] - (E[X])^2 = 27/10 - (3/2)^2 = .45$$

   (d) Suppose Y be the gain of the game (Y takes on the negative value if you loss), then we have $Y = 100 - 100(X-1)$ and

$$E[Y] = E[100 - 100(X-1)] = 100 - 100E[X-1] = 50$$

Moment Generating Function

5.13, 5.15, 5.17, 5.19

1. Exercise 5.13:

$$M_Y(t) = \frac{1}{2} + \frac{1}{16}e^{-34t} + \frac{1}{8}e^{-5t} + \frac{1}{100}e^{3t} + \frac{121}{400}e^{100t}$$

   (a)

$$M_Y'(t) = \frac{-34}{16}e^{-34t} + \frac{-5}{8}e^{-5t} + \frac{3}{100}e^{3t} + \frac{121 * 100}{400}e^{400t}$$

   So

$$E[Y] = M_Y'(0) = -\frac{34}{16} - \frac{5}{8} + \frac{3}{100} + \frac{12100}{400} = 27.53$$

   (b) Recover from MGF to p.m.f, we have

X	0	-34	-5	3	100
P	1/2	1/16	1/8	1/100	121/400

By definition of expected value, we have $E[X] = (-34)\cdot\frac{1}{16} + (-5)\cdot\frac{1}{8} + 3\cdot\frac{1}{100} + 100\cdot\frac{121}{400} = 27.53$

2. Exercise 5.15:

   (a)
$$M_X(t) = \frac{1}{10}e^{-2t} + \frac{1}{5}e^{-t} + \frac{3}{10} + \frac{2}{5}e^{t}$$

   (b) Given information on X, we have

X	-2	-1	0	1
$Y = \vert X+1\vert$	1	0	1	2
P	1/10	1/5	3/10	2/5

   so p.m.f of Y is

Y	0	1	2
P	1/5	2/5	2/5

3. Exercise 5.17: P.D.F of X is
$$f(x) = \begin{cases} 2x & x \in (0,2) \\ 0 & \text{otherwise} \end{cases}$$

   (a) When $t = 0$, $M_X(0) = E[e^0] = 1$; when $t \neq 0$, we have
$$\begin{aligned} M_X(t) &= \int_0^2 \frac{x}{2}e^{tx}dx \\ &= \frac{xe^{tx}}{2t}\bigg|_0^2 - \frac{1}{2t}\int_0^2 e^{tx}dx \\ &= \frac{e^{2t}}{t} - \frac{1}{2t^2}e^{tx}\bigg|_0^2 \\ &= \frac{e^{2t}}{t} - \frac{e^{2t}}{2t^2} + \frac{1}{2t^2} = \frac{2te^{2t} - e^{2t} + 1}{2t^2} \end{aligned}$$

   (b) Use the taylor series: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, we have
$$\begin{aligned} M_X(t) &= \frac{2te^{2t} - e^{2t} + 1}{2t^2} \\ &= \frac{1}{2t^2}\left(\sum_{k=0}^{\infty}\frac{(2t)^{k+1}}{k!} - \sum_{k=0}^{\infty}\frac{(2t)^k}{k!} + 1\right) \\ &= \frac{1}{2t^2}\left(\sum_{k=1}^{\infty}\frac{(2t)^k}{(k-1)!} - \sum_{k=1}^{\infty}\frac{(2t)^k}{k!} - \frac{((2t)^0}{0!} + 1\right) \\ &= \frac{1}{2t^2}\sum_{k=1}^{\infty}\left(\frac{(2t)^k}{(k-1)!} - \frac{(2t)^k}{k!}\right) \\ &= \frac{1}{2t^2}\sum_{k=2}^{\infty}\frac{2^k t^k (k-1)}{k!} \\ &= \frac{1}{2t^2}\sum_{k=0}^{\infty}\frac{2^{k+2}t^{k+2}(k+1)}{(k+2)!} \\ &= \sum_{k=0}^{\infty}\frac{2^{k+1}(k+1)t^k}{(k+2)!} = \sum_{k=0}^{\infty}\frac{2^{k+1}}{k+2}\cdot\frac{t^k}{k!} = \sum_{k=0}^{\infty}M_X^{(k)}(0)\cdot\frac{t^k}{k!} \end{aligned}$$

   Thus we have
$$E[X^n] = M_X^{(n)}(0) = \frac{2^{n+1}}{n+2}$$

(c)

$$E[X^n] = \frac{1}{2}\int_0^2 x \cdot x^n dx = \frac{1}{2(n+2)} x^{n+2}\Big|_0^2 = \frac{2^{n+1}}{n+2}$$

4. Exercise 5.19: The p.m.f of X is

- $P(X = 0) = 2/5$
- For $k \geq 1$, we have

$$P(X = k) = \left(\frac{3}{4}\right)^k \frac{1}{5}$$

Extra exercise: show that this is a valid p.m.f.

Proof. Check:

- $P(X = k) \geq 0$
-

$$\sum_{k=0}^{\infty} P(X = k) = \frac{2}{5} + \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k \frac{1}{5} = \frac{2}{5} + \frac{1}{5}\cdot\left[\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k - 1\right] = \frac{2}{5} + \frac{1}{5}\cdot\left(\frac{1}{1-3/4} - 1\right) = 1$$

□

(a)

$$\begin{aligned} M_X(t) &= \frac{2}{5} + \sum_{k=1}^{\infty} e^{kt}\left(\frac{3}{4}\right)^k \frac{1}{5} \\ &= \frac{2}{5} + \frac{1}{5}\sum_{k=1}^{\infty}\left(\frac{3}{4}\cdot e^t\right)^k \end{aligned}$$

The series converges if and only if $(3/4)e^t < 1$, i.e. $t < \log(4/3)$, and thus

$$M_X(t) = \begin{cases} \frac{2}{5} + \frac{1}{5}\cdot\left(\frac{1}{1-(3/4)e^t} - 1\right) & t < \log(4/3) \\ \infty & \text{otherwise} \end{cases} = \begin{cases} \frac{8-3e^t}{20-15e^t} & t < \log(4/3) \\ \infty & \text{otherwise} \end{cases}$$

(b) Once we obtain $M_X(t)$ from part a, we have

$$M_X'(t) = \frac{1}{5}\cdot\frac{3e^t(4-3e^t) + 9e^{2t}}{(4-3e^t)^2}$$

thus

$$E[X] = M_X'(0) = \frac{1}{5}\cdot\frac{3(4-3)+9}{(4-3)^2} = \frac{12}{5}$$

Similarly, $E[X^2] = \frac{84}{5}$ and $Var(X) = \frac{276}{25}$

Transformation of Random variables

1. Exercise 5.7: Let $X \sim exp(\lambda)$, then

$$\begin{aligned} P(Y \leq y) &= P(\log(X) \leq y) \\ &= P(X \leq e^y) \\ &= 1 - e^{-\lambda x}\Big|_{x=e^y} = 1 - \exp\{-\lambda e^y\} \end{aligned}$$

Thus density function is $f_Y(y) = \lambda e^y \exp\{-\lambda e^y\}$

2. Exercise 5.8: Let $X \sim Unif[-1, 2]$, then for $y > 0$, let F be the cumulative density function of X, we have

$$\begin{aligned} P(Y \leq y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F(\sqrt{y}) - F(-\sqrt{y}) \\ &= \begin{cases} 0 & y \leq 0 \\ \frac{2}{3}\sqrt{y} & y \in (0, 1] \\ \frac{1}{3}(\sqrt{y} + 1) & y \in (1, 4) \\ 1 & y \geq 4 \end{cases} \end{aligned}$$

Thus density function of Y is

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & y \in (0, 1] \\ \frac{1}{6\sqrt{y}} & y \in (1, 4) \\ 0 & \text{Otherwise} \end{cases}$$

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu

Solution for Suggested Problems (Joint Distributions)

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Discrete Cases

1. Exercise 6.1: Given the joint distribution of (X, Y), we have

   (a) Marginal of X is

X	1	2	3
P	.3	.5	.2

   (b) For $Z = XY$:

   - $P(Z = 0) = P(Y = 0) = .35$
   - $P(Z = 1) = P(X = 1, Y = 1) = .15$
   - $P(Z = 2) = P(X = 1, Y = 2) + P(X = 2, Y = 1) = .05$
   - $P(Z = 3) = P(X = 1, Y = 3) + P(X = 3, Y = 1) = .05$
   - $P(Z = 4) = P(X = 2, Y = 2) = .05$
   - $P(Z = 6) = P(X = 2, Y = 3) + P(X = 3, Y = 2) = .2 + .1 = .3$
   - $P(Z = 9) = P(X = 3, Y = 3) = .05$

   (c)
   $$E[Xe^Y] = \sum_{x=1}^{3}\sum_{y=0}^{3} xe^y P(X = x, Y = y) \approx 16.3365$$

2. Exercise 6.19:

   (a) Marginal distribution of X is

X	0	1
P	1/3	2/3

   and marginal distribution of Y is

Y	0	1	2
P	1/6	1/3	1/2

   (b) $p(z, w) = P(Z = z, W = w) = f_X(z)f_Y(w)$ for f_X and f_Y are marginal p.m.f of X and Y, respectively.

Continuous Cases

1. Exercise 6.5: $f(x, y) = \frac{12}{7}(xy + y^2)$ for $0 \le x \le 1$ and $0 \le y \le 1$.

   (a) Just to check that $\iint_{\mathbb{R}^2} f(x, y)dxdy = 1$

Proof.

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)dxdy &= \int_0^1\int_0^1 \frac{12}{7}(xy+y^2)dxdy \\ &= \frac{12}{7}\int_0^1 \frac{1}{2}x^2y+xy^2\Big|_{x=0}^{x=1} dy \\ &= \frac{12}{7}\int_0^1 \frac{1}{2}y+y^2dy \\ &= \frac{12}{7}\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\right)=1\end{aligned}$$

□

(b) Marginal for X:

$$f_X(x)=\int_{\mathbb{R}} f(x,y)dy = \int_0^1 \frac{12}{7}(xy+y^2)dy = \frac{12}{7}(x/2+1/3) = \frac{6x}{7}+\frac{4}{7}$$

for $x \in (0,1)$
Marginal for Y:

$$f_Y(y)=\int_{\mathbb{R}} f(x,y)dx = \int_0^1 \frac{12}{7}(xy+y^2)dx = \frac{12}{7}(y/2+y^2) = \frac{6y}{7}+\frac{12y^2}{7}$$

for $y \in (0,1)$

(c)

$$\begin{aligned}P(X<Y) &= \int_0^1\int_0^y \frac{12}{7}(xy+y^2)dxdy \\ &= \frac{12}{7}\int_0^1 \frac{x^2}{2}y+xy^2\Big|_{x=0}^{x=y} dy \\ &= \frac{12}{7}\cdot\frac{3}{2}\int_0^1 y^3dy \\ &= \frac{12}{7}\cdot\frac{3}{2}\cdot\frac{1}{4}=\frac{9}{14}\end{aligned}$$

(d)

$$\begin{aligned}E[X^2Y] &= \frac{12}{7}\int_0^1\int_0^1 x^2y(xy+y^2)dxdy \\ &= \frac{12}{7}\int_0^1\int_0^1 (x^3y^2+x^2y^3)dxdy \\ &= \frac{12}{7}\int_0^1 \frac{1}{4}y^2+\frac{1}{3}y^3dy \\ &= \frac{12}{7}\left(\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}\right)=\frac{2}{7}\end{aligned}$$

2. Exercise 6.35: $f_{X,Y}(x,y)=\frac{1}{4}(x+y)$ for $0 \le x \le y \le 2$
   (a) Check that f satisfies $\iint_{\mathbb{R}} f(x,y)dxdy=1$.

Proof.

$$
\begin{aligned}
\iint_{\mathbb{R}} f(x,y)dxdy &= \frac{1}{4}\int_0^2\int_0^y x+ydxdy \\
&= \frac{1}{4}\int_0^2 \frac{1}{2}x^2+xy\Big|_{x=0}^{x=y} dy \\
&= \frac{1}{4}\cdot\frac{3}{2}\int_0^2 y^2dy \\
&= \frac{1}{4}\cdot\frac{3}{2}\cdot\frac{8}{3} = 1
\end{aligned}
$$

□

(b)

$$
\begin{aligned}
P(Y<2X) &= \frac{1}{4}\int_0^2\int_{y/2}^y x+ydxdy \\
&= \frac{1}{4}\int_0^2 xy+\frac{1}{2}x^2\Big|_{x=y/2}^{x=y} dy \\
&= \frac{1}{4}\int_0^2 y^2+\frac{1}{2}y^2-\frac{y^2}{2}-\frac{1}{2}(y/2)^2dy \\
&= \frac{1}{4}\int_0^2 \frac{7}{8}y^2dy \\
&= \frac{1}{4}\cdot\frac{7}{8}\cdot\frac{8}{3} = \frac{7}{12}
\end{aligned}
$$

(c) Marginal of Y is

$$
f_Y(y) = \frac{1}{4}\int_0^y x+ydx = \frac{1}{4}\left(\frac{x^2}{2}+xy\right)\Big|_0^y = \frac{3}{8}y^2
$$

for $0<y<2$.

Remark. When you obtain the (marginal) PDF of one random variable, you need to specify the support $\{x : f(x)\neq 0\}$ of the random variable to receive full credits in the exam.

Independence

1. Exercise 6.12: Note that for any $\alpha>0$, we have $\int_0^\infty e^{-\alpha x}dx = \frac{1}{\alpha}$, thus

$$
g(x) = \begin{cases} \alpha e^{-\alpha} & x>0 \\ 0 & x\leq 0 \end{cases}
$$

is a valid p.d.f function. Using this argument, for $x>0$ and $y>0$ we have

$$
f(x,y) = e^{-x}\cdot 2e^{-2y} = f_X(x)f_Y(y)
$$

implies that X and Y are independent, where $f_X(x)$ and $f_Y(y)$ are two marginal density of X and Y, respectively.

2. Exercise 6.27: X_1 and X_2 satisfy $P(X_1=1)=P(X_1=-1)=1/2$, $P(X_2=1)=p$ and $P(X_2=-1)=1-p$. Also, X_1 and X_2 are independent. Let $Y=X_1X_2$.

(a)

$$
P(Y=1) = P(X_1=1,X_2=1)+P(X_1=-1,X_2=-1) = \frac{1}{2}(p+q) = \frac{1}{2}
$$

(b)

$$
P(Y=1) = P(X_1=1,X_2=-1)+P(X_1=-1,X_2=-1) = \frac{1}{2}(p+q) = \frac{1}{2}
$$

(c)

$$P(X_2 = 1, Y = 1) = P(X_2 = 1, X_1 = 1) = \frac{1}{2}p = P(X_2 = 1)P(Y = 1)$$

(d)

$$P(X_2 = 1, Y = -1) = P(X_2 = 1, X_1 = -1) = \frac{1}{2}p = P(X_2 = 1)P(Y = -1)$$

(e)

$$P(X_2 = -1, Y = 1) = P(X_2 = -1, X_1 = -1) = \frac{1}{2}q = P(X_2 = -1)P(Y = 1)$$

(f)

$$P(X_2 = -1, Y = 1) = P(X_2 = 1-, X_1 = -1) = \frac{1}{2}q = P(X_2 = -1)P(Y = 1)$$

Based on the formulas from (c)-(f), we have X_2 and Y are independent.

Covariance

Exercise 8.14 – 8.16

1. Exercise 8.14: The Marginal density of X is

X	1	2	3
P	1/3	1/2	1/6

The Marginal density of Y is

Y	0	1	2	3
P	1/5	1/5	1/3	4/15

Thus $E[X] = 11/6$, $E[X^2] = 23/6$, $Var(X) = 17/36$ and
$E[Y] = 5/3$, $E[Y^2] = 59/15$, $Var(Y) = 52/45$.

$$E[XY] = \sum_{x=1}^{3}\sum_{y=0}^{3} xyP(X = x, Y = y) = 47/15$$

Thus $Cov(X, Y) = E[XY] - E[X]E[Y] = 47/15 - (11/6)(5/3) = 7/90$ and

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \approx .1053$$

2. Exercise 8.15: Given the information on D, we have the area of D is equal to 3/2, then we derive that the density of (X, Y) should be

$$f(x, y) = \begin{cases} 2/3 & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$$

Then we have

$$E[X] = \iint_{\mathbb{R}} xf(x, y)dxdy = \frac{2}{3}\int_0^2\int_0^{2-y} xdxdy = \frac{7}{9}$$
$$E[Y] = \iint_{\mathbb{R}} yf(x, y)dxdy = \frac{2}{3}\int_0^2\int_0^{2-y} ydxdy = \frac{4}{9}$$
$$E[XY] = \iint_{\mathbb{R}} xyf(x, y)dxdy = \frac{2}{3}\int_0^2\int_0^{2-y} xydxdy = \frac{11}{36}$$

Thus

$$Cov(X, Y) = E[XY] - E[X]E[Y] = \frac{11}{36} - \frac{7}{9}\cdot\frac{4}{9} = -\frac{13}{324}$$

Thus, X and Y are negatively corrected.

3. Exercise 8.16: $E[X] = 1$, $E[X^2] = 3$, $E[XY] = -4$, and $E[Y] = 2$.

$$\begin{aligned} Cov(X, 2X + Y - 3) &= 2Cov(X, X) + Cov(X, Y) - Cov(X, 3) \\ &= 2Var(X) + Cove(X, Y) \\ &= 2(E[X^2] - (E[X])^2) + E[XY] - E[X]E[Y] \\ &= 2 \cdot (3 - 1) + (-4) - 1 \cdot 2 = -2 \end{aligned}$$

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Solutions for Suggested Problems (Binomial Approximation and Conditional Distributions)

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Binomial Approximation

1. Exercise 4.35: $X \sim bin(n, p)$, for $n = 365$ and

$$p = P(\text{All ten are heads or tails}) = P(\text{All heads}) + P(\text{All tails}) = \frac{1}{2^{10}} + \frac{1}{2^{10}} = \frac{1}{512}$$

(a) Given the information on the random variable X, we have

$$P(X > 1) = \sum_{k=2}^{365} \binom{365}{k} (1/512)^k \left(1 - \frac{1}{512}\right)^{365-k}$$

or

$$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - \left(1 - \frac{1}{512}\right)^{365} - \frac{365}{512}\left(1 - \frac{1}{512}\right)^{364}$$

(b) Since p is relatively small, consider Poisson approximation is better. $\lambda = np = \frac{365}{512}$ implies

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - e^{-\lambda} - \lambda e^{\lambda} \\ &= 1 - e^{-365/512} - (365/512)e^{365/512} = .1603 \end{aligned}$$

2. Exercise 4.41: Suppose X be the number of sixes in the experiment. Then $X \sim Bin(72, 1/6)$.

(a) Poisson approximation: Since $\lambda = np = 12$, so X can be approximate as a Poisson distribution with parameter $\lambda = 12$. So

$$P(X = 3) \approx e^{-12}\frac{12^3}{3!} = .0018$$

(b) Normal approximation (Continuity correction is needed): $\mu = np = 12$ and $\sigma^2 = np(1-p) = 10$

$$\begin{aligned} P(X = 3) &= P(2.5 \leq X \leq 3.5) \\ &\approx \Phi\left(\frac{3.5 - 12}{\sqrt{10}}\right) - \Phi\left(\frac{2.5 - 12}{\sqrt{10}}\right) \approx \Phi\left(\frac{12 - 2.5}{\sqrt{10}}\right) - \Phi\left(\frac{12 - 3.5}{\sqrt{10}}\right) \approx .0023 \end{aligned}$$

Remark. For $z_1 < z_2$, we have

$$\Phi(-z_1) - \Phi(-z_2) = (1 - \Phi(z_1)) - (1 - \Phi(z_2)) = \Phi(z_2) - \Phi(z_1)$$

Conditional Distribution

1. Exercise 10.1: The marginal distribution of Y is

Y	0	1	2
P	1/3	4/9	2/9

Conditional distribution of X given $Y = y$ is $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$, thus we have

- $p_{X|Y}(2|0) = 1$
- $p_{X|Y}(1|1) = 1/4$, $p_{X|Y}(2|1) = 1/2$, and $p_{X|Y}(3|1) = 1/4$
- $p_{X|Y}(2|2) = 1/2$ and $p_{X|Y}(3|2) = 1/2$

Also, since $E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$, then we have

- $E[X|Y = 0] = 2$
- $E[X|Y = 1] = 1(1/4) + 2(1/2) + 3(1/4) = 2$
- $E[X|Y = 2] = 2(1/2) + 3(1/2) = 5/2$

2. Exercise 10.2: Fill in the blank of the joint distribution table of (X, Y):

    (a) Given $X = 1$, Y is uniformly distributed, and this implies that

    $$P(X = 1, Y = 0) = P(X = 1, Y = 1) = P(X = 2, Y = 1) = 1/8$$

    (b) $p_{X|Y}(0|0) = 2/3$ implies that

    $$p_{X|Y}(1|0) = \frac{1}{3}$$

    and

    $$p_Y(0) = \frac{p(1,0)}{p_{X|Y}(1|0)} = \frac{1/8}{1/3} = \frac{3}{8}$$

    So

    $$p(0,0) = p_Y(0) - p(1,0) = \frac{1}{4}$$

    (c) $P(X = 0) = 1 - P(X = 1) = 1 - 3(1/8) = 5/8$ and $P(X = 0, Y = 0) = 1/4$ implies that

    $$P(X = 0, Y = 1) + P(X = 0, Y = 2) = \frac{3}{8}$$

    (d) $E[Y|X = 0] = 1P(Y = 1|X = 0) + 2P(Y = 2|X = 0) = 4/5$ implies that

    $$P(X = 0, Y = 1) + 2P(X = 0, Y = 2) = \frac{4}{5}P(X = 0) = \frac{1}{2}$$

    (e) We can solve for

    $$P(X = 0, Y = 1) = \frac{1}{4}$$
    $$P(X = 0, Y = 2) = \frac{1}{8}$$

3. Exercise 10.5: The joint density of (X, Y) is

    $$f(x,y) = \begin{cases} \frac{12}{5}x(2 - x - y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

    (a) To figure out $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ for $y \in (0,1)$, we need to obtain marginal density f_Y of Y, which is

    $$f_Y(y) = \int f(x,y)dx = \frac{12}{5}\int_0^1 x(2 - x - y)dx = \frac{8}{5} - \frac{6}{5}y$$

    for $0 < y < 1$. Then we have

    $$f_{X|Y}(x|y) = \frac{12x(2 - x - y)}{8 - 6y}$$

(b) First we obtain

$$f_{X|Y}\left(x\middle|\frac{3}{4}\right)=\frac{24}{7}\left(\frac{5}{4}x-x^2\right)$$

by taking $y=3/4$ in the formula of conditional density. Then the calculation should be

$$P\left(x>\frac{1}{2}\middle|Y=\frac{3}{4}\right)=\int_{1/2}^{1}\frac{24}{7}\left(\frac{5}{4}x-x^2\right)dx=\frac{17}{28}$$

and

$$E\left[X\middle|Y=\frac{3}{4}\right]=\frac{24}{7}\int_0^1 x\left(\frac{5}{4}x-x^2\right)dx=\frac{4}{7}$$

4. Exercise 10.9: Joint density of (X,Y) is

$$f(x,y)=\begin{cases}\frac{1}{y}e^{-x/y}e^{-y} & x>0, y>0\\ 0 & \text{Otherwise}\end{cases}$$

Note that for any $\alpha>0$, we have $\int_0^\infty \alpha e^{-\alpha x}dx=1(*)$ and $\int_0^\infty \alpha x e^{-\alpha x}dx=1/\alpha(**)$, and this formula can make the calculation faster.

(a) Using the formula $(*)$ above, we have

$$f_Y(y)=e^{-y}$$

for $y>0$, so conditional density is

$$f_{x|y}(x|y)=\frac{1}{y}e^{-x/y}$$

for $x,y>0$.

(b) Using $(**)$:

$$E[X|Y=y]=\int_0^\infty \frac{x}{y}e^{-x/y}dx=y$$

So $E[X|Y]=Y$.

(c) Using $(**)$:

$$E[X]=E[E[X|Y]]=E[Y]=\int_0^\infty ye^{-y}dy=1$$

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