Counting Methods

1. Basic Principle of Counting

- Suppose there are k jobs to be done.

$$\begin{cases} \text{first job: be done in } \underline{n_1} \text{ ways} \\ \text{second job: be done in } \underline{n_2} \text{ ways} \\ \quad \vdots \\ k\text{-th job: be done in } \underline{n_k} \text{ ways} \end{cases}$$

Total number of ways doing k jobs is

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i \quad (\text{Multiplication Rules})$$

e.g. 1: Consider an experiment where we roll a fair 4-sided die (1, 2, 3, 4) for three times.

Q: How many possible outcomes in total?

A: roll for three times

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three jobs to be done $\begin{cases} \text{first: } 4 \text{ ways} \\ \text{second: } 4 \text{ ways} \\ \text{third: } 4 \text{ ways} \end{cases}$

total number of outcomes: $4 \cdot 4 \cdot 4 = 4^3 = 64$ //

e.g. 2: In a country, licence plates have three letters followed by three digits.

Three letters $\Leftrightarrow$ three jobs $\begin{cases} ① : 26 \\ ② : 26 \\ ③ : 26 \end{cases}$

Three digits $\Leftrightarrow$ three jobs $\begin{cases} ① : 10 \\ ② : 10 \\ ③ : 10 \end{cases}$

by multiplication rules: this country can construct

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^3$ different licence plates. //.

e.g. 3: Suppose that $A_1, A_2, \cdots, A_k$ are finite sets.

$\# A_i$ (the number of elements of A_i) $= n_i$ $\quad i = 1, 2, \cdots, k$

Cartesian product:

$$A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \cdots, x_k) : x_i \in A_i, \ i = 1, 2, \cdots, k\}$$

by multiplication rule: $\begin{cases} x_1 : n_1 \text{ possibilities} \\ x_2 : n_2 \quad \text{〃} \\ \vdots \\ x_k : n_k \quad \text{----} \end{cases}$

$$\# (A_1 \times A_2 \times \cdots \times A_k) = n_1 \cdot n_2 \cdot \cdots \cdot n_k = \prod_{i=1}^{k} n_i$$ //

2. Random Sampling (Chapter 1.2 in the textbook)

Suppose an urn has n balls (numbered $1, 2, \cdots, n$), retrieve the ball for k times.

outcome $\omega = (s_1, s_2, \cdots, s_k)$ $\begin{cases} s_1: \text{the number of the first ball} \\ s_2: \text{the number of the second ball.} \\ \vdots \end{cases}$

Note: $\omega \in S^k = \underbrace{S \times S \times \cdots \times S}_{k \text{ times}}$ $\quad S = \{1, 2, \cdots, n\}$.

① Sampling with replacement: retrieve a ball, record its number, and put it back into the urn.

(possible for the ball to be retrieved again)

Sample space $\Omega = S^k = \{1, 2, \cdots, n\}^k$

Number of outcomes: $\# \Omega = n^k$

② Sampling without replacement: retrieve a ball, record its number, and put it aside.

(the same ball cannot be drawn twice, and the number $s_1, s_2, \cdots, s_k$ are distinct)

$\begin{cases} \text{(1) Order matters} \\ \text{(2) Order doesn't matters} \end{cases}$

e.g. 5 balls (1, 2, 3, 4, 5)

retrieve the ball for three times w/o replacement.

(1) (1, 2, 5) and (2, 1, 5) are different outcomes.

(2) (1, 2, 5) and (2, 1, 5) are the same.

(1) Order matters

Sample space:

$$\Omega = \{(s_1, s_2, \cdots, s_k): \text{each } s_i \in S \text{ but } s_i \neq s_j \text{ for } i \neq j\}$$

$$\#\Omega = n \cdot (n-1) \cdot \cdots \cdot (n-k+1) \quad (\text{Check by multiplication rule})$$

$$= \frac{n!}{(n-k)!} \quad (n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n)$$

(2) Order doesn't matter

Sample space: $\Omega = \{A : A \subseteq S, \#A = k\}$

$$\#\Omega = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

e.g. 1: Find the number of six-letter words (not need to be meaningful) constructed from the letters B, A, D, G, E, R?

Soln: It is equivalent to retrieve 6 letter w/o replacement, order matters: $6! = 720$ possibilities.

e.g. 2: Find the number of 5-letter words

from the letters A, P, P, L, E? (no need to be meaningful)

(MTD 1) Order 5 letters: $5! = 120$ ways,

but two Ps can be in two different orders,

we counted each word twice, so the number

of different words is $\frac{120}{2} = 60$.

(MTD 2) 5-letter words has the format

_ _ _ _ _.

Step 1: Choose the position of two Ps: $\binom{5}{2} = 10$ ways

Step 2: Arrange the remaining three words: $3! = 6$ ways

By multiplication rules:

$10 \cdot 6 = 60$ different words.

3. Addition Principle

Two event A and B. If $A \cap B = \phi$ (disjoint),

$\#(A \cup B) = (\# A) + (\# B)$.

Moreover, $\Omega = A \cup A^c$, $\# \Omega = \# A + \# A^c$

If $\# A$ is hard to calculate, $\# \Omega$ and $\# A^c$ are relatively easier, then use $\# A = \# \Omega - \# A^c$.

e.g. 10 people : each person flips a coin and rolls a die.

- each person : $2 \times 6 = 12$ outcomes
- $\# \Omega = 12^{10}$ outcomes
- event $A =$ "no people rolled a 5"
  - ↪ each person : $2 \times 5 = 10$ outcomes
  - ↪ $\# A = 10^{10}$
- event $B =$ "At least 1 person rolled a 5"
  - ↪ $B = A^c$
  - ↪ $\# B = \# \Omega - \# A = 12^{10} - 10^{10}$.