Combined Section on Distributions and Joint Distributions

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See exercise: $6.6, 6.11, 6.12, 7.5, 8.15$

1. Exercise 6.6: In this problem, you can use the fact that $\int_0^\infty xe^{-\alpha x}dx = \frac{1}{\alpha^2}$ and $\int_0^\infty x^2e^{-\alpha x}dx = \frac{2}{\alpha^3}$ for $\alpha > 0$.

   (a) The marginal density for X is: for $x > 0$

   $$f_X(x) = \int_0^\infty xe^{-x(1+y)}dy = xe^{-x}\int_0^\infty e^{-xy}dy = e^{-x}$$

   and zero density if $x \leq 0$. The marginal density for Y is: for $y > 0$,

   $$f_Y(y) = \int_0^\infty xe^{-x(1+y)}dy = \frac{1}{(1+y)^2}$$

   and zero density if $y \leq 0$.

   (b) The expected value of XY can be obtained by

   $$\begin{aligned} E[XY] &= \int_0^\infty \int_0^\infty xy \cdot xe^{-x(1+y)}dxdy \\ &= \int_0^\infty x^2e^{-x}\left(\int_0^\infty ye^{-xy}dy\right)dx \\ &= \int_0^\infty x^2e^{-x}\cdot\frac{1}{x^2}dx \\ &= \int_0^\infty e^{-x}dx = 1 \end{aligned}$$

   (c) The expected value of $\frac{X}{1+Y}$ can be obtained by

   $$\begin{aligned} E\left[\frac{X}{1+Y}\right] &= \int_0^\infty \int_0^\infty \frac{x}{1+y}\cdot xe^{-x(1+y)}dxdy \\ &= \int_0^\infty \frac{1}{1+y}\left(\int_0^\infty x^2e^{-x(1+y)}dx\right)dy \\ &= \int_0^\infty \frac{1}{1+y}\cdot\frac{2}{(1+y)^3}dy \\ &= 2\int_0^\infty (1+y)^{-4}dy = \frac{2}{3} \end{aligned}$$

2. Exercise 6.11: Given the density of X is

   $$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & o.w. \end{cases}$$

   $Y \sim \text{Unif}[1, 2]$, and X and Y are independent, the joint distribution of X and Y is

   $$f(x, y) = f_X(x)f_Y(y) = \begin{cases} 2x, & 0 < x < 1, 1 < y < 2 \\ 0, & o.w. \end{cases}$$

Then, we have

$$
\begin{aligned}
P\left(Y - X \geq \frac{3}{2}\right) &= \iint_{y-x>3/2} f(x,y)\,dxdy \\
&= \int_0^{1/2}\int_{x+3/2}^{1} 2x\,dy\,dx \\
&= \int_0^{1/2} 2x\left(\frac{1}{2} - x\right)\,dx \\
&= \frac{1}{24}
\end{aligned}
$$

3. Exercise 6.12: Note that for any $\alpha > 0$, we have $\int_0^\infty e^{-\alpha x}dx = \frac{1}{\alpha}$, thus

$$
g(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & x \leq 0 \end{cases}
$$

is a valid p.d.f function. Using this argument, for $x > 0$ and $y > 0$ we have

$$
f(x,y) = e^{-x} \cdot 2e^{-2y} = f_X(x)f_Y(y)
$$

implies that X and Y are independent, where $f_X(x)$ and $f_Y(y)$ are two marginal density of X and Y, respectively.

4. Exercise 7.5: Since X, Y, Z are three independent random variables with $X \sim \mathcal{N}(1,2)$, $Y \sim \mathcal{N}(2,1)$,and $Z \sim \mathcal{N}(0,7)$, then W is still a normal random variable with

$$
E[W] = E[X] - 4E[Y] + E[Z] = 1 - 4 \times 2 + 0 = -7
$$

and

$$
\mathrm{Var}(W) = \mathrm{Var}(X) + (-4)^2\mathrm{Var}(Y) + \mathrm{Var}(Z) = 2 + 16 + 7 = 25
$$

We have

(a) $W \sim \mathcal{N}(-7, 25)$

(b) Let Φ be distribution function of standard normal random variables with $\mathcal{N}(0,1)$, we have

$$
\begin{aligned}
P(W > -2) &= P\left(\frac{W-(-7)}{\sqrt{25}} > \frac{-2+7}{\sqrt{25}}\right) \\
&= 1 - \Phi(1) \\
&= 1 - .8413 = .1587
\end{aligned}
$$

5. Exercise 8.15: Given the information on D, we have the area of D is equal to $3/2$, then we derive that the density of (X, Y) should be

$$
f(x,y) = \begin{cases} 2/3 & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}
$$

Then we have

$$
\begin{aligned}
E[X] &= \iint_{\mathbb{R}} xf(x,y)dxdy = \frac{2}{3}\int_0^1\int_0^{2-y} xdxdy = \frac{7}{9} \\
E[Y] &= \iint_{\mathbb{R}} yf(x,y)dxdy = \frac{2}{3}\int_0^1\int_0^{2-y} ydxdy = \frac{4}{9} \\
E[XY] &= \iint_{\mathbb{R}} xyf(x,y)dxdy = \frac{2}{3}\int_0^1\int_0^{2-y} xydxdy = \frac{11}{36}
\end{aligned}
$$

Thus

$$
Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{11}{36} - \frac{7}{9}\cdot\frac{4}{9} = -\frac{13}{324}
$$

Thus, X and Y are negatively corrected.
