Solution for Suggested Problems (Joint Distributions)

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Discrete Cases

1. Exercise 6.2: We have

   (a) The marginal p.m.f of X is

X	1	2	3
P	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

   and the marginal p.m.f of Y is

Y	0	1	2	3
P	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{4}{15}$

   (b) We need to find the ordered pairs (x, y) in the support of X and Y such that $x+y^2 \leq 2$. Thus,

   $$\begin{aligned} P(X+Y^2 \leq 2) &= P(X=1, Y=0) + P(X=1, Y=1) + P(X=2, Y=0) \\ &= \frac{1}{15} + \frac{1}{15} + \frac{1}{10} \\ &= \frac{7}{30} \end{aligned}$$

2. Exercise 6.19:

   (a) Marginal distribution of X is

X	0	1
P	1/3	2/3

   and marginal distribution of Y is

Y	0	1	2
P	1/6	1/3	1/2

   (b) $p(z, w) = P(Z = z, W = w) = f_X(z) f_Y(w)$ for f_X and f_Y are marginal p.m.f of X and Y, respectively.

Continuous Cases

1. Exercise 6.5: $f(x, y) = \frac{12}{7}(xy + y^2)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

   (a) Just to check that $\iint_{\mathbb{R}^2} f(x, y) dx dy = 1$

Proof.

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)dxdy &= \int_0^1 \int_0^1 \frac{12}{7}(xy+y^2)dxdy \\ &= \frac{12}{7}\int_0^1 \frac{1}{2}x^2y + xy^2\bigg|_{x=0}^{x=1} dy \\ &= \frac{12}{7}\int_0^1 \frac{1}{2}y + y^2 dy \\ &= \frac{12}{7}\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\right) = 1\end{aligned}$$

□

(b) Marginal for X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy = \int_0^1 \frac{12}{7}(xy+y^2)dy = \frac{12}{7}(x/2+1/3) = \frac{6x}{7}+\frac{4}{7}$$

for $x \in (0,1)$

Marginal for Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)dx = \int_0^1 \frac{12}{7}(xy+y^2)dx = \frac{12}{7}(y/2+y^2) = \frac{6y}{7}+\frac{12y^2}{7}$$

for $y \in (0,1)$

(c)

$$\begin{aligned}P(X<Y) &= \int_0^1 \int_0^y \frac{12}{7}(xy+y^2)dxdy \\ &= \frac{12}{7}\int_0^1 \frac{x^2}{2}y + xy^2\bigg|_{x=0}^{x=y} dy \\ &= \frac{12}{7}\cdot\frac{3}{2}\int_0^1 y^3 dy \\ &= \frac{12}{7}\cdot\frac{3}{2}\cdot\frac{1}{4} = \frac{9}{14}\end{aligned}$$

(d)

$$\begin{aligned}E[X^2Y] &= \frac{12}{7}\int_0^1\int_0^1 x^2y(xy+y^2)dxdy \\ &= \frac{12}{7}\int_0^1\int_0^1 (x^3y^2+x^2y^3)dxdy \\ &= \frac{12}{7}\int_0^1 \frac{1}{4}y^2+\frac{1}{3}y^3 dy \\ &= \frac{12}{7}\left(\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}\right) = \frac{2}{7}\end{aligned}$$

2. Exercise 6.35: $f_{X,Y}(x,y) = \frac{1}{4}(x+y)$ for $0 \le x \le y \le 2$

   (a) Check that f satisfies $\iint_{\mathbb{R}} f(x,y)dxdy = 1$.

Proof.

$$\begin{aligned}
\iint_{\mathbb{R}} f(x,y)dxdy &= \frac{1}{4}\int_0^2\int_0^y x+ydxdy \\
&= \frac{1}{4}\int_0^2 \frac{1}{2}x^2+xy\Big|_{x=0}^{x=y} dy \\
&= \frac{1}{4}\cdot\frac{3}{2}\int_0^2 y^2dy \\
&= \frac{1}{4}\cdot\frac{3}{2}\cdot\frac{8}{3} = 1
\end{aligned}$$

□

(b)

$$\begin{aligned}
P(Y<2X) &= \frac{1}{4}\int_0^2\int_{y/2}^y x+ydxdy \\
&= \frac{1}{4}\int_0^2 xy+\frac{1}{2}x^2\Big|_{x=y/2}^{x=y} dy \\
&= \frac{1}{4}\int_0^2 y^2+\frac{1}{2}y^2-\frac{y^2}{2}-\frac{1}{2}(y/2)^2dy \\
&= \frac{1}{4}\int_0^2 \frac{7}{8}y^2dy \\
&= \frac{1}{4}\cdot\frac{7}{8}\cdot\frac{8}{3} = \frac{7}{12}
\end{aligned}$$

(c) for $0<y<2$, the marginal density of Y is

$$f_Y(y) = \frac{1}{4}\int_0^y x+ydx = \frac{1}{4}\left(\frac{x^2}{2}+xy\right)\Big|_0^y = \frac{3}{8}y^2$$

and 0 otherwise.

Independence

See exercise 6.27, 6.32, 7.3, 8.9

1. Exercise 6.27: X_1 and X_2 satisfy $P(X_1=1)=P(X_1=-1)=1/2$, $P(X_2=1)=p$ and $P(X_2=-1)=1-p$. Also, X_1 and X_2 are independent. Let $Y=X_1X_2$.

   (a)
   $$P(Y=1)=P(X_1=1,X_2=1)+P(X_1=-1,X_2=-1)=\frac{1}{2}(p+q)=\frac{1}{2}$$

   (b)
   $$P(Y=1)=P(X_1=1,X_2=-1)+P(X_1=-1,X_2=-1)=\frac{1}{2}(p+q)=\frac{1}{2}$$

   (c)
   $$P(X_2=1,Y=1)=P(X_2=1,X_1=1)=\frac{1}{2}p=P(X_2=1)P(Y=1)$$

   (d)
   $$P(X_2=1,Y=-1)=P(X_2=1,X_1=-1)=\frac{1}{2}p=P(X_2=1)P(Y=-1)$$

   (e)
   $$P(X_2=-1,Y=1)=P(X_2=-1,X_1=-1)=\frac{1}{2}q=P(X_2=-1)P(Y=1)$$

(f)

$$P(X_2 = -1, Y = 1) = P(X_2 = 1-, X_1 = -1) = \frac{1}{2}q = P(X_2 = -1)P(Y = 1)$$

Based on the formulas from (c)-(f), we have X_2 and Y are independent.

2. Exercise 6.32: $p = \frac{7}{9}$ is the probability of one draws resulting in yellow or green balls. Note that $N \sim Geo(p)$, then the probability mass function for N is $P(N = k) = (2/9)^{k-1}(7/9)$ for $k \geq 1$. The joint distribution of (N, Y) is

$$\begin{aligned} P(N = k, Y = 1) &= P(k-1 \text{ white balls followed by a green ball}) \\ &= \left(\frac{2}{9}\right)^{k-1}\left(\frac{4}{9}\right) \end{aligned}$$

and

$$\begin{aligned} P(N = k, Y = 2) &= P(k-1 \text{ white balls followed by a yellow ball}) \\ &= \left(\frac{2}{9}\right)^{k-1}\left(\frac{3}{9}\right) \end{aligned}$$

By law of total probability, we have

$$\begin{aligned} P(Y = 1) &= \sum_{k=1}^{\infty} P(Y = 1, N = k) \\ &= \sum_{k=1}^{\infty}\left(\frac{2}{9}\right)^{k-1}\left(\frac{4}{9}\right) \\ &= \frac{4}{9}\sum_{k=1}^{\infty}\left(\frac{2}{9}\right)^{k-1} \\ &= \frac{4}{9}\cdot\frac{1}{1-(2/9)} = \frac{4}{7} \end{aligned}$$

Here, we use the geometric series $\sum_{k=1}^{\infty} p^{k-1} = \frac{1}{1-p}$ for any $|p| < 1$. Using the same argument (Law of Total Probability), we obtained $P(Y = 2) = \frac{3}{7}$. Thus, we can see N and Y are independent: for any $k \geq 1$,

$$P(N = k, Y = 1) = \left(\frac{2}{9}\right)^{k-1}\left(\frac{4}{9}\right) = \left(\frac{2}{9}\right)^{k-1}\left(\frac{7}{9}\right)\left(\frac{4}{7}\right) = P(N = k)P(Y = 1)$$

$$P(N = k, Y = 2) = \left(\frac{2}{9}\right)^{k-1}\left(\frac{3}{9}\right) = \left(\frac{2}{9}\right)^{k-1}\left(\frac{7}{9}\right)\left(\frac{3}{7}\right) = P(N = k)P(Y = 2)$$

3. Exercise 7.3: Let X_1 and X_2 be the change in price tomorrow and the day after tomorrow, with X_1 and X_2 being independent and their p.m.f given. Then,

$$\begin{aligned} P(X_1 + X_2 = 2) =& P(X = -1, Y = 3) + P(X = 0, Y = 2) + P(X = 1, Y = 1) \\ &+ P(X = 2, Y = 0) + P(X = 3, Y = 0) \\ =& \frac{1}{64} + \frac{1}{64} + \frac{1}{16} + \frac{1}{64} + \frac{1}{64} = \frac{1}{8} \end{aligned}$$

Expectation

Exercise $8.4, 8.7, 8.11, 8.9$

1. Exercise 8.4: Let I_k be the indicator of the event that hat the number 4 is showing on the k-sided die. Then $Z = I_4 + I_6 + I_{12}$ with $E[I_k] = \frac{1}{k}$. Thus, by linearity of expectation:

$$E[Z] = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$$

2. Exercise 8.7: Let X_B be the event that Ben calls Adam, and similar for X_C and X_D. Then, $X = X_C + X_D + X_B$ with

$$E[X] = E[X_B] + E[X_C] + E[X_D] = .3 + .4 + .7 = 1.4$$

and

$$\text{Var}(X) = \text{Var}(X_B) + \text{Var}(X_C) + \text{Var}(X_D) = (.3)(1 - .3) + (.4)(1 - .4) + (.7)(1 - .7) = .66$$

3. Exercise 8.11: Continue our argument from exercise 8.4:

$$M_X(t) = M_{X_B}(t) \cdot M_{X_C}(t) \cdot M_{X_D}(t) = (.3e^t + .7)(.4e^t + .6)(.7e^t + .3)$$

Remark. We know that $X_B \sim \text{Ber}(.3)$, which implies that its moment generating function $M_{X_B}(t) = .3e^t + .7$, and similar argument for X_C and X_D.

4. Exercise 8.9: X and Y are independent random variables with $E[X] = 3$, $E[Y] = 5$, $\text{Var}(X) = 2$, and $\text{Var}(Y) = 3$. Thus, $E[X^2] = 11$ and $E[Y^2] = 28$

   (a) $E[3X - 2Y + 7] = 3E[X] - 2E[Y] + 7 = 3 \cdot 3 - 2 \cdot 5 + 7 = 6$.

   (b) $\text{Var}(3X - 2Y + 7) = 9 \cdot \text{Var}(X) + 4 \cdot \text{Var}(Y) = 18 + 12 = 30$

   (c) We have

$$\begin{aligned}\text{Var}(XY) &= E[(XY)^2] - (E[XY])^2 \\ &= E[X^2]E[Y^2] - (E[X]E[Y])^2 \\ &= 11 \cdot 28 - (3 \cdot 5)^2 = 83\end{aligned}$$

Covariance

Exercise 8.14, 8.16, 8.17

1. Exercise 8.14: The Marginal density of X is

X	1	2	3
P	1/3	1/2	1/6

The Marginal density of Y is

Y	0	1	2	3
P	1/5	1/5	1/3	4/15

Thus $E[X] = 11/6$, $E[X^2] = 23/6$, $Var(X) = 17/36$ and
$E[Y] = 5/3$, $E[Y^2] = 59/15$, $Var(Y) = 52/45$.

$$E[XY] = \sum_{x=1}^{3} \sum_{y=0}^{3} xyP(X = x, Y = y) = 47/15$$

Thus $Cov(X, Y) = E[XY] - E[X]E[Y] = 47/15 - (11/6)(5/3) = 7/90$ and

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \approx .1053$$

2. Exercise 8.16: $E[X] = 1$, $E[X^2] = 3$, $E[XY] = -4$, and $E[Y] = 2$.

$$\begin{aligned}Cov(X, 2X + Y - 3) &= 2Cov(X, X) + Cov(X, Y) - Cov(X, 3) \\ &= 2Var(X) + Cov(X, Y) \\ &= 2(E[X^2] - (E[X])^2) + E[XY] - E[X]E[Y] \\ &= 2 \cdot (3 - 1) + (-4) - 1 \cdot 2 = -2\end{aligned}$$

3. Exercise 8.17: Given $P(A) = .5$, $P(B) = .2$, and $P(AB) = .1$, we have

$$\mathrm{Var}(X) = \mathrm{Var}(I_A) + \mathrm{Var}(I_B) + 2\mathrm{Cov}(I_A, I_B)$$

where $\mathrm{Var}(I_A) = (.5)(1 - .5) = .25$, $\mathrm{Var}(I_B) = (.2)(1 - .2) = .16$, and $\mathrm{Cov}(I_A, I_B) = E[I_A I_B] - E[I_A]E[I_B] = P(AB) - P(A)P(B) = 0$. Thus, $\mathrm{Var}(X) = .41$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu