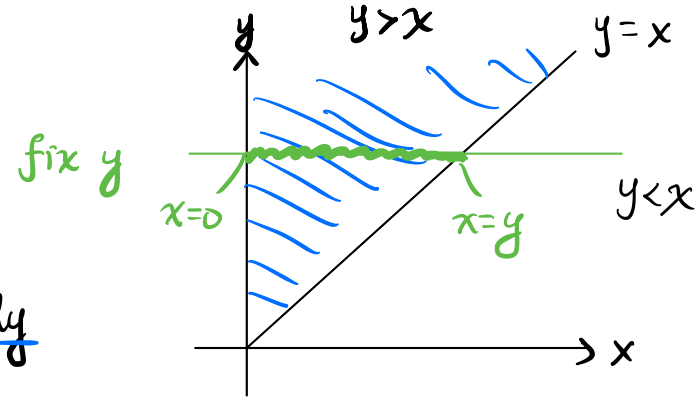
Problem 2. Let (X, Y) be a random vector with joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} ce^{-2(x+y)}, & \underline{0 < x < y} \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c that makes $f_{X,Y}$ a valid joint probability density function.

Hint: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1.$



Soln: If we want

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\underline{dx}\,\underline{dy}$$

$$= \int_0^{\infty}\int_0^{y} c\,e^{-2(x+y)}\,dx\,dy$$

$$= c\int_0^{\infty} e^{-2y} \boxed{\int_0^{y} e^{-2x}\,dx}\,dy$$

$$= c\int_0^{\infty} e^{-2y}\,\frac{1}{2}(1-e^{-2y})\,dy$$

$$= \frac{c}{2}\left(\int_0^{\infty} e^{-2y}\,dy - \int_0^{\infty} e^{-4y}\,dy\right)$$

$$= \frac{c}{2}\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \cancel{\frac{c}{4}}\ \frac{c}{8}$$

i.e. $\frac{c}{8} = 1 \Rightarrow c = 8$

Problem 3. Let (X, Y, Z) be a random vector with joint probability density function given by

$$f_{X,Y,Z}(x, y, z) = \begin{cases} c, & 0 < x < y < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c that makes $f_{X,Y,Z}$ a valid joint probability density function.

Hint: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z)dxdydz = 1.$

Soln: If we want $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dxdy\,dz = 1$

(Step 1): Do the integration for x first

$0 < x < y \Rightarrow$ the bounds for x is $0 \to y$

(Step 2): Do the integration for y:

$0 < y < z \Rightarrow$ the bounds for y is $0 \to z$

(We do the integration for x in step 1, x disappears)

(Step 3): Do the integration for z:

$0 < z < 2 \Rightarrow$ the bounds for z is $0 \to 2$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^z\int_0^y c\,dx\,dy\,dz$$

$$= \int_0^2\int_0^z cy\,dy\,dz$$

$$= \int_0^2 \frac{c}{2}z^2\,dz = \frac{c}{6}z^3\Big|_0^2 = \frac{8c}{6}$$

$$\Rightarrow c = 3/4$$

Extra exercise: Do the integration for y first, then do it for z, and lastly for x.

(Step 1): Do integration on y first:

$x<y<z \Rightarrow$ bounds on y: $x \to z$

(Step 2): Do integration on z:

$x<z<2 \Rightarrow$ bounds on z: $x \to 2$

(y disappears)

(Step 3): Do integration on x:

$0<x<2 \Rightarrow$ bounds on x: $0 \to 2$

(y, z disappears)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dy\,dz\,dx$$

$$= c\int_0^2\int_x^2\int_x^z 1\,dy\,dz\,dx$$

$$= c\int_0^2\int_x^2 (z-x)\,dz\,dx$$

$$= c\int_0^2 \left(\tfrac{1}{2}z^2 - xz\right)\Big|_{z=x}^{z=2} dx$$

$$= c\int_0^2 (2-2x) - \left(\tfrac{1}{2}x^2 - x^2\right) dx$$

$$= c\left(2x - x^2 + \frac{x^3}{6}\right)\Big|_0^2 = c\,\frac{8}{6} \Rightarrow \quad c = \frac{3}{4}$$

$\mathbb{E}[X] = 1/\lambda$

Problem 7.6 The waiting time for bus 24X follows an exponential distribution with expected value 18, measured in minutes.

X $\quad X \sim \exp(1/18)$

(a) Compute the probability that you wait more than 22 minutes. $\{X > 22\}$

(b) Compute the probability that your waiting time is between 16 minutes and 25 minutes. $\{16 < X \le 25\}$

Recall that $X \sim \exp(\lambda) \Rightarrow$ density $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$

To calculate $\underline{\mathbb{P}(X > t)} = \int_t^\infty f_X(x)dx$

$$= \int_t^\infty \lambda e^{-\lambda x} dx \quad (t > 0)$$

$$= -e^{-\lambda x}\Big|_{x=t}^{x=\infty} = -(0 - e^{-\lambda t}) = \underline{e^{-\lambda t}}$$

Soln: (a) $\mathbb{P}(X > 22) \quad \lambda = 1/18,\ t = 22$

$$= e^{-22/18} = \cdots \quad (\text{Calculator})$$

(b) $\mathbb{P}(16 < X \le 25) = \mathbb{P}(X > 25) - \mathbb{P}(X > 16)$

$$= e^{-\frac{25}{18}} - e^{-\frac{16}{18}} = \cdots \quad (\text{Calculator})$$

(c) Suppose the waiting time for bus 12X is also exponentially distributed, and the probability that a person waits more than 22 minutes is 0.33287. Find λ, the parameter of the waiting time for bus 12X.

Y

Let's assume $Y \sim \exp(\lambda)$

$$\mathbb{P}(Y > 22) = .33287$$

$$\| $$

$$e^{-22\lambda}$$

$$e^{-22\lambda} = .33287$$

log

$$-22\lambda = \log(.33287)$$

4-digit decimals

$$\lambda = -\frac{\log(.33287)}{22} = \cdots \quad (\text{Calculator})$$