

Conditional Probability

Def: $\mathbb{P}(A|B)$ — Conditional probability of event A given event B has occurred,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{for } \mathbb{P}(B) > 0.$$

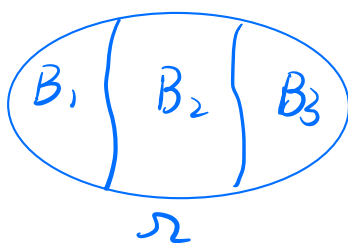
1. Law of Total Probability

Def: $\{B_1, B_2, \dots, B_n\}$ is a partition of Ω if

$$B_1 \cup B_2 \cup \dots \cup B_n = \Omega \Rightarrow \text{make up } \Omega$$

$$B_i \cap B_j = \emptyset \quad \text{for } i \neq j \Rightarrow \text{Pairwise disjoint}$$

For example: $n=3$

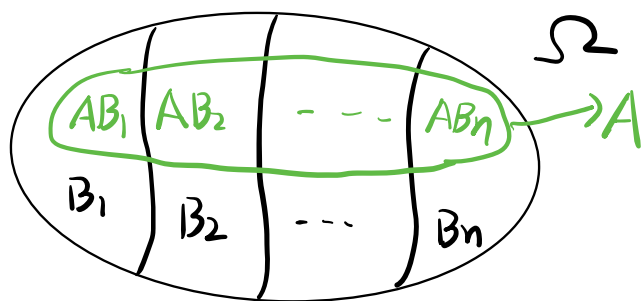


$\{B_1, B_2, B_3\}$:
partition of Ω

Theorem 1: Let $\{B_1, B_2, \dots, B_n\}$ be a partition of Ω ,

then $\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A|B_i) \mathbb{P}(B_i)$ — Law of total Prob.

Pf:



From the graph, we can see

$$A = (AB_1) \cup (AB_2) \cup \dots \cup (AB_n),$$

$\{B_1, B_2, \dots, B_n\}$ forms
a partition of Ω

$\{AB_1, AB_2, \dots, AB_n\}$ forms
a partition of A

and they are pairwise disjoint: $(AB_i) \cap (AB_j) = \emptyset \quad i \neq j$

$$\mathbb{P}(A) = \mathbb{P}(AB_1) + \mathbb{P}(AB_2) + \dots + \mathbb{P}(AB_n)$$

$$= \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \dots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)$$

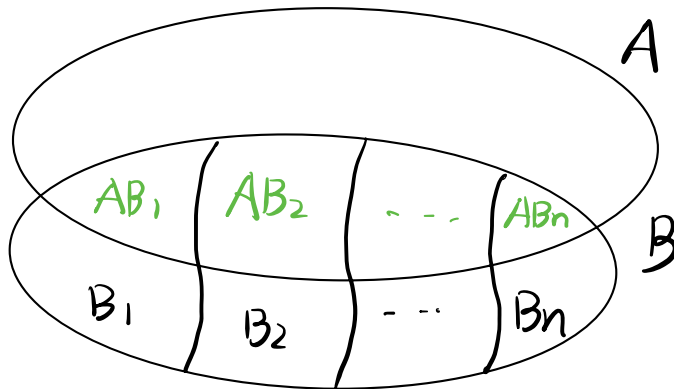
$$= \sum_{i=1}^n \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

□

Theorem 2: Let $\{B_1, B_2, \dots, B_n\}$ be a partition of $B \subseteq \Omega$,

then $\mathbb{P}(A|B) = \sum_{i=1}^n \mathbb{P}(A|B_i)\mathbb{P}(B_i|B)$ — HW2 P6

Pf:



$\{AB_1, AB_2, \dots, AB_n\}$ forms
a partition of A

$\{B_1, B_2, \dots, B_n\}$ forms
a partition of B

From the graph, we can see

$$A \cap B = (AB_1) \cup (AB_2) \cup \dots \cup (AB_n)$$

and they pairwise disjoint: $(AB_i) \cap (AB_j) = \emptyset \quad i \neq j$

Note: $\mathbb{P}(A \cap B) = \mathbb{P}(AB_1) + \mathbb{P}(AB_2) + \dots + \mathbb{P}(AB_n)$

and $B_i \subseteq B$, implies $B_i = B_i \cap B$, and

$$\mathbb{P}(B_i|B) = \frac{\mathbb{P}(B_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1}{P(B)} \sum_{i=1}^n P(A \cap B_i)$$

$$= \sum_{i=1}^n P(A|B_i) \underbrace{\frac{P(B_i)}{P(B)}}_{P(B_i|B)} = \sum_{i=1}^n P(A|B_i) P(B_i|B).$$

2. Bayes' Theorem

$\{B_1, B_2, \dots, B_n\}$ partition of Ω .

$$P(B_k|A) = \frac{P(A \cap B_k)}{P(A)} = \frac{P(B_k|A) P(A)}{P(A)}$$

where $P(A) = \sum_{i=1}^n P(A|B_i) P(B_i)$ — Law of Total Prob.

Example: 3 fair dice $\begin{cases} 4 \text{ sides } (1, 2, 3, 4) \\ 6 \text{ sides } (1, 2, \dots, 6) \\ 12 \text{ sides } (1, 2, \dots, 12) \end{cases}$

pick one dice at random, roll it twice.

$$\begin{cases} B_4 = \{4\text{-sided dice is chosen}\} & P(B_4) = P(B_6) \\ B_6 = \{6\text{-sided dice is chosen}\} & = P(B_{12}) = 1/3 \\ B_{12} = \{12\text{-sided dice is chosen}\} \\ A_1 = \{\text{The first roll is 3}\}, A_2 = \{\text{The second roll is 4}\} \end{cases}$$

(a) Calculate $P(B_6|A)$, where $A = A_1 \cap A_2$.

By Law of total probability:

$$P(A) = \underbrace{P(A|B_4)}_{(1/4)^2} \underbrace{P(B_4)}_{1/3} + \underbrace{P(A|B_6)}_{(1/6)^2} \underbrace{P(B_6)}_{1/3} + \underbrace{P(A|B_{12})}_{(1/12)^2} \underbrace{P(B_{12})}_{1/3}$$

$$P(B_6|A) = \frac{P(A \cap B_6)}{P(A)} = \frac{P(A|B_6) P(B_6)}{P(A)} \quad (\text{by Bayes' thm}).$$

(b) Are A_1 and A_2 independent? Why? **No.**

Def: Two events $A, B \subset \Omega$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$$

Pf: by Law of total prob:

$$\mathbb{P}(A) = \frac{1}{3} \left(\frac{1}{16} + \frac{1}{36} + \frac{1}{144} \right) = \frac{7}{216}$$

$$\begin{aligned} \mathbb{P}(A_1) &= \mathbb{P}(A_1 | B_4) \mathbb{P}(B_4) + \mathbb{P}(A_1 | B_6) \mathbb{P}(B_6) + \mathbb{P}(A_1 | B_{12}) \mathbb{P}(B_{12}) \\ &= \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{12} \cdot \frac{1}{3} = \frac{1}{3} \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{12} \right) = \frac{1}{6} \end{aligned}$$

Similarly, $\mathbb{P}(A_2) = \frac{1}{6}$, so $\mathbb{P}(A_1 \cap A_2) \neq \mathbb{P}(A_1) \mathbb{P}(A_2)$

Thus, A_1 and A_2 : **NOT** indep. □

Remark: A_1 and A_2 are not independent, but

they are conditionally independent given B_4, B_6

and B_{12} , i.e.,

$$\mathbb{P}(A_1 \cap A_2 | B_4) = \mathbb{P}(A_1 | B_4) \cdot \mathbb{P}(A_2 | B_4) = (1/4)^2$$

$$\mathbb{P}(A_1 \cap A_2 | B_6) = \mathbb{P}(A_1 | B_6) \cdot \mathbb{P}(A_2 | B_6) = (1/6)^2$$

$$\mathbb{P}(A_1 \cap A_2 | B_{12}) = \mathbb{P}(A_1 | B_{12}) \cdot \mathbb{P}(A_2 | B_{12}) = (1/12)^2 \quad //$$

Extra Exercise from CLAs:

Problem 2.2 You are given $\mathbb{P}(A \cup B) = 0.8$ and $\mathbb{P}(A \cup B^c) = 0.9$. Compute $\mathbb{P}(A)$.

$$0.8 = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$0.9 = \mathbb{P}(A \cup B^c) = \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c)$$

Add two equations up, we have $1.7 = 2 \cdot \mathbb{P}(A) + [\mathbb{P}(B) + \mathbb{P}(B^c)] - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)]$

$$\text{Then } 1.7 = 2 \cdot \mathbb{P}(A) + 1 - \mathbb{P}(A) \implies \mathbb{P}(A) = 0.7$$

Problem 3.3 If two events A, B are such that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.3$, and $\mathbb{P}(A \cap B) = 0.1$. Compute $\mathbb{P}(A|A \cup B)$, $\mathbb{P}(A|A \cap B)$, and $\mathbb{P}(A \cap B|A \cup B)$.

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.4 + 0.3 - 0.1 = 0.6$$

$$\mathbb{P}(A|A \cup B) = \frac{\mathbb{P}(A \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A \cup B)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$\mathbb{P}(A|A \cap B) = \frac{\mathbb{P}(A \cap (A \cap B))}{\mathbb{P}(A \cap B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cap B)} = 1$$

$$\mathbb{P}(A \cap B|A \cup B) = \frac{\mathbb{P}((A \cap B) \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} = \frac{0.1}{0.6} = \frac{1}{6}$$

Problem 3.13 Jeremy usually checks his emails several times a day. Let X be the number of times Jeremy checks his email per day. Some probabilities are given below:

$$\mathbb{P}(X = 0) = \frac{1}{10}, \quad \mathbb{P}(X = 1) = \frac{1}{5}, \quad \mathbb{P}(X = 2) = \frac{2}{5}, \quad \mathbb{P}(X = 3) = \frac{1}{5}, \quad \mathbb{P}(X \geq 4) = \frac{1}{10}$$

(a) Compute $\mathbb{P}(X \leq 2)$. Hint: $\mathbb{P}(X \geq 3) = \mathbb{P}(X = 3) + \mathbb{P}(X \geq 4)$, and $\mathbb{P}(X < 1) = \mathbb{P}(X = 0)$.

$$\mathbb{P}(X \leq 2) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \frac{1}{10} + \frac{1}{5} + \frac{2}{5} = \frac{7}{10}$$

(b) Compute $\mathbb{P}(X \geq 4|X \geq 2)$. Hint: $\mathbb{P}(X = 0|X \neq 1) = \frac{\mathbb{P}(X = 0 \text{ and } X \neq 1)}{\mathbb{P}(X \neq 1)} = \frac{\mathbb{P}(X = 0)}{\mathbb{P}(X \neq 1)}$.

$$\mathbb{P}(X \geq 2) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X \geq 4) = \frac{2}{5} + \frac{1}{5} + \frac{1}{10} = \frac{7}{10}$$

$$\mathbb{P}(X \geq 4|X \geq 2) = \frac{\mathbb{P}(\{X \geq 4\} \cap \{X \geq 2\})}{\mathbb{P}(X \geq 2)} = \frac{\mathbb{P}(X \geq 4)}{\mathbb{P}(X \geq 2)} = \frac{\frac{1}{10}}{\frac{7}{10}} = \frac{1}{7}$$

(c) Compute $\mathbb{P}(X = 2|X < 3)$.

$$\mathbb{P}(X < 3) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \mathbb{P}(X \leq 2) = \frac{7}{10}$$

$$\mathbb{P}(X = 2|X < 3) = \frac{\mathbb{P}(X = 2 \text{ and } X < 3)}{\mathbb{P}(X < 3)} = \frac{\mathbb{P}(X = 2)}{\mathbb{P}(X < 3)} = \frac{\frac{2}{5}}{\frac{7}{10}} = \frac{4}{7}$$

(d) Compute $\mathbb{P}(X \geq 4|X \neq 2)$.

$$\mathbb{P}(X \geq 4|X \neq 2) = \frac{\mathbb{P}(X \geq 4, X \neq 2)}{\mathbb{P}(X \neq 2)} = \frac{\mathbb{P}(X \geq 4)}{1 - \mathbb{P}(X = 2)} = \frac{\frac{1}{10}}{1 - \frac{2}{5}} = \frac{1}{6}$$

Remark. The following three different notations are all acceptable:

$$\mathbb{P}(X \geq 4, X \neq 2) = \mathbb{P}(X \geq 4 \text{ and } X \neq 2) = \mathbb{P}(\{X \geq 4\} \cap \{X \neq 2\})$$