Conditional Probability

Def: $\mathbb{P}(A|B)$ — Conditional probability of event A given event B has occurred,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{for } \mathbb{P}(B) > 0.$$

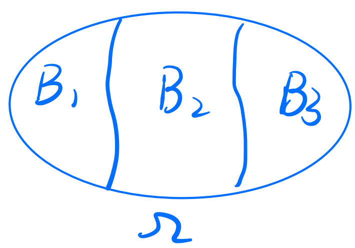
1. Law of Total Probability

Def: $\{B_1, B_2, \cdots, B_n\}$ is a partition of Ω if

$B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ ⇒ make up Ω

$B_i \cap B_j = \phi$ for $i \neq j$ ⇒ Pairwise disjoint

For example: $n=3$

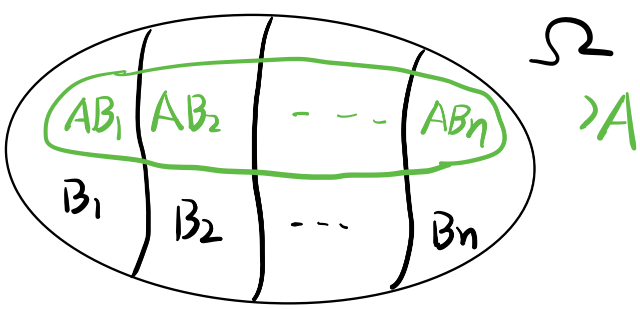


$\{B_1, B_2, B_3\}$: partition of Ω

Theorem 1: Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of Ω,

then $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$ — Law of total Prob.

Pf:



$\{B_1, B_2, \cdots, B_n\}$ forms a partition of Ω

$\{AB_1, AB_2, \cdots, AB_n\}$ forms a partition of A

From the graph, we can see

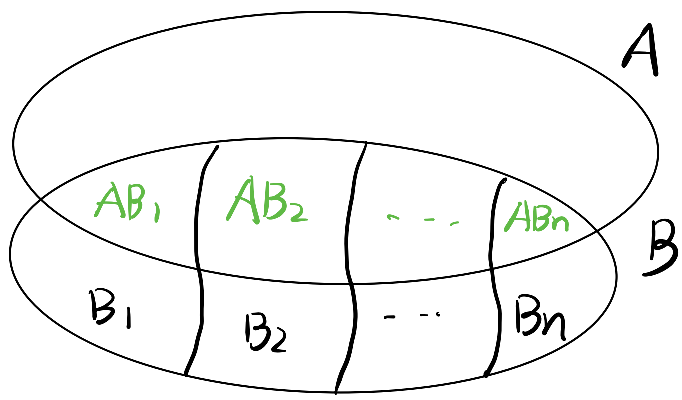
$A = (AB_1) \cup (AB_2) \cup \cdots \cup (AB_n)$, and they are pairwise disjoint: $(AB_i)\cap(AB_j)=\phi \quad i\neq j$

$$\mathbb{P}(A)=\mathbb{P}(AB_1)+\mathbb{P}(AB_2)+\cdots+\mathbb{P}(AB_n)$$
$$=\mathbb{P}(A|B_1)\mathbb{P}(B_1)+\mathbb{P}(A|B_2)\mathbb{P}(B_2)+\cdots+\mathbb{P}(A|B_n)\mathbb{P}(B_n)$$
$$=\sum_{i=1}^{n}\mathbb{P}(A|B_i)\mathbb{P}(B_i) \qquad \boxtimes$$

<u>Theorem 2</u>: Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of $B\subseteq\Omega$,

then $\mathbb{P}(A|B)=\sum_{i=1}^{n}\mathbb{P}(A|B_i)\mathbb{P}(B_i|B)$ — HW2 P6

<u>Pf</u>:



$\{AB_1, AB_2, \cdots, AB_n\}$ forms a partition of A

$\{B_1, B_2, \cdots, B_n\}$ forms a partition of B

From the graph, we can see

$$A\cap B=(AB_1)\cup(AB_2)\cup\cdots\cup(AB_n)$$

and they pairwise disjoint: $(AB_i)\cap(AB_j)=\phi \quad i\neq j$

Note: $\mathbb{P}(A\cap B)=\mathbb{P}(AB_1)+\mathbb{P}(AB_2)+\cdots+\mathbb{P}(AB_n)$

and $B_i\subseteq B$, implies $B_i=B_i\cap B$, and

$$\mathbb{P}(B_i|B)=\frac{\mathbb{P}(B_i\cap B)}{\mathbb{P}(B)}=\frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

$$= \frac{1}{\mathbb{P}(B)} \sum_{i=1}^{n} \mathbb{P}(A\cap B_i)$$

$$= \sum_{i=1}^{n} \mathbb{P}(A|B_i) \underbrace{\frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}}_{\mathbb{P}(B_i|B)} = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\, \mathbb{P}(B_i|B).$$

2. Baye's Theorem

$\{B_1, B_2, \cdots, B_n\}$: partition of Ω.

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A \cap B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k|A)\,\mathbb{P}(A)}{\mathbb{P}(A)}$$

where $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$ — Law of Total Prob.

<u>Example</u>: 3 fair dice
$$\begin{cases} 4 \text{ sides} & (1, 2, 3, 4) \\ 6 \text{ sides} & (1, 2, \cdots, 6) \\ 12 \text{ sides} & (1, 2, \cdots, 12) \end{cases}$$

pick one dice at random, roll it twice.

$$\begin{cases} B_4 = \{4\text{-sided dice is chosen}\} \\ B_6 = \{6\text{-sided dice is chosen}\} \\ B_{12} = \{12\text{-sided dice is chosen}\} \\ \\ A_1 = \{\text{The first roll is } 3\}, \quad A_2 = \{\text{The second roll is } 4\} \end{cases}$$

$\mathbb{P}(B_4) = \mathbb{P}(B_6) = \mathbb{P}(B_{12}) = 1/3$

(a) Calculate $\mathbb{P}(B_6|A)$, where $A_1 \cap A_2$.

By Law of total probability:

$$\mathbb{P}(A) = \underbrace{\mathbb{P}(A|B_4)}_{(1/4)^2}\underbrace{\mathbb{P}(B_4)}_{1/3} + \underbrace{\mathbb{P}(A|B_6)}_{(1/6)^2}\underbrace{\mathbb{P}(B_6)}_{1/3} + \underbrace{\mathbb{P}(A|B_{12})}_{(1/12)^2}\underbrace{\mathbb{P}(B_{12})}_{1/3}$$

$$\mathbb{P}(B_6|A) = \frac{\mathbb{P}(A \cap B_6)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_6)\,\mathbb{P}(B_6)}{\mathbb{P}(A)} \quad \text{(by Bayes thm).}$$

(b) Are A_1 and A_2 independent? Why? No.

Def: Two events $A, B \subset \Omega$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Pf: by Law of total prob:

$$\mathbb{P}(A) = \frac{1}{3}\left(\frac{1}{16} + \frac{1}{36} + \frac{1}{144}\right) = \frac{7}{216}$$

$$\mathbb{P}(A_1) = \mathbb{P}(A_1 \mid B_4)\,\mathbb{P}(B_4) + \mathbb{P}(A_1 \mid B_6)\,\mathbb{P}(B_6) + \mathbb{P}(A_1 \mid B_{12})\,\mathbb{P}(B_{12})$$

$$= \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{3} + \frac{1}{12}\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{12}\right) = \frac{1}{6}$$

Similarly, $\mathbb{P}(A_2) = \frac{1}{6}$, so $\mathbb{P}(A_1 \cap A_2) \neq \mathbb{P}(A_1)\,\mathbb{P}(A_2)$

Thus, A_1 and A_2: NOT indep. $\square$

Remark: A_1 and A_2 are not independent, but they are conditionally independent given B_4, B_6 and B_{12}, i.e.

$$\mathbb{P}(A_1 \cap A_2 \mid B_4) = \mathbb{P}(A_1 \mid B_4)\cdot\mathbb{P}(A_2 \mid B_4) = (1/4)^2$$

$$\mathbb{P}(A_1 \cap A_2 \mid B_6) = \mathbb{P}(A_1 \mid B_6)\cdot\mathbb{P}(A_2 \mid B_6) = (1/6)^2$$

$$\mathbb{P}(A_1 \cap A_2 \mid B_{12}) = \mathbb{P}(A_1 \mid B_{12})\cdot\mathbb{P}(A_2 \mid B_{12}) = (1/12)^2$$

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Extra Exercise from CLAS:

Problem 2.2 You are given $\mathbb{P}(A \cup B) = 0.8$ and $\mathbb{P}(A \cup B^c) = 0.9$. Compute $\mathbb{P}(A)$.

$0.8 = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$0.9 = \mathbb{P}(A \cup B^c) = \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c)$

Add two equations up, we have $1.7 = 2 \cdot \mathbb{P}(A) + [\mathbb{P}(B) + \mathbb{P}(B^c)] - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)]$

Then $1.7 = 2 \cdot \mathbb{P}(A) + 1 - \mathbb{P}(A) \implies \mathbb{P}(A) = 0.7$

Problem 3.3 If two events A, B are such that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.3$, and $\mathbb{P}(A \cap B) = 0.1$. Compute $\mathbb{P}(A|A \cup B)$, $\mathbb{P}(A|A \cap B)$, and $\mathbb{P}(A \cap B|A \cup B)$.

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.4 + 0.3 - 0.1 = 0.6$

$$\mathbb{P}(A|A \cup B) = \frac{\mathbb{P}(A \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A \cup B)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$\mathbb{P}(A|A \cap B) = \frac{\mathbb{P}(A \cap (A \cap B))}{\mathbb{P}(A \cap B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cap B)} = 1$$

$$\mathbb{P}(A \cap B|A \cup B) = \frac{\mathbb{P}((A \cap B) \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} = \frac{0.1}{0.6} = \frac{1}{6}$$

Problem 3.13 Jeremy usually checks his emails several times a day. Let X be the number of times Jeremy checks his email per day. Some probabilities are given below:

$$\mathbb{P}(X = 0) = \frac{1}{10}, \quad \mathbb{P}(X = 1) = \frac{1}{5}, \quad \mathbb{P}(X = 2) = \frac{2}{5}, \quad \mathbb{P}(X = 3) = \frac{1}{5}, \quad \mathbb{P}(X \geq 4) = \frac{1}{10}$$

(a) Compute $\mathbb{P}(X \leq 2)$. Hint: $\mathbb{P}(X \geq 3) = \mathbb{P}(X = 3) + \mathbb{P}(X \geq 4)$, and $\mathbb{P}(X < 1) = \mathbb{P}(X = 0)$.

$$\mathbb{P}(X \leq 2) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \frac{1}{10} + \frac{1}{5} + \frac{2}{5} = \frac{7}{10}$$

(b) Compute $\mathbb{P}(X \geq 4|X \geq 2)$. Hint: $\mathbb{P}(X = 0|X \neq 1) = \dfrac{\mathbb{P}(X = 0 \text{ and } X \neq 1)}{\mathbb{P}(X \neq 1)} = \dfrac{\mathbb{P}(X = 0)}{\mathbb{P}(X \neq 1)}$.

$$\mathbb{P}(X \geq 2) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X \geq 4) = \frac{2}{5} + \frac{1}{5} + \frac{1}{10} = \frac{7}{10}$$

$$\mathbb{P}(X \geq 4|X \geq 2) = \frac{\mathbb{P}(\{X \geq 4\} \cap \{X \geq 2\})}{\mathbb{P}(X \geq 2)} = \frac{\mathbb{P}(X \geq 4)}{\mathbb{P}(X \geq 2)} = \frac{\frac{1}{10}}{\frac{7}{10}} = \frac{1}{7}$$

(c) Compute $\mathbb{P}(X = 2|X < 3)$.

$$\mathbb{P}(X < 3) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \mathbb{P}(X \leq 2) = \frac{7}{10}$$

$$\mathbb{P}(X = 2|X < 3) = \frac{\mathbb{P}(X = 2 \text{ and } X < 3)}{\mathbb{P}(X < 3)} = \frac{\mathbb{P}(X = 2)}{\mathbb{P}(X < 3)} = \frac{\frac{2}{5}}{\frac{7}{10}} = \frac{4}{7}$$

(d) Compute $\mathbb{P}(X \geq 4|X \neq 2)$.

$$\mathbb{P}(X \geq 4|X \neq 2) = \frac{\mathbb{P}(X \geq 4, X \neq 2)}{\mathbb{P}(X \neq 2)} = \frac{\mathbb{P}(X \geq 4)}{1 - \mathbb{P}(X = 2)} = \frac{\frac{1}{10}}{1 - \frac{2}{5}} = \frac{1}{6}$$

Remark. The following three different notations are all acceptable:

$$\mathbb{P}(X \geq 4, X \neq 2) = \mathbb{P}(X \geq 4 \text{ and } X \neq 2) = \mathbb{P}(\{X \geq 4\} \cap \{X \neq 2\})$$