1. A Brief Summary of Expectation

For a random variable X,

$$\mathbb{E}[X] \overset{\text{def}}{=} \begin{cases} \sum_{k \in S_X} k\,\mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{P.D.F}} dx & X: \text{continuous} \end{cases}$$

$E[X]$: the expected value X.

More generally, let $g: \mathbb{R} \to \mathbb{R}$, then $g(X)$ is also a R.V.

with

$$\mathbb{E}[g(X)] \overset{\text{def}}{=} \begin{cases} \sum_{k \in S_X} g(k)\,\mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\, dx & X: \text{continuous} \end{cases}$$

Moreover, $\mathrm{Var}(X) \overset{\text{def}}{=} \mathbb{E}[(X-\mathbb{E}[X])^2]$

$$= \underbrace{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}_{\text{easier way to calculate}}$$

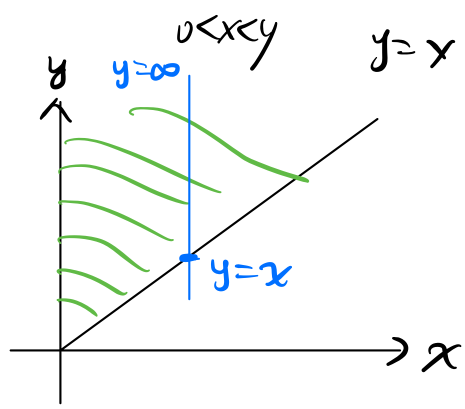
2. Tutorial on Multivariate Calculus

Problem 2. Let (X, Y) be a random vector with joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} ce^{-2(x+y)}, & 0 < x < y \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c that makes $f_{X,Y}$ a valid joint probability density function.

Hint: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1.$



$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$= c\int_0^{\infty}\int_x^{\infty} e^{-2x}\cdot e^{-2y}\,dy\,dx$$

$$= c\int_0^{\infty} e^{-2x}\cdot\left(-\tfrac{1}{2}\right)e^{-2y}\Big|_{y=x}^{y=\infty}\,dx$$

$$= \frac{c}{2}\int_0^{\infty} e^{-4x}\,dx$$

$$= \frac{c}{2}\cdot\left(-\tfrac{1}{4}\right)e^{-4x}\Big|_{x=y}^{x=\infty}$$

$$= \frac{c}{2}\cdot\frac{1}{4}$$

$$= \frac{c}{8}$$

So $c = 8$

Problem 3. Let (X, Y, Z) be a random vector with joint probability density function given by

$$f_{X,Y,Z}(x,y,z) = \begin{cases} c, & 0 < x < y < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c that makes $f_{X,Y,Z}$ a valid joint probability density function.

Hint: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)dxdydz = 1.$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$$

$$\overset{(*)}{=} \int_0^2\int_0^z\int_0^y c\,dx\,dy\,dz$$

To see $(*)$:

① take the integration of x, which satisfies $0<x<y$,

so the bound for x is 0 and y.

② take the integration of y, which satisfies $0<y<z$

(since we've already taken the integration of x, so the bound of y is independent of x)

③ take the integration of z, which satisfies $0<z<2$

so $(*)$

$$= \int_0^2\int_0^z cy\,dy\,dz$$

$$= \frac{c}{2}\int_0^2 z^2 dz$$

$$= \frac{c}{2}\cdot\frac{8}{3} = \frac{4c}{3}$$

so $c = \frac{3}{4}$

3. Gamma Function

Gamma Function $\Gamma(\alpha) \overset{def}{=} \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \alpha > 0$

Facts: ① $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$

(Useful)

② $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \qquad \alpha > 1$

(Proof using Integration by parts)

③ $\forall n \in \mathbb{N}, \ \Gamma(n) = (n-1)!$ (Mathematical Induction)

④ $\alpha, \beta > 0: \ \int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$

HW5 P4 (b): Let $X \sim \exp(1)$, $Y = X^\beta$, then calculate the n-th moment of Y, which is $\mathbb{E}[Y^n]$, when $\beta = 3$?

Soln: $X \sim \exp(1) \Rightarrow$ PDF $f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & o.w. \end{cases}$

$$\mathbb{E}[Y^n] = \mathbb{E}[X^{\beta n}]$$

$$= \int_0^\infty x^{\beta n} e^{-x} dx$$

$$= \int_0^\infty x^{(\beta n+1)-1} e^{-x} dx$$

$$= \Gamma(\beta n + 1)$$

when $\beta = 3$, $3n+1 \in \mathbb{N}$, so by Facts ②: $\mathbb{E}[Y^n] = (3n)!$ $\boxed{B}$

4. Exponential Distribution

From Table of Distribution, we can see for

$X \sim \exp(\lambda)$

PDF $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$

CDF $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Mean $\mathbb{E}[X] = \frac{1}{\lambda}$

Variance $Var(X) = \frac{1}{\lambda^2}$

<u>Q</u>: How to apply what you learnt from prob. class to verify $\int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$ and $\int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{2}{\alpha^3}$ without using integration by parts, for $\alpha > 0$?

<u>A</u>: $\int_0^\infty x e^{-\alpha x}\,dx$

$= \frac{1}{\alpha} \int_0^\infty x \alpha e^{-\alpha x}\,dx \quad (*)$

Now, Define a Random variable $Y \sim \exp(\alpha)$, then its density

$$f_Y(y) = \begin{cases} \alpha e^{-\alpha y} & y > 0 \\ 0 & \text{o.w} \end{cases}$$

Distribution table

and mean $\mathbb{E}[Y] = \int_0^\infty y \alpha e^{-\alpha y} dy = \frac{1}{\alpha}$

So (*)

$$= \frac{1}{\alpha} \int_0^\infty x \alpha e^{-\alpha x} dx = \frac{1}{\alpha} \cdot \frac{1}{\alpha} = \frac{1}{\alpha^2}$$

$$\int_0^\infty x^2 e^{-\alpha x} dx$$

$$= \frac{1}{\alpha} \int_0^\infty x^2 \alpha e^{-\alpha x} dx$$

$$= \frac{1}{\alpha} \mathbb{E}[Y^2]$$

$$= \frac{1}{\alpha} \left(\mathbb{E}[Y]^2 + \mathrm{Var}(Y) \right)$$

$$= \frac{1}{\alpha} \left(\frac{1}{\alpha^2} + \frac{1}{\alpha^2} \right) = \frac{2}{\alpha^3}$$

We will see a lot of integration like this later in the future.