Extra Exercise (Similar to some past midterm Questions):

Now, suppose there are 20 distinct items

20 items

Group A $\swarrow$ $\searrow$ Group B

$N_A = 4$ items $\qquad\qquad$ $N - N_A = 16$ items

Select the items at random.

(a) If I select the items for 9 times, one item each time, with replacement, define

X = # of time I select the items from Group A.

calculate $\mathbb{P}(X \geq 1)$?

<u>Soln</u>: $X \sim Bin(9, p)$ $\qquad p = \frac{4}{20} = \frac{1}{5}$

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0) = 1 - \binom{9}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^9 = 1 - \left(\frac{4}{5}\right)^9 \quad //$$

(b) If I select 9 distinct items at random, calculate $\mathbb{P}(X \geq 1)$?

<u>Soln</u>: $\{X=0\} = \{$All 9 items come from Group B$\}$

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0) = 1 - \frac{\binom{4}{0}\binom{16}{9}}{\binom{20}{9}} = \text{simplify it} \quad //$$

HW4, P1:

80 patients: test a new drug w.p. p to be effective.

with probability

X: # of patients for whom the drug is effective.

Event $S = \{$Trial is successful for two friends$\}$

Note $X \sim Bin(80, p)$ $\quad \mathbb{P}(S) = p^2$

then $S \cap \{X = 55\}$

$= \{$Trial is successful for two of your friends and the other 53 patients out of 78$\}$

Given the trial is successful for 55 patients,

$$\mathbb{P}(S \mid X=55) = \frac{\mathbb{P}(S \cap \{X=55\})}{\mathbb{P}(X=55)}$$

$$= \frac{\cancel{p^2} \binom{78}{53} \cancel{p^{53}} \cancel{(1-p)^{25}}}{\binom{80}{55} \cancel{p^{55}} \cancel{(1-p)^{25}}}$$

Please simplify the fraction.

$$= \frac{78!}{53! \cdot 25!} \cdot \frac{55! \cdot 25!}{80!} = \frac{55 \cdot 54}{80 \cdot 79}$$

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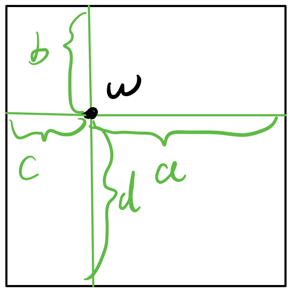
HW4, P4: A point w is randomly chosen in a unit square Ω.

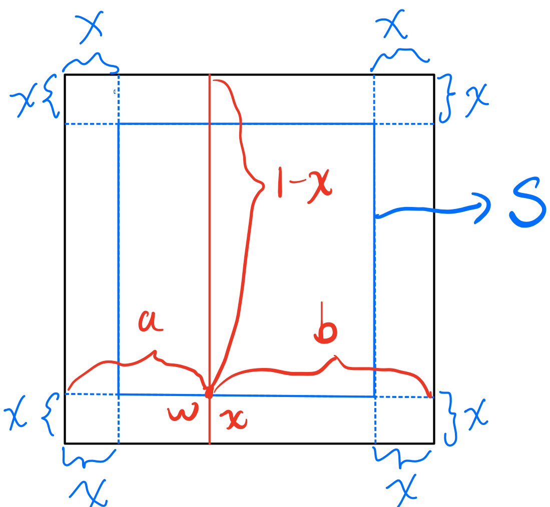


unit square Ω

Then the random variable

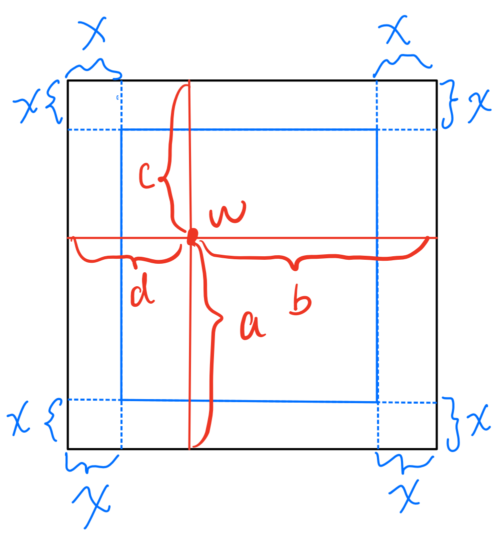
$$X(w) \overset{\text{def}}{=} \min\{a, b, c, d\}$$

Remark: For one number $x \in [0, \frac{1}{2}]$, cut down the unit square Ω as follows, and denote the edge of small square as S.



(1) $S = \{w \in \Omega : X(w) = x\}$ since (see the graph in the left), $w \in S \Rightarrow a > x,\ b > x$, and $1 - x \geq \frac{1}{2} \geq x$, so

$X(w) = \min\{a, b, x, 1-x\} = x$

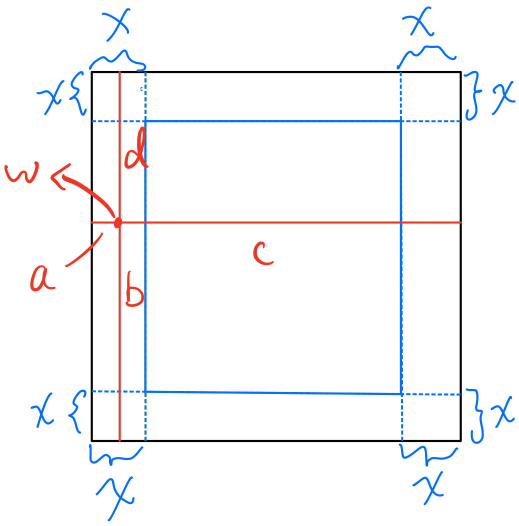


(2) If w is inside S, then

$a > x,\ b > x,\ c > x,\ d > x$

$\Rightarrow X(w) = \min\{a, b, c, d\}$

$> x$



(3) If w is outside S, then

$a < x$ implies

$X(w) = \min\{a, b, c, d\}$

$\leq a < x$

Based on the argument (1), (2), (3) above, we have $\{X(w) \leq x\} \Leftrightarrow \{w \text{ is outside } S\}$

$$\text{Area}(S) = (1-2x)^2$$

$$\mathbb{P}(X \leq x) = \frac{\text{Area}(\Omega \setminus S)}{\text{Area}(\Omega)}$$

$$= \frac{1-(1-2x)^2}{1 \times 1} \quad \text{for} \quad 0 \leq x \leq \frac{1}{2}$$

So C.D.F $F_X(x) = 1-(1-2x)^2 \quad 0 \leq x \leq \frac{1}{2}$

P.D.F $f_X(x) = 4-8x \quad 0 \leq x \leq \frac{1}{2}$

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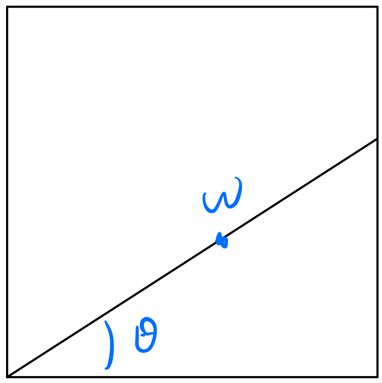
HW4, P8: A point w is randomly chose in a unit square Ω.



$X(w) =$ slop of line

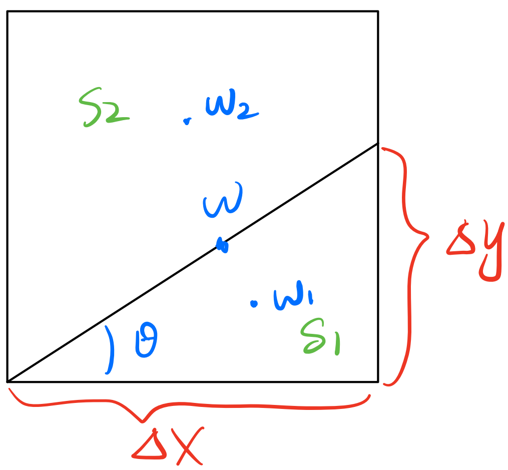
$= \tan\theta$

Support: $S_X = [0, \infty]$

Unit Square Ω

Two cases:

(1) Slope $X(w) = x \le 1$



$w_1 \in S_1$: Slope $X(w_1) \le X(w) = x$

$w_2 \in S_2$: slope $X(w_2) \ge X(w) = x$

So $X(w) \le x \Leftrightarrow w \in S_1$

$\text{slope} = x = \frac{\Delta y}{\Delta x}$ $\Delta x = 1$, $\Delta y = x$

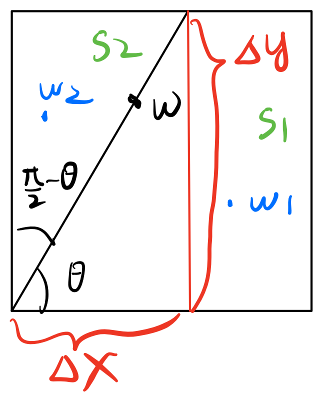
$$\mathbb{P}(X \le x) = \frac{\text{Area}(S_1)}{\text{Area}(\Omega)}$$

$\Rightarrow \text{Area}(S_1) = \frac{(\Delta x)(\Delta y)}{2} = \frac{x}{2}$

$$= \frac{1 \cdot x \cdot (1/2)}{1 \times 1} = x/2$$

C.D.F. $F_X(x) = x/2 \Rightarrow$ P.D.F $f_X(x) = \frac{1}{2}$ $0 \le x \le 1$.

(2) Slope $X(\omega) = x > 1$



$\omega_1 \in S_1$: Slope $X(\omega_1) \le X(\omega) = x$

$\omega_2 \in S_2$: slope $X(\omega_2) \ge X(\omega) = x$

So $X(\omega) \le x \Leftarrow \omega \in S_1$ and

slope $= x = \dfrac{\Delta y}{\Delta X}$, $\Delta y = 1 \Rightarrow \Delta X = \dfrac{1}{x}$

$\Rightarrow \text{Area}(S_2) = \dfrac{(\Delta X)(\Delta y)}{2} = \dfrac{1}{2X}$

$$\mathbb{P}(X \le x) = \frac{\text{Area}(S_1)}{\text{Area}(\Omega)}$$

$$= \frac{\text{Area}(\Omega) - \text{Area}(S_2)}{\text{Area}(\Omega)}$$

$$= \frac{1 - 1 \cdot \frac{1}{X} \cdot \frac{1}{2}}{1 \cdot 1} = 1 - \frac{1}{2X}$$

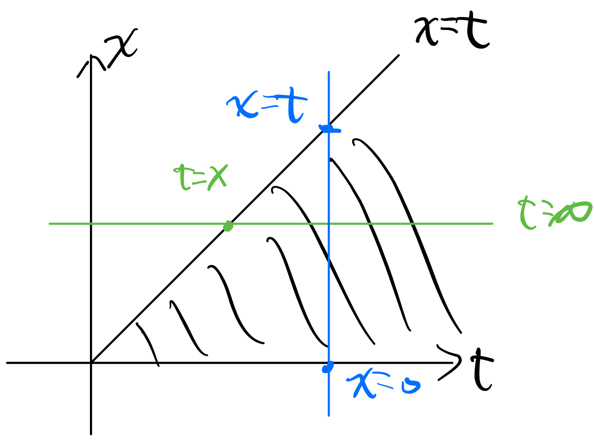
C.D.F. $F_X(X) = 1 - \dfrac{1}{2X}$ $\quad x > 1$

$\Rightarrow$ P.D.F. $f_X(X) = \dfrac{1}{2x^2}$ $\quad x > 1$

so $$f_X(X) = \begin{cases} 1/2 & 0 \le X \le 1 \\ \dfrac{1}{2X^2} & x > 1 \\ 0 & \text{o.w.} \end{cases}$$

HW4 P7 :

$\mathbb{P}(X \geq 0) = 1 \Rightarrow x > 0, \quad F_X(x) = \int_0^x f_X(t)\,dt$



$$\int_0^\infty [1 - F_X(x)]\,dx$$

$$= \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx \quad \text{by } (*)$$

$$= \int_0^\infty \int_0^t f_X(t)\,dx\,dt$$

$$= \int_0^\infty t f_X(t)\,dt$$

$$(*) \quad 1 - F_X(x) = \int_{-\infty}^{\infty} f(t)\,dt - \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_x^\infty f(t)\,dt$$