

Some Examples for Conditional Prob / Independence / Random Variables:

Ex 1, 3 Jurors & 1 Defendant

$$\text{Defendant } \begin{cases} G = \{\text{Guilty}\} & \text{w.p. } 70\% \\ G^c = \{\text{Innocent}\} & \text{w.p. } 30\% \end{cases}$$

• Given G (he is guilty):

Each juror declares guilty w.p. .7, independently

• Given G^c (he is innocent):

Each juror declares guilty w.p. .2, independently,

$$A_i = \{\text{Juror } i \text{ declares guilty}\} \quad i=1,2,3$$

Remark: Given the information, $\{A_1, A_2, A_3\}$ are conditionally independent given G or G^c , but it doesn't mean they are independent.

(a) Check that A_1, A_2, A_3 are NOT independent.

Soln: By Law of Total Prob: in fact

$$\mathbb{P}(A_1) = \mathbb{P}(A_1|G) \mathbb{P}(G) + \mathbb{P}(A_1|G^c) \mathbb{P}(G^c)$$

$$= (.7)(.7) + (.2)(.3) = .49 + .06 = .55$$

$$\text{Similarly, } \mathbb{P}(A_2) = \mathbb{P}(A_3) = .55$$

$$\begin{aligned} & \mathbb{P}(A_1 A_2 A_3) \\ &= \mathbb{P}(A_1 A_2 A_3 | G) \underbrace{\mathbb{P}(G)}_{.7} + \mathbb{P}(A_1 A_2 A_3 | G^c) \underbrace{\mathbb{P}(G^c)}_{.3} \end{aligned}$$

$$= (.7) \underbrace{\mathbb{P}(A_1|G)}_{.7} \mathbb{P}(A_2|G) \mathbb{P}(A_3|G) + (.3) \underbrace{\mathbb{P}(A_1|G^c)}_{.2} \mathbb{P}(A_2|G^c) \mathbb{P}(A_3|G^c)$$

$$= (.7)^3 (.7) + (.2)^3 (.3)$$

(conditionally Independent)

Thus, $\mathbb{P}(A_1 A_2 A_3) = .2425 \neq (.55)^3 = \mathbb{P}(A_1) \mathbb{P}(A_2) \mathbb{P}(A_3)$.

(b) what is the prob. that Juror 3 declares guilty given the other two declare?

Soln: $\mathbb{P}(A_3 | A_1 A_2) = \frac{\mathbb{P}(A_1 A_2 A_3)}{\mathbb{P}(A_1 A_2)} = \frac{.2425}{.355}$

$$\mathbb{P}(A_1 A_2) = (.7)^2 (.7) + (.2)^2 (.3) = .355$$

(c) What is the prob. that exactly TWO of them voted guilty?

Soln: $\mathbb{P}(\text{Exactly TWO of them voted guilty})$

$$= \mathbb{P}(A_1^c A_2 A_3) + \mathbb{P}(A_1 A_2^c A_3) + \mathbb{P}(A_1 A_2 A_3^c)$$

$$= 3 \times .1125 = .3375$$

where $\mathbb{P}(A_1^c A_2 A_3)$

$$= \mathbb{P}(A_1^c A_2 A_3 | G) \mathbb{P}(G) + \mathbb{P}(A_1^c A_2 A_3 | G^c) \mathbb{P}(G^c)$$

$$(C.I.) = (.3)(.7)^2 (.7) + (.8)(.2)^2 (.3) = .1125$$

Similar argument for $P(A_1 A_2^c A_3)$, $P(A_1 A_2 A_3^c)$.

(d) Calculate the prob that the defendant is guilty given juror 1 declares guilty and juror 3 declares non-guilty?

Soln: $P(G | A_1 A_3^c) = \frac{P(A_1 A_3^c | G) P(G)}{P(A_1 A_3^c)}$

where $P(A_1 A_3^c)$

$$= \underbrace{P(A_1 A_3^c | G)}_{(.7)(.3)} \underbrace{P(G)}_{(.7)} + \underbrace{P(A_1 A_3^c | G^c)}_{(.2)(.8)} \underbrace{P(G^c)}_{(.3)} \quad \boxed{13}$$

Ex2. Unreliable COVID Test

• A person $\begin{cases} H = \{\text{healthy}\} & P(H) = .95 \\ H^c = \{\text{COVID}\} & P(H^c) = .05 \end{cases}$

• Given H : $\begin{cases} \text{misclassify as COVID w.p. } .2 \\ \text{correctly classify as healthy w.p. } 1 - (.2) = .8 \end{cases}$

Given H^c : misclassify as healthy w.p. $.15$

• Doing the COVID test for three times.

$\begin{cases} H_i = \{\text{The } i\text{-th test indicates healthy}\} \quad (i=1, 2, 3) \\ N = \# \text{ of tests indicating he is healthy} \end{cases}$

(a) Does N follow Binomial Distribution? **No.**

Set up for Binomial distribution:

(1) n independent trials.

(2) For each trial: 2 outcomes $\begin{cases} \text{Success} \\ \text{Failure} \end{cases}$

(3) $\mathbb{P}(\text{success}) = p$ — same for all trials

Let $X = \#$ of successes in these n indep trials,

$$X \sim \text{Bin}(n, p), \quad \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad //$$

In this problem, you can check these three trials

H_1, H_2, H_3 are NOT indep, i.e.

$$\mathbb{P}(H_1 H_2 H_3) \neq \mathbb{P}(H_1) \mathbb{P}(H_2) \mathbb{P}(H_3).$$

which contradicts w. statement (1) in the set-up.

Thus, N does NOT follow $\text{Bin}(3, p)$.

(b) Derive the P.M.F of N .

Note that H_1, H_2, H_3 are conditionally indep given H and H^c , thus

$$N|H \sim \text{Bin}(3, .8) \quad \text{and} \quad N|H^c \sim \text{Bin}(3, .15)$$

$k = 0, 1, 2, 3$, by Law of total Prob:

$$\begin{aligned}
 \mathbb{P}(N=k) &= \mathbb{P}(N=k|H) \mathbb{P}(H) + \mathbb{P}(N=k|H^c) \mathbb{P}(H^c) \\
 &= \binom{3}{k} (.8)^k (.2)^{3-k} (.95) + \binom{3}{k} (.15)^k (.85)^{3-k} (.05)
 \end{aligned}$$

(c) Calculate $\mathbb{P}(H|N=2)$.

By Bayes Thm:

$$\mathbb{P}(H|N=2) = \frac{\mathbb{P}(N=2|H) \mathbb{P}(H)}{\mathbb{P}(N=2)} \quad \square$$