1. $X_1 \sim \exp(\beta_1)$, $X_2 \sim \exp(\beta_2)$, $X_1 \perp X_2$, $k > 0$

independent

(a) distribution of $\min\{X_1, kX_2\}$? For $x>0$:

$$\mathbb{P}(\min\{X_1, kX_2\} > x)$$

CDF: $\mathbb{P}(X_1 \le x) = 1 - e^{-\beta_1 x}$

$\mathbb{P}(X_2 \le x) = 1 - e^{-\beta_2 x}$

$x > 0$

$$= \mathbb{P}(X_1 > x,\ kX_2 > x)$$

$$= \mathbb{P}(X_1 > x) \cdot \mathbb{P}(X_2 > x/k)$$

$$= e^{-\beta_1 x} \cdot e^{-\beta_2 x/k}$$

$$= e^{-(\beta_1 + \beta_2/k)x}$$

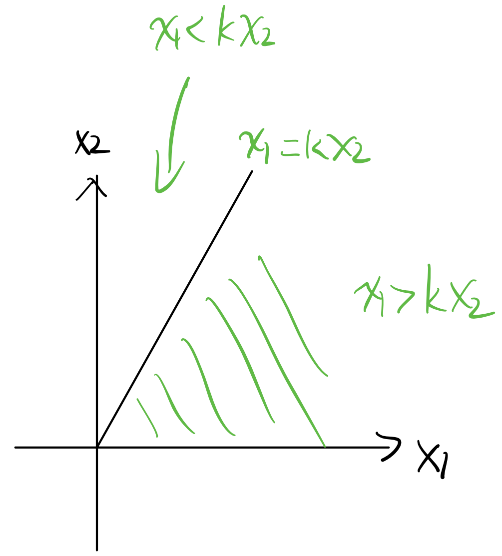
So CDF of $\min\{X_1, kX_2\}$ is

$$\mathbb{P}(\min\{X_1, kX_2\} \le x) = 1 - e^{-(\beta_1 + \beta_2/k)x}$$

Hence, $\min\{X_1, kX_2\} \sim \exp\left(\beta_1 + \frac{\beta_2}{k}\right)$

(b) Joint density of (X_1, X_2) is

$$f(x_1, x_2) = \begin{cases} \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} & x_1, x_2 > 0 \\ 0 & \text{o.w} \end{cases}$$



$$\mathbb{P}(X_1 > kX_2)$$

$$= \iint_{x_1 > kx_2} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^\infty \int_{kx_2}^\infty \beta_1 e^{-\beta_1 x_1} dx_1 \ \beta_2 e^{-\beta_2 x_2} dx_2$$

$$= \int_0^\infty e^{-\beta_1 (kx_2)} \beta_2 e^{-\beta_2 x_2} dx_2$$

$$= \beta_2 \int_0^\infty e^{-(k\beta_1 + \beta_2) x_2} dx_2$$

$$= \frac{\beta_2}{k\beta_1 + \beta_2}$$

2. X = number of toss needed to obtain five heads

$X \sim Negbin(r=5,\ p=1/30)$

(a) $\mathbb{E}[X] = \frac{r}{p} = \frac{5}{1/30} = 150$

(b) $Var(X) = \frac{r(1-p)}{p^2} = \frac{5 \times 29/30}{(1/30)^2} = 4350$

3. misprints in one page $\sim Poi(\lambda)$

$p = \mathbb{P}(\text{Exactly } k \text{ misprints}) = e^{-\lambda} \frac{\lambda^k}{k!}$

$\mathbb{P}$(At least one page contains exactly k misprints)

$= 1 - \mathbb{P}$(no pages contains exactly k misprints)

$$= 1 - (1-p)^n = 1 - \left(1 - e^{-\lambda} \frac{\lambda^k}{k!}\right)^n$$

4. X has a density

$$f(x) = \begin{cases} c x^2 e^{-4x} & x>0 \\ 0 & \text{o.w.} \end{cases}$$

MTD 1: Use Integration by part.

MTD 2: Relate this problem w. exponential dist.

Assume another R.V. $Y \sim \exp(4)$, then

$\mathbb{E}[Y] = 1/4$ $\quad$ $\mathrm{Var}(Y) = (1/4)^2$ by Table of Dist

(a) $1 = \int_0^\infty f(y)\,dy$

$$= c\int_0^\infty y^2 e^{-4y}\,dy$$

$$= \frac{1}{4}c\int_0^\infty y^2 \underbrace{4e^{-4y}}_{\text{density of } Y}\,dy$$

$$= \frac{1}{4}c\,\mathbb{E}[Y^2] = \frac{1}{4}c\left(\mathbb{E}[Y]^2 + \mathrm{Var}(Y)\right)$$

$$= \frac{c}{32}$$

so $c = 32$

(b) $\mathbb{E}\left[\frac{1}{X}\right] = 32\int_0^\infty \frac{1}{x}\cdot x^2 e^{-4x}\,dx$

$$= 8\int_0^\infty x \underbrace{4e^{-4x}}_{\text{density for } Y}\,dx$$

$$= 8\,\mathbb{E}[Y] = 8\cdot\frac{1}{4} = 2$$

$$\mathbb{E}\left[\left(\frac{1}{X}\right)^2\right] = 32\int_0^\infty \frac{1}{x^2}\cdot x^2 e^{-4x}\,dx$$

$$= 32\int_0^\infty e^{-4x}\,dx = 8$$

$$\operatorname{Var}\left(\frac{1}{X}\right) = \mathbb{E}\left[\left(\frac{1}{X}\right)^2\right] - \left(\mathbb{E}\left[\frac{1}{X}\right]\right)^2$$

$$= 8 - 2^2$$

$$= 4$$

<u>Rmk</u>: in part (a), you can see that $X \sim \text{Gamma}(3, 4)$,

so $c = \dfrac{4^3}{\Gamma(3)} = \dfrac{64}{(3-1)!} = 32$

5. For a R.V. $Z = X+Y$, with

$X \sim \text{Bern}(p=1/2)$, Y: $\mathbb{E}[Y]=10$, $Var(Y) = 2^2 = 4$

(a) $M_Z(t) = M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}]$

$= \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$ b/c X and Y are indep

$= \left(\frac{1}{2}+\frac{1}{2}e^t\right) M(t)$

(b) mean: $\frac{d}{dt}M_Z(t) = \frac{1}{2}e^t M(t) + \left(\frac{1}{2}+\frac{1}{2}e^t\right) M'(t)$

$\boxed{\mathbb{E}[Z]} = \frac{d}{dt}M_Z(t)\Big|_{t=0}$ $(\mathbb{E}[Y] = M'(0) = 10)$

$= \frac{1}{2}M(0) + \left(\frac{1}{2}+\frac{1}{2}\right)\mathbb{E}[Y] \boxed{= 10.5}$

Variance: $\mathbb{E}[Z^2] = \frac{d^2}{dt^2}M_Z(t)\Big|_{t=0}$, where

$\frac{d^2}{dt^2}M_Z(t) = \frac{1}{2}\left(e^t M(t) + e^t M'(t)\right) + \frac{1}{2}e^t M'(t)$

$+ \left(\frac{1}{2}+\frac{1}{2}e^t\right) M''(t)$

$(M''(0) = \mathbb{E}[Y^2] = Var(Y) + (\mathbb{E}Y)^2 = 4+10^2 = 104$

$$\mathbb{E}[Z^2] = \frac{1}{2}(1+10) + \frac{1}{2}\cdot 10 + \left(\frac{1}{2}+\frac{1}{2}\right)\cdot 104$$

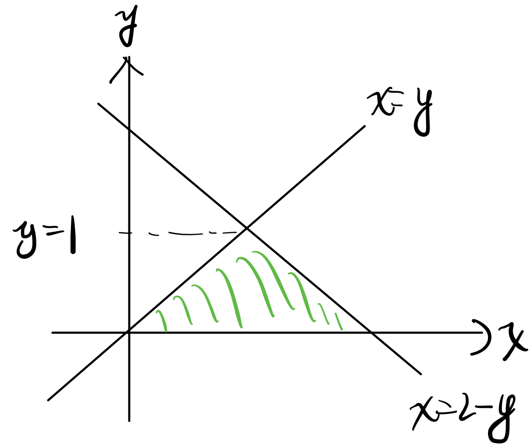
$$= 5.5 + 5 + 104 = 114.5$$

$$\boxed{\mathrm{Var}(Z)} = \mathbb{E}[Z^2] - \left(\mathbb{E}[Z]\right)^2$$

$$= 114.5 - 10.5^2 \boxed{= 4.25}$$

//

6. $f(x,y)=\begin{cases} 3(2-x)y & 0<y<1,\ y<x<2-y \\ 0 & \text{o.w.} \end{cases}$



(a) $\iint f(x,y)\,dx\,dy$

$= \int_0^1 \int_y^{2-y} (2-x)dx\, 3y\ dy$

$= \int_0^1 \{2x-\frac{1}{2}x^2\Big|_y^{2-y}\}\, 3y\ dy$

$= \int_0^1 \{2(2-y)-\frac{1}{2}(2-y)^2-2y+\frac{1}{2}y^2\}\, 3y\,dy$

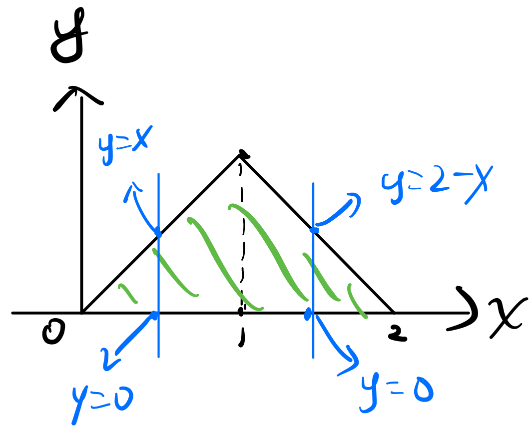
$= \int_0^1 (2-2y)\,3y\,dy$

$= \int_0^1 6y-6y^2\,dy = 3y^2-2y^3\Big|_0^1 = 1$

(b) Marginal density of X:

we derive

$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy$ piecewisely



① $x\in(0,1)$:

$\int_0^x f(x,y)dy$

$=\int_0^x 3(2-x)y\,dy = 3(2-x)\int_0^x y\,dy = \frac{3}{2}(2-x)x^2$

② $x\in(1,2)$:

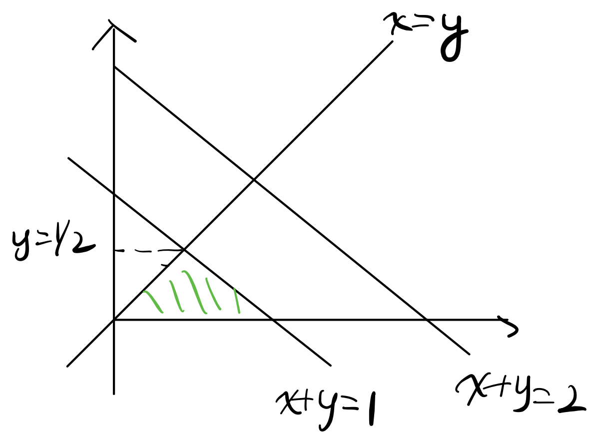
$$\int_0^{2-x} f(x,y)\,dy$$

$$= \int_0^{2-x} 3(2-x)\,y\,dy$$

$$= 3(2-x)\int_0^{2-x} y\,dy = \frac{3}{2}(2-x)^3$$

So $f_X(x) = \begin{cases} \frac{3}{2}(2-x)x^2 & x\in(0,1) \\ \frac{3}{2}(2-x)^3 & x\in(1,2) \\ 0 & \text{o.w.} \end{cases}$

(c) $\displaystyle\iint_{x+y\le 1} f(x,y)\,dx\,dy$



$$= \int_0^{1/2}\int_y^{1-y} (2-x)\,dx\,(3y)\,dy$$

$$= 3\int_0^{1/2} \left\{ 2x - \frac{1}{2}x^2 \Big|_{x=y}^{x=1-y} \right\} y\,dy$$

$$= 3\int_0^{1/2} \left(2(1-y) - \frac{1}{2}(1-y)^2 - 2y + \frac{1}{2}y^2 \right) y\,dy$$

$$= 3\int_0^{1/2} \left(\frac{3}{2} - 3y\right) y\,dy$$

$$= 3\left(\frac{3}{4}y^2 - y^3\right)\Big|_{y=0}^{y=1/2}$$

$$= \frac{3}{16}$$

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7. Let A, B, C represent each school.

X = the score of one student

$\mathbb{E}[X|A] = 80$

$\mathbb{E}[X|B] = 76$

$\mathbb{E}[X|C] = 84$

by Law of total Expectation:

$$\mathbb{E}[X]$$

$$= \mathbb{E}[X|A]\,\mathbb{P}(A) + \mathbb{E}[X|B]\,\mathbb{P}(B) + \mathbb{E}[X|C]\,\mathbb{P}(C)$$

$$= 80(.2) + 76(.3) + 84(.5)$$

$$= 80.8$$