Solutions for Suggested Problems (Conditional Distributions)

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Conditional Distribution

1. Exercise 10.1: The marginal distribution of Y is

Y	0	1	2
P	1/3	4/9	2/9

   Conditional distribution of X given $Y = y$ is $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$, thus we have

   - $p_{X|Y}(2|0) = 1$
   - $p_{X|Y}(1|1) = 1/4$, $p_{X|Y}(2|1) = 1/2$, and $p_{X|Y}(3|1) = 1/4$
   - $p_{X|Y}(2|2) = 1/2$ and $p_{X|Y}(3|2) = 1/2$

   Also, since $E[X|Y = y] = \sum_x xp_{X|Y}(x|y)$, then we have

   - $E[X|Y = 0] = 2$
   - $E[X|Y = 1] = 1(1/4) + 2(1/2) + 3(1/4) = 2$
   - $E[X|Y = 2] = 2(1/2) + 3(1/2) = 5/2$

2. Exercise 10.2: Fill in the blank of the joint distribution table of (X, Y):

   (a) Given $X = 1$, Y is uniformly distributed, and this implies that

   $$P(X = 1, Y = 0) = P(X = 1, Y = 1) = P(X = 2, Y = 1) = 1/8$$

   (b) $p_{X|Y}(0|0) = 2/3$ implies that

   $$p_{X|Y}(1|0) = \frac{1}{3}$$

   and

   $$p_Y(0) = \frac{p(1,0)}{p_{X|Y}(1|0)} = \frac{1/8}{1/3} = \frac{3}{8}$$

   So

   $$p(0,0) = p_Y(0) - p(1,0) = \frac{1}{4}$$

   (c) $P(X = 0) = 1 - P(X = 1) = 1 - 3(1/8) = 5/8$ and $P(X = 0, Y = 0) = 1/4$ implies that

   $$P(X = 0, Y = 1) + P(X = 0, Y = 2) = \frac{3}{8}$$

   (d) $E[Y|X = 0] = 1P(Y = 1|X = 0) + 2P(Y = 2|X = 0) = 4/5$ implies that

   $$P(X = 0, Y = 1) + 2P(X = 0, Y = 2) = \frac{4}{5}P(X = 0) = \frac{1}{2}$$

   (e) We can solve for

   $$P(X = 0, Y = 1) = \frac{1}{4}$$

   $$P(X = 0, Y = 2) = \frac{1}{8}$$

3. Exercise 10.5: The joint density of (X, Y) is

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

(a) To figure out $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ for $y \in (0,1)$, we need to obtain marginal density f_Y of Y, which is

$$f_Y(y) = \int f(x,y)dx = \frac{12}{5}\int_0^1 x(2-x-y)dx = \frac{8}{5} - \frac{6}{5}y$$

for $0 < y < 1$. Then we have

$$f_{X|Y}(x|y) = \frac{12x(2-x-y)}{8-6y}$$

(b) First we obtain

$$f_{X|Y}\left(x\middle|\frac{3}{4}\right) = \frac{24}{7}\left(\frac{5}{4}x - x^2\right)$$

by taking $y = 3/4$ in the formula of conditional density. Then the calculation should be

$$P\left(x > \frac{1}{2}\middle|Y = \frac{3}{4}\right) = \int_{1/2}^1 \frac{24}{7}\left(\frac{5}{4}x - x^2\right)dx = \frac{17}{28}$$

and

$$E\left[X\middle|Y = \frac{3}{4}\right] = \frac{24}{7}\int_0^1 x\left(\frac{5}{4}x - x^2\right)dx = \frac{4}{7}$$

4. Exercise 10.9: Joint density of (X, Y) is

$$f(x,y) = \begin{cases} \frac{1}{y}e^{-x/y}e^{-y} & x > 0, y > 0 \\ 0 & \text{Otherwise} \end{cases}$$

Note that for any $\alpha > 0$, we have $\int_0^\infty \alpha e^{-\alpha x}dx = 1(*)$ and $\int_0^\infty \alpha x e^{-\alpha x}dx = 1/\alpha(**)$, and this formula can make the calculation faster.

(a) Using the formula $(*)$ above, we have

$$f_Y(y) = e^{-y}$$

for $y > 0$, so conditional density is

$$f_{x|y}(x|y) = \frac{1}{y}e^{-x/y}$$

for $x, y > 0$.

(b) Using $(**)$:

$$E[X|Y = y] = \int_0^\infty \frac{x}{y}e^{-x/y}dx = y$$

So $E[X|Y] = Y$.

(c) Using $(**)$:

$$E[X] = E[E[X|Y]] = E[Y] = \int_0^\infty ye^{-y}dy = 1$$

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu