PSTAT 120A Quiz 3

1. (4 points) Consider the event of rolling two (fair) 6-sided dice, one time. Calculate the expectation for the larger (maximum) value of the two dice.

See the next Page !!!

$X \sim N(50, 15^2)$

2. (4 points) For a certain type of computer, the length of time between needing to charge of the battery is normally distributed with a mean of 50 hours and a standard deviation of 15 hours. John owns one of these computers and the length of time is recorded.

$\sigma = 15 \rightarrow \sigma^2 = 15^2$

   (a) (2 points) What is the probability that the time is more than 40 hours?

$$\mathbb{P}(X \geq 40) = \mathbb{P}\left(\frac{X-50}{15} \geq \frac{40-50}{15}\right) = \mathbb{P}\left(Z \geq -\frac{2}{3}\right) = .7475$$

(Fact: $Z \sim N(0,1)$, $\mathbb{P}(Z \geq z) = \mathbb{P}(Z \leq -z)$)

   (b) (2 points) What is the probability that the time is between 50 and 70 hours?

$$\mathbb{P}(X \in [50, 70])$$

$$= \mathbb{P}\left(\frac{50-50}{15} \leq \frac{X-50}{15} \leq \frac{70-50}{15}\right)$$

$$= \Phi\left(\frac{4}{3}\right) - \Phi(0) = .4088$$

Solution to P1: X = Maximum value of two dices

Step 1: Derive C.D.F of X

for $k \in S_X = \{1, 2, 3, 4, 5, 6\}$

$$\{X \le k\} = \{\text{Both values} \le k\}$$
$$= \underbrace{\{\#\text{ of } 1^{st}\text{ dice} \le k\} \cap \{\#\text{ of } 2^{nd}\text{ dice} \le k\}}_{\text{indep}}$$

$$\mathbb{P}(X \le k) = \left(\frac{k}{6}\right)^2 \quad k \in S_X$$

Step 2: CDF ⇒ PMF

$k=1$: $\mathbb{P}(X=1) = \mathbb{P}(X \le 1) = (1/6)^2 = \frac{1}{36}$

$k \in S_X \setminus \{1\}$: $\mathbb{P}(X=k) = \mathbb{P}(X \le k) - \mathbb{P}(X \le k-1)$

$$\underset{\text{(Step 1)}}{=} \left(\frac{k}{6}\right)^2 - \left(\frac{k-1}{6}\right)^2$$

$$\mathbb{P}(X=2) = \frac{4}{36} - \frac{1}{36} = \frac{3}{36}$$

$$\mathbb{P}(X=3) = \frac{9}{36} - \frac{4}{36} = \frac{5}{36}$$

$$\mathbb{P}(X=4) = \frac{16}{36} - \frac{9}{36} = \frac{7}{36}$$

$$\mathbb{P}(X=5) = \frac{25}{36} - \frac{16}{36} = \frac{9}{36}$$

$$\mathbb{P}(X=6) = 1 - \frac{25}{36} = \frac{11}{36}$$

Step 3: PMF ⇒ $\mathbb{E}[X]$

Use formula $\mathbb{E}[X] = \sum_{k=1}^{6} k\,\mathbb{P}(X=k) = \frac{161}{36}$ (Check)

3. (3 points) Five friends ordered a pizza and do not want to pay separately. They decide to play a game. Everybody flips a fair coin and if one player has a different result than the four others, then he/she has to pay. If there is not exactly one person with a differing result, then the game will be repeated. After how many rounds will a decision be reached on average?

In each round, denote $E = \{$ one player has a different result than four others $\} = \{4H1T \text{ or } 1H4T\}$.

(Step 1) $\mathbb{P}(E) = \mathbb{P}(4H1T) + \mathbb{P}(1H4T)$

$$= \binom{5}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^1 + \binom{5}{1}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^1 = \frac{2\times 5}{32} = \frac{5}{16}$$

(Step 2) Denote X = # of rounds we needed until E happens

$X \sim Geo(\mathbb{P}(E) = 5/16)$

$\Rightarrow$ by Table of Distribution: $\mathbb{E}[X] = \frac{1}{5/16} = \frac{16}{5}$ //

4. (2 points) The random variable X has the following probability mass function:

$$P(X=0) = \tfrac{3}{4}, \qquad P(X=k) = \tfrac{1}{2}\left(\tfrac{1}{3}\right)^k \text{ for integer } 1 \le k$$

Compute $E(X)$. (Hint : $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$)

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \frac{1}{2}\left(\frac{1}{3}\right)^k = \frac{1}{2}\cdot\frac{1}{3}\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k-1} = \frac{1}{6}\cdot\frac{1}{\left(1-\frac{1}{3}\right)^2} = \frac{3}{8}$$

Rmk; Geometric series : for $|x|<1$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

take derivative for both sides, we obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

Optional Exercise: Show that $\sum_{k=0}^{\infty} \mathbb{P}(X=k) = 1$

i.e. Show that this is a valid p.m.f

Hint: Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\overbrace{28, 23, 32, 19 \text{ and } 18}^{120 \text{ in total}}$$

5. (3 points) At UCSB, there are five sports clubs, with 28, 23, 32, 19 and 18 members respectively. We choose one of the club members at random and denote the number of members in her club by X. Find the expectation of X.

$$\mathbb{P}(X=28)=\frac{28}{120}$$

$$\mathbb{P}(X=23)=\frac{23}{120}$$

$$\mathbb{P}(X=32)=\frac{32}{120}$$

$$\mathbb{P}(X=19)=\frac{19}{120}$$

$$\mathbb{P}(X=18)=\frac{18}{120}$$

$$\mathbb{E}[X]=28\times\frac{28}{120}+23\times\frac{23}{120}+32\times\frac{32}{120}$$

$$+19\times\frac{19}{120}+18\times\frac{18}{120}=\frac{3022}{120}$$

$$=\frac{1511}{60}=25.1833 \quad //$$