Some Examples for Conditional Prob / Independence / Random Variables:

__Ex 1__. 3 Jurors & 1 Defendant

$$\text{Defendant} \begin{cases} G = \{\text{Guilty}\} & \text{w.p. } 70\% \\ G^c = \{\text{Innocent}\} & \text{w.p. } 30\% \end{cases}$$

- Given G (he is guilty):

  Each juror declares guilty w.p. .7, independently

- Given G^c (he is innocent):

  Each juror declares guilty w.p. .2, independently.

$A_1 = \{\text{Juror 1 declares guilty}\}$

$A_2 = \{\text{Juror 2 declares guilty}\}$

$A_3 = \{\text{Juror 3 declares guilty}\}$

(a) Check that A_1, A_2, A_3 are NOT independent.

__Soln__: By Law of Total Prob:

$$\mathbb{P}(A_1) = \mathbb{P}(A_1|G)\,\mathbb{P}(G) + \mathbb{P}(A_1|G^c)\,\mathbb{P}(G^c)$$

$$= (.7)(.7) + (.2)(.3) = .49 + .06 = .55$$

Similarly, $\mathbb{P}(A_2) = \mathbb{P}(A_3) = .55$

$$\mathbb{P}(A_1A_2A_3)$$

$$= \mathbb{P}(A_1A_2A_3|G)\,\overbrace{\mathbb{P}(G)}^{.7} + \mathbb{P}(A_1A_2A_3|G^c)\,\overbrace{\mathbb{P}(G^c)}^{.3}$$

$= (.7)\underbrace{\mathbb{P}(A_1|G)}_{.7}\mathbb{P}(A_2|G)\mathbb{P}(A_3|G) + (.3)\underbrace{\mathbb{P}(A_1|G^c)}_{.2}\mathbb{P}(A_2|G^c)\mathbb{P}(A_3|G^c)$

$= (.7)^3(.7) + (.2)^3(.3)$

(conditionally Independent)

Thus, $\mathbb{P}(A_1A_2A_3) \doteq .2425 \neq (.55)^3 = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.

(b) What is the prob. that Juror 3 declares guilty given the other two declare?

Soln: $\mathbb{P}(A_3|A_1A_2) = \dfrac{\mathbb{P}(A_1A_2A_3)}{\mathbb{P}(A_1A_2)} = \dfrac{.2425}{.355}$

$\mathbb{P}(A_1A_2) = (.7)^2(.7) + (.2)^2(.3) = .355$

(c) What is the prob. that exactly TWO of them voted guilty?

Soln: $\mathbb{P}$(Exactly TWO of them voted guilty)

$= \mathbb{P}(A_1^cA_2A_3) + \mathbb{P}(A_1A_2^cA_3) + \mathbb{P}(A_1A_2A_3^c)$

$= 3 \times .1125 = .3375$

where $\mathbb{P}(A_1^cA_2A_3)$

$= \mathbb{P}(A_1^cA_2A_3|G)\mathbb{P}(G) + \mathbb{P}(A_1^cA_2A_3|G^c)\mathbb{P}(G^c)$

(C.I.) $= (.3)(.7)^2(.7) + (.8)(.2)^2(.3) = .1125$

Similar argument for $\mathbb{P}(A_1A_2^cA_3)$, $\mathbb{P}(A_1A_2A_3^c)$.

(d) Calculate the prob that the defendant is guilty given juror 1 declares guilty and juror 3 declares non-guilty?

Soln: $$\mathbb{P}(G \mid A_1A_3^c) = \frac{\mathbb{P}(A_1A_3^c \mid G)\,\mathbb{P}(G)}{\mathbb{P}(A_1A_3^c)}$$

where $\mathbb{P}(A_1A_3^c)$

$$= \underbrace{\mathbb{P}(A_1A_3^c \mid G)}_{(.7)(.3)}\,\underbrace{\mathbb{P}(G)}_{(.7)} + \underbrace{\mathbb{P}(A_1A_3^c \mid G^c)}_{(.2)(.8)}\,\underbrace{\mathbb{P}(G^c)}_{(.3)}$$

Ex2. Unreliable COVID Test

- A person $\begin{cases} H = \{\text{healthy}\} & \mathbb{P}(H) = .95 \\ H^c = \{\text{COVID}\} & \mathbb{P}(H^c) = .05 \end{cases}$

- Given H: $\begin{cases} \text{misclassify as COVID w.p. } .2 \\ \text{correctly classify as healthy w.p. } 1-(.2) = .8 \end{cases}$

  Given H^c: misclassify as healthy w.p. $.15$

- Doing the COVID test for three times.

$$\begin{cases} H_i = \{\text{The } i\text{-th test indicates healthy}\} \ (i=1,2,3) \\ N = \#\text{ of tests indicating he is healthy} \end{cases}$$

(a) Dose N follows Binomial Distribution? No.

Set up for Binomial distribution:

(1) n independent trials.

(2) For each trial: 2 outcomes $\begin{cases} \text{Success} \\ \text{Failure} \end{cases}$

(3) $\mathbb{P}(\text{success}) = p$ — same for all trials

Let X = # of successes in these n indep trials,

$X \sim Bin(n, p)$, $\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ "

In this problem, you can check these three trials

H_1, H_2, H_3 are NOT indep, i.e.

$$\mathbb{P}(H_1 H_2 H_3) \neq \mathbb{P}(H_1)\,\mathbb{P}(H_2)\,\mathbb{P}(H_3).$$

which contradicts w. statement (1) in the set-up.

Thus, N does NOT follow $Bin(3, p)$.

(b) Derive the P.M.F of N.

Note that H_1, H_2, H_3 are conditionally indep given H and H^c, thus

$$N \mid H \sim Bin(3, .8) \quad \text{and} \quad N \mid H^c \sim Bin(3, .15)$$

$k = 0, 1, 2, 3$, by Law of total Prob:

$$\mathbb{P}(N=k) = \mathbb{P}(N=k \mid H)\,\mathbb{P}(H) + \mathbb{P}(N=k \mid H^c)\,\mathbb{P}(H^c)$$

$$= \binom{3}{k} (.8)^k (.2)^{3-k} (.95) + \binom{3}{k} (.15)^k (.85)^{3-k} (.05)$$

(c) Calculate $\mathbb{P}(H \mid N=2)$.

By Bayes Thm:

$$\mathbb{P}(H \mid N=2) = \frac{\mathbb{P}(N=2 \mid H)\,\mathbb{P}(H)}{\mathbb{P}(N=2)}$$