

Random Variables (Solutions)

Botao Jin

University of California, Santa Barbara — October 26, 2024

Discrete Random Variables

See exercise 3.5, 2.38

1. Exercise 3.5: The support for the random variable X is $\{1, 4/3, 3/2, 9/5\}$ with p.m.f:

X	1	$4/3$	$3/2$	$9/5$
P	$1/3$	$1/6$	$1/4$	$1/4$

2. Exercise 2.38:

- (a) Use the law of total probability:

$$\begin{aligned}P(R) &= P(R|SOME)P(SOME) + P(R|DOGS)P(DOGS) \\&\quad + P(R|ARE)P(ARE) + P(R|BROWN)P(BROWN) \\&= 0 + 0 + \frac{1}{3} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{2}{15}\end{aligned}$$

- (b) Support of X is $S_X = \{3, 4, 5\}$, the probability mass function is

X	3	4	5
P	$1/4$	$1/2$	$1/4$

- (c) Note that

- $\{X > 3\} = \{SOME, DOGS, BROWN\}$
- $\{X = 4\} = \{SOME, DOGS\}$
- $\{X = 5\} = \{BROWN\}$

By definition of conditional probability, we obtain

$$P(X = 3|X > 3) = 0$$

$$P(X = 4|X > 3) = \frac{2}{3}$$

$$P(X = 5|X > 3) = \frac{1}{3}$$

- (d) Note that $\{X = k\}$ for $k = 3, 4, 5$ form a partition for the whole sample space Ω , then followed from hint, we have

$$\begin{aligned}P(R|X > 3) &= \sum_{k=3}^5 P(R \cap \{X = k\}|X > 3) \\&= P(R \cap \{X = 4\}|X > 3) + P(R \cap \{X = 5\}|X > 3) \\&= P(R|X = 4)P(X = 4|X > 3) + P(R|X = 5)P(X = 5|X > 3)\end{aligned}$$

where

$$\begin{aligned} P(R|X=4) &= P(R \cap \text{SOME}|X=4) + P(R \cap \text{DOGS}|X=4) \\ &= P(R|\text{SOME})P(\text{SOME}|X=4) + P(R|\text{DOGS})P(\text{DOGS}|X=4) \\ &= 0 \end{aligned}$$

and

$$P(R|X=5) = P(R|\text{BROWN}) = \frac{1}{5}$$

Using the result from part (c), we have

$$P(R|X > 3) = \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}$$

(e) By Bayes formula:

$$\begin{aligned} P(\text{BROWN}|R) &= \frac{P(R|\text{BROWN})P(\text{BROWN})}{P(R)} \\ &= \frac{(1/5)(1/4)}{2/15} \\ &= \frac{3}{8} \end{aligned}$$

Continuous Random Variables

See exercise 3.7, 3.25

1. Exercise 3.7: Since X is a continuous random variable, we have $P(a \leq X \leq b) = F(b) - F(a)$

(a) $F(b) - F(a) = P(a \leq X \leq b) = 1$ implies that $F(b) = 1$ and $F(a) = 0$, which further implies $b \geq \sqrt{3}$ and $a \leq \sqrt{2}$, thus the smallest interval is $[\sqrt{2}, \sqrt{3}]$

(b)

$$\begin{aligned} P(X = 1.6) &= P(X \leq 1.6) - P(X < 1.6) \\ &= F(1.6) - \lim_{x \rightarrow 1.6^-} F(x) = 0 \end{aligned}$$

The last equality holds since the cumulative density function F is continuous for any x .

Remark. For any continuous random variable X , the probability that X takes on one special value is equal to zero.

(c) Since X is a continuous Random Variable, we have $P(X = 1) = 0$ so that $P(1 \leq X \leq 3/2) = P(1 < X \leq 3/2)$, hence

$$P(1 \leq X \leq 3/2) = F(3/2) - F(1) = (3/2)^2 - 2 - 0 = .25$$

(d) Since F is continuous, and it is differential apart from $\sqrt{2}$ and $\sqrt{3}$. So, we can differentiate F to obtain the density:

$$f(x) = \begin{cases} 2x & x \in (\sqrt{2}, \sqrt{3}) \\ 0 & \text{otherwise} \end{cases}$$

2. Exercise 3.25:

(a) If f is a p.d.f, then

$$1 = \int_{-\infty}^{\infty} f(x) dx = \int_1^3 (x^2 - b) dx = \frac{1}{3}x^3 - bx \Big|_1^3 = \frac{26}{3} - 2b$$

Then, we solve for $b = \frac{23}{6}$. However, if $x \in [1, \sqrt{23/6})$, then $f(x) < 0$, which volates with the definition the probability density function. Thus, there is no b such that f is a valid density.

- (b) Now, we know that $\cos x \geq 0$ whenever $-\pi/2 \leq x \leq \pi/2$, so if we want f is a valid density function, we need $b \in (0, \pi/2]$. By computing the integral, we have

$$1 = \int_{-\infty}^{\infty} h(x) dx = \int_{-b}^b \cos x dx = 2 \sin b$$

Thus, we have $\sin b = \frac{1}{2}$, and therefore, $b = \frac{\pi}{6}$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu