Combined Section on Distributions and Expectations

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See exercise: $3.18, 3.51 - 52, 5.12, 5.16, 5.24$

1. Exercise 3.18: $X \sim N(3,4)$ and let Φ be the distribution function (C.D.F) of $Z = \frac{X-3}{2} \sim N(0,1)$:

   (a) We have

$$
\begin{aligned}
P(2 < X < 6) &= P\left(\frac{2-3}{2} < \frac{X-3}{2} < \frac{6-3}{2}\right) \\
&= \Phi(1.5) - \Phi(-.5) \\
&= \Phi(1.5) - 1 + \Phi(.5) \\
&= .6247
\end{aligned}
$$

   (b) We have

$$
P(X > c) = P\left(\frac{X-3}{2} > \frac{c-3}{2}\right) = 1 - \Phi\left(\frac{c-3}{2}\right) = .33
$$

   By normal table, we solved for $c = 3.88$

   (c) $E[X^2] = \mathrm{Var}(X) + (E[X])^2 = 4 + 9 = 13$

2. Exercise $3.51 - 52$: For $X \sim Geo(p)$, we can calculate the expected values as follows:

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kP(X = k) \\
&= \sum_{k=1}^{n} k(1-p)^{k-1} p \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k} (1-p)^{k-1} p \\
&= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} (1-p)^{k-1} p \\
&= \sum_{j=1}^{\infty} p(1-p)^{j-1} \sum_{k=j}^{\infty} (1-p)^{k-j} \\
&= \sum_{j=1}^{\infty} p(1-p)^{j-1} \frac{1}{1-(1-p)} \\
&= \sum_{j=1}^{\infty} (1-p)^{j-1} \\
&= \frac{1}{1-(1-p)} = \frac{1}{p}
\end{aligned}
$$

Here, we used the formula $\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}$ for $|p| < 1$. In general, the tailed formula for expected value can be derived as follows:

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} P(X = k) \\ &= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} P(X = k) \\ &= \sum_{j=1}^{\infty} P(X \geq j) \end{aligned}$$

Extra practice: see exercise 3.54.

3. Exercise 5.12: Please use the formula

$$\int_0^{\infty} x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3}$$

for all $\alpha > 0$.

Now, to calculate the moment generating function of X, set $\alpha = 1 - t$, we have:

$$\begin{aligned} M(t) &= \int_0^{\infty} f(x) e^{tx} dx \\ &= \int_0^{\infty} \frac{1}{2} x^2 e^{-x} e^{tx} dx \\ &= \int_0^{\infty} \frac{1}{2} x^2 e^{(t-1)x} dx \\ &= \begin{cases} \frac{1}{(1-t)^3} & t < 1 \\ \infty & t \geq 1 \end{cases} \end{aligned}$$

4. Exercise 5.16:

(a) We have $E[X^n] = \int_0^1 x^n dx = \frac{1}{n+1}$.

(b) By exercise 5.3 (or check by yourself), the moment generating function is

$$M(t) = \begin{cases} 1 & t = 0 \\ \frac{e^t - 1}{t} & t \neq 0 \end{cases}$$

Given the Taylor expansion of exponential function is $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!}$. Thus,

$$\begin{aligned} M(t) &= \frac{e^t - 1}{t} \\ &= \frac{1}{t} \sum_{k=1}^{\infty} \frac{t^k}{k!} \\ &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} = \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \frac{t^k}{k!} \end{aligned}$$

Note that the expansion of moment generating function is

$$M(t) = E[e^{tX}] = E\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^k E[X^k]}{k!}$$

We can derive that $E[X^k] = \frac{1}{k+1}$ by comparing the coeffient of t^k.

5. Exercise 5.24: $X \sim \mathcal{N}(0,1)$ and $Y = e^X$. Then

   (a) The distribution function of Y can be calculated as

$$\begin{aligned} F_Y(t) &= P(Y \le t) \\ &= P(e^X \le t) \\ &= P(X \le \log(t)) \\ &= F_X(\log(t)) \end{aligned}$$

   Then, the density of Y can be obtained by

$$\begin{aligned} f_Y(y) &= \frac{d}{dt} F_X(\log(t)) \\ &= \frac{1}{t} f_X(\log(t)) \\ &= \begin{cases} \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{(\log(t))^2}{2}\right) & t > 0 \\ 0 & t \le 0 \end{cases} \end{aligned}$$

   (b) We can see that $E[Y^n] = E[e^{nX}] = M_X(n)$ is the moment generating function of X evaluate at n, then $E[Y^n] = e^{n^2/2}$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu