Week 10 Study Guide (Solution)

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Covariance

1. Exercise **8.14**: The Marginal density of X is

X	1	2	3
P	1/3	1/2	1/6

The Marginal density of Y is

Y	0	1	2	3
P	1/5	1/5	1/3	4/15

Thus $E[X] = 11/6$, $E[X^2] = 23/6$, $Var(X) = 17/36$ and
$E[Y] = 5/3$, $E[Y^2] = 59/15$, $Var(Y) = 52/45$.

$$E[XY] = \sum_{x=1}^{3} \sum_{y=0}^{3} xyP(X = x, Y = y) = 47/15$$

Thus $Cov(X, Y) = E[XY] - E[X]E[Y] = 47/15 - (11/6)(5/3) = 7/90$ and

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \approx .1053$$

2. Exercise **8.16**: $E[X] = 1$, $E[X^2] = 3$, $E[XY] = -4$, and $E[Y] = 2$.

$$\begin{aligned} Cov(X, 2X + Y - 3) &= 2Cov(X, X) + Cov(X, Y) - Cov(X, 3) \\ &= 2Var(X) + Cov(X, Y) \\ &= 2(E[X^2] - (E[X])^2) + E[XY] - E[X]E[Y] \\ &= 2 \cdot (3 - 1) + (-4) - 1 \cdot 2 = -2 \end{aligned}$$

3. Exercise **8.17**: Given $P(A) = .5$, $P(B) = .2$, and $P(AB) = .1$, we have

$$\mathrm{Var}(X) = \mathrm{Var}(I_A) + \mathrm{Var}(I_B) + 2\mathrm{Cov}(I_A, I_B)$$

where $\mathrm{Var}(I_A) = (.5)(1 - .5) = .25$, $\mathrm{Var}(I_B) = (.2)(1 - .2) = .16$, and $\mathrm{Cov}(I_A, I_B) = E[I_A I_B] - E[I_A]E[I_B] = P(AB) - P(A)P(B) = 0$. Thus, $\mathrm{Var}(X) = .41$.

Conditional Distributions

See **Exercise 10.2, 10.8** in the textbook and the **Extra Practice Problem(s) below**:

1. Suppose that 20 percent of the students who took a certain test were from school A and that the arithmetic average of their scores on the test was 80. Suppose also that 30 percent of the students were from school B and that the arithmetic average of their scores was 76. Suppose, finally, that the other 50 percent of the students were from school C and that the arithmetic average of their scores was 84. If a student is selected at random from the entire group that took the test, what is the expected value of her score? Hint: Use conditional expectation.

   Solution: By Law of total expectation (or tower properties of expectation), let X be a score of a randomly chosen student, we have

$$\begin{aligned} E[X|A] &= 80, \\ E[X|B] &= 76, \\ E[X|C] &= 84. \end{aligned}$$

   Also, we have 20 percent of students come from school A, 30 percent of students come from school B, and 50 percent of students come from school C, which implies that $P(A) = .2$, $P(B) = .3$, and $P(C) = .5$. Therefore,

$$\begin{aligned} E[X] &= E[X|A]P(A) + E[X|B]P(B) + E[X|C]P(C) \\ &= 80(.2) + 76(.3) + 84(.5) \\ &= 80.8. \end{aligned}$$

2. Exercise **10.2**: Fill in the blank of the joint distribution table of (X, Y):

   (a) Given $X = 1$, Y is uniformly distributed, and this implies that

$$P(X = 1, Y = 0) = P(X = 1, Y = 1) = P(X = 2, Y = 1) = 1/8$$

   (b) $p_{X|Y}(0|0) = 2/3$ implies that

$$p_{X|Y}(1|0) = \frac{1}{3}$$

   and

$$p_Y(0) = \frac{p(1,0)}{p_{X|Y}(1|0)} = \frac{1/8}{1/3} = \frac{3}{8}$$

   So

$$p(0,0) = p_Y(0) - p(1,0) = \frac{1}{4}$$

   (c) $P(X = 0) = 1 - P(X = 1) = 1 - 3(1/8) = 5/8$ and $P(X = 0, Y = 0) = 1/4$ implies that

$$P(X = 0, Y = 1) + P(X = 0, Y = 2) = \frac{3}{8}$$

   (d) $E[Y|X = 0] = 1P(Y = 1|X = 0) + 2P(Y = 2|X = 0) = 4/5$ implies that

$$P(X = 0, Y = 1) + 2P(X = 0, Y = 2) = \frac{4}{5}P(X = 0) = \frac{1}{2}$$

   (e) We can solve for

$$\begin{aligned} P(X = 0, Y = 1) &= \frac{1}{4} \\ P(X = 0, Y = 2) &= \frac{1}{8} \end{aligned}$$

3. Exercise **10.8**:

   (a) Given $X = x$, we have Y follows from Binomial distribution with parameter x (we sample the ball for x times) and $4/9$ (4 green balls and 5 red balls). Thus, $E[Y|X = x] = \frac{4x}{9}$, and thus $E[Y|X] = \frac{4}{9}X$.

   (b) By law of total expectation (tower properties), we have

$$E[Y] = E[E[Y|X]] = \frac{4}{9}E[X] = \frac{4}{9} \cdot 6 = \frac{8}{3}$$

   since X follows from geometric distribution with $p = 1/6$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu