Week 9 Study Guide (Solution)

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Joint Distributions

1. Discrete case:

   (1) Exercise **6.2**: We have

   a. The marginal p.m.f of X is

X	1	2	3
P	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

   and the marginal p.m.f of Y is

Y	0	1	2	3
P	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{4}{15}$

   b. We need to find the ordered pairs (x, y) in the support of X and Y such that $x + y^2 \leq 2$. Thus,

$$\begin{aligned}P(X + Y^2 \leq 2) &= P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 2, Y = 0) \\ &= \frac{1}{15} + \frac{1}{15} + \frac{1}{10} \\ &= \frac{7}{30}\end{aligned}$$

   (2) Exercise **6.19**:

   a. Marginal distribution of X is

X	0	1
P	1/3	2/3

   and marginal distribution of Y is

Y	0	1	2
P	1/6	1/3	1/2

   b. $p(z, w) = P(Z = z, W = w) = f_X(z)f_Y(w)$ for f_X and f_Y are marginal p.m.f of X and Y, respectively.

2. Continuous case:

   (1) Let the random variables X, Y have joint density

$$f(x, y) = \begin{cases} 3(2 - x)y & \text{if } 0 < y < 1 \text{ and } y < x < 2 - y \\ 0 & \text{otherwise} \end{cases}$$

   a. Verify that it is a valid joint density function.
   b. Derive the marginal density for X.
   c. Calculate

$$P(X + Y \leq 1)$$

   Solution:

a. By properties of joint density functions:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1 \left(\int_y^{2-y} (2-x)dx\right) 3ydy \\
&= \int_0^1 \left(2x - \frac{1}{2}x^2 \Big|_{x=y}^{x=2-y}\right) \\
&= \int_0^1 (2-2y)\cdot 3ydy \\
&= \int_0^1 6y - 6y^2 dy = 1.
\end{aligned}$$

b. To derive the marginal density for X, we use the formula $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$ and separate our argument by cases:

i. When $x \in (0,1)$,

$$\begin{aligned}
f_X(x) &= \int_0^x f(x,y)dy \\
&= \int_0^x 3(2-x)ydy \\
&= 3(2-x)\int_0^x ydy \\
&= \frac{3}{2}(2-x)x^2.
\end{aligned}$$

ii. When $x \in (1,2)$,

$$\begin{aligned}
f_X(x) &= \int_0^{2-x} f(x,y)dy \\
&= \int_0^{2-x} 3(2-x)ydy \\
&= 3(2-x)\int_0^{2-x} ydy \\
&= \frac{3}{2}(2-x)^3.
\end{aligned}$$

Please note that you can check the bounds for integration by drawing a graph.

c. To find the value of $P(X+Y \leq 1)$, we need

$$\begin{aligned}
P(X+Y\leq 1) &= \iint_{x+y\leq 1} f(x,y)dxdy \\
&= 3\int_0^{1/2}\left(\int_y^{1-y} 2 - xdx\right) ydy \\
&= 3\int_0^{1/2}\left(2x - \frac{1}{2}\Big|_{x=y}^{x=1-y}\right) ydy \\
&= 3\int_0^{1/2}\left(\frac{3}{2} - 3y\right) ydy \\
&= \frac{3}{16}
\end{aligned}$$

(2) Exercise **6.5**: $f(x,y) = \frac{12}{7}(xy+y^2)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and zero otherwise.

a. Just to check that $\iint_{\mathbb{R}^2} f(x,y)dxdy = 1$

Proof.

$$\begin{aligned}
\iint_{\mathbb{R}^2} f(x,y)dxdy &= \int_0^1 \int_0^1 \frac{12}{7}(xy+y^2)dxdy \\
&= \frac{12}{7}\int_0^1 \frac{1}{2}x^2y + xy^2\Big|_{x=0}^{x=1} dy \\
&= \frac{12}{7}\int_0^1 \frac{1}{2}y + y^2 dy \\
&= \frac{12}{7}\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\right) = 1
\end{aligned}$$

□

b. Marginal for X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy = \int_0^1 \frac{12}{7}(xy+y^2)dy = \frac{12}{7}(x/2+1/3) = \frac{6x}{7}+\frac{4}{7}$$

for $x \in (0,1)$

Marginal for Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)dx = \int_0^1 \frac{12}{7}(xy+y^2)dx = \frac{12}{7}(y/2+y^2) = \frac{6y}{7}+\frac{12y^2}{7}$$

for $y \in (0,1)$

c.

$$\begin{aligned}
P(X<Y) &= \int_0^1 \int_0^y \frac{12}{7}(xy+y^2)dxdy \\
&= \frac{12}{7}\int_0^1 \frac{x^2}{2}y + xy^2\Big|_{x=0}^{x=y} dy \\
&= \frac{12}{7}\cdot\frac{3}{2}\int_0^1 y^3 dy \\
&= \frac{12}{7}\cdot\frac{3}{2}\cdot\frac{1}{4} = \frac{9}{14}
\end{aligned}$$

d.

$$\begin{aligned}
E[X^2Y] &= \frac{12}{7}\int_0^1 \int_0^1 x^2y(xy+y^2)dxdy \\
&= \frac{12}{7}\int_0^1 \int_0^1 (x^3y^2+x^2y^3)dxdy \\
&= \frac{12}{7}\int_0^1 \frac{1}{4}y^2+\frac{1}{3}y^3 dy \\
&= \frac{12}{7}\left(\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}\right) = \frac{2}{7}
\end{aligned}$$

(3) Exercise **6.35**: $f_{X,Y}(x,y) = \frac{1}{4}(x+y)$ for $0 \le x \le y \le 2$

a. Check that f satisfies $\iint_{\mathbb{R}} f(x,y)dxdy = 1$.

Proof.

$$\begin{aligned}
\iint_{\mathbb{R}} f(x,y)dxdy &= \frac{1}{4}\int_0^2 \int_0^y x+ydxdy \\
&= \frac{1}{4}\int_0^2 \frac{1}{2}x^2+xy\Big|_{x=0}^{x=y} dy \\
&= \frac{1}{4}\cdot\frac{3}{2}\int_0^2 y^2dy \\
&= \frac{1}{4}\cdot\frac{3}{2}\cdot\frac{8}{3} = 1
\end{aligned}$$

□

b.

$$\begin{aligned}P(Y<2X) &= \frac{1}{4}\int_0^2\int_{y/2}^{y} x+y dx dy\\ &= \frac{1}{4}\int_0^2 xy+\frac{1}{2}x^2\Big|_{x=y/2}^{x=y} dy\\ &= \frac{1}{4}\int_0^2 y^2+\frac{1}{2}y^2-\frac{y^2}{2}-\frac{1}{2}(y/2)^2 dy\\ &= \frac{1}{4}\int_0^2 \frac{7}{8}y^2 dy\\ &= \frac{1}{4}\cdot\frac{7}{8}\cdot\frac{8}{3}=\frac{7}{12}\end{aligned}$$

c. for $0<y<2$, the marginal density of Y is

$$f_Y(y)=\frac{1}{4}\int_0^y x+y dx=\frac{1}{4}\left(\frac{x^2}{2}+xy\right)\Big|_0^y=\frac{3}{8}y^2$$

and 0 otherwise.

3. Independence:

(1) Suppose that Y is a random variable with mean 10 and standard deviation 2. Let X be a Bernoulli random variable with $p=\frac{1}{2}$, independent of Y. Now, consider a random variable $Z=X+Y$.

a. Let $M(t)$ be the moment generating function (MGF) of Y, calculate the moment generating function of Z, denoted as $M_Z(t)$.

b. Using the result from part a, derive the mean and variance of Z.

Solution: The standard deviation for Y is 2 implies that its variance is $Var(X)=4$.

a. Since X and Y are independent, we have

$$M_{X+Y}(t)=E\left[e^{t(X+Y)}\right]=E\left[e^{tX}\cdot e^{tY}\right]=E\left[e^{tX}\right]E\left[e^{tY}\right]=M_X(t)M_Y(t)$$

where $M_X(t)=\frac{1}{2}(1+e^t)$ and $M_Y(t)=M(t)$ is the MGF of X and Y, respectively. Therefore,

$$M_{X+Y}(t)=\frac{1}{2}(1+e^t)M(t).$$

b. To derive the mean and variance of $X+Y$ by its MGF from part a, we need the result

$$M'(0)=E[Y]=10$$

and

$$M''(0)=E[Y^2]=Var(Y)+(E[Y])^2=4+10=104$$

Thus, we have

$$\frac{d}{dt}M_{X+Y}(t)=\frac{1}{2}\left[e^tM(t)+(1+e^t)M'(t)\right]$$

and

$$M''_{X+Y}(t)=\frac{1}{2}\frac{d}{dt}\left[e^tM(t)+(1+e^t)M'(t)\right]=\frac{1}{2}(e^tM(t)+2e^tM'(t)+(1+e^t)M''(t))$$

Thus

$$E[X+Y]=M'_{X+Y}(0)=\frac{1}{2}\left[1+(1+1)\cdot 10\right]=10.5$$

and

$$E[(X+Y)^2]=M''_{X+Y}(0)=\frac{1}{2}\left[1+2\cdot 10+2\cdot 104\right]=114.5$$

and thus

$$Var(X+Y)=E\left[(X+Y)^2\right]-(E[X+Y])^2=114.5-10.5^2=4.25$$

(2) Exercise **6.27**: X_1 and X_2 satisfy $P(X_1 = 1) = P(X_1 = -1) = 1/2$, $P(X_2 = 1) = p$ and $P(X_2 = -1) = 1 - p$. Also, X_1 and X_2 are independent. Let $Y = X_1 X_2$.

a.
$$P(Y = 1) = P(X_1 = 1, X_2 = 1) + P(X_1 = -1, X_2 = -1) = \frac{1}{2}(p + q) = \frac{1}{2}$$

b.
$$P(Y = 1) = P(X_1 = 1, X_2 = -1) + P(X_1 = -1, X_2 = -1) = \frac{1}{2}(p + q) = \frac{1}{2}$$

c.
$$P(X_2 = 1, Y = 1) = P(X_2 = 1, X_1 = 1) = \frac{1}{2}p = P(X_2 = 1)P(Y = 1)$$

d.
$$P(X_2 = 1, Y = -1) = P(X_2 = 1, X_1 = -1) = \frac{1}{2}p = P(X_2 = 1)P(Y = -1)$$

e.
$$P(X_2 = -1, Y = 1) = P(X_2 = -1, X_1 = -1) = \frac{1}{2}q = P(X_2 = -1)P(Y = 1)$$

f.
$$P(X_2 = -1, Y = 1) = P(X_2 = 1-, X_1 = -1) = \frac{1}{2}q = P(X_2 = -1)P(Y = 1)$$

Based on the formulas from (c)-(f), we have X_2 and Y are independent.

(3) Exercise **6.32**: $p = \frac{7}{9}$ is the probability of one draws resulting in yellow or green balls. Note that $N \sim Geo(p)$, then the probability mass function for N is $P(N = k) = (2/9)^{k-1}(7/9)$ for $k \geq 1$. The joint distribution of (N, Y) is

$$\begin{aligned} P(N = k, Y = 1) &= P(k - 1 \text{ white balls followed by a green ball}) \\ &= \left(\frac{2}{9}\right)^{k-1} \left(\frac{4}{9}\right) \end{aligned}$$

and

$$\begin{aligned} P(N = k, Y = 2) &= P(k - 1 \text{ white balls followed by a yellow ball}) \\ &= \left(\frac{2}{9}\right)^{k-1} \left(\frac{3}{9}\right) \end{aligned}$$

By law of total probability, we have

$$\begin{aligned} P(Y = 1) &= \sum_{k=1}^{\infty} P(Y = 1, N = k) \\ &= \sum_{k=1}^{\infty} \left(\frac{2}{9}\right)^{k-1} \left(\frac{4}{9}\right) \\ &= \frac{4}{9} \sum_{k=1}^{\infty} \left(\frac{2}{9}\right)^{k-1} \\ &= \frac{4}{9} \cdot \frac{1}{1 - (2/9)} = \frac{4}{7} \end{aligned}$$

Here, we use the geometric series $\sum_{k=1}^{\infty} p^{k-1} = \frac{1}{1-p}$ for any $|p| < 1$. Using the same argument (Law of Total Probability), we obtained $P(Y = 2) = \frac{3}{7}$. Thus, we can see N and Y are independent: for any $k \geq 1$,

$$P(N = k, Y = 1) = \left(\frac{2}{9}\right)^{k-1} \left(\frac{4}{9}\right) = \left(\frac{2}{9}\right)^{k-1} \left(\frac{7}{9}\right) \left(\frac{4}{7}\right) = P(N = k)P(Y = 1)$$

$$P(N = k, Y = 2) = \left(\frac{2}{9}\right)^{k-1} \left(\frac{3}{9}\right) = \left(\frac{2}{9}\right)^{k-1} \left(\frac{7}{9}\right) \left(\frac{3}{7}\right) = P(N = k)P(Y = 2)$$

(4) Exercise **7.3**: Let X_1 and X_2 be the change in price tomorrow and the day after tomorrow, with X_1 and X_2 being independent and their p.m.f given. Then,

$$\begin{aligned} P(X_1 + X_2 = 2) =& P(X = -1, Y = 3) + P(X = 0, Y = 2) + P(X = 1, Y = 1) \\ &+ P(X = 2, Y = 0) + P(X = 3, Y = 0) \\ =& \frac{1}{64} + \frac{1}{64} + \frac{1}{16} + \frac{1}{64} + \frac{1}{64} = \frac{1}{8} \end{aligned}$$

(5) Exercise **7.5**: Since X, Y, Z are three independent random variables with $X \sim \mathcal{N}(1, 2)$, $Y \sim \mathcal{N}(2, 1)$,and $Z \sim \mathcal{N}(0, 7)$, then W is still a normal random variable with

$$E[W] = E[X] - 4E[Y] + E[Z] = 1 - 4 \times 2 + 0 = -7$$

and

$$\mathrm{Var}(W) = \mathrm{Var}(X) + (-4)^2\mathrm{Var}(Y) + \mathrm{Var}(Z) = 2 + 16 + 7 = 25$$

We have

a. $W \sim \mathcal{N}(-7, 25)$

b. Let Φ be distribution function of standard normal random variables with $\mathcal{N}(0, 1)$, we have

$$\begin{aligned} P(W > -2) &= P\left(\frac{W - (-7)}{\sqrt{25}} > \frac{-2 + 7}{\sqrt{25}}\right) \\ &= 1 - \Phi(1) \\ &= 1 - .8413 = .1587 \end{aligned}$$

(6) Exercise **8.9**: X and Y are independent random variables with $E[X] = 3$, $E[Y] = 5$, $\mathrm{Var}(X) = 2$, and $\mathrm{Var}(Y) = 3$. Thus, $E[X^2] = 11$ and $E[Y^2] = 28$.

a. $E[3X - 2Y + 7] = 3E[X] - 2E[Y] + 7 = 3 \cdot 3 - 2 \cdot 5 + 7 = 6$.

b. $\mathrm{Var}(3X - 2Y + 7) = 9 \cdot \mathrm{Var}(X) + 4 \cdot \mathrm{Var}(Y) = 18 + 12 = 30$

c. We have

$$\begin{aligned} \mathrm{Var}(XY) &= E[(XY)^2] - (E[XY])^2 \\ &= E[X^2]E[Y^2] - (E[X]E[Y])^2 \\ &= 11 \cdot 28 - (3 \cdot 5)^2 = 83 \end{aligned}$$

4. Expectation:

(1) Exercise **8.4**: Let I_k be the indicator of the event that hat the number 4 is showing on the k-sided die. Then $Z = I_4 + I_6 + I_{12}$ with $E[I_k] = \frac{1}{k}$. Thus, by linearity of expectation:

$$E[Z] = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$$

(2) Exercise **8.7**: Let X_B be the event that Ben calls Adam, and similar for X_C and X_D. Then, $X = X_C + X_D + X_B$ with

$$E[X] = E[X_B] + E[X_C] + E[X_D] = .3 + .4 + .7 = 1.4$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(X_B) + \mathrm{Var}(X_C) + \mathrm{Var}(X_D) = (.3)(1 - .3) + (.4)(1 - .4) + (.7)(1 - .7) = .66$$

(3) Exercise **8.11**: Continue our argument from exercise 8.4:

$$M_X(t) = M_{X_B}(t) \cdot M_{X_C}(t) \cdot M_{X_D}(t) = (.3e^t + .7)(.4e^t + .6)(.7e^t + .3)$$

Remark. We know that $X_B \sim \mathrm{Ber}(.3)$, which implies that its moment generating function $M_{X_B}(t) = .3e^t + .7$, and similar argument for X_C and X_D.

5. Special Distributions:

(1) Suppose that X_1 and X_2 are independent random variables, and X_1 and X_2 have the exponential distribution with parameters β_1 and β_2, respectively. Then,

a. Identify the joint density function $f(x_1, x_2)$ of X_1 and X_2.

b. Use part a to show that the probability

$$P(X_1 > X_2) = \frac{\beta_2}{\beta_1 + \beta_2}$$

Solution:

a. Given that the marginal density for X_1 and X_2 are

$$f_{X_1}(x_1) = \begin{cases} \beta_1 e^{-\beta_1 x_1} & x_1 > 0 \\ 0 & x_1 \leq 0 \end{cases}$$

and

$$f_{X_2}(x_2) = \begin{cases} \beta_2 e^{-\beta_2 x_2} & x_2 > 0 \\ 0 & x_2 \leq 0 \end{cases}$$

Since X_1 and X_2 are independent, the joint density is

$$f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \begin{cases} \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} & x_1, x_2 > 0 \\ 0 & \textit{Otherwise} \end{cases}$$

b. To verify the equation above, we need to do the integration on the joint density over the area $\{x_1 > x_2\}$. In other words,

$$\begin{aligned} P(X_1 > X_2) &= \iint_{x_1 > x_2} f(x_1, x_2) dx_1 dx_2 \\ &= \int_0^\infty \int_{x_2}^\infty \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} dx_1 dx_2 \\ &= \beta_1 \beta_2 \int_0^\infty e^{-\beta_2 x_2} \left(\frac{1}{\beta_1} e^{-\beta_1 x_2} \right) dx_2 \\ &= \frac{\beta_2}{\beta_1 + \beta_2} \end{aligned}$$

Note. You can check the bounds for integration by drawing the graph.

(2) Exercise **6.6**: In this problem, you can use the fact that $\int_0^\infty x e^{-\alpha x} dx = \frac{1}{\alpha^2}$ and $\int_0^\infty x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3}$ for $\alpha > 0$.

i. The marginal density for X is: for $x > 0$

$$f_X(x) = \int_0^\infty x e^{-x(1+y)} dy = x e^{-x} \int_0^\infty e^{-xy} dy = e^{-x}$$

and zero density if $x \leq 0$. The marginal density for Y is: for $y > 0$,

$$f_Y(y) = \int_0^\infty x e^{-x(1+y)} dy = \frac{1}{(1+y)^2}$$

and zero density if $y \leq 0$.

ii. The expected value of XY can be obtained by

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^\infty xy \cdot x e^{-x(1+y)} dx dy \\ &= \int_0^\infty x^2 e^{-x} \left(\int_0^\infty y e^{-xy} dy \right) dx \\ &= \int_0^\infty x^2 e^{-x} \cdot \frac{1}{x^2} dx \\ &= \int_0^\infty e^{-x} dx = 1 \end{aligned}$$

iii. The expected value of $\frac{X}{1+Y}$ can be obtained by

$$\begin{aligned}
E\left[\frac{X}{1+Y}\right] &= \int_0^\infty \int_0^\infty \frac{x}{1+y} \cdot xe^{-x(1+y)} dxdy \\
&= \int_0^\infty \frac{1}{1+y} \left(\int_0^\infty x^2 e^{-x(1+y)} dx\right) dy \\
&= \int_0^\infty \frac{1}{1+y} \cdot \frac{2}{(1+y)^3} dy \\
&= 2\int_0^\infty (1+y)^{-4} dy = \frac{2}{3}
\end{aligned}$$

(3) Exercise **6.11**: Given the density of X is

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & o.w. \end{cases}$$

$Y \sim \text{Unif}[1, 2]$, and X and Y are independent, the joint distribution of X and Y is

$$f(x, y) = f_X(x) f_Y(y) = \begin{cases} 2x, & 0 < x < 1, 1 < y < 2 \\ 0, & o.w. \end{cases}$$

Then, we have

$$\begin{aligned}
P\left(Y - X \geq \frac{3}{2}\right) &= \iint_{y-x>3/2} f(x, y)\, dxdy \\
&= \int_0^{1/2} \int_{x+3/2}^{2} 2x \, dy \, dx \\
&= \int_0^{1/2} 2x\left(\frac{1}{2} - x\right) dx \\
&= \frac{1}{24}
\end{aligned}$$

(4) Exercise **6.12**: Note that for any $\alpha > 0$, we have $\int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha}$, thus

$$g(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is a valid p.d.f function. Using this argument, for $x > 0$ and $y > 0$ we have

$$f(x, y) = e^{-x} \cdot 2e^{-2y} = f_X(x) f_Y(y)$$

implies that X and Y are independent, where $f_X(x)$ and $f_Y(y)$ are two marginal density of X and Y, respectively.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu