Random Variables (Solutions)

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Discrete Random Variables

See exercise $3.5, 2.38$

1. Exercise 3.5: The support for the random variable X is $\{1, 4/3, 3/2, 9/5\}$ with p.m.f:

X	1	$4/3$	$3/2$	$9/5$
P	$1/3$	$1/6$	$1/4$	$1/4$

2. Exercise 2.38:

   (a) Use the law of total probability:

$$\begin{aligned} P(R) &= P(R|SOME)P(SOME) + P(R|DOGS)P(DOGS) \\ &\quad + P(R|ARE)P(ARE) + P(R|BROWN)P(BROWN) \\ &= 0 + 0 + \frac{1}{3} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{2}{15} \end{aligned}$$

   (b) Support of X is $S_X = \{3, 4, 5\}$, the probability mass function is

X	3	4	5
P	$1/4$	$1/2$	$1/4$

   (c) Note that

   - $\{X > 3\} = \{$SOME, DOGS, BROWN$\}$
   - $\{X = 4\} = \{$SOME, DOGS$\}$
   - $\{X = 5\} = \{$BROWN$\}$

   By definition of conditional probability, we obtain

$$P(X = 3|X > 3) = 0$$

$$P(X = 4|X > 3) = \frac{2}{3}$$

$$P(X = 5|X > 3) = \frac{1}{3}$$

   (d) Note that $\{X = k\}$ for $k = 3, 4, 5$ form a partition for the whole sample space Ω, then followed from hint, we have

$$\begin{aligned} P(R|X > 3) &= \sum_{k=3}^{5} P(R \cap \{X = k\}|X > 3) \\ &= P(R \cap \{X = 4\}|X > 3) + P(R \cap \{X = 5\}|X > 3) \\ &= P(R|X = 4)P(X = 4|X > 3) + P(R|X = 5)P(X = 5|X > 3) \end{aligned}$$

where

$$\begin{aligned}P(R|X=4) &= P(R\cap SOME|X=4)+P(R\cap DOGS|X=4)\\ &= P(R|SOME)P(SOME|X=4)+P(R|DOGS)P(DOGS|X=4)\\ &= 0\end{aligned}$$

and

$$P(R|X=5)=P(R|BROWN)=\frac{1}{5}$$

Using the result from part (c), we have

$$P(R|X>3)=\frac{1}{5}\times\frac{1}{3}=\frac{1}{15}$$

(e) By Bayes formula:

$$\begin{aligned}P(BROWN|R) &= \frac{P(R|BROWN)P(BROWN)}{P(R)}\\ &= \frac{(1/5)(1/4)}{2/15}\\ &= \frac{3}{8}\end{aligned}$$

Continuous Random Variables

See exercise $3.7, 3.25$

1. Exercise 3.7: Since X is a continuous random variable, we have $P(a \le X \le b) = F(b) - F(a)$

   (a) $F(b)-F(a) = P(a \le X \le b) = 1$ implies that $F(b) = 1$ and $F(a) = 0$, which further implies $b \ge \sqrt{3}$ and $a \le \sqrt{2}$, thus the smallest interval is $[\sqrt{2}, \sqrt{3}]$

   (b)

   $$\begin{aligned}P(X=1.6) &= P(X\le 1.6)-P(X<1.6)\\ &= F(1.6)-\lim_{x\to 1.6^-}F(x)=0\end{aligned}$$

   The last equality holds since the cumulative density function F is continuous for any x.
   Remark. For any continuous random variable X, the probability that X takes on one special value is equal to zero.

   (c) Since X is a continuous Random Variable, we have $P(X=1)=0$ so that $P(1 \le X \le 3/2) = P(1 < X \le 3/2)$, hence

   $$P(1\le X\le 3/2)=F(3/2)-F(1)=(3/2)^2-2-0=.25$$

   (d) Since F is continuous, and it is differential apart from $\sqrt{2}$ and $\sqrt{3}$. So, we can differentiate F to obtain the density:

   $$f(x)=\begin{cases}2x & x\in(\sqrt{2},\sqrt{3})\\ 0 & \text{otherwise}\end{cases}$$

2. Exercise 3.25:

   (a) If f is a p.d.f, then

   $$1=\int_{-\infty}^{\infty}f(x)\,dx=\int_1^3(x^2-b)\,dx=\frac{1}{3}x^3-bx\Big|_1^3=\frac{26}{3}-2b$$

   Then, we solve for $b=\frac{23}{6}$. However, if $x\in[1,\sqrt{23/6})$, then $f(x)<0$, which volates with the definition the probability density function. Thus, there is no b such that f is a valid density.

(b) Now, we know that $\cos x \geq 0$ whenever $-\pi/2 \leq x \leq \pi/2$, so if we want f is a valid density function, we need $b \in (0, \pi/2]$. By computing the integral, we have

$$1 = \int_{-\infty}^{\infty} h(x)\,dx = \int_{-b}^{b} \cos x\,dx = 2\sin b$$

Thus, we have $\sin b = \frac{1}{2}$, and therefore, $b = \frac{\pi}{6}$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu