1. Expectation

For a random variable X,

$$\mathbb{E}[X] \overset{\text{def}}{=} \begin{cases} \sum_k k\,\mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{P.D.F}}\,dx & X: \text{continuous} \end{cases}$$

$E[X]$: the expected value X.

In general, let $g: \mathbb{R} \to \mathbb{R}$, then $g(X)$ is also a R.V.

$$\mathbb{E}[g(X)] \overset{\text{def}}{=} \begin{cases} \sum_k g(k)\,\mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx & X: \text{continuous} \end{cases}$$

<u>Ex 3.32</u>. X has density

$$f_X(x) = \begin{cases} \frac{1}{2} x^{-3/2} & 1 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

(a) Find $\mathbb{P}(X > 10)$.

<u>Soln</u>: $\mathbb{P}(X>10) = \int_{10}^{\infty} \frac{1}{2} x^{-3/2}\,dx$

$$= -x^{-1/2} \Big|_{x=10}^{x=\infty}$$

$$= -(0 - 10^{-1/2}) = 1/\sqrt{10}$$

(b) Find the CDF F_X of X.

<u>Soln</u>: Use formula $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

① $x \le 1$, $\int_{-\infty}^{x} \cancel{f_X(t)}^{0}\,dt = 0$

② $x > 1$, $\int_{-\infty}^{x} f_X(t)\,dt$

$$= \int_{-\infty}^{1} \cancel{f_X(t)}^{0}\,dt + \int_{1}^{x} f_X(t)\,dt$$

$$= \int_{1}^{x} \frac{1}{2} t^{-3/2}\,dt = \text{(exercise)}$$

(c) Find $\mathbb{E}[X]$ and $\mathbb{E}[X^{1/4}]$.

<u>Soln</u>: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_{1}^{\infty} \frac{1}{2} x^{-3/2} \cdot x\,dx$$

$$= x^{1/2} \Big|_{x=1}^{x=\infty} = \lim_{x\to\infty} (\sqrt{x} - 1) = \infty$$

$\mathbb{E}[X^{1/4}] = \int_{-\infty}^{\infty} x^{1/4} f_X(x)\,dx$

$$= \frac{1}{2} \int_{1}^{\infty} x^{-3/2} \cdot x^{\frac{1}{4}}\,dx$$

$$= \frac{1}{2} \int_{1}^{\infty} x^{-5/4}\,dx$$

$$= \frac{1}{2}(-4)\, x^{-\frac{1}{4}} \Big|_{x=1}^{x=\infty} = 2$$

☒

2. Variance

For a R.V. X,

$$\mathrm{Var}(X) \stackrel{\text{def}}{=} \mathbb{E}\left[(X-\mathbb{E}[X])^2\right]$$

Two useful facts we need to know:

(1) $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ — easier to compute

(2) If $a, b \in \mathbb{R}$,

$$\mathbb{E}[aX+b] = a\,\mathbb{E}[X] + b$$ — linearity of expectation

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

Proof of (1) & (2): See lecture notes. $\boxtimes$

Exercise 3.18 (c): X — a normal random variable with mean 3 and variance 4.

Find $\mathbb{E}[X^2]$.

Soln: by formula: $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\Rightarrow \mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2$$

$$= 4 + 9$$

$$= 13$$

Exercise 3.67: Let $Z \sim N(0,1)$ and $X \sim N(\mu, \sigma^2)$

(a) Calculate $\mathbb{E}[Z^3]$ (the 3^{rd} moment of Z)

Soln: $Z \sim N(0,1)$ implies its PDF

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \qquad z \in \mathbb{R}$$

$$\mathbb{E}[Z^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z^3 \exp\left\{-\frac{z^2}{2}\right\}}_{\text{odd function}} dz = 0$$

z^3 – odd func, $\exp\left\{-\frac{z^2}{2}\right\}$ – even func

(b) Calculate $\mathbb{E}[X^3]$ (the 3^{rd} moment of X).

Soln: $X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$

in other words: $\frac{X-\mu}{\sigma} \overset{(d)}{=} Z$ and $X \overset{(d)}{=} \sigma Z + \mu$

same distribution

$$\mathbb{E}[X^3] = \mathbb{E}[(\sigma Z + \mu)^3]$$

$$= \mathbb{E}\left[(\sigma Z)^3 + 3(\sigma Z)^2 \cdot \mu + 3(\sigma Z) \cdot \mu^2 + \mu^3\right]$$

linearity

$$= \sigma^3 \overset{0}{\cancel{\mathbb{E}[Z^3]}} + 3\mu\sigma^2 \underbrace{\mathbb{E}[Z^2]}_{(\mathbb{E}[Z])^2 + Var(Z) = 1} + 3\sigma\mu^2 \overset{0}{\cancel{\mathbb{E}[Z]}} + \mu^3$$

$$= 3\mu\sigma^2 + \mu^3$$

∎

3. Moment Generating Function

For a R.V. X, the M.G.F is defined by

$$M_X(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_k e^{tk}\, \mathbb{P}(X=k) & X \text{ -discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & X \text{ -continuous} \end{cases}$$

Useful Results:

$$\mathbb{E}[X^n] = M_X^{(n)}(0)$$

In particular, given M.G.F. $M_X(t)$:

$$\mathbb{E}[X] = M_X'(0), \quad \mathbb{E}[X^2] = M_X''(0)$$

$$Var(X) = M_X''(0) - (M_X'(0))^2$$

Pf: see the next Page.

Pf: By Taylor series: $e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$

$$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right]$$

$$(\text{linearity}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n] \quad (*)$$

Recall in Calculus class: Taylor expansion of the function $M_X(t)$ at $t=0$ is

$$M_X(t) = \sum_{n\geq 0}^{\infty} \frac{t^n}{n!} M_X^{(n)}(0) \quad (**)$$

Two series $(*)$ and $(**)$ have the same value, which implies that coefficients of x^n should be the same for each n,

i.e.
$$\mathbb{E}[X^n] = M_X^{(n)}(0)$$

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