PSTAT 120A OH

Possible topics:

Intro
1. Set Theory / Sample Space
2. Counting Methods / Hypeo-Geometric
3. Axiom of Probabilities / Conditional Probabilities

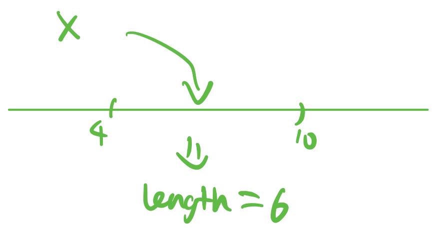
Discrete R.V.s
4. PMF $\rightleftarrows$ CDF
5. Special Distributions → Binomial / Bernoulli, Geometric, Negative Binomial, Poisson

Continuous R.V.s
6. PDF $\underset{\frac{d}{dx}F_X(x)}{\overset{\int_{-\infty}^{x} f(t)dt}{\rightleftarrows}}$ CDF

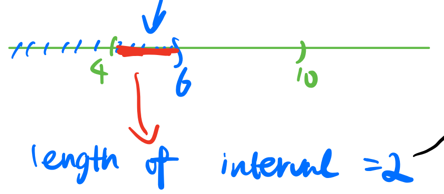
PDF, CDF

7. Uniform: $f_X(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & o.w. \end{cases}$ — Ex 3.4
8. Exponential: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & o.w. \end{cases}$ — Ex 4.13–14
9. Normal — Ex 3.17, 3.18 (a–b)

Exercise 3.4: $X \sim Unif(4,10)$



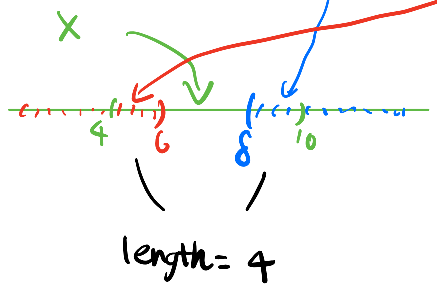
(a) $\mathbb{P}(X<6) = \frac{2}{6} = \frac{1}{3}$



(b) $\mathbb{P}(|X-7|>1)$

Soln: $|X-7|>1 \iff X-7>1$ or $X-7<-1$

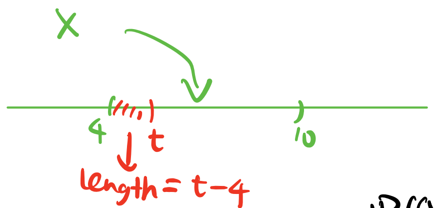
$\Leftrightarrow$ $X > 8$ or $X < 6$



$\Rightarrow \mathbb{P}(|X-7|>1) = \frac{4}{6} = \frac{2}{3}$

(c) for $4 \le t \le 6$, calculate $\mathbb{P}(X<t \mid X<6)$.

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$



$$\mathbb{P}(X<t \mid X<6) = \frac{\mathbb{P}(\{X<t\} \cap \{X<6\})}{\mathbb{P}(X<6) \overset{(a)}{=} 1/3} = \frac{\mathbb{P}(X<t)}{\mathbb{P}(X<6)} = \frac{t-4}{2}$$

$t \le 6$: $\{X<t\} \subseteq \{X<6\} \Rightarrow \{X<t\} \cap \{X<6\} = \{X<t\}$

$\mathbb{P}(X<t) = \frac{t-4}{6}$ (If $A \Rightarrow B$, then $A \subseteq B$)

Review of Normal Distribution:

- Suppose that $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ Standard Normal
- We can find the values of CDF using z-table

※ Normal: Ex 3.18 (a–b)

When X be a normal distr w. mean 3 and variance 4.

(a) Find Prob. that $\mathbb{P}(2<X<6)$.

(b) Find the value c s.t. $\mathbb{P}(X>c) = .33$

Soln: $\underset{\mu\ \ \sigma^2}{X \sim N(3,4)} \Rightarrow Z = \frac{X-3}{\sqrt{4}} \sim N(0,1)$

(a) $\mathbb{P}(2<X<6)$

$= \mathbb{P}\left(\frac{2-3}{2} < \frac{X-3}{2} < \frac{6-3}{2}\right)$

$= \mathbb{P}\left(-\frac{1}{2} < Z < \frac{3}{2}\right) = N\left(\frac{3}{2}\right) - N\left(-\frac{1}{2}\right)$

CDF of $N(0,1)$ (from Z-table)

Rmk: Z-table provides $N(z)$ in which $z \geq 0$. if you want to compute $N(z)$ for $z<0$, you need to use formula

$N(z) = 1 - N(-z)$

(b) Find the value c s.t. $\mathbb{P}(X>c) = .33$

How to solve for c?

$\underset{.33}{\mathbb{P}(X>c)} = \mathbb{P}\left(\underset{Z \sim N(0,1)}{\frac{X-3}{2}} > \underset{d}{\frac{c-3}{2}}\right) = \mathbb{P}(Z>d) = 1 - \mathbb{P}(Z \leq d)$

$= 1 - N(d)$

$\Rightarrow N(d) = .67 \underset{\text{Z-table}}{\Rightarrow} d = ? \Rightarrow c = ?$

3

Example of PDF ⇒ CDF: Consider the PDF $f_Y(y)$ of a R.V. Y:
(CLAS)

$$f_Y(y) = \begin{cases} y^2 & 0 \le y < 1 \\ cy^2 & 1 \le y < 2 \\ 0 & \text{o.w.} \end{cases}$$

o.w.: $y<0$, $y \ge 2$

(a) Find the value of c to make f_Y a valid P.D.F.

<u>Soln</u>: 2 criteria: $\begin{cases} f_Y(y) \ge 0 \quad \forall y & \text{— trivial} \\ \int_{-\infty}^{\infty} f_Y(y)dy = 1 & \text{— *} \end{cases}$

$$1 = \int_{-\infty}^{+\infty} f_Y(y)\,dy$$

$$= \underbrace{\cancel{\int_{-\infty}^{0} 0\,dy}}_{0} + \underbrace{\int_0^1 y^2\,dy}_{\frac{1}{3}y^3\Big|_{y=0}^{y=1} = 1/3} + \underbrace{\int_1^2 cy^2\,dy}_{\frac{c}{3}y^3\Big|_{y=1}^{y=2} = 7c/3} + \underbrace{\cancel{\int_2^{+\infty} 0\,dy}}_{0} = \frac{1}{3} + \frac{7c}{3} \Rightarrow c = \frac{2}{7}$$

$$f_Y(y) = \begin{cases} y^2 & 0 \le y < 1 \\ \frac{2}{7}y^2 & 1 \le y < 2 \\ 0 & \text{o.w.} \end{cases}$$

o.w.: $y<0$, $y \ge 2$

(b) Derive the C.D.F F_Y. (Use formula: $F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt$)

By cases:

Case 1: $y<0 \Rightarrow F_Y(y) = \int_{-\infty}^{y} \underbrace{\cancel{f_Y(t)}}_{0}\,dt = 0$

Case 2: $0 \le y < 1 \Rightarrow F_Y(y) = \int_{-\infty}^{0} \underbrace{\cancel{f_Y(t)}}_{0}\,dt + \int_0^y \underbrace{f_Y(t)}_{t^2}\,dt = y^3/3$

Case 3: $1 \le y < 2 \Rightarrow F_Y(y) = \int_{-\infty}^{0} \underbrace{\cancel{f_Y(t)}}_{0}\,dt + \underbrace{\int_0^1 \cancel{f_Y(t)}_{t^2}\,dt}_{1/3} + \underbrace{\int_1^y \cancel{f_Y(t)}_{\frac{2}{7}t^2}\,dt}_{\frac{2}{21}t^3\Big|_{t=1}^{t=y}} = \frac{1}{3} + \frac{2}{21}(y^3 - 1)$

Case 4: $y \ge 2 \Rightarrow F_Y(y) = 1$

Exponential Distribution: $X \sim \exp(\lambda)$, then

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & \text{o.w.} \end{cases}$$

Useful result: 1. C.D.F $F_X(x) = \begin{cases} 1-e^{-\lambda x} & x>0 \\ 0 & x\le 0 \end{cases}$ (Check by yourself)

2. Memoryless property:

$$\mathbb{P}(X \ge t+s \mid X \ge t) = \mathbb{P}(X \ge s) = e^{-\lambda s}$$ (Check)

Refer to Ex. 4.13 – 4.14 (Textbook)

Discrete R.V.s (General):

Q: Find the PMF of X.

ex 3.5:

CDF of X $\swarrow$ $F_X(x) = \begin{cases} 0 & x<1 \\ 1/3 & 1\le x<4/3 \\ 1/2 & 4/3 \le x < 3/2 \\ 3/4 & 3/2 \le x < 9/5 \\ 1 & x \ge 9/5 \end{cases}$

Jump point	Jump size
1	$1/3$
$4/3$	$1/2 - 1/3 = 1/6$
$3/2$	$3/4 - 1/2 = 1/4$
$9/5$	$1 - 3/4 = 1/4$

$\Rightarrow$ PMF of X:

X	1	4/3	3/2	9/5
$\mathbb{P}$	1/3	1/6	1/4	1/4

Special Distribution <
- Discrete: You have to identify distributions by yourself.
- Continuous: They will tell you the name of Distribution.

Discrete R.V.s (Special Distribution):

Binomial	V.S. Negative Binomial
Fix the number of trials, count the number of successes	Fix the number of successes, count the number of trials you needed.

Bernoulli
||
1 trial

(Bernoulli(p) = Binomial(1, p))

Geometric
||
wait for the first success

(Geo(p) = NegBin(k=1, p))