

1. Expectation

For a random variable X ,

$$\mathbb{E}[X] \stackrel{\text{def}}{=} \begin{cases} \sum_k k \mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{P.D.F}} dx & X: \text{continuous} \end{cases}$$

$\mathbb{E}[X]$: the expected value X .

In general, let $g: \mathbb{R} \rightarrow \mathbb{R}$, then $g(X)$ is also a R.V.

$$\mathbb{E}[g(X)] \stackrel{\text{def}}{=} \begin{cases} \sum_k g(k) \mathbb{P}(X=k) & X: \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f(x) dx & X: \text{continuous} \end{cases}$$

Ex 3.32. X has density

$$f_X(x) = \begin{cases} \frac{1}{2} x^{-3/2} & 1 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

(a) Find $\mathbb{P}(X > 10)$.

Soln: $\mathbb{P}(X > 10) = \int_{10}^{\infty} \frac{1}{2} x^{-3/2} dx$

$$= -x^{-1/2} \Big|_{x=10}^{x=\infty}$$

$$= -(0 - 10^{-1/2}) = 1/\sqrt{10}$$

(b) Find the CDF F_X of X .

Soln: Use formula $F_X(x) = \int_{-\infty}^x f_X(t) dt$

① $x \leq 1$, $\int_{-\infty}^x \cancel{f_X(t)} dt = 0$

② $x > 1$, $\int_{-\infty}^x f_X(t) dt$

$$= \int_{-\infty}^1 \cancel{f_X(t)} dt + \int_1^x f_X(t) dt$$

$$= \int_1^x \frac{1}{2} t^{-3/2} dt = (\text{exercise})$$

(c) Find $E[X]$ and $E[X^{1/4}]$.

Soln: $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

$$= \int_1^{\infty} \frac{1}{2} x^{-3/2} \cdot x dx$$

$$= x^{1/2} \Big|_{x=1}^{x=\infty} = \lim_{x \rightarrow \infty} (\sqrt{x} - 1) = \infty$$

$$E[X^{1/4}] = \int_{-\infty}^{\infty} x^{1/4} f_X(x) dx$$

$$= \frac{1}{2} \int_1^{\infty} x^{-3/2} \cdot x^{1/4} dx$$

$$= \frac{1}{2} \int_1^{\infty} x^{-5/4} dx$$

$$= \frac{1}{2} (-4) x^{-1/4} \Big|_{x=1}^{x=\infty} = 2$$

2. Variance

For a R.V. X ,

$$\text{Var}(X) \stackrel{\text{def}}{=} \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Two useful facts we need to know:

(1) $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ — easier to compute

(2) If $a, b \in \mathbb{R}$,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b \quad \text{— linearity of expectation}$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Proof of (1) & (2): See lecture notes. \square

Exercise 3.18(c): X — a normal random variable with mean 3 and variance 4.

Find $\mathbb{E}[X^2]$.

Soln: by formula: $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\Rightarrow \mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2$$

$$= 4 + 9$$

$$= 13$$

Exercise 3.67: Let $Z \sim N(0, 1)$ and $X \sim N(\mu, \sigma^2)$

(a) Calculate $\mathbb{E}[Z^3]$ (the 3rd moment of Z)

Soln: $Z \sim N(0, 1)$ implies its PDF

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \quad z \in \mathbb{R}$$

$$\mathbb{E}[Z^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z^3 \exp\left\{-\frac{z^2}{2}\right\}}_{\substack{\text{odd function} \\ \uparrow}} dz = 0$$

z^3 - odd func, $\exp\left\{-\frac{z^2}{2}\right\}$ - even func

(b) Calculate $\mathbb{E}[X^3]$ (the 3rd moment of X).

Soln: $Z \sim N(0, 1) \Rightarrow \sigma Z \sim N(0, \sigma^2)$

$$\Rightarrow \mu + \sigma Z \sim N(\mu, \sigma^2),$$

i.e. X and $\mu + \sigma Z$ have the same distribution

$$\mathbb{E}[X^3] = \mathbb{E}[(\sigma Z + \mu)^3]$$

$$= \mathbb{E}[(\sigma Z)^3 + 3(\sigma Z)^2 \mu + 3(\sigma Z) \mu^2 + \mu^3]$$

linearity

$$= \overset{0}{\cancel{\sigma^3 \mathbb{E}[Z^3]}} + 3\mu\sigma^2 \underbrace{\mathbb{E}[Z^2]}_{=1} + 3\sigma\mu^2 \overset{0}{\cancel{\mathbb{E}[Z]}} + \mu^3$$

$$= 3\mu\sigma^2 + \mu^3 \quad (\mathbb{E}[Z]^2 + \text{Var}(Z) = 1)$$

3. Moment Generating Function

For a R.V. X , the M.G.F is defined by

$$M_X(t) = \mathbb{E}[e^{tx}] = \begin{cases} \sum_k e^{tk} \mathbb{P}(X=k) & X \text{ - discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & X \text{ - continuous} \end{cases}$$

Useful Results:

$$\mathbb{E}[X^n] = M_X^{(n)}(0)$$

In particular, given M.G.F. $M_X(t)$:

$$\mathbb{E}[X] = M_X'(0), \quad \mathbb{E}[X^2] = M_X''(0)$$

$$\text{Var}(X) = M_X''(0) - (M_X'(0))^2$$

Pf: see the next Page.

Pf: By Taylor series: $e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$

$$M_X(t) = \mathbb{E}[e^{tx}] = \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right]$$

$$(\text{linearity}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n] \quad (*)$$

Recall in Calculus class: Taylor expansion of the function $M_X(t)$ at $t=0$ is

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} M_X^{(n)}(0) \quad (**)$$

Two series $(*)$ and $(**)$ have the same value, which implies that coefficients of x^n should be the same for each n ,

i.e.

$$\mathbb{E}[X^n] = M_X^{(n)}(0)$$

□