Distributions for Continuous Random Variables

Botao Jin

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Uniform Distribution

1. Exercise 1.9: To get the shorter piece less than 1/5 of the original, the distance between the location we choose and left/right end of the stick should be lower than 1/5 of the origin, thus the probability is 2/5.

2. Exercise 1.11: $p = \frac{2^2\pi}{20^2} = \frac{\pi}{100}$

3. Exercise 3.4: $X \sim Unif[4, 10]$, then the p.d.f of X is

$$f(x) = \begin{cases} 1/6 & 4 < x < 10 \\ 0 & \text{Otherwise} \end{cases}$$

   (a) $P(X < 6) = \int_4^6 1/6\,dx = 1/3$

   (b) $P(|X - 7| > 1) = P(X < 6) + P(X > 8) = 2/3$

   (c) For $4 < t < 6$, we have

$$P(X < t | X < 6) = \frac{P(X < t)}{P(X < 6)} = \frac{t - 4}{6} \cdot 3 = \frac{t - 4}{2}$$

4. Exercise 3.20: For $y \in (0, c)$, we have $P(Y \leq y) = P(c - X \leq y) = P(X \geq c - y) = \frac{c - (c-y)}{c} = \frac{c}{y}$. We can show that Y follows from Uniform distribution by taking the derivative for the density.

5. Exercise 3.41: Note that given $Y = k$, $X \sim Unif(0, k]$ for $k = 1, 2, 3, 4, 5, 6$. Given $s \in (3, 4)$, we have $P(X \leq s | Y = k) = 1$ for $k = 1, 2, 3$ since $X \leq k < s$ in this case. For $k = 4, 5, 6$, $s < k$ implies $P(X \leq s | Y = k) = s/k$. Thus by law of total probability, we have

$$P(X \leq s) = \sum_{k=1}^{6} P(X \leq s | Y = k)P(Y = k) = \sum_{k=1}^{3} 1 \cdot \frac{1}{6} + \sum_{k=4}^{6} \frac{s}{k} \cdot \frac{1}{6} = \frac{1}{2} + \frac{s}{6}\left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = \frac{1}{2} + \frac{37s}{360}$$

Thus $F(s) = \frac{1}{2} + \frac{37s}{360}$ and $f(s) = 37/360$

Exponential Distribution

1. Exercise 4.50: By memoryless property of $T \sim exp(1/3)$, we have

$P(\text{Wait for at least addition three hours}|\text{already waited for 7 hours}) = P(T \geq 3) = e^{-1}$

More generally,

$P(\text{Wait for at lesat x additional hours}|\text{already waited for 7 hours}) = P(T \geq x) = e^{-x/3}$

Normal Distribution

1. Exercise 3.17: $X \sim N(-2, 7)$ and let Φ be the distribution function of $Z \sim N(0, 1)$

   (a)
   $$P(X > 3.5) = P\left(\frac{X+2}{\sqrt{7}} > \frac{3.5+2}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{5.5}{\sqrt{7}}\right) = .0188$$

   (b)
   $$P(-2.1 < X < -1.9) = P\left(\frac{-2.1+2}{\sqrt{7}} < \frac{X+2}{\sqrt{7}} < \frac{-1.9+2}{\sqrt{7}}\right) = \Phi\left(\frac{-.1}{\sqrt{7}}\right) - \Phi\left(\frac{.1}{\sqrt{7}}\right) = .032$$

   Note that $\Phi(-z) = 1 - \Phi(z)$ and $\Phi(z) - \Phi(-z) = \Phi(z) - (1 - \Phi(z)) = 2\Phi(z) - 1$.

   (c)
   $$P(X < 2) = P\left(\frac{X+2}{\sqrt{7}} 2 \frac{2+2}{\sqrt{7}}\right) = \Phi\left(\frac{4}{\sqrt{7}}\right) = .9345$$

   (d)
   $$P(X < -10) = P\left(\frac{X+2}{\sqrt{7}} < \frac{-10+2}{\sqrt{7}}\right) = \Phi\left(\frac{-8}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{8}{\sqrt{7}}\right) = .0013$$

   (e)
   $$P(X > 4) = P\left(\frac{X+2}{\sqrt{7}} > \frac{4+2}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{6}{\sqrt{7}}\right) = .0116$$

2. Exercise 3.18: $X \sim N(3, 4)$ and let Φ be the distribution function (C.D.F) of $Z = \frac{X-3}{2} \sim N(0, 1)$:

   (a) We have
   $$P(2 < X < 6) = P\left(\frac{2-3}{2} < \frac{X-3}{2} < \frac{6-3}{2}\right)$$
   $$= \Phi(1.5) - \Phi(-.5)$$
   $$= \Phi(1.5) - 1 + \Phi(.5)$$
   $$= .6247$$

   (b) We have
   $$P(X > c) = P\left(\frac{X-3}{2} > \frac{c-3}{2}\right) = 1 - \Phi\left(\frac{c-3}{2}\right) = .33$$

   By normal table, we solved for $c = 3.88$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu