Counting Methods

1. Basic Principle of Counting

- Suppose there are k jobs to be done.

$$\begin{cases} \text{first job: be done in } \underline{n_1} \text{ ways} \\ \text{second job: be done in } \underline{n_2} \text{ ways} \\ \vdots \\ k\text{-th job: be done in } \underline{n_k} \text{ ways} \end{cases}$$

Total number of ways doing k jobs is

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i \quad \text{(Multiplication Rules)}$$

Moreover, suppose that $A_1, A_2, \cdots, A_k$ are finite sets.

$\# A_i$ (the number of elements of A_i) $= n_i \quad i = 1, 2, \cdots, k$

Cartesian product:

$$A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \cdots, x_k) : x_1 \in A_1, x_2 \in A_2, \cdots, x_k \in A_k\}$$

by multiplication rule:
$$\begin{cases} x_1 : n_1 \text{ possibilities} \\ x_2 : n_2 \quad \text{〃} \\ \vdots \\ x_k : n_k \quad \text{〃} \end{cases}$$

$$\#(A_1 \times A_2 \times \cdots \times A_k) = n_1 \cdot n_2 \cdot \cdots \cdot n_k = \prod_{i=1}^{k} n_i \quad //$$

Example: Consider an experiment where we roll a fair 4-sided die (1, 2, 3, 4) for three times.

Q: How many possible outcomes in total?

Step 1: Write down the sample space

$$\Omega = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \{1, 2, 3, 4\}\}$$

$$= A_1 \times A_2 \times A_3$$

$A_i = \{1, 2, 3, 4\}$ — the outcome of the i-th roll

Step 2: $\# \Omega = \# A_1 \times \# A_2 \times \# A_3 = 4^3 = 64$ //

2. Addition Principle

Two event A and B. If $A \cap B = \phi$ (disjoint),

$\#(A \cup B) = (\# A) + (\# B)$

Moreover, $\Omega = A \cup A^c$, $\# \Omega = \# A + \# A^c$

$\Rightarrow$ Given $\# \Omega$ and $\# A^c$, then $\# A = \# \Omega - \# A^c$.

Example: In an experiment: 10 participants.

Each flips a coin and rolls a die.

(a) Describe the sample space and calculate the # of possible outcomes?

Soln: Set A — the outcome of a single participants.

$$A = \{(a_1, a_2) : a_1 \in \{H, T\},\ a_2 \in \{1, 2, \cdots, 6\}\}$$

$$= \{H, T\} \times \{1, 2, \cdots, 6\}$$

$\# A = 12$

Ω — the outcome of all 10 participants

$\Omega = A^{10}$ $\qquad \# \Omega = (\# A)^{10} = 12^{10}$

(b) $B = \{$At least one person rolled a 5$\}$. $\# B = ?$

Soln: $B^c = \{$None of them gets a 5$\}$.

$$B^c = (\{H, T\} \times \{1, 2, 3, 4, 6\})^{10}$$

$$\# B^c = (2 \times 5)^{10} = 10^{10} \Rightarrow \# B = 12^{10} - 10^{10}$$

//.

3. Random Sampling (Chapter 1.2 in the textbook)

Suppose an urn has n balls (numbered $1, 2, \cdots, n$), retrieve the ball for k times.

① Sampling with replacement:

retrieve a ball, record its number, and put it back into the urn.

(possible for the ball to be retrieved again)

Sample space

$$\Omega = \{ (\underset{\text{1st round}}{w_1}, \underset{\text{2nd round}}{w_2}, \cdots, w_k) : \text{each } w_i \in \{1, 2, \cdots, n\} \}$$

$$= \{1, 2, \cdots, n\}^k$$

$$\# \Omega = n^k$$

② Sampling without replacement: retrieve a ball, record its number, and put it aside.

(the same ball cannot be drawn twice)

(1) Order matters (1,2) & (2,1) — different outcomes

$$\# \Omega = \frac{n!}{(n-k)!} \qquad (n! = 1 \times 2 \times \cdots \times n)$$

(2) Order doesn't matter (1,2) & (2,1) — same outcomes

$$\# \Omega = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Pf: See Appendix C. ☒

Exercise: 4 balls (1, 2, 3, 4). Sample 2 balls w/o replacement. Write down the sample spaces and count the number of element in each cases:

(1) Order matters: $(1,2)$ & $(2,1)$ – different outcomes

$$\Omega = \{(s_1, s_2): \text{each } s_1, s_2 \in \{1,2,3,4\}, \; s_1, s_2 - \text{distinct}\}$$

$$= \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$$

$\Rightarrow \#\Omega = 12$

(2) Order doesn't matter: $(1,2)$ & $(2,1)$ – same outcomes

$$\Omega = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$$

$\Rightarrow \#\Omega = 6$

Extra Exercise:

e.g. 1: Find the number of six-letter words (not need to be meaningful) constructed from the letters B, A, D, G, E, R?

Soln: It is equivalent to retrieve 6 letter w/o replacement, order matters: $6! = 720$ possibilities.

e.g. 2: Find the number of 5-letter words from the letters A, P, P, L, E? (no need to be meaningful)

(MTD 1) Order 5 letters: $5! = 120$ ways,

but two Ps can be in two different orders,

we counted each word twice, so the number of different words is $\frac{120}{2} = 60$.

(MTD 2) 5-letter words has the format

_ _ _ _ _

Step 1: Choose the position of two Ps: $\binom{5}{2} = 10$ ways

Step 2: Arrange the remaining three words: $3! = 6$ ways

By multiplication rules:

$10 \cdot 6 = 60$ different words.