Midterm Refresher:

P1: X — a R.V. with possible values $\{1, 2, 3, 4, 5\}$

and P.M.F $\mathbb{P}(X=k) = ck$

(a) Find the value of $\underline{c}$ and write out the dist of X.

Soln: Use formula: $\sum_k \mathbb{P}(X=k) = 1$

i.e. $1 = \sum_{k=1}^{5} ck = c(1+2+3+4+5) = 15c$

$\Rightarrow c = 1/15$

P.M.F:

X	1	2	3	4	5
$\mathbb{P}$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$

(b) Find the C.D.F $\mathbb{P}(X \le k)$ for $k = 1, 2, 3, 4, 5$.

Soln: $k=1$: $\mathbb{P}(X \le 1) = \mathbb{P}(X=1) = 1/15$

$k=2$: $\mathbb{P}(X \le 2) = \mathbb{P}(X=1) + \mathbb{P}(X=2) = 1/15 + 2/15 = 3/15$

$k=3$: $\mathbb{P}(X \le 3) = \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3)$

$= 1/15 + 2/15 + 3/15 = 6/15$

$k=4$: $\mathbb{P}(X \le 4) = \frac{1}{15}(1+2+3+4) = 10/15$

$k=5$: $\mathbb{P}(X \le 5) = 1$

Extra Exercise for P1: Find the mean and variance of X.

Soln: mean $\mathbb{E}[X] = \sum_{k=1}^{5} k\,\mathbb{P}(X=k)$

$$= \sum_{k=1}^{5} k \cdot ck$$

$$= \frac{1}{15}(1^2+2^2+3^2+4^2+5^2) = 3.67$$

variance $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{(result)}$

where $\mathbb{E}[X^2] = \sum_{k=1}^{5} k^2\,\mathbb{P}(X=k)$

$$= \sum_{k=1}^{5} k^2 \cdot ck$$

$$= \frac{1}{15}(1^3+2^3+3^3+4^3+5^3) = 15 \qquad //$$

P2. Let Y have the density function

$$f(y) = \begin{cases} 0.1 & -2 < y \le 0 \\ 0.1 + cy & 0 < y \le 2 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the value of c s.t. f is a valid PDF.

Soln: By properties of P.D.F: $\int_{-\infty}^{+\infty} f(y)\,dy = 1$

i.e. $1 = \int_{-\infty}^{-2} \overset{0}{\cancel{f(y)}}\,dy + \int_{-2}^{0} \underbrace{f(y)}_{0.1}\,dy + \int_{0}^{2} \underbrace{f(y)}_{0.1+cy}\,dy + \int_{2}^{\infty} \overset{0}{\cancel{f(y)}}\,dy$

$$= \int_{-2}^{0} (.1)\,dy + \int_{0}^{2} .1+cy\,dy$$

$$= (.1)y\Big|_{y=-2}^{y=0} + \left\{(.1)y + c\frac{y^2}{2}\right\}\Big|_{y=0}^{y=2}$$

$$= (.1)(0-(-2)) + (.1)\cdot 2 + 2c \quad = (.4)+2c$$

$$\Rightarrow \quad c = .3$$

(b) Find the C.D.F $F(y)$.

PDF from (a)

Soln: PDF → CDF, use formula $F(y) = \int_{-\infty}^{y} f(t)\,dt$

Case 1: $y<-2$

$$F(y) = \int_{-\infty}^{y} \overset{0}{\cancel{f(t)}}\,dt = 0$$

Case 2: $-2 \le y < 0$

$$F(y) = \int_{-\infty}^{-2} \overset{0}{\cancel{f(t)}}\,dt + \int_{-2}^{y} \overbrace{f(t)}^{(.1)}\,dt = (.1)(y-(-2))$$

$$= (.1)(y+2)$$

Case 3: $0 \le y < 2$

$$F(y) = \int_{-\infty}^{-2} \overset{0}{\cancel{f(t)}}\,dt + \int_{-2}^{0} \overbrace{f(t)}^{(.1)}\,dt + \int_{0}^{y} \overbrace{f(t)}^{(.1)+(.3)t}\,dt$$

$$= 0 + (.1)\cdot(0-(-2)) + \left(0.1t + (.3)\frac{t^2}{2}\right)\Big|_{t=0}^{t=y}$$

$$= (.2) + 0.1y + 0.3\cdot\frac{y^2}{2}$$

Case 4: $y>2 \Rightarrow F(y) = 1$

Extra Exercise for P2: Calculate $\mathbb{E}[Y]$ and $\mathbb{E}[Y^3]$.

Soln: $\mathbb{E}[Y] = \int_{-\infty}^{+\infty} y f(y) dy$

$$= \int_{-2}^{0} (0.1) y dy + \int_{0}^{2} (0.1 + 0.3y) y dy$$

$$= \cancel{(0.1) \frac{y^2}{2}\Big|_{-2}^{0}} + \cancel{0.1 \frac{y^2}{2}\Big|_{0}^{2}} + 0.3 \cdot \frac{y^3}{3}\Big|_{0}^{2} = 0.8$$

$$\mathbb{E}[Y^3] = \int_{-\infty}^{+\infty} y^3 f(y) dy$$

$$= \int_{-2}^{0} (0.1) y^3 dy + \int_{0}^{2} (0.1 + 0.3y) y^3 dy$$

$$= \cancel{(0.1) \cdot \frac{y^4}{4}\Big|_{-2}^{0}} + \cancel{(0.1) \frac{y^4}{4}\Big|_{-2}^{0}} + (0.3) \frac{y^5}{5}\Big|_{0}^{2} = 1.92 \quad \blacksquare$$

P3. Suppose a soccer player's chest size $\sim N(41.6, 5.0625)$, independent of other players'. We selected 25 soccer players.

Q: Let Y be the number of players having chest size of at least 42.275, calculate $\mathbb{P}(Y=7)$ and $\mathbb{E}[Y]$.

Soln: (Step 1) X = a player's chest size $\sim N(41.6, 5.0625)$.

Calculate p: prob. that {the chest size ≥ 42.275}

$$p = \mathbb{P}(X \geq 42.275)$$

$$= \mathbb{P}\left(\frac{X - 41.6}{\sqrt{5.0625}} \geq \frac{42.275 - 41.6}{\sqrt{5.0625}} \right)$$

$= \mathbb{P}(Z \geq 0.3) \qquad Z = \frac{X - 41.6}{\sqrt{5.0625}} \sim N(0,1)$

$= 1 - \Phi(0.3)$

$= .3821$

(Step 2) $Y \sim Bin(n=25, \ p = .3821)$

$\mathbb{P}(Y=7) = \binom{25}{7} p^7 (1-p)^{25-7} =$ (result)

$\mathbb{E}[Y] = np$ (by table of Distribution)

$= (25)(.3821)$

$= 9.5525$

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