1. $X_1 \sim \exp(\beta_1)$, $X_2 \sim \exp(\beta_2)$, $X_1 \perp X_2$, $k > 0$

$\hookrightarrow$ independent

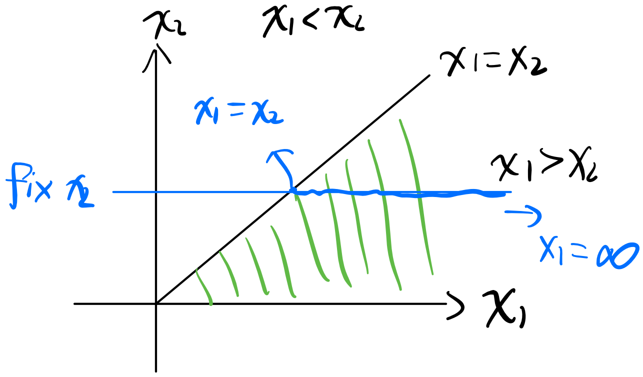
(a) $X_1 \sim \exp(\beta_1) \Rightarrow f_{X_1}(x_1) = \begin{cases} \beta_1 e^{-\beta_1 x_1} & x_1 > 0 \\ 0 & \text{o.w.} \end{cases}$

$X_2 \sim \exp(\beta_2) \Rightarrow f_{X_2}(x_2) = \begin{cases} \beta_2 e^{-\beta_2 x_2} & x_2 > 0 \\ 0 & \text{o.w.} \end{cases}$

Joint density

$$f(x_1, x_2) \underset{\substack{\downarrow \\ \text{independence}}}{=} f_{X_1}(x_1)\, f_{X_2}(x_2) = \begin{cases} \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} & x_1, x_2 > 0 \\ 0 & \text{o.w.} \end{cases}$$

(b) $\mathbb{P}(X_1 > X_2)$



$$= \iint_{x_1 > x_2} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^{+\infty} \int_{x_2}^{+\infty} \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)}\, dx_1\, dx_2$$

$$= \beta_1 \beta_2 \int_0^{+\infty} e^{-\beta_2 x_2} \int_{x_2}^{+\infty} e^{-\beta_1 x_1}\, dx_1\, dx_2$$

$$= \text{(exercise)}$$

$$= \frac{\beta_2}{\beta_1 + \beta_2}$$

2. X = number of toss needed to obtain five heads

$X \sim \text{Negbin}(r=5,\ p=1/30)$

(a) $\mathbb{E}[X] = \dfrac{r}{p} = \dfrac{5}{1/30} = 150$

(b) $\text{Var}(X) = \dfrac{r(1-p)}{p^2} = \dfrac{5 \times 29/30}{(1/30)^2} = 4350$

3. misprints in one page $\sim \text{Poi}(\lambda)$

$p = \mathbb{P}(\text{no misprints}) = e^{-\lambda} \dfrac{\lambda^k}{k!} \Big|_{k=0} = e^{-\lambda}$

$\mathbb{P}(\text{At least one page contains no misprints})$

$= 1 - \mathbb{P}(\underbrace{\text{no pages contains no misprints}})$

or we can say: each pages contain at least one misprint.

$= 1 - (1-p)^n$

$= 1 - (1-e^{-\lambda})^n$

4. X : a discrete unif R.V. w. PMF

$\mathbb{P}(X=k) = 1/6 \quad k = -2, -1, 0, 1, 4, 6$

Let $Y = (X+1)^2$, to find the M.G.F of Y:

X	-2	-1	0	1	4	6
$Y=(X+1)^2$	1	0	1	4	25	36
$\mathbb{P}$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$\Rightarrow$ P.M.F of Y: $\mathbb{P}(Y=k) = \begin{cases} 1/6 & k = 0, 4, 25, 36 \\ 1/3 & k=1 \end{cases}$

$$M_Y(t) = \mathbb{E}[e^{tY}]$$

$$= \sum_k e^{tk}\,\mathbb{P}(Y=k)$$

$$= \frac{1}{6}(1 + e^{4t} + e^{25t} + e^{36t}) + \frac{1}{3}e^{t}$$

Another MTD:

$$M_{(X+1)^2}(t) = \mathbb{E}\left[e^{t(X+1)^2}\right] = \sum_k e^{t(k+1)^2}\,\mathbb{P}(X=k)$$

5. For a R.V. $Z = X+Y$, with

$X \sim Bern(p=1/2)$, Y: $\mathbb{E}[Y] = 10$, $Var(Y) = 2^2 = 4$

(a) $M_Z(t) = M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}]$

$= \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$ b/c X and Y are indep

$= \left(\frac{1}{2} + \frac{1}{2}e^t\right) M(t)$

(b) mean: $\frac{d}{dt} M_Z(t) = \frac{1}{2}e^t M(t) + \left(\frac{1}{2}+\frac{1}{2}e^t\right) M'(t)$

$\boxed{\mathbb{E}[Z]} = \frac{d}{dt} M_Z(t)\Big|_{t=0}$ $(\mathbb{E}[Y] = M'(0) = 10)$

$= \frac{1}{2} M(0) + \left(\frac{1}{2}+\frac{1}{2}\right)\mathbb{E}[Y] \boxed{= 10.5}$

Variance: $\mathbb{E}[Z^2] = \frac{d^2}{dt^2} M_Z(t)\Big|_{t=0}$, where

$\frac{d^2}{dt^2} M_Z(t) = \frac{1}{2}\left(e^t M(t) + e^t M'(t)\right) + \frac{1}{2} e^t M'(t)$

$+ \left(\frac{1}{2} + \frac{1}{2}e^t\right) M''(t)$

$(M''(0) = \mathbb{E}[Y^2] = Var(Y) + (\mathbb{E}Y)^2 = 4 + 10^2 = 104)$

$$\mathbb{E}[Z^2] = \frac{1}{2}(1+10) + \frac{1}{2}\cdot 10 + \left(\frac{1}{2}+\frac{1}{2}\right)\cdot 104$$

$$= 5.5 + 5 + 104 = 114.5$$

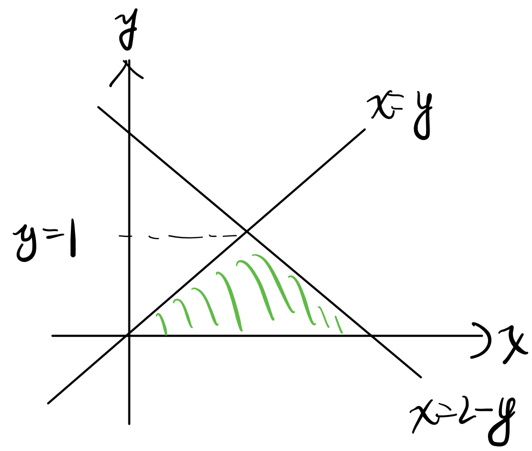
$$\boxed{\mathrm{Var}(Z)} = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$

$$= 114.5 - 10.5^2 \boxed{= 4.25}$$

//

6.

$$f(x,y)=\begin{cases} 3(2-x)y & 0<y<1,\ y<x<2-y \\ 0 & \text{o.w.} \end{cases}$$



(a) $\iint f(x,y)\,dx\,dy$

$$= \int_0^1 \int_y^{2-y} (2-x)dx\, 3y\; dy$$

$$= \int_0^1 \left\{ 2x - \tfrac{1}{2}x^2 \Big|_y^{2-y} \right\} 3y\; dy$$

$$= \int_0^1 \left\{ 2(2-y) - \tfrac{1}{2}(2-y)^2 - 2y + \tfrac{1}{2}y^2 \right\} 3y\,dy$$

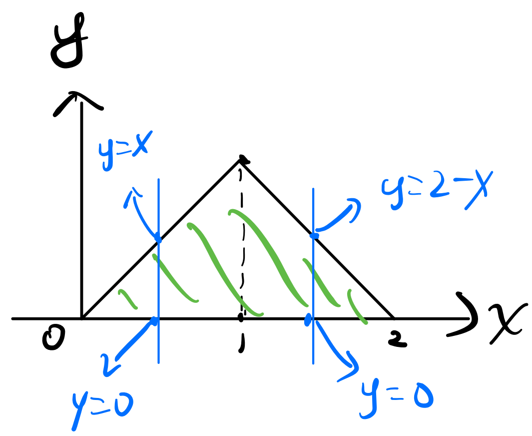
$$= \int_0^1 (2-2y)\,3y\,dy$$

$$= \int_0^1 6y - 6y^2\, dy = 3y^2 - 2y^3 \Big|_0^1 = 1$$

(b) Marginal density of X:

we derive

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$ piecewisely



① $x \in (0,1)$:

$$\int_0^x f(x,y)dy$$

$$= \int_0^x 3(2-x)y\,dy = 3(2-x)\int_0^x y\,dy = \frac{3}{2}(2-x)x^2$$

② $x \in (1,2)$:

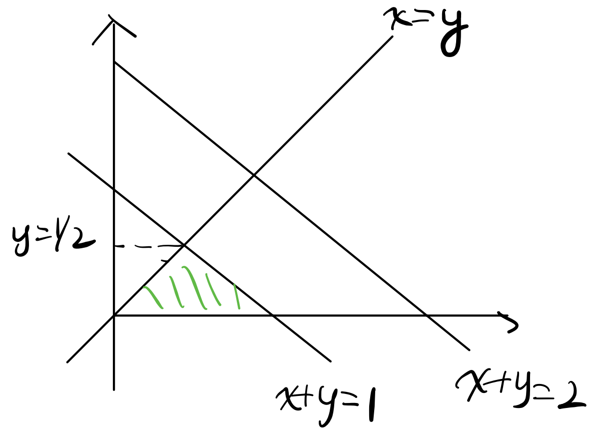
$$\int_0^{2-x} f(x,y)\,dy$$

$$= \int_0^{2-x} 3(2-x)\,y\,dy$$

$$= 3(2-x)\int_0^{2-x} y\,dy = \frac{3}{2}(2-x)^3$$

So $f_X(x) = \begin{cases} \frac{3}{2}(2-x)x^2 & x \in (0,1) \\ \frac{3}{2}(2-x)^3 & x \in (1,2) \\ 0 & \text{o.w.} \end{cases}$

(c) $\displaystyle\iint_{x+y\le 1} f(x,y)\,dx\,dy$



$$= \int_0^{1/2}\int_y^{1-y} (2-x)\,dx\,(3y)\,dy$$

$$= 3\int_0^{1/2}\left\{2x-\frac{1}{2}x^2\Big|_{x=y}^{x=1-y}\right\} y\,dy$$

$$= 3\int_0^{1/2}\left(2(1-y)-\frac{1}{2}(1-y)^2-2y+\frac{1}{2}y^2\right) y\,dy$$

$$= 3\int_0^{1/2}\left(\frac{3}{2}-3y\right) y\,dy$$

$$= 3\left(\frac{3}{4}y^2-y^3\right)\Big|_{y=0}^{y=1/2}$$

$$= \frac{3}{16}$$

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7. Let A, B, C represent each school.

X = the score of one student

$\mathbb{E}[X|A] = 80$

$\mathbb{E}[X|B] = 76$

$\mathbb{E}[X|C] = 84$

by Law of total Expectation:

$\mathbb{E}[X]$

$= \mathbb{E}[X|A]\,\mathbb{P}(A) + \mathbb{E}[X|B]\,\mathbb{P}(B) + \mathbb{E}[X|C]\,\mathbb{P}(C)$

$= 80(.2) + 76(.3) + 84(.5)$

$= 80.8$