Expectations

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Expected values

See exercise $3.30, 3.32, 3.37$

1. Exercise 3.30: For X be the number of missing shots

   - $P(X=0) = P(\text{success in the first shot}) = 1/2$
   - $P(X=1) = P(\text{miss the first shot, but make the second shot}) = (1-1/2)(1/3) = 1/6$
   - $P(X=2) = P(\text{miss the first two shots, but make the third shot}) = (1-1/2)(1-1/3)(1/4) = 1/12$
   - $P(X=3) = P(\text{miss the first three shots, but make the four shot}) = (1-1/2)(1-1/3)(1-1/4)(1/5) = 1/20$
   - $P(X=4) = P(\text{miss all the shot}) = (1-1/2)(1-1/3)(1-1/4)(1-1/5) = 1/5$

   Thus the p.m.f of X is

X	0	1	2	3	4
P	$1/2$	$1/6$	$1/12$	$1/20$	$1/5$

   and the expected value is

   $$E[X] = 0 \times \frac{1}{2} + 1 \times \frac{1}{6} + 2 \times \frac{1}{12} + 3 \times \frac{1}{20} + 4 \times \frac{1}{5} = \frac{77}{60}$$

2. Exercise 3.32:

   (a) We have

   $$P(X > 10) = \int_{10}^{\infty} \frac{1}{2} x^{-3/2}\, dx = -x^{-1/2}\Big|_{10}^{\infty} = \frac{1}{\sqrt{10}}$$

   (b) For $t < 1$, we have $F_X(t) = 0$. For $t \geq 1$,

   $$F_X(t) = P(X \leq t) = \int_1^t \frac{1}{2} x^{-3/2}\, dx = 1 - \frac{1}{\sqrt{t}}$$

   Thus,

   $$F_x(t) = \begin{cases} 0 & t \leq 1 \\ 1 - \frac{1}{\sqrt{t}} & t \geq 1 \end{cases}$$

   (c) We have

   $$E[X] = \int_1^{\infty} \frac{1}{2} x \cdot x^{-3/2} dx = \infty$$

   (d) We have

   $$E[X^{1/4}] = \frac{1}{2} \int_1^{\infty} x^{1/4} \cdot x^{-3/2} dx = \frac{1}{2} \int_1^{\infty} x^{-5/4} dx = (-2) x^{-1/4} = 2\Big|_1^{\infty} =$$

3. Exercise 3.37: Cumulative Distribution Function of X is:

$$F(x) = \begin{cases} \frac{x}{1+x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

   (a) The density of X is

$$f(x) = \begin{cases} \frac{1}{(1+x)^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (b) The probability is

$$P(2 < X < 3) = F(3) - F(2) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

   (c) Given the probability density in part a, we have

$$E[(1+X)^2 e^{-2X}] = \int_0^\infty (1+x)^2 e^{-2x}(1+x)^{-2} dx = \int_0^\infty e^{-2x} dx = \frac{1}{2}$$

Variance

See exercise 3.15, 3.31

1. Exercise 3.31: Note that $\int_1^\infty x^{-4} dx = 1/3$

   (a) By properties of p.d.f, $\int_1^\infty cx^{-4} dx = 1$. It implies that $c = 3$.
   (b) We have $P(0.5 < X < 1) = 0$
   (c) We have $P(0.5 < X < 2) = \int_1^2 3x^{-4}\, dx = 7/8$
   (d) We have $P(2 < X < 4) = \int_2^4 3x^{-4}\, dx = 7/64$
   (e) For $x > 1$: $\int_1^x 3t^{-4} dx = 1 - x^{-3}$, thus the c.d.f of X should be

$$F(x) = \begin{cases} 0 & x \leq 1 \\ 1 - x^{-3} & x > 1 \end{cases}$$

   (f) We have

$$E[X] = \int_1^\infty 3x \times x^{-4}\, dx = \int_1^\infty 3x^{-3}\, dx = (-3/2)x^{-2}|_1^\infty = 3/2$$
$$E[X^2] = \int_1^\infty 3x^2 \times x^{-4}\, dx = \int_1^\infty 3x^{-2}\, dx = (-3)x^{-1}|_1^\infty = 3$$
$$Var(X) = E[X^2] - (E[X])^2 = 3 - (3/2)^2 = 3/4$$

   (g) We have

$$E[5X^2 + 3X] = 5E[X^2] + 3E[X] = 5 \times 3 + 3 \times (3/2) = 19.5$$

   (h) Note that when $\alpha \geq -1$, we have

$$\int_1^\infty x^- 1 dx = \log x|_1^\infty = \infty$$

   and

$$\int_1^\infty x^\alpha dx = \frac{1}{\alpha + 1} x^{\alpha+1}|_1^\infty = \infty$$

   for $\alpha > -1$ (so $\alpha + 1 > 0$), therefore, we have

$$E[X^n] = \begin{cases} 3/2 & n = 1 \\ 3 & n = 2 \\ \infty & n \geq 3 \end{cases} = \begin{cases} \frac{3}{3-n} & n \leq 2 \\ \infty & n \geq 3 \end{cases}$$

2. Exercise 3.15: $E[X] = 3$, $\text{Var}(X) = 4$.

   (a) $E[3X + 2] = 3E[X] + 2 = 3 \times 3 + 2 = 11$
   (b) $E[X^2] = Var(X) + (E[X])^2 = 4 + 3^2 = 13$
   (c) $E[(2X+3)^2] = E[4X^2 + 12X + 9] = 4E[X^2] + 12E[X] + 9 = 4 \times 13 + 12 \times 3 + 9 = 97$
   (d) $\text{Var}(4X - 2) = 16\text{Var}(X) = 16 \times 4 = 64$

Moment Generating Function

See exercise $5.13, 5.15$

1. Exercise 5.13:
$$M_Y(t) = \frac{1}{2} + \frac{1}{16}e^{-34t} + \frac{1}{8}e^{-5t} + \frac{1}{100}e^{3t} + \frac{121}{400}e^{100t}$$

   (a)
$$M'_Y(t) = \frac{-34}{16}e^{-34t} + \frac{-5}{8}e^{-5t} + \frac{3}{100}e^{3t} + \frac{121*100}{400}e^{400t}$$
   So
$$E[Y] = M'_Y(0) = -\frac{34}{16} - \frac{5}{8} + \frac{3}{100} + \frac{12100}{400} = 27.53$$

   (b) Recover from MGF to p.m.f, we have

X	0	-34	-5	3	100
P	1/2	1/16	1/8	1/100	121/400

   By definition of expected value, we have $E[X] = (-34)\cdot\frac{1}{16} + (-5)\cdot\frac{1}{8} + 3\cdot\frac{1}{100} + 100\cdot\frac{121}{400} = 27.53$

2. Exercise 5.15:

   (a)
$$M_X(t) = \frac{1}{10}e^{-2t} + \frac{1}{5}e^{-t} + \frac{3}{10} + \frac{2}{5}e^{t}$$

   (b) Given information on X, we have

X	-2	-1	0	1
$Y = \|X+1\|$	1	0	1	2
P	1/10	1/5	3/10	2/5

   so p.m.f of Y is

Y	0	1	2
P	1/5	2/5	2/5

Transformation of Random variables

1. Exercise 5.7: Let $X \sim \exp(\lambda)$, then
$$\begin{aligned} P(Y \le y) &= P(\log(X) \le y) \\ &= P(X \le e^y) \\ &= 1 - e^{-\lambda x}\Big|_{x=e^y} = 1 - \exp\{-\lambda e^y\} \end{aligned}$$
   Thus density function is $f_Y(y) = \lambda e^y \exp\{-\lambda e^y\}$

2. Exercise 5.8: Let $X \sim \text{Unif}[-1, 2]$, then for $y > 0$, let F be the cumulative density function of X, we have
$$\begin{aligned} P(Y \le y) &= P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= F(\sqrt{y}) - F(-\sqrt{y}) \end{aligned}$$

Thus density function of Y is

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}F_Y(y) \\
&= \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) \\
&= \begin{cases} \frac{1}{3\sqrt{y}} & 0 < y < 1 \\ \frac{1}{6\sqrt{y}} & 1 < y < 4 \\ 0 & \text{Otherwise} \end{cases}
\end{aligned}
$$

Where f_X is the density of X, which is

$$
f_X(x) = \begin{cases} \frac{1}{3} & -1 < x < 2 \\ 0 & \text{o.w.} \end{cases}
$$

The last step comes from these three cases:

- When $0 < \sqrt{y} < 1$ $(0 < y < 1)$, then $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{3}$.
- When $1 < \sqrt{y} < 2$ $(1 < y < 4)$, then $f_X(\sqrt{y}) = \frac{1}{3}$ and $f_X(-\sqrt{y}) = 0$.
- Otherwise, $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = 0$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu