Conditional Probability

<u>Def</u>: $\mathbb{P}(A|B)$ — Conditional probability of event A given event B has occurred,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} \quad \text{for } \mathbb{P}(B) > 0.$$

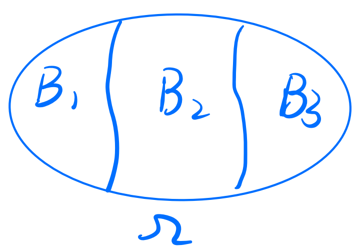
1. Law of Total Probability

<u>Def</u>: $\{B_1, B_2, \cdots, B_n\}$ is a partition of Ω if

$B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ $\Rightarrow$ make up Ω

$B_i \cap B_j = \phi$ for $i \neq j$ $\Rightarrow$ Pairwise disjoint

For example: $n=3$

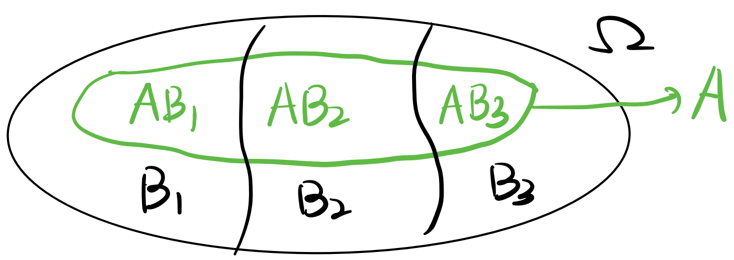


$\{B_1, B_2, B_3\}$: partition of Ω

<u>Theorem 1</u>: Let $\{B_1, B_2, B_3\}$ be a partition of Ω,

then $\mathbb{P}(A) = \sum_{i=1}^{3} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$ — Law of total Prob.

<u>Pf</u>:



$\{B_1, B_2, B_3\}$ forms a partition of Ω

$\{AB_1, AB_2, AB_3\}$ forms a partition of A $\Rightarrow$ $A = (AB_1) \cup (AB_2) \cup (AB_3)$

and they are pairwise disjoint

So $\mathbb{P}(A) = \mathbb{P}(AB_1) + \mathbb{P}(AB_2) + \mathbb{P}(AB_3)$

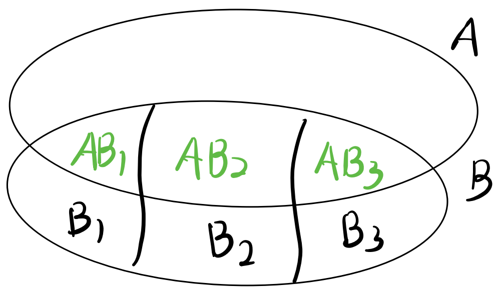
$$= \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3)$$

$$= \sum_{i=1}^{3} \mathbb{P}(A|B_i)\mathbb{P}(B_i) \qquad \square$$

<u>Theorem 2</u>: Let $\{B_1, B_2, B_3\}$ be a partition of $B \subseteq \Omega$,

then $\mathbb{P}(A|B) = \sum_{i=1}^{3} \mathbb{P}(A|B_i)\mathbb{P}(B_i|B)$ — HW2 P6

<u>Pf</u>:



$\{B_1, B_2, B_3\}$ forms a partition of B

$\{AB_1, AB_2, AB_3\}$ forms a partition of $A \cap B$

$\Rightarrow \quad A \cap B = (AB_1) \cup (AB_2) \cup (AB_3)$

and they pairwise disjoint

So, $\mathbb{P}(A \cap B) = \mathbb{P}(AB_1) + \mathbb{P}(AB_2) + \mathbb{P}(AB_3)$

$$= \sum_{i=1}^{3} \mathbb{P}(AB_i) = \sum_{i=1}^{3} \mathbb{P}(A|B_i)\mathbb{P}(B_i) \qquad (*)$$

Note: $B_i \subseteq B$, implies $B_i = B_i \cap B$, and

$$\mathbb{P}(B_i|B) = \frac{\mathbb{P}(B_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

$$= \frac{1}{\mathbb{P}(B)} \sum_{i=1}^{3} \mathbb{P}(A\cap B_i)$$

$$= \frac{1}{\mathbb{P}(B)} \sum_{i=1}^{3} \mathbb{P}(A|B_i)\, \mathbb{P}(B_i) \qquad (*)$$

$$= \sum_{i=1}^{3} \mathbb{P}(A|B_i) \underbrace{\frac{\mathbb{P}(B_i)}{\mathbb{P}(B)}}_{\mathbb{P}(B_i|B)} = \sum_{i=1}^{3} \mathbb{P}(A|B_i)\, \mathbb{P}(B_i|B).$$

2. Baye's Theorem

$\{B_1, B_2, \cdots, B_n\}$: partition of Ω.

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A\cap B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k|A)\,\mathbb{P}(A)}{\mathbb{P}(A)}$$

where $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)$ — Law of Total Prob.

Example: 3 fair dice $\begin{cases} 4 \text{ sides } (1,2,3,4) \\ 6 \text{ sides } (1,2,\cdots,6) \\ 12 \text{ sides } (1,2,\cdots,12) \end{cases}$

pick one dice at random, roll it twice.

$B_4 = \{$4-sided dice is chosen$\}$

$B_6 = \{$6-sided dice is chosen$\}$

$B_{12} = \{$12-sided dice is chosen$\}$

$\mathbb{P}(B_4) = \mathbb{P}(B_6) = \mathbb{P}(B_{12}) = 1/3$

$A_1 = \{$The first roll is 3$\}$, $A_2 = \{$The second roll is 4$\}$

(a) Calculate $\mathbb{P}(B_6|A)$, where $A_1 \cap A_2$.

By Law of total probability:

$$\mathbb{P}(A) = \underbrace{\mathbb{P}(A|B_4)}_{(1/4)^2}\underbrace{\mathbb{P}(B_4)}_{1/3} + \underbrace{\mathbb{P}(A|B_6)}_{(1/6)^2}\underbrace{\mathbb{P}(B_6)}_{1/3} + \underbrace{\mathbb{P}(A|B_{12})}_{(1/12)^2}\underbrace{\mathbb{P}(B_{12})}_{1/3}$$

$$\mathbb{P}(B_6|A) = \frac{\mathbb{P}(A\cap B_6)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_6)\,\mathbb{P}(B_6)}{\mathbb{P}(A)} \quad \text{(by Bayes thm).}$$

(b) Are A_1 and A_2 independent? Why? No.

Def: Two events $A, B \subset \Omega$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Pf: by Law of total prob:

$$\mathbb{P}(A) = \frac{1}{3}\left(\frac{1}{16} + \frac{1}{36} + \frac{1}{144}\right) = \frac{7}{216}$$

$$\mathbb{P}(A_1) = \mathbb{P}(A_1 \mid B_4)\,\mathbb{P}(B_4) + \mathbb{P}(A_1 \mid B_6)\,\mathbb{P}(B_6) + \mathbb{P}(A_1 \mid B_{12})\,\mathbb{P}(B_{12})$$

$$= \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{3} + \frac{1}{12}\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{12}\right) = \frac{1}{6}$$

Similarly, $\mathbb{P}(A_2) = \frac{1}{6}$, so $\mathbb{P}(A_1 \cap A_2) \neq \mathbb{P}(A_1)\,\mathbb{P}(A_2)$

Thus, A_1 and A_2: NOT indep. 证

Remark: A_1 and A_2 are not independent, but they are conditionally independent given B_4, B_6 and B_{12}, i.e.

$$\mathbb{P}(A_1 \cap A_2 \mid B_4) = \mathbb{P}(A_1 \mid B_4)\cdot\mathbb{P}(A_2 \mid B_4) = (1/4)^2$$

$$\mathbb{P}(A_1 \cap A_2 \mid B_6) = \mathbb{P}(A_1 \mid B_6)\cdot\mathbb{P}(A_2 \mid B_6) = (1/6)^2$$

$$\mathbb{P}(A_1 \cap A_2 \mid B_{12}) = \mathbb{P}(A_1 \mid B_{12})\cdot\mathbb{P}(A_2 \mid B_{12}) = (1/12)^2 \quad //$$

Some Examples for Conditional Prob / Independence / Random Variables:

Ex 1. 3 Jurors & 1 Defendant

Defendant $\begin{cases} G = \{\text{Guilty}\} & \text{w.p. } 70\% \\ G^c = \{\text{Innocent}\} & \text{w.p. } 30\% \end{cases}$

- Given G (he is guilty):

  Each juror declares guilty w.p. .7, independently

- Given G^c (he is innocent):

  Each juror declares guilty w.p. .2, independently.

$A_i = \{\text{Juror } i \text{ declares guilty}\}$ $i = 1, 2, 3$

Remark: Given the information, $\{A_1, A_2, A_3\}$ are conditionally independent given G or G^c, but it doesn't mean they are independent.

(a) Check that A_1, A_2, A_3 are in fact NOT independent.

Soln: By Law of Total Prob:

$\mathbb{P}(A_1) = \mathbb{P}(A_1|G)\,\mathbb{P}(G) + \mathbb{P}(A_1|G^c)\,\mathbb{P}(G^c)$

$= (.7)(.7) + (.2)(.3) = .49 + .06 = .55$

Similarly, $\mathbb{P}(A_2) = \mathbb{P}(A_3) = .55$

$\mathbb{P}(A_1A_2A_3)$

$= \mathbb{P}(A_1A_2A_3|G)\,\overbrace{\mathbb{P}(G)}^{.7} + \mathbb{P}(A_1A_2A_3|G^c)\,\overbrace{\mathbb{P}(G^c)}^{.3}$

$$= (.7)\,\underbrace{\mathbb{P}(A_1|G)}_{.7}\,\mathbb{P}(A_2|G)\,\mathbb{P}(A_3|G) + (.3)\,\underbrace{\mathbb{P}(A_1|G^c)}_{.2}\,\mathbb{P}(A_2|G^c)\,\mathbb{P}(A_3|G^c)$$

$$= (.7)^3(.7) + (.2)^3(.3)$$

(conditionally Independent)

Thus, $\mathbb{P}(A_1A_2A_3) \doteq .2425 \neq (.55)^3 = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$.

(b) What is the prob. that Juror 3 declares guilty given the other two declare?

Soln: $\mathbb{P}(A_3|A_1A_2) = \dfrac{\mathbb{P}(A_1A_2A_3)}{\mathbb{P}(A_1A_2)} = \dfrac{.2425}{.355}$

$$\mathbb{P}(A_1A_2) = \mathbb{P}(A_1A_2|G)\,\mathbb{P}(G) + \mathbb{P}(A_1A_2|G^c)\,\mathbb{P}(G^c)$$

$$= (.7)^2(.7) + (.2)^2(.3) = .355$$

(c) What is the prob. that exactly TWO of them voted Guilty?

Soln: $\mathbb{P}(\text{Exactly TWO voted guilty})$

$$= \mathbb{P}(A_1^cA_2A_3) + \mathbb{P}(A_1A_2^cA_3) + \mathbb{P}(A_1A_2A_3^c)$$

where $\mathbb{P}(A_1^cA_2A_3)$

$$= \underbrace{\mathbb{P}(A_1^cA_2A_3|G)}_{(.3)(.7)(.7)}\,\mathbb{P}(G) + \underbrace{\mathbb{P}(A_1^cA_2A_3|G^c)}_{(.8)(.2)(.2)}\,\mathbb{P}(G) = .1125$$

(c.I.)

Similarly, $\mathbb{P}(A_1 A_2^c A_3) = \mathbb{P}(A_1 A_2 A_3^c) = .1125$ $\boxtimes$

(d) Calculate the prob that the defendant is guilty given juror 1 declares guilty and juror 3 declares non-guilty?

Soln: $\mathbb{P}(G \mid A_1 A_3^c) = \dfrac{\mathbb{P}(A_1 A_3^c \mid G)\,\mathbb{P}(G)}{\mathbb{P}(A_1 A_3^c)}$ (Bayes Thm)

where $\mathbb{P}(A_1 A_3^c)$

$$= \underbrace{\mathbb{P}(A_1 A_3^c \mid G)}_{(.7)(.3)}\,\underbrace{\mathbb{P}(G)}_{(.7)} + \underbrace{\mathbb{P}(A_1 A_3^c \mid G^c)}_{(.2)(.8)}\,\underbrace{\mathbb{P}(G^c)}_{(.3)}$$ $\boxtimes$

(e) Calculate the prob. that at least one juror voted guilty?

Soln: $\mathbb{P}($At least one juror voted guilty$)$

$= 1 - \mathbb{P}($No jurors declared guilty$)$

$= 1 - \mathbb{P}(A_1^c A_2^c A_3^c)$

$=$ exercise $\begin{cases} \text{Law of total Prob.} \\ \text{Cond. Indep} \end{cases}$ $\boxtimes$