

Counting Methods

1. Basic Principle of Counting

• Suppose there are k jobs to be done.

{ first job: be done in $\underline{n_1}$ ways
second job: be done in $\underline{n_2}$ ways
⋮
k-th job: be done in $\underline{n_k}$ ways

Total number of ways doing k jobs is

$$n_1 \times n_2 \times \dots \times n_k = \prod_{i=1}^k n_i \quad (\text{Multiplication Rules})$$

Moreover, suppose that A_1, A_2, \dots, A_k are finite sets.

$\#A_i$ (the number of elements of A_i) = $n_i \quad i=1, 2, \dots, k$

Cartesian product:

$$A_1 \times A_2 \times \dots \times A_k = \{(x_1, x_2, \dots, x_k) : x_i \in A_i, i=1, 2, \dots, k\}$$

by multiplication rule: { $x_1 : n_1$ possibilities
 $x_2 : n_2$ ⋮
⋮
 $x_k : n_k$ ⋯

$$\#(A_1 \times A_2 \times \dots \times A_k) = n_1 \cdot n_2 \cdot \dots \cdot n_k = \prod_{i=1}^k n_i$$

//

Example: Consider an experiment where we roll a fair 4-sided die $(1, 2, 3, 4)$ for three times.

Q: How many possible outcomes in total?

Step 1: Write down the sample space

$$\Omega = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \{1, 2, 3, 4\}\}$$

$$= A_1 \times A_2 \times A_3$$

$A_i = \{1, 2, 3, 4\}$ — the outcome of the i -th roll

Step 2: $\#\Omega = \#A_1 \times \#A_2 \times \#A_3 = 4^3 = 64$ //

2. Addition Principle

Two event A and B . If $A \cap B = \emptyset$ (disjoint),

$$\#(A \cup B) = (\#A) + (\#B).$$

Moreover, $\Omega = A \cup A^c$, $\#\Omega = \#A + \#A^c$

\Rightarrow Given $\#\Omega$ and $\#A^c$, then $\#A = \#\Omega - \#A^c$.

Example: In an experiment: 10 participants.

Each flips a coin and rolls a die.

(a) Describe the sample space and calculate the # of possible outcomes?

Soln: Set A — the outcome of a single participants.

$$A = \{(a_1, a_2): a_1 \in \{H, T\}, a_2 \in \{1, 2, \dots, 6\}\}$$
$$= \{H, T\} \times \{1, 2, \dots, 6\}$$

$$\#A = 12$$

Ω — the outcome of all 10 participants

$$\Omega = A^{10} \quad \# \Omega = (\#A)^{10} = 12^{10}$$

(b) $B = \{\text{At least one person rolled a 5}\}$. $\#B = ?$

Soln: $B^c = \{\text{None of them gets a 5}\}$.

$$B^c = (\{H, T\} \times \{1, 2, 3, 4, 6\})^{10}$$

$$\#B^c = (2 \times 5)^{10} = 10^{10} \Rightarrow \#B = 12^{10} - 10^{10} \quad //$$

3. Random Sampling (Chapter 1.2 in the textbook)

Suppose an urn has n balls (numbered $1, 2, \dots, n$),
retrieve the ball for k times. Let $S = \{1, 2, \dots, n\}$.

① Sampling with replacement: retrieve a ball, record its number, and put it back into the urn.

(possible for the ball to be retrieved again)

$$\text{Sample space } \Omega = S^k = \{1, 2, \dots, n\}^k$$

Number of outcomes: $\# \Omega = n^k$

② Sampling without replacement: retrieve a ball, record its number, and put it aside.

(the same ball cannot be drawn twice)

Example: 4 balls (1, 2, 3, 4). Sample 2 balls w/o replacement.

Two ways to see this example:

(1) Order matters: (1, 2) & (2, 1) — different outcomes

$$\Omega = \{ (s_1, s_2) : \text{each } s_1, s_2 \in \{1, 2, 3, 4\}, \\ s_1, s_2 \text{ — distinct} \}$$

$$= \{ (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), \\ (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), \\ (4, 2), (4, 3) \}$$

$$\Rightarrow \# \Omega = 12$$

(2) Order doesn't matter: (1, 2) & (2, 1) — same outcomes

$$\Omega = \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \\ \{3, 4\} \}$$

$$\Rightarrow \# \Omega = 6$$

In general, n balls $(1, 2, \dots, n)$, sample k times

(1) Order matters

Sample space:

$$\Omega = \{(s_1, s_2, \dots, s_k) : \text{each } s_i \in S \text{ but } s_i \neq s_j \text{ for } i \neq j\}$$

$$\#\Omega = n \cdot (n-1) \cdot \dots \cdot (n-k+1) \quad (\text{Check by multiplication rule})$$

$$= \frac{n!}{(n-k)!} \quad (n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n)$$

(2) Order doesn't matter

$$\text{Sample space: } \Omega = \{A : A \subseteq S, \#A = k\}$$

$$\#\Omega = \frac{n!}{k! (n-k)!} = \binom{n}{k}$$

Extra Exercise (NOT covered in the section):

e.g. 1: Find the number of six-letter words

(not need to be meaningful)

constructed from the letters B, A, D, G, E, R?

Soln: It is equivalent to retrieve 6 letter w/o replacement, order matters: $6! = 720$ possibilities.

e.g. 2: Find the number of 5-letter words

from the letters A, P, P, L, E? (no need to be meaningful)
↓

(MTD 1) Order 5 letters: $5! = 120$ ways,

but two Ps can be in two different orders,

we counted each word twice, so the number of different words is $\frac{120}{2} = 60$.

(MTD 2) 5-letter words has the format

— — — — —

Step 1: Choose the position of two Ps: $\binom{5}{2} = 10$ ways

Step 2: Arrange the remaining three words: $3! = 6$ ways

By multiplication rules:

$10 \cdot 6 = 60$ different words.