Distributions for Continuous Random Variables

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Uniform Distribution

1. Exercise 1.9: To get the shorter piece less than 1/5 of the original, the distance between the location we choose and left/right end of the stick should be lower than 1/5 of the origin, thus the probability is 2/5.

2. Exercise 1.11: $p = \frac{2^2\pi}{20^2} = \frac{\pi}{100}$

3. Exercise 3.4: $X \sim Unif[4, 10]$, then the p.d.f of X is

$$f(x) = \begin{cases} 1/6 & 4 < x < 10 \\ 0 & \text{Otherwise} \end{cases}$$

   (a) $P(X < 6) = \int_4^6 1/6\, dx = 1/3$
   (b) $P(|X - 7| > 1) = P(X < 6) + P(X > 8) = 2/3$
   (c) For $4 < t < 6$, we have

$$P(X < t|X < 6) = \frac{P(X < t)}{P(X < 6)} = \frac{t-4}{6} \cdot 3 = \frac{t-4}{2}$$

4. Exercise 3.20: For $y \in (0, c)$, we have $P(Y \leq y) = P(c - X \leq y) = P(X \geq c - y) = \frac{c-(c-y)}{c} = \frac{c}{y}$. We can show that Y follows from Uniform distribution by taking the derivative for the density.

5. Exercise 3.41: Note that given $Y = k$, $X \sim Unif(0, k]$ for $k = 1, 2, 3, 4, 5, 6$. Given $s \in (3, 4)$, we have $P(X \leq s|Y = k) = 1$ for $k = 1, 2, 3$ since $X \leq k < s$ in this case. For $k = 4, 5, 6$, $s < k$ implies $P(X \leq s|Y = k) = s/k$. Thus by law of total probability, we have

$$P(X \leq s) = \sum_{k=1}^{6} P(X \leq s|Y = k)P(Y = k) = \sum_{k=1}^{3} 1 \cdot \frac{1}{6} + \sum_{k=4}^{6} \frac{s}{k} \cdot \frac{1}{6} = \frac{1}{2} + \frac{s}{6}(\frac{1}{4} + \frac{1}{5} + \frac{1}{6}) = \frac{1}{2} + \frac{37s}{360}$$

   Thus $F(s) = \frac{1}{2} + \frac{37s}{360}$ and $f(s) = 37/360$

Exponential Distribution

1. Exercise 4.13: Since $T \sim \exp(1/3)$, the probability density function of T is

$$f_T(t) = \begin{cases} \frac{1}{3}e^{-\frac{1}{3}t} & t > 0 \\ 0 & \text{Otherwise} \end{cases}$$

   It is easy to derive that its CDF is $F_T(t) = 1 - e^{-t/3}$ for $t > 0$ and zero otherwise. Then we can compute the following quantity:

   - $P(T > 3) = 1 - F_T(1/3) = e^{-1} = .3679$
   - $P(1 \leq T < 8) = F_T(8) - F_T(1) = e^{-1/3} - e^{-8/3} = .6470$
   - By memoryless properties: $P(T > 4|T > 1) = P(T > 3) = .3679$

2. Exercise 4.14: Denote the lifetime of lightbulb by T, since T is exponential distributed with expected value 1000, we have $T \sim \exp(1/1000)$. Then, the CDF of T is $F_T(t) = 1 - e^{t/1000}$.

   - $P(T > 2000) = 1 - F_T(2000) = e^{-\frac{1}{2}} = .6065$.
   - By memoryless properties: $P(T > 2000|T > 500) = P(T > 1500) = e^{-3/2} = .2231$.

Normal Distribution

1. Exercise 3.17: $X \sim N(-2, 7)$ and let Φ be the distribution function of $Z \sim N(0, 1)$

   (a)
$$P(X > 3.5) = P\left(\frac{X+2}{\sqrt{7}} > \frac{3.5+2}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{5.5}{\sqrt{7}}\right) = .0188$$

   (b)
$$P(-2.1 < X < -1.9) = P\left(\frac{-2.1+2}{\sqrt{7}} < \frac{X+2}{\sqrt{7}} < \frac{-1.9+2}{\sqrt{7}}\right) = \Phi\left(\frac{-.1}{\sqrt{7}}\right) - \Phi\left(\frac{.1}{\sqrt{7}}\right) = .032$$

   Note that $\Phi(-z) = 1 - \Phi(z)$ and $\Phi(z) - \Phi(-z) = \Phi(z) - (1 - \Phi(z)) = 2\Phi(z) - 1$.

   (c)
$$P(X < 2) = P\left(\frac{X+2}{\sqrt{7}}2\frac{2+2}{\sqrt{7}}\right) = \Phi\left(\frac{4}{\sqrt{7}}\right) = .9345$$

   (d)
$$P(X < -10) = P\left(\frac{X+2}{\sqrt{7}} < \frac{-10+2}{\sqrt{7}}\right) = \Phi\left(\frac{-8}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{8}{\sqrt{7}}\right) = .0013$$

   (e)
$$P(X > 4) = P\left(\frac{X+2}{\sqrt{7}} > \frac{4+2}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{6}{\sqrt{7}}\right) = .0116$$

2. Exercise 3.18: $X \sim N(3, 4)$ and let Φ be the distribution function (C.D.F) of $Z = \frac{X-3}{2} \sim N(0, 1)$:

   (a) We have
$$\begin{aligned} P(2 < X < 6) &= P\left(\frac{2-3}{2} < \frac{X-3}{2} < \frac{6-3}{2}\right) \\ &= \Phi(1.5) - \Phi(-.5) \\ &= \Phi(1.5) - 1 + \Phi(.5) \\ &= .6247 \end{aligned}$$

   (b) We have
$$P(X > c) = P\left(\frac{X-3}{2} > \frac{c-3}{2}\right) = 1 - \Phi\left(\frac{c-3}{2}\right) = .33$$

   By normal table, we solved for $c = 3.88$.

Note: This study guide is used for Botao Jin's sections only. Comments, bug reports: b_jin@ucsb.edu